Hardy{\textendash}Rellich and second order Poincar{\'{e}} identities on the hyperbolic space via Bessel pairs

HARDY-RELLICH AND SECOND ORDER POINCARÉ IDENTITIES ON THE HYPERBOLIC SPACE VIA BESSEL PAIRS

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Abstract. We prove a family of Hardy-Rellich and Poincaré identities and inequalities on the hyperbolic space having, as particular cases, improved Hardy-Rellich, Rellich and second order Poincaré inequalities. All remainder terms provided improve those already known in literature, and all identities hold with same constants for radial operators also. Furthermore, as applications of the main results, second order versions of the uncertainty principle on the hyperbolic space are derived.

1. Introduction

Let $\mathbb{H}^N$ with $N \geq 2$ denote the hyperbolic space, namely the most important example of Cartan-Hadamard manifold (i.e., a manifold which is complete, simply-connected, and has everywhere non-positive sectional curvature) and let $\lambda_1(\mathbb{H}^N)$ denote the bottom of the spectrum of $-\Delta_{\mathbb{H}^N}$ which is explicitly given by

$$\lambda_1(\mathbb{H}^N) = \inf_{u \in C_c^\infty(\mathbb{H}^N)\setminus\{0\}} \frac{\int_{\mathbb{H}^N} |\nabla_{\mathbb{H}^N} u|^2 \, \mathrm{d}v_{\mathbb{H}^N}}{\int_{\mathbb{H}^N} u^2 \, \mathrm{d}v_{\mathbb{H}^N}} = \left(\frac{N-1}{2}\right)^2 . \tag{1.1}$$

The present paper takes its origin from the following family of Hardy-Poincaré inequalities recently proved in [5]: for all $N-2 \leq \lambda \leq \lambda_1(\mathbb{H}^N)$ and all $u \in C_c^\infty(\mathbb{H}^N \setminus \{x_0\})$ there holds

$$\begin{aligned} \int_{\mathbb{H}^N} |\nabla_{\mathbb{H}^N} u|^2 \mathrm{d}v_{\mathbb{H}^N} &\geq \lambda \int_{\mathbb{H}^N} u^2 \mathrm{d}v_{\mathbb{H}^N} + h_N^2(\lambda) \int_{\mathbb{H}^N} \frac{u^2}{r^2} \mathrm{d}v_{\mathbb{H}^N} \\ &+ \left[\left(\frac{N-2}{2}\right)^2 - h_N^2(\lambda)\right] \int_{\mathbb{H}^N} \frac{u^2}{\sinh^2 r} \mathrm{d}v_{\mathbb{H}^N} + \gamma_N(\lambda)\, h_N(\lambda) \int_{\mathbb{H}^N} \frac{r\coth r - 1}{r^2} u^2 \mathrm{d}v_{\mathbb{H}^N} \end{aligned} \tag{1.2}$$

where $\gamma_N(\lambda) := \sqrt{(N-1)^2 - 4\lambda}$, $h_N(\lambda) := \frac{\gamma_N(\lambda)+1}{2}$ and $r := \mathrm{d}(x, x_0)$ is the geodesic distance from a fixed pole $x_0 \in \mathbb{H}^N$. We notice that the function $\frac{r\coth r - 1}{r^2}$ is positive while the map $[N-2, \lambda_1(\mathbb{H}^N)] \ni \lambda \mapsto h_N(\lambda)$ is decreasing. Furthermore, for $N \geq 3$, there holds $\frac{1}{4} \leq h_N^2(\lambda) \leq \left(\frac{N-2}{2}\right)^2$ and, for all λ, one locally recovers the optimal Hardy weight: $\left(\frac{N-2}{2}\right)^2 \frac{1}{r^2}$. Besides, denoted with V_λ the positive potential at the r.h.s. of (1.2), the operator $-\Delta_{\mathbb{H}^N} - V_\lambda(r)$ is *critical* in $\mathbb{H}^N \setminus \{x_0\}$ in the sense that the inequality $\int_{\mathbb{H}^N} |\nabla_{\mathbb{H}^N} u|^2 \, dv_{\mathbb{H}^N} \geq \int_{\mathbb{H}^N} V u^2 \, dv_{\mathbb{H}^N}$ is not valid for all $u \in C_c^\infty(\mathbb{H}^N \setminus \{x_0\})$ if $V \gneqq V_\lambda$.

The interest of (1.2) relies on the fact that it provides in a single inequality, proved by means of a unified approach, an optimal improvement (in the sense of adding nonnegative terms in the right side of the inequality) of the Poincaré inequality (1.1) and an optimal

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improvement of the Hardy inequality. Indeed, for $\lambda = \lambda_1(\mathbb{H}^N)$ ($\gamma_N = 0$) inequality (1.2) becomes the improved Poincaré inequality:

$$\begin{aligned}\int_{\mathbb{H}^N} |\nabla_{\mathbb{H}^N} u|^2 \, \mathrm{d}v_{\mathbb{H}^N} &\geq \left(\frac{N-1}{2}\right)^2 \int_{\mathbb{H}^N} u^2 \, \mathrm{d}v_{\mathbb{H}^N} \\ &+ \frac{1}{4}\int_{\mathbb{H}^N} \frac{u^2}{r^2} \, \mathrm{d}v_{\mathbb{H}^N} + \frac{(N-1)(N-3)}{4} \int_{\mathbb{H}^N} \frac{u^2}{\sinh^2 r} \, \mathrm{d}v_{\mathbb{H}^N} \,, \end{aligned} \tag{1.3}$$

for all $u \in C_c^\infty(\mathbb{H}^N \setminus \{x_0\})$ with $N \geq 2$. Instead, for $\lambda = N-2$ ($\gamma_N = N-3$) (1.2) becomes the improved Hardy inequality:

$$\begin{aligned}\int_{\mathbb{H}^N} |\nabla_{\mathbb{H}^N} u|^2 \, \mathrm{d}v_{\mathbb{H}^N} &\geq \left(\frac{N-2}{2}\right)^2 \int_{\mathbb{H}^N} \frac{u^2}{r^2} \, \mathrm{d}v_{\mathbb{H}^N} \\ &+ (N-2)\int_{\mathbb{H}^N} u^2 \, \mathrm{d}v_{\mathbb{H}^N} + \frac{(N-2)(N-3)}{2} \int_{\mathbb{H}^N} \frac{r\coth r - 1}{r^2} u^2 \, \mathrm{d}v_{\mathbb{H}^N} \,, \end{aligned} \tag{1.4}$$

for all $u \in C_c^\infty(\mathbb{H}^N \setminus \{x_0\})$ with $N \geq 3$. As concerns inequality (1.3), we recall that it has been shown first in [1] and then, with different methods, adapted to larger classes of manifolds in [4] where criticality has also been shown. Very recently, another improvement has been reached in [17] where, by using the notion of Bessel pairs, it has been proved that a further positive term of the form $\int_{\mathbb{H}^N} \frac{r}{\sinh^{N-1} r} \big|\nabla_{\mathbb{H}^N}\big(u \frac{\sinh^{\frac{N-1}{2}} r}{r}\big)\big|^2 \mathrm{d}v_{\mathbb{H}^N}$ can be added at the r.h.s. of (1.3) so that the inequality becomes an equality. Clearly, this is not in contrast with the criticality proved in [4] since the added term is not of the form Vu^2. We refer the interested reader to [2] for the L^p version of (1.3), to [8] for remainder terms of (1.1) involving the Green's function of the Laplacian, and to [7] for the analogous of (1.3) in the non-local realm of homogeneous trees.

Regarding (1.4), it's worth recalling that generalizations to Riemannian manifolds of the classical Euclidean Hardy inequality have been intensively pursued after the seminal work of Carron [10]. In particular, on Cartan-Hadamard manifolds the optimal constant is known to be $\big(\frac{N-2}{2}\big)^2$ and improvements of the Hardy inequality have been given e.g., in [12, 17, 20, 21, 22, 34]. This is in contrast to what happens in the Euclidean setting where the operator $-\Delta_{\mathbb{R}^N} - \big(\frac{N-2}{2}\big)^2 \frac{1}{|x|^2}$ is known to be *critical* in $\mathbb{R}^N \setminus \{0\}$ (see [13]). In particular, in inequality (1.4) the effect of the curvature allows to provide a remainder term of L^2-type, therefore of the same kind of that given in the seminal paper by Brezis-Vazquez [9] for the Hardy inequality on Euclidean *bounded* domains.

The above mentioned results make it natural to investigate the existence of a family of inequalities extending (1.2) to the second order. That is, with the convention $\nabla^0_{\mathbb{H}^N} u = u$ and $\nabla^1_{\mathbb{H}^N} u = \nabla_{\mathbb{H}^N} u$, we look for an inequality including either improvements of the second order Poincaré inequalities:

$$\int_{\mathbb{H}^N} (\Delta_{\mathbb{H}^N} u)^2 \, \mathrm{d}v_{\mathbb{H}^N} \geq \left(\frac{N-1}{2}\right)^{2(2-l)} \int_{\mathbb{H}^N} |\nabla^l_{\mathbb{H}^N} u|^2 \, \mathrm{d}v_{\mathbb{H}^N} \quad (l = 0 \text{ or } l = 1) \tag{1.5}$$

for all $u \in C_c^\infty(\mathbb{H}^N)$ ($N \geq 2$), and improvements of the second order Hardy inequalities:

$$\int_{\mathbb{H}^N} (\Delta_{\mathbb{H}^N} u)^2 \, \mathrm{d}v_{\mathbb{H}^N} \geq \frac{N^2}{4}\left(\frac{N-4}{2}\right)^{2(1-l)} \int_{\mathbb{H}^N} \frac{|\nabla^l_{\mathbb{H}^N} u|^2}{r^{4-2l}} \, \mathrm{d}v_{\mathbb{H}^N} \quad (l = 0 \text{ or } l = 1) \tag{1.6}$$

for all $u \in C_c^\infty(\mathbb{H}^N)$ ($N \geq 5$), i.e. the Rellich inequality which comes for $l = 0$ and the Hardy-Rellich inequality for $l = 1$. We recall that inequalities (1.5) are known from [29] and

[32] with optimal constants, while improvements have been provided in [3],[4] and, for radial operators, in [6],[31]. Instead, inequalities (1.6) were firstly studied in [20] and in [34], where the optimality of the constants was proved together with the existence of some remainder terms. More recently, a stronger version of (1.6), only involving radial operators and still holding with the same constants, has been obtained in [30]. See also [23] for improved versions of (1.6) in the general framework of Finsler-Hadamard manifolds.

In the present paper we complete the picture of results in $\mathbb{H}^N$ by proving a family of inequalities including either an improved version of (1.5) and an improved version of (1.6) when $l = 1$, therefore extending (1.2) to the second order, see Theorem 2.2 below. Furthermore, in Theorem 2.1, we show that the obtained family of inequalities reads as a family of *identities* for radial operators (also for non radial functions) giving a more precise understanding of the remainder terms provided. A fine exploitation of these results also allows to obtain improved versions of (1.5) and of (1.6) for $l = 0$ in such a way to exhaust the second order scenario, see Corollaries 2.3 and 2.4. As far we are aware, all the improvements provided have larger remainder terms than those already known in literature, see Remark 2.2 in the following.

We notice that (1.2) was proved in [5] by means of a unified approach based on criticality theory, well established for *second* order operators only (see [13]); therefore, a similar approach seems not applicable in the higher order case. Here, drawing primary motivation from the seminal paper [18], we extend (1.2) to the second order by using the notion of *Bessel pair*. This notion has been very recently developed in [17] on Cartan-Hadamard manifolds to establish several interesting Hardy identities and inequalities which, in particular, generalise many well-known Hardy inequalities on Cartan-Hadamard manifolds. By combining some ideas from [17],[18], and through some computations with spherical harmonics, in the present article we develop the method of Bessel pairs to derive general abstract Rellich inequalities and identities on $\mathbb{H}^N$ that we employ to prove our main results, i.e., Theorems 2.1 and 2.2. In this way, we get either Poincaré and Hardy-Rellich identities, and improved inequalities, by means of a unified proof where the key ingredient is the construction of a family of Bessel pairs, see (4.1) in the following. Finally, as applications of the obtained inequalities, we derive quantitative versions of the second order Heisenberg-Pauli-Weyl uncertainty principle, see Section 2.3. As far as we know, the results provided represent the first example of second order Heisenberg-Pauli-Weyl uncertainty principle in the hyperbolic context.

The paper is organized as follows: in Section 2 we introduce some of the notations and we state our main results, i.e. Poincaré and Hardy-Rellich identities and related improved inequalities; furthermore, in this section, we also state second order versions of the Heisenberg-Pauli-Weyl uncertainty principle. In Section 3 we provide abstract Rellich identities and inequalities via Bessel pairs together with a related Heisenberg-Pauli-Weyl uncertainty principle. Section 4 is devoted to the proofs of the results stated in Section 2 by exploiting the results stated in Section 3, while Section 5 contains the proofs of the results stated in Section 3. In Section 6 we discuss possible extensions of our proofs and results to more general manifolds. Finally, in the Appendix we present a family of improved Hardy-Poincaré identities which follows as a corollary from [17, Theorem 3.2], see Lemma 3.1 below, by exploiting the family of Bessel pairs introduced in Section 4. In particular, these identities give a deeper understanding of (1.2) and include [17, Theorem 1.4] as a particular case.

2. Main results

2.1. **Notations.** From now onward, if nothing is specified, we will always assume $N \geq 2$. It is well known that the N-dimensional hyperbolic space $\mathbb{H}^N$ admits a polar coordinate decomposition structure. Namely, for $x \in \mathbb{H}^N$ we can write $x = (r, \Theta) = (r, \theta_1, \ldots, \theta_{N-1}) \in (0, \infty) \times \mathbb{S}^{N-1}$, where r denotes the geodesic distance between the point x and a fixed pole x_0 in $\mathbb{H}^N$ and $\mathbb{S}^{N-1}$ is the unit sphere in the N-dimensional euclidean space $\mathbb{R}^N$. Recall that the Riemannian Laplacian of a scalar function u on $\mathbb{H}^N$ is given by

$$\Delta_{\mathbb{H}^N} u(r,\Theta) = \frac{1}{\sinh^2 r}\frac{\partial}{\partial r}\left[(\sinh r)^{N-1}\frac{\partial u}{\partial r}(r,\Theta)\right] + \frac{1}{\sinh^2 r}\Delta_{\mathbb{S}^{N-1}} u(r,\Theta), \tag{2.1}$$

where $\Delta_{\mathbb{S}^{N-1}}$ is the Riemannian Laplacian on the unit sphere $\mathbb{S}^{N-1}$. In particular, the radial contribution of the Riemannian Laplacian $\Delta_{r,\mathbb{H}^N} u$ reads as

$$\Delta_{r,\mathbb{H}^N} u = \frac{1}{(\sinh r)^{N-1}}\frac{\partial}{\partial r}\left[(\sinh r)^{N-1}\frac{\partial u}{\partial r}\right] = u'' + (N-1)\coth r\, u',$$

where from now on a prime will denote, for radial functions, derivative w.r.t .r. Also, let us recall the Gradient in terms of the polar coordinate decomposition is given by

$$\nabla_{\mathbb{H}^N} u(r,\Theta) = \left(\frac{\partial u}{\partial r}(r,\Theta), \frac{1}{\sinh r}\nabla_{\mathbb{S}^{N-1}} u(r,\Theta)\right),$$

where $\nabla_{\mathbb{S}^{N-1}}$ denotes the Gradient on the unit sphere $\mathbb{S}^{N-1}$. Again, the radial contribution of the Gradient, $\nabla_{r,\mathbb{H}^N} u$, is defined as

$$\nabla_{r,\mathbb{H}^N} u = \left(\frac{\partial u}{\partial r}, 0\right).$$

2.2. **Hardy-Rellich and Poincaré identities and improved inequalities.** Our main result for radial operators reads as follows

Theorem 2.1. *For all $0 \leq \lambda \leq \lambda_1(\mathbb{H}^N) = \left(\frac{N-1}{2}\right)^2$ and all $u \in C_c^\infty(\mathbb{H}^N \setminus \{x_0\})$ there holds*

$$\begin{aligned}\int_{\mathbb{H}^N} |\Delta_{r,\mathbb{H}^N} u|^2 \, dv_{\mathbb{H}^N} &= \lambda \int_{\mathbb{H}^N} |\nabla_{r,\mathbb{H}^N} u|^2 \, dv_{\mathbb{H}^N} + h_N^2(\lambda) \int_{\mathbb{H}^N} \frac{|\nabla_{r,\mathbb{H}^N} u|^2}{r^2} \, dv_{\mathbb{H}^N} \\ &+ \left[\left(\frac{N}{2}\right)^2 - h_N^2(\lambda)\right] \int_{\mathbb{H}^N} \frac{|\nabla_{r,\mathbb{H}^N} u|^2}{\sinh^2 r} \, dv_{\mathbb{H}^N} + \gamma_N(\lambda)\, h_N(\lambda) \int_{\mathbb{H}^N} \frac{r\coth r - 1}{r^2} |\nabla_{r,\mathbb{H}^N} u|^2 \, dv_{\mathbb{H}^N} \\ &+ \int_{\mathbb{H}^N} (\Psi_\lambda(r))^2 \left|\nabla_{r,\mathbb{H}^N}\left(\frac{u_r}{\Psi_\lambda(r)}\right)\right|^2 dv_{\mathbb{H}^N}\end{aligned}$$

where $\gamma_N(\lambda) := \sqrt{(N-1)^2 - 4\lambda}$, $h_N(\lambda) := \frac{\gamma_N(\lambda)+1}{2}$ and $\Psi_\lambda(r) := r^{-\frac{N-2}{2}}\left(\frac{\sinh r}{r}\right)^{-\frac{N-1+\gamma_N(\lambda)}{2}}$. Furthermore, for $N \geq 5$ and λ given, the constants $h_N^2(\lambda)$ and $\left[\left(\frac{N}{2}\right)^2 - h_N^2(\lambda)\right]$ are jointly sharp in the sense that, fixed $h_N^2(\lambda)$, the inequality does not hold if we replace $\left[\left(\frac{N}{2}\right)^2 - h_N^2(\lambda)\right]$ with a larger constant.

Remark 2.1. *We remark that the the function $\frac{r\coth r - 1}{r^2}$ is positive, strictly decreasing and satisfies*

$$\frac{r\coth r - 1}{r^2} \sim \frac{1}{3} \quad as \quad r \to 0^+ \qquad and \qquad \frac{r\coth r - 1}{r^2} \sim \frac{1}{r} \quad as \quad r \to +\infty\,.$$

Besides, the map $[0, \lambda_1(\mathbb{H}^N)] \ni \lambda \mapsto h_N(\lambda)$ *is decreasing and* $\frac{1}{4} \le h_N(\lambda) \le \left(\frac{N}{2}\right)^2$.

Furthermore, for non radial operators we obtain the second order analogous to (1.2):

Theorem 2.2. *Let* $N \ge 5$. *For all* $0 \le \lambda \le \lambda_1(\mathbb{H}^N) = \left(\frac{N-1}{2}\right)^2$ *and all* $u \in C_c^\infty(\mathbb{H}^N \setminus \{x_0\})$ *there holds*

$$\int_{\mathbb{H}^N} |\Delta_{\mathbb{H}^N} u|^2 \,\mathrm{d}v_{\mathbb{H}^N} \ge \lambda \int_{\mathbb{H}^N} |\nabla_{\mathbb{H}^N} u|^2 \,\mathrm{d}v_{\mathbb{H}^N} + h_N^2(\lambda) \int_{\mathbb{H}^N} \frac{|\nabla_{\mathbb{H}^N} u|^2}{r^2} \,\mathrm{d}v_{\mathbb{H}^N}$$
$$+ \left[\left(\frac{N}{2}\right)^2 - h_N^2(\lambda)\right] \int_{\mathbb{H}^N} \frac{|\nabla_{\mathbb{H}^N} u|^2}{\sinh^2 r} \,\mathrm{d}v_{\mathbb{H}^N} + \gamma_N(\lambda)\, h_N(\lambda) \int_{\mathbb{H}^N} \frac{r \coth r - 1}{r^2} |\nabla_{\mathbb{H}^N} u|^2 \,\mathrm{d}v_{\mathbb{H}^N}$$
$$+ \int_{\mathbb{H}^N} (\Psi_\lambda(r))^2 \left|\nabla_{\mathbb{H}^N}\left(\frac{u_r}{\Psi_\lambda(r)}\right)\right|^2 \mathrm{d}v_{\mathbb{H}^N}$$

where $\gamma_N(\lambda), h_N(\lambda)$ *and* $\Psi_\lambda(r)$ *are as given in Theorem 2.1. Furthermore, for any given* λ, *the constants* $h_N^2(\lambda)$ *and* $\left[\left(\frac{N}{2}\right)^2 - h_N^2(\lambda)\right]$ *are jointly sharp in the sense explained in Theorem 2.1.*

We notice that the dimension restriction $N \ge 5$ in Theorem 2.2 comes from assumption (3.4) in Theorem 3.2 below where we state our abstract Rellich inequalities, see also Remark 3.1 for some comments about this assumption that naturally comes when passing from the radial to the non radial framework. Theorems 2.1 and 2.2 yield a number of improved Poincaré and Hardy-Rellich inequalities that we state here below; a comparison with previous results is provided in Remark 2.2. More precisely, for $\lambda = 0$ we readily got the following improved Hardy-Rellich identity and inequality:

Corollary 2.1. *For all* $u \in C_c^\infty(\mathbb{H}^N \setminus \{x_0\})$ *there holds*

$$\int_{\mathbb{H}^N} |\Delta_{r,\mathbb{H}^N} u|^2 \,\mathrm{d}v_{\mathbb{H}^N} = \left(\frac{N}{2}\right)^2 \int_{\mathbb{H}^N} \frac{|\nabla_{r,\mathbb{H}^N} u|^2}{r^2} \,\mathrm{d}v_{\mathbb{H}^N}$$
$$+ \frac{N(N-1)}{2} \int_{\mathbb{H}^N} \frac{r \coth r - 1}{r^2} |\nabla_{r,\mathbb{H}^N} u|^2 \,\mathrm{d}v_{\mathbb{H}^N}$$
$$+ \int_{\mathbb{H}^N} \frac{r^N}{(\sinh r)^{2(N-1)}} \left|\nabla_{r,\mathbb{H}^N}\left(\frac{(\sinh r)^{N-1}\, u_r}{r^{\frac{N}{2}}}\right)\right|^2 \mathrm{d}v_{\mathbb{H}^N}\,.$$

Moreover, if $N \ge 5$, *for all* $u \in C_c^\infty(\mathbb{H}^N \setminus \{x_0\})$ *there holds*

$$\int_{\mathbb{H}^N} |\Delta_{\mathbb{H}^N} u|^2 \,\mathrm{d}v_{\mathbb{H}^N} \ge \left(\frac{N}{2}\right)^2 \int_{\mathbb{H}^N} \frac{|\nabla_{\mathbb{H}^N} u|^2}{r^2} \,\mathrm{d}v_{\mathbb{H}^N}$$
$$+ \frac{N(N-1)}{2} \int_{\mathbb{H}^N} \frac{r \coth r - 1}{r^2} |\nabla_{\mathbb{H}^N} u|^2 \,\mathrm{d}v_{\mathbb{H}^N}$$
$$+ \int_{\mathbb{H}^N} \frac{r^N}{(\sinh r)^{2(N-1)}} \left|\nabla_{\mathbb{H}^N}\left(\frac{(\sinh r)^{N-1}\, u_r}{r^{\frac{N}{2}}}\right)\right|^2 \mathrm{d}v_{\mathbb{H}^N}\,,$$

and the constant $\left(\frac{N}{2}\right)^2$ *appearing in the L.H.S of both equations is the sharp constant.*

For $\lambda = \lambda_1(\mathbb{H}^N)$ we got an improvement of the second order Poincaré identity (1.5) with $l = 0$, and the related inequality:

Corollary 2.2. *For all $u \in C_c^\infty(\mathbb{H}^N \setminus \{x_0\})$ there holds*

$$\begin{aligned}\int_{\mathbb{H}^N} |\Delta_{r,\mathbb{H}^N} u|^2 \, \mathrm{d}v_{\mathbb{H}^N} &= \left(\frac{N-1}{2}\right)^2 \int_{\mathbb{H}^N} |\nabla_{r,\mathbb{H}^N} u|^2 \, \mathrm{d}v_{\mathbb{H}^N} \\ &\quad + \frac{1}{4} \int_{\mathbb{H}^N} \frac{|\nabla_{r,\mathbb{H}^N} u|^2}{r^2} \, \mathrm{d}v_{\mathbb{H}^N} + \frac{N^2-1}{4} \int_{\mathbb{H}^N} \frac{|\nabla_{r,\mathbb{H}^N} u|^2}{\sinh^2 r} \, \mathrm{d}v_{\mathbb{H}^N} \\ &\quad + \int_{\mathbb{H}^N} \frac{r}{(\sinh r)^{N-1}} \left| \nabla_{r,\mathbb{H}^N} \left(\frac{(\sinh r)^{\frac{N-1}{2}} u_r}{r^{\frac{1}{2}}} \right) \right|^2 \mathrm{d}v_{\mathbb{H}^N}.\end{aligned}$$

Moreover, if $N \geq 5$, for all $u \in C_c^\infty(\mathbb{H}^N \setminus \{x_0\})$ there holds

$$\begin{aligned}\int_{\mathbb{H}^N} |\Delta_{\mathbb{H}^N} u|^2 \, \mathrm{d}v_{\mathbb{H}^N} &\geq \left(\frac{N-1}{2}\right)^2 \int_{\mathbb{H}^N} |\nabla_{\mathbb{H}^N} u|^2 \, \mathrm{d}v_{\mathbb{H}^N} \\ &\quad + \frac{1}{4} \int_{\mathbb{H}^N} \frac{|\nabla_{\mathbb{H}^N} u|^2}{r^2} \, \mathrm{d}v_{\mathbb{H}^N} + \frac{N^2-1}{4} \int_{\mathbb{H}^N} \frac{|\nabla_{\mathbb{H}^N} u|^2}{\sinh^2 r} \, \mathrm{d}v_{\mathbb{H}^N} \\ &\quad + \int_{\mathbb{H}^N} \frac{r}{(\sinh r)^{N-1}} \left| \nabla_{\mathbb{H}^N} \left(\frac{(\sinh r)^{\frac{N-1}{2}} u_r}{r^{\frac{1}{2}}} \right) \right|^2 \mathrm{d}v_{\mathbb{H}^N}.\end{aligned}$$

The constant $\left(\frac{N-1}{2}\right)^2$ appearing in the L.H.S of both equations is the sharp constant. Moreover, for $N \geq 5$, the constants $\frac{1}{4}$ and $\frac{N^2-1}{4}$ are jointly sharp in the sense explained in Theorem 2.1.

By combining Corollary 2.1 with [17, Corollary 3.2] we also get an improved Rellich inequality:

Corollary 2.3. *For all $u \in C_c^\infty(\mathbb{H}^N \setminus \{x_0\})$ there holds*

$$\begin{aligned}\int_{\mathbb{H}^N} |\Delta_{r,\mathbb{H}^N} u|^2 \, \mathrm{d}v_{\mathbb{H}^N} &= \frac{N^2}{4} \left(\frac{N-4}{2}\right)^2 \int_{\mathbb{H}^N} \frac{u^2}{r^4} \, \mathrm{d}v_{\mathbb{H}^N} \\ &\quad + \frac{N^2(N-4)(N-1)}{8} \int_{\mathbb{H}^N} \frac{r \coth r - 1}{r^4} u^2 \, \mathrm{d}v_{\mathbb{H}^N} \\ &\quad + \frac{N(N-1)}{2} \int_{\mathbb{H}^N} \frac{r \coth r - 1}{r^2} |\nabla_{r,\mathbb{H}^N} u|^2 \, \mathrm{d}v_{\mathbb{H}^N} \\ &\quad + \frac{N^2}{4} \int_{\mathbb{H}^N} \frac{1}{r^{N-2}} \left| \nabla_{r,\mathbb{H}^N} \left(r^{\frac{N-4}{2}} u \right) \right|^2 \mathrm{d}v_{\mathbb{H}^N} \\ &\quad + \int_{\mathbb{H}^N} \frac{r^N}{(\sinh r)^{2(N-1)}} \left| \nabla_{r,\mathbb{H}^N} \left(\frac{(\sinh r)^{N-1} u_r}{r^{\frac{N}{2}}} \right) \right|^2 \mathrm{d}v_{\mathbb{H}^N}.\end{aligned}$$

Moreover, if $N \geq 5$, for all $u \in C_c^\infty(\mathbb{H}^N \setminus \{x_0\})$ there holds

$$\begin{aligned}\int_{\mathbb{H}^N} |\Delta_{\mathbb{H}^N} u|^2 \, \mathrm{d}v_{\mathbb{H}^N} &\geq \frac{N^2}{4} \left(\frac{N-4}{2}\right)^2 \int_{\mathbb{H}^N} \frac{u^2}{r^4} \, \mathrm{d}v_{\mathbb{H}^N} \\ &\quad + \frac{N^2(N-4)(N-1)}{8} \int_{\mathbb{H}^N} \frac{r \coth r - 1}{r^4} u^2 \, \mathrm{d}v_{\mathbb{H}^N} \\ &\quad + \frac{N(N-1)}{2} \int_{\mathbb{H}^N} \frac{r \coth r - 1}{r^2} |\nabla_{\mathbb{H}^N} u|^2 \, \mathrm{d}v_{\mathbb{H}^N}\end{aligned}$$

$$+\frac{N^2}{4}\int_{\mathbb{H}^N}\frac{1}{r^{N-2}}\left|\nabla_{\mathbb{H}^N}\left(r^{\frac{N-4}{2}}u\right)\right|^2\,\mathrm{d}v_{\mathbb{H}^N}$$
$$+\int_{\mathbb{H}^N}\frac{r^N}{(\sinh r)^{2(N-1)}}\left|\nabla_{\mathbb{H}^N}\left(\frac{(\sinh r)^{N-1}\,u_r}{r^{\frac{N}{2}}}\right)\right|^2\,\mathrm{d}v_{\mathbb{H}^N}\,,$$

and the constant $\frac{N^2}{4}\,\left(\frac{N-4}{2}\right)^2$ *appearing in the L.H.S of both equations is the sharp constant.*

Instead, by combining Corollary 2.2 with [17, Theorem 1.4 and Corollary 3.2], we improve (1.5) with $l=0$, i.e. we complete the second order scenario about Poincaré identities and inequalities :

Corollary 2.4. *For all* $u\in C_c^\infty(\mathbb{H}^N\setminus\{x_0\})$ *there holds*

$$\begin{aligned}\int_{\mathbb{H}^N}|\Delta_{r,\mathbb{H}^N}u|^2\,\mathrm{d}v_{\mathbb{H}^N}&=\left(\frac{N-1}{2}\right)^4\int_{\mathbb{H}^N}u^2\,\mathrm{d}v_{\mathbb{H}^N}\\&+\left(\frac{N-1}{4}\right)^2\int_{\mathbb{H}^N}\frac{u^2}{r^2}\,\mathrm{d}v_{\mathbb{H}^N}+\frac{(N-1)^3(N-3)}{16}\int_{\mathbb{H}^N}\frac{u^2}{\sinh^2 r}\,\mathrm{d}v_{\mathbb{H}^N}\\&+\frac{1}{4}\int_{\mathbb{H}^N}\frac{|\nabla_{r,\mathbb{H}^N}u|^2}{r^2}\,\mathrm{d}v_{\mathbb{H}^N}+\frac{N^2-1}{4}\int_{\mathbb{H}^N}\frac{|\nabla_{r,\mathbb{H}^N}u|^2}{\sinh^2 r}\,\mathrm{d}v_{\mathbb{H}^N}\\&+\left[\left(\frac{N-1}{2}\right)^2+1\right]\int_{\mathbb{H}^N}\frac{r}{(\sinh r)^{N-1}}\left|\nabla_{r,\mathbb{H}^N}\left(\frac{(\sinh r)^{\frac{N-1}{2}}u_r}{r^{\frac{1}{2}}}\right)\right|^2\,\mathrm{d}v_{\mathbb{H}^N}.\end{aligned}$$

Moreover, if $N\geq 5$*, for all* $u\in C_c^\infty(\mathbb{H}^N\setminus\{x_0\})$ *there holds*

$$\begin{aligned}\int_{\mathbb{H}^N}|\Delta_{\mathbb{H}^N}u|^2\,\mathrm{d}v_{\mathbb{H}^N}&\geq\left(\frac{N-1}{2}\right)^4\int_{\mathbb{H}^N}u^2\,\mathrm{d}v_{\mathbb{H}^N}\\&+\left(\frac{N-1}{4}\right)^2\int_{\mathbb{H}^N}\frac{u^2}{r^2}\,\mathrm{d}v_{\mathbb{H}^N}+\frac{(N-1)^3(N-3)}{16}\int_{\mathbb{H}^N}\frac{u^2}{\sinh^2 r}\,\mathrm{d}v_{\mathbb{H}^N}\\&+\frac{1}{4}\int_{\mathbb{H}^N}\frac{|\nabla_{\mathbb{H}^N}u|^2}{r^2}\,\mathrm{d}v_{\mathbb{H}^N}+\frac{N^2-1}{4}\int_{\mathbb{H}^N}\frac{|\nabla_{\mathbb{H}^N}u|^2}{\sinh^2 r}\,\mathrm{d}v_{\mathbb{H}^N}\\&+\left[\left(\frac{N-1}{2}\right)^2+1\right]\int_{\mathbb{H}^N}\frac{r}{(\sinh r)^{N-1}}\left|\nabla_{\mathbb{H}^N}\left(\frac{(\sinh r)^{\frac{N-1}{2}}u_r}{r^{\frac{1}{2}}}\right)\right|^2\,\mathrm{d}v_{\mathbb{H}^N}.\end{aligned}$$

The constant $\left(\frac{N-1}{2}\right)^4$ *appearing in the L.H.S of both equations is the sharp constant. Moreover, for* $N\geq 5$*, the constants* $\frac{1}{4}$ *and* $\frac{N^2-1}{4}$ *in both equations are jointly sharp in the sense explained in Theorem 2.1.*

Remark 2.2. *As far as we are aware, improved second order Poincaré and Hardy-Rellich* equalities *in* $\mathbb{H}^N$ *were not known in literature. As concerns the Hardy-Rellich and Rellich* inequalities*, improved versions were already known from* [23]*,* [30] *and* [34] *on general manifolds but with fewer and smaller remainder terms. As a matter of example, if we compare Corollary 2.1 with* [30, Theorem 4.2]*, the improvement of the Hardy-Rellich inequality provided there reads as* $\frac{3N(N-1)}{2}\int_{\mathbb{H}^N}\frac{|\nabla_{r,\mathbb{H}^N}u|^2}{\pi^2+r^2}\,\mathrm{d}v_{\mathbb{H}^N}$*, therefore it decays more rapidly, both as* $r\to 0^+$ *and as* $r\to+\infty$*, than the term* $\frac{N(N-1)}{2}\int_{\mathbb{H}^N}\frac{r\coth r-1}{r^2}\,|\nabla_{r,\mathbb{H}^N}u|^2\,\mathrm{d}v_{\mathbb{H}^N}$ *provided in Corollary 2.1. Similarly, if we compare Corollary 2.2 with* [30, Theorem 4.3]*, again, the corrections of the Rellich inequality provided there decays more rapidly than ours, either as*

$r \to 0^+$ and as $r \to +\infty$. As concerns the improved second order Poincaré inequalities *given by Corollaries 2.3 and 2.4, the gain with respect to the inequalities already known in* [6] *is in the adding of a further remainder term.*

2.3. **Second order Heisenberg-Pauli-Weyl uncertainty principle.** Another remarkable consequence of Theorem 2.2 is the following quantitative version of HPW principle in $\mathbb{H}^N$:

Theorem 2.3. *Let $N \geq 5$. For all $0 \leq \lambda \leq \lambda_1(\mathbb{H}^N)$ and all $u \in C_c^\infty(\mathbb{H}^N \setminus \{x_0\})$ there holds*

$$\left(\int_{\mathbb{H}^N} \left(|\Delta_{\mathbb{H}^N} u|^2 - \lambda |\nabla_{\mathbb{H}^N} u|^2\right) \mathrm{d}v_{\mathbb{H}^N}\right) \left(\int_{\mathbb{H}^N} r^2 |\nabla_{\mathbb{H}^N} u|^2 \,\mathrm{d}v_{\mathbb{H}^N}\right) \geq h_N^2(\lambda) \left(\int_{\mathbb{H}^N} |\nabla_{\mathbb{H}^N} u|^2 \,\mathrm{d}v_{\mathbb{H}^N}\right)^2 \tag{2.2}$$

where $h_N(\lambda)$ is as defined as in Theorem 2.1. In particular, for $\lambda = 0$, we obtain

$$\left(\int_{\mathbb{H}^N} |\Delta_{\mathbb{H}^N} u|^2 \,\mathrm{d}v_{\mathbb{H}^N}\right) \left(\int_{\mathbb{H}^N} r^2 |\nabla_{\mathbb{H}^N} u|^2 \,\mathrm{d}v_{\mathbb{H}^N}\right) \geq \frac{N^2}{4} \left(\int_{\mathbb{H}^N} |\nabla_{\mathbb{H}^N} u|^2 \,\mathrm{d}v_{\mathbb{H}^N}\right)^2 , \tag{2.3}$$

for all $u \in C_c^\infty(\mathbb{H}^N \setminus \{x_0\})$.

Remark 2.3. *In the Euclidean context the second order Heisenberg-Pauli-Weyl uncertainty principle has been only recently studied in* [11, Theorem 2.1-2.2] *where it is proved that the best constant switches from $\frac{N^2}{4}$ to $\frac{(N+2)^2}{4}$ when passing to the second order. Instead, in* [14, Theorem 1.1] *a weighted version of inequality* (2.3) *in $\mathbb{R}^N$ is studied together with the sharpness of the constants and the existence of extremals.*

As far as we know, inequality (2.2) *is the first example of second order Heisenberg-Pauli-Weyl uncertainty principle in the hyperbolic context. For the first order case, we refer to* [19] *and* [22] *where the authors fully describe the influence of curvature to uncertainty principles in the Riemannian and Finslerian settings. It's worth mentioning that a straightforward modification of the proof of Theorem 2.3, by exploiting appropriately Theorem 2.2, yields the improved version of* (2.2) *below which supports the conjecture that the sharp constant* (2.2) *should be larger than $h_N^2(\lambda)$. More precisely, for all $0 \leq \lambda \leq \lambda_1(\mathbb{H}^N)$ and all $u \in C_c^\infty(\mathbb{H}^N \setminus \{x_0\})$, there holds*

$$\begin{aligned}
&\left(\int_{\mathbb{H}^N} \left(|\Delta_{\mathbb{H}^N} u|^2 - \lambda |\nabla_{\mathbb{H}^N} u|^2\right) \mathrm{d}v_{\mathbb{H}^N}\right) \left(\int_{\mathbb{H}^N} r^2 |\nabla_{\mathbb{H}^N} u|^2 \mathrm{d}v_{\mathbb{H}^N}\right) \\
&\quad \geq h_N^2(\lambda) \left(\int_{\mathbb{H}^N} |\nabla_{\mathbb{H}^N} u|^2 \mathrm{d}v_{\mathbb{H}^N}\right)^2 + \left(\int_{\mathbb{H}^N} r^2 |\nabla_{\mathbb{H}^N} u|^2 \mathrm{d}v_{\mathbb{H}^N}\right) \times \\
&\times \left\{ \left[\left(\frac{N}{2}\right)^2 - h_N^2(\lambda)\right] \int_{\mathbb{H}^N} \frac{|\nabla_{\mathbb{H}^N} u|^2}{\sinh^2 r} \,\mathrm{d}v_{\mathbb{H}^N} + \gamma_N(\lambda)\, h_N(\lambda) \int_{\mathbb{H}^N} \frac{r \coth r - 1}{r^2} |\nabla_{\mathbb{H}^N} u|^2 \,\mathrm{d}v_{\mathbb{H}^N} \right\}
\end{aligned}$$

where $\gamma_N(\lambda)$ and $h_N(\lambda)$ are defined as in Theorem 2.1. Therefore, for $\lambda = 0$, we obtain the improved version of (2.3)*:*

$$\begin{aligned}
\left(\int_{\mathbb{H}^N} |\Delta_{\mathbb{H}^N} u|^2 \mathrm{d}v_{\mathbb{H}^N}\right) \left(\int_{\mathbb{H}^N} r^2 |\nabla_{\mathbb{H}^N} u|^2 \,\mathrm{d}v_{\mathbb{H}^N}\right) &\geq \frac{N^2}{4} \left(\int_{\mathbb{H}^N} |\nabla_{\mathbb{H}^N} u|^2 \,\mathrm{d}v_{\mathbb{H}^N}\right)^2 \\
+ \left(\int_{\mathbb{H}^N} r^2 |\nabla_{\mathbb{H}^N} u|^2 \mathrm{d}v_{\mathbb{H}^N}\right) &\left(\frac{N(N-1)}{2} \int_{\mathbb{H}^N} \frac{r \coth r - 1}{r^2} |\nabla_{\mathbb{H}^N} u|^2 \,\mathrm{d}v_{\mathbb{H}^N}\right)
\end{aligned}$$

for all $u \in C_c^\infty(\mathbb{H}^N \setminus \{x_0\})$. *The above inequality should be compared with inequality* (3.6) *provided in Section 3 which also improves* (2.3).

We conclude the section by stating the counterpart of Theorem 2.3 for radial operators:

Theorem 2.4. *For all* $0 \le \lambda \le \lambda_1(\mathbb{H}^N)$ *and all* $u \in C_c^\infty(\mathbb{H}^N \setminus \{x_0\})$ *there holds*

$$\left(\int_{\mathbb{H}^N} \left(|\Delta_{r,\mathbb{H}^N} u|^2 - \lambda |\nabla_{r,\mathbb{H}^N} u|^2\right) \mathrm{d}v_{\mathbb{H}^N}\right) \left(\int_{\mathbb{H}^N} r^2 |\nabla_{r,\mathbb{H}^N} u|^2 \mathrm{d}v_{\mathbb{H}^N}\right) \ge h_N^2(\lambda) \left(\int_{\mathbb{H}^N} |\nabla_{r,\mathbb{H}^N} u|^2 \mathrm{d}v_{\mathbb{H}^N}\right)^2$$

where $h_N(\lambda)$ *is as defined as in Theorem 2.1. In particular, for* $\lambda = 0$, *we obtain*

$$\left(\int_{\mathbb{H}^N} |\Delta_{r,\mathbb{H}^N} u|^2 \mathrm{d}v_{\mathbb{H}^N}\right) \left(\int_{\mathbb{H}^N} r^2 |\nabla_{r,\mathbb{H}^N} u|^2 \,\mathrm{d}v_{\mathbb{H}^N}\right) \ge \frac{N^2}{4} \left(\int_{\mathbb{H}^N} |\nabla_{r,\mathbb{H}^N} u|^2 \,\mathrm{d}v_{\mathbb{H}^N}\right)^2$$

for all $u \in C_c^\infty(\mathbb{H}^N \setminus \{x_0\})$.

3. Abstract Rellich identities and inequalities via Bessel pairs

Ghoussoub-Moradifam in [18] provided a very general framework to obtain various Hardy-type inequalities and their improvements on the Euclidean space (or bounded domain). Their approach was based on the notion of Bessel pair that we recall in the following

Definition 3.1. *We say that a pair* (V, W) *of* C^1*-functions is a Bessel pair on* $(0, R)$ *for some* $0 < R \le \infty$ *if the ordinary differential equation:*

$$(Vy')' + Wy = 0$$

admits a positive solutions f *on the interval* $(0, R)$.

In [18] the authors proved the following inequality for some positive constant $C > 0$:

$$\int_{B_R} V(x) |\nabla u|^2 \,\mathrm{d}x \ge C \int_{B_R} W(x)\, |u|^2 \,\mathrm{d}x \quad \forall\, u \in C_c^\infty(B_R), \tag{3.1}$$

subject to the constraints that the functions V and W are positive radial functions defined on the euclidean ball B_R and such that: $(r^{N-1}V, r^{N-1}W)$ is a Bessel pair $\int_0^R \frac{1}{r^{N-1}V(r)} \,\mathrm{d}r = \infty$ and $\int_0^R r^{N-1} V(r) \,\mathrm{d}r < \infty$ where $0 < R \le \infty$ is the radius of the ball B_R.

In view of (3.1), with particular choices of (V, W), the results in [18] improved several known results concerning Hardy inequalities. Recently, the notion of Bessel pair has been exploited in [24] to establish improved Hardy inequalities involving general distance functions, in [26] to sharpen several Hardy type inequalities on half spaces, and in [25] to prove Hardy inequalities on homogeneous groups.

Regarding Cartan-Hadamard manifolds, the notion of Bessel pair has been very recently emploied to obtain improved Hardy inequalities in [17]; to our future purposes, we recall their Theorem 3.2 on $\mathbb{H}^N$:

Lemma 3.1. [17, Theorem 3.2] *Let* $(r^{N-1}V, r^{N-1}W)$ *be a Bessel pair on* $(0, R)$ *with positive solution* f *on* $(0, R)$. *Then for all* $u \in C_c^\infty(\mathbb{H}^N \setminus \{x_0\})$, *there holds*

$$\int_{B_R} V(r) |\nabla_{\mathbb{H}^N} u|^2 \,\mathrm{d}v_{\mathbb{H}^N} = \int_{B_R} W(r) |u|^2 \,\mathrm{d}v_{\mathbb{H}^N} + \int_{B_R} V(r) (f(r))^2 \left|\nabla_{\mathbb{H}^N} \left(\frac{u}{f(r)}\right)\right|^2 \mathrm{d}v_{\mathbb{H}^N}$$

$$- (N-1) \int_{B_R} V(r) \frac{f'(r)}{f(r)} \left(\coth r - \frac{1}{r} \right) u^2 \, \mathrm{d}v_{\mathbb{H}^N}.$$

and

$$\int_{B_R} V(r) |\nabla_{r,\mathbb{H}^N} u|^2 \, \mathrm{d}v_{\mathbb{H}^N} = \int_{B_R} W(r) |u|^2 \, \mathrm{d}v_{\mathbb{H}^N} + \int_{B_R} V(r) (f(r))^2 \left| \nabla_{r,\mathbb{H}^N} \left(\frac{u}{f(r)} \right) \right|^2 \mathrm{d}v_{\mathbb{H}^N}$$
$$- (N-1) \int_{B_R} V(r) \frac{f'(r)}{f(r)} \left(\coth r - \frac{1}{r} \right) u^2 \, \mathrm{d}v_{\mathbb{H}^N}.$$

In view of Lemma 3.1 a subsequent natural issue is to study wether the notion of Bessel pair can be adopted to treat higher order Hardy type inequalities in $\mathbb{H}^N$. In the Euclidean space (or in bounded euclidean domains) this topic was faced in [18]. One of their results read as follows: let $0 < R \leq \infty, V$ and W be positive C^1-functions on $B_R \setminus \{0\}$ such that $(r^{N-1}V, r^{N-1}W)$ forms a Bessel pair; then for all radial functions $u \in C_c^\infty(B_R)$ there holds

$$\int_{B_R} V(x) |\Delta u|^2 \, \mathrm{d}x \geq \int_B W(x) \, |\nabla u|^2 \, \mathrm{d}x + (N-1) \int_{B_R} \left(\frac{V(x)}{|x|^2} - \frac{V_r(x)}{|x|} \right) |\nabla u|^2 \, \mathrm{d}x, \tag{3.2}$$

where $r = |x|$. In addition, if $W(x) - 2\frac{V(x)}{|x|^2} + 2\frac{V_r(x)}{|x|} - V_{rr}(x) \geq 0$ on $(0, R)$, the above inequality is true for non radial function as well (we refer [18, Theorem 3.1-3.3] for more insight). We also refer to [15, 16, 27] for recent results on Hardy-Rellich inequalities and their improvements on the Euclidean space using the approach of Bessel pairs.

In the present article, we extend (3.2) to $\mathbb{H}^N$ by showing first the following:

Theorem 3.1. *Let $(r^{N-1}V, r^{N-1}W)$ be a Bessel pair on $(0, R)$ with positive solution f on $(0, R)$. Then for all radial function $u \in C_c^\infty(B_R \setminus \{x_0\})$ there holds*

$$\begin{aligned} \int_{B_R} V(r) |\Delta_{\mathbb{H}^N} u|^2 \, \mathrm{d}v_{\mathbb{H}^N} &= \int_{B_R} W(r) |\nabla_{\mathbb{H}^N} u|^2 \, \mathrm{d}v_{\mathbb{H}^N} \\ &+ (N-1) \int_{B_R} \left(\frac{V(r)}{\sinh^2 r} - \frac{V_r(r) \cosh r}{\sinh r} \right) |\nabla_{\mathbb{H}^N} u|^2 \, \mathrm{d}v_{\mathbb{H}^N} \\ &- (N-1) \int_{B_R} V(r) \frac{f'}{f} \left(\coth r - \frac{1}{r} \right) |\nabla_{\mathbb{H}^N} u|^2 \, \mathrm{d}v_{\mathbb{H}^N} \\ &+ \int_{B_R} V(r) (f(r))^2 \left| \nabla_{\mathbb{H}^N} \left(\frac{u_r}{f(r)} \right) \right|^2 \mathrm{d}v_{\mathbb{H}^N} \, . \end{aligned} \tag{3.3}$$

As a direct consequence of the above result, we tackle the non-radial scenario by the spherical harmonic method and we prove:

Corollary 3.1. *Let $(r^{N-1}V, r^{N-1}W)$ be a Bessel pair on $(0, \infty)$ with positive solution f on $(0, \infty)$. Then for all $u \in C_c^\infty(\mathbb{H}^N \setminus \{x_0\})$ there holds*

$$\begin{aligned} \int_{\mathbb{H}^N} V(r) |\Delta_{r,\mathbb{H}^N} u|^2 \, \mathrm{d}v_{\mathbb{H}^N} &= \int_{\mathbb{H}^N} W(r) |\nabla_{r,\mathbb{H}^N} u|^2 \, \mathrm{d}v_{\mathbb{H}^N} \\ &+ (N-1) \int_{\mathbb{H}^N} \left(\frac{V(r)}{\sinh^2 r} - \frac{V_r(r) \cosh r}{\sinh r} \right) |\nabla_{r,\mathbb{H}^N} u|^2 \, \mathrm{d}v_{\mathbb{H}^N} \\ &- (N-1) \int_{\mathbb{H}^N} V(r) \frac{f'}{f} \left(\coth r - \frac{1}{r} \right) |\nabla_{r,\mathbb{H}^N} u|^2 \, \mathrm{d}v_{\mathbb{H}^N} \end{aligned}$$

$$+\int_{\mathbb{H}^N} V(r)(f(r))^2\left|\nabla_{r,\mathbb{H}^N}\left(\frac{u_r}{f(r)}\right)\right|^2 \mathrm{d}v_{\mathbb{H}^N}$$

In Theorem 3.2 below we state the counterpart of Theorem 3.1 for functions not necessarily radial, under the extra condition (3.4) below:

Theorem 3.2. *Let $(r^{N-1}V, r^{N-1}W)$ be a Bessel pair on $(0,\infty)$ with positive solution f on $(0,\infty)$. Also assume $N \geq 5$ and V satisfies*

$$(N-5)\frac{V(r)}{\sinh^2 r} + 3\frac{V_r(r)\cosh r}{\sinh r} - V_{rr}(r) + (N-4)V(r) \geq 0. \tag{3.4}$$

Then for all $u \in C_c^\infty(\mathbb{H}^N \setminus \{x_0\})$ there holds

$$\begin{aligned}\int_{\mathbb{H}^N} V(r)|\Delta_{\mathbb{H}^N} u|^2 \,\mathrm{d}v_{\mathbb{H}^N} \geq& \int_{\mathbb{H}^N} W(r)|\nabla_{\mathbb{H}^N} u|^2 \,\mathrm{d}v_{\mathbb{H}^N}\\ &+ (N-1)\int_{\mathbb{H}^N}\left(\frac{V(r)}{\sinh^2 r} - \frac{V_r(r)\cosh r}{\sinh r}\right)|\nabla_{\mathbb{H}^N} u|^2 \,\mathrm{d}v_{\mathbb{H}^N}\\ &- (N-1)\int_{\mathbb{H}^N} V(r)\frac{f'}{f}\left(\coth r - \frac{1}{r}\right)|\nabla_{\mathbb{H}^N} u|^2 \,\mathrm{d}v_{\mathbb{H}^N}\\ &+ \int_{\mathbb{H}^N} V(r)(f(r))^2\left|\nabla_{\mathbb{H}^N}\left(\frac{u_r}{f(r)}\right)\right|^2 \mathrm{d}v_{\mathbb{H}^N}\,.\end{aligned} \tag{3.5}$$

Remark 3.1. *We remark that assumption (3.4) in Theorem 3.2 is not too restrictive to our purposes: we shall provide a remarkable family of (V,W) for which the assumption holds true in the proof of Theorem 2.1. On the other hand, an analogous assumption was required in the Euclidean space as well, see (3.2) and the comments just below.*

We conclude the section by stating an abstract version of Heisenberg-Pauli-Weyl uncertainty principle involving Bessel pairs which follows as a corollary from Corollary 3.1 (for radial operators) and from Theorem 3.2:

Theorem 3.3. *Let $(r^{N-1}V, r^{N-1}W)$ be a Bessel pair on $(0,\infty)$ with positive solution f on $(0,\infty)$ and set*

$$\tilde{W}(r) := W(r) + (N-1)\left(\frac{V(r)}{\sinh^2 r} - \frac{V_r(r)\cosh r}{\sinh r}\right) - (N-1)V(r)\frac{f'}{f}\left(\coth r - \frac{1}{r}\right).$$

Assume that $\tilde{W}(r) > 0$ for all $r > 0$, then there holds

$$\left(\int_{\mathbb{H}^N} V(r)|\Delta_{r,\mathbb{H}^N} u|^2 \,\mathrm{d}v_{\mathbb{H}^N}\right)\left(\int_{\mathbb{H}^N} \frac{|\nabla_{r,\mathbb{H}^N} u|^2}{\tilde{W}(r)} \,\mathrm{d}v_{\mathbb{H}^N}\right) \geq \left(\int_{\mathbb{H}^N} |\nabla_{r,\mathbb{H}^N} u|^2 \,\mathrm{d}v_{\mathbb{H}^N}\right)^2,$$

for all $u \in C_c^\infty(\mathbb{H}^N \setminus \{x_0\})$. Furthermore, if $N \geq 5$ and V satisfies (3.4), there holds

$$\left(\int_{\mathbb{H}^N} V(r)|\Delta_{\mathbb{H}^N} u|^2 \,\mathrm{d}v_{\mathbb{H}^N}\right)\left(\int_{\mathbb{H}^N} \frac{|\nabla_{\mathbb{H}^N} u|^2}{\tilde{W}(r)} \,\mathrm{d}v_{\mathbb{H}^N}\right) \geq \left(\int_{\mathbb{H}^N} |\nabla_{\mathbb{H}^N} u|^2 \,\mathrm{d}v_{\mathbb{H}^N}\right)^2,$$

for all $u \in C_c^\infty(\mathbb{H}^N \setminus \{x_0\})$.

Remark 3.2. *A non trivial example of pairs satisfying the assumptions of Theorem 3.3 is given by the family of Bessel pairs $(r^{N-1}, r^{N-1}W_\lambda)$, for all $0 \leq \lambda \leq \lambda_1(\mathbb{H}^N)$, defined in*

(4.1) *below and emploied in the proof of Theorem 2.1. Indeed, they satisfy condition* (3.4) *and give the function* $\tilde{W}$ *below:*

$$\tilde{W}_\lambda(r) = \lambda + h_N^2(\lambda)\frac{1}{r^2} + \left(\left(\frac{N}{2}\right)^2 - h_N^2(\lambda)\right)\frac{1}{\sinh^2 r} + \frac{\gamma_N(\lambda)\,h_N(\lambda)}{r}\left(\coth r - \frac{1}{r}\right)$$

which is positive in $(0,+\infty)$ *for all* $0 \le \lambda \le \lambda_1(\mathbb{H}^N)$. *In particular, with this pair, taking* $\lambda = 0$ *for simplicity, Theorem 3.3 yields*

(3.6)

$$\left(\int_{\mathbb{H}^N} |\Delta_{\mathbb{H}^N} u|^2\,\mathrm{d}v_{\mathbb{H}^N}\right)\left(\int_{\mathbb{H}^N} \frac{|\nabla_{\mathbb{H}^N} u|^2}{\frac{N^2}{4}\frac{1}{r^2} + \frac{N(N-1)}{2r}(\coth r - \frac{1}{r})}\,\mathrm{d}v_{\mathbb{H}^N}\right) \ge \left(\int_{\mathbb{H}^N} |\nabla_{\mathbb{H}^N} u|^2\,\mathrm{d}v_{\mathbb{H}^N}\right)^2,$$

for all $u \in C_c^\infty(\mathbb{H}^N \setminus \{x_0\})$. *The above inequality turns out to be more stringent than* (2.3) *thereby confirming the conjecture that* $\frac{N^2}{4}$ *is not the sharp constant in* (2.3).

4. Proofs of Theorems 2.1, 2.2, 2.3 and Corollaries 2.3,2.4

Proofs of Theorems 2.1 and 2.2. The proof follows, respectively, by applying Corollary 3.1 and Theorem 3.2 with the family of Bessel pairs $(r^{N-1}, r^{N-1}W_\lambda)$ with $0 \le \lambda \le \lambda_1(\mathbb{H}^N)$ and

$$\begin{aligned} W_\lambda(r) &:= \lambda + h_N^2(\lambda)\frac{1}{r^2} + \left(\left(\frac{N-2}{2}\right)^2 - h_N^2(\lambda)\right)\frac{1}{\sinh^2 r} \\ &+ \left(\frac{\gamma_N(\lambda)\,h_N(\lambda)}{r} + (N-1)\frac{\Psi_\lambda'(r)}{\Psi_\lambda(r)}\right)\left(\coth r - \frac{1}{r}\right) \qquad (r>0)\,, \end{aligned} \tag{4.1}$$

where $\gamma_N(\lambda)$ and $h_N(\lambda)$ are as defined in the statement of Theorem 3.1 and

$$\Psi_\lambda(r) := r^{-\frac{N-2}{2}}\left(\frac{\sinh r}{r}\right)^{-\frac{N-1+\gamma_N(\lambda)}{2}} \qquad (r>0)\,.$$

In particular, by noticing that

$$\Psi_\lambda'(r) = \Psi_\lambda(r)\left[\frac{h_N(\lambda)}{r} + \frac{1-N-\gamma_N(\lambda)}{2}\coth r\right],$$

$$\begin{aligned} \Psi_\lambda''(r) = \Psi_\lambda(r)\Big[&\frac{(1-N-\gamma_N(\lambda))^2}{4} + \frac{\gamma_N^2(\lambda)-1}{r^2} \\ &- \frac{(1-N-\gamma_N(\lambda))(1+N+\gamma_N(\lambda))}{4\sinh^2 r} + \frac{(1-N-\gamma_N(\lambda))h_N(\lambda)\coth r}{r}\Big] \end{aligned}$$

and recalling the definition of $\gamma_N(\lambda)$, it follows that $\Psi_\lambda(r)$ satisfies

$$(r^{N-1}\Psi_\lambda'(r))' + r^{N-1}W_\lambda(r)\Psi_\lambda(r) = 0 \quad \text{for } r>0\,,$$

namely $(r^{N-1}, r^{N-1}W_\lambda)$ is a Bessel pair with positive solution $\Psi_\lambda(r)$. See also [5, Lemma 6.2] where the functions Ψ_λ were originally introduced but exploited with different purposes. Finally, from Corollary 3.1 we deduce that, for all function $u \in C_c^\infty(B_R \setminus \{x_0\})$, there holds

$$\begin{aligned} \int_{B_R} |\Delta_{\mathbb{H}^N} u|^2\,\mathrm{d}v_{\mathbb{H}^N} &= \int_{B_R} W_\lambda(r)|\nabla_{\mathbb{H}^N} u|^2\,\mathrm{d}v_{\mathbb{H}^N} \\ &+ (N-1)\int_{B_R}\left(\frac{1}{\sinh^2 r}\right)|\nabla_{\mathbb{H}^N} u|^2\,\mathrm{d}v_{\mathbb{H}^N} \end{aligned}$$

$$- (N-1) \int_{B_R} \frac{\Psi_\lambda'(r)}{\Psi_\lambda(r)} \left(\coth r - \frac{1}{r} \right) |\nabla_{\mathbb{H}^N} u|^2 \, dv_{\mathbb{H}^N}$$
$$+ \int_{B_R} (\Psi_\lambda(r))^2 \left| \nabla_{\mathbb{H}^N} \left(\frac{u_r}{\Psi_\lambda(r)} \right) \right|^2 dv_{\mathbb{H}^N} \, .$$

By this, recalling (4.1), the proof of Theorem 2.1 follows. The proof of Theorem 2.2 works similarly by applying Theorem 3.2 since condition (3.4) holds for the Bessel pair $(r^{N-1}, r^{N-1}W_\lambda)$ if $N \geq 5$.

As concerns the proof of the fact that the constants $h_N^2(\lambda)$ and $\left[\left(\frac{N}{2}\right)^2 - h_N^2(\lambda) \right]$ are jointly sharp when $N \geq 5$, this follows by noticing that as $r \to 0$ we have

$$h_N^2(\lambda) \int_{\mathbb{H}^N} \frac{|\nabla_{\mathbb{H}^N} u|^2}{r^2} dv_{\mathbb{H}^N} + \left[\left(\frac{N}{2} \right)^2 - h_N^2(\lambda) \right] \int_{\mathbb{H}^N} \frac{|\nabla_{\mathbb{H}^N} u|^2}{\sinh^2 r} dv_{\mathbb{H}^N} \sim \frac{N^2}{4} \int_{\mathbb{H}^N} \frac{|\nabla_{\mathbb{H}^N} u|^2}{r^2} dv_{\mathbb{H}^N} \, .$$

Therefore, locally, we recover inequality (1.6) for $l = 1$; by this we readily infer that, for $h_N^2(\lambda)$ fixed, any larger constant in front of the term $\frac{|\nabla_{\mathbb{H}^N} u|^2}{\sinh^2 r}$ would contradict the optimality of the constant $\frac{N^2}{4}$ in (1.6) (when $l = 1$). □

Proof of Corollary 2.3. The proof follows from Corollary 2.1 by evaluating the term $\int_{\mathbb{H}^N} \frac{|\nabla_{\mathbb{H}^N} u|^2}{r^2} \, dv_{\mathbb{H}^N}$ with the aid of [17, Corollary 3.2] from which we know that

$$\int_{\mathbb{H}^N} \frac{|\nabla_{\mathbb{H}^N} u|^2}{r^2} \, dv_{\mathbb{H}^N} = \left(\frac{N-4}{2} \right)^2 \int_{\mathbb{H}^N} \frac{u^2}{r^4} \, dv_{\mathbb{H}^N}$$
$$+ \frac{(N-4)(N-1)}{2} \int_{\mathbb{H}^N} \frac{r \coth r - 1}{r^4} u^2 \, dv_{\mathbb{H}^N}$$
$$+ \int_{\mathbb{H}^N} \frac{1}{r^{N-2}} \left| \nabla_{\mathbb{H}^N} \left(r^{\frac{N-4}{2}} u \right) \right|^2 dv_{\mathbb{H}^N} \, .$$

for all $u \in C_c^\infty(\mathbb{H}^N \setminus \{x_0\})$. The proof for radial operators follows similarly since the above identity holds with the same constants for radial operators too. □

Proof of Corollary 2.4. Here the proof follows by combining Corollary 2.2 with [17, Theorem 1.4] according to which we know that

$$\int_{\mathbb{H}^N} |\nabla_{\mathbb{H}^N} u|^2 \, dv_{\mathbb{H}^N} = \left(\frac{N-1}{2} \right)^2 \int_{\mathbb{H}^N} u^2 \, dv_{\mathbb{H}^N}$$
$$+ \frac{1}{4} \int_{\mathbb{H}^N} \frac{u^2}{r^2} \, dv_{\mathbb{H}^N} + \frac{(N-1)(N-3)}{4} \int_{\mathbb{H}^N} \frac{u^2}{\sinh^2 r} \, dv_{\mathbb{H}^N}$$
$$+ \int_{\mathbb{H}^N} \frac{r}{(\sinh r)^{N-1}} \left| \nabla_{\mathbb{H}^N} \left(\frac{(\sinh r)^{\frac{N-1}{2}} u_r}{r^{\frac{1}{2}}} \right) \right|^2 dv_{\mathbb{H}^N} \, .$$

for all $u \in C_c^\infty(\mathbb{H}^N \setminus \{x_0\})$ and similarly for radial operators since the above identity holds with the same constants for radial operators too. □

Proof of Theorem 2.3. The proof is a simple application of Cauchy-Schwartz inequality combined with Theorem 2.2:

$$\int_{\mathbb{H}^N} |\nabla_{\mathbb{H}^N} u|^2 \, dv_{\mathbb{H}^N} = \int_{\mathbb{H}^N} r |\nabla_{\mathbb{H}^N} u| \frac{|\nabla_{\mathbb{H}^N} u|}{r} \, dv_{\mathbb{H}^N}$$

$$\leq \left(\int_{\mathbb{H}^N} r^2|\nabla_{\mathbb{H}^N} u|^2 \, \mathrm{d}v_{\mathbb{H}^N}\right)^{\frac{1}{2}} \underbrace{\left(\int_{\mathbb{H}^N} \frac{|\nabla_{\mathbb{H}^N} u|^2}{r^2} \, \mathrm{d}v_{\mathbb{H}^N}\right)^{\frac{1}{2}}}_{Using\ Theorem\ 2.2}$$

$$\leq \frac{1}{h_N(\lambda)} \left(\int_{\mathbb{H}^N} \left(|\Delta_{\mathbb{H}^N} u|^2 - \lambda|\nabla_{\mathbb{H}^N} u|^2\right) \mathrm{d}v_{\mathbb{H}^N}\right)^{\frac{1}{2}} \left(\int_{\mathbb{H}^N} r^2|\nabla_{\mathbb{H}^N} u|^2 \mathrm{d}v_{\mathbb{H}^N}\right)^{\frac{1}{2}}.$$

□

5. Proofs of Theorem 3.1, Corollary 3.1, Theorem 3.2 and Theorem 3.3

We shall begin with the proof of Theorem 3.1.

Proof of Theorem 3.1.
Let $u \in C_c^\infty(B_R \setminus \{x_0\})$ be a radial function, in terms of polar coordinates we have

$$\int_{B_R} V(r)|\Delta_{\mathbb{H}^N} u|^2 \, \mathrm{d}v_{\mathbb{H}^N} = N\omega_N \Bigg[\int_0^R V(r) u_{rr}^2 (\sinh r)^{N-1} \, \mathrm{d}r$$
$$+ (N-1)^2 \int_0^R V(r)(\coth r)^2 u_r^2 (\sinh r)^{N-1} \, \mathrm{d}r$$
$$+ 2(N-1) \int_0^R V(r) u_{rr} u_r (\coth r)(\sinh r)^{N-1} \, \mathrm{d}r \Bigg].$$

Now, applying integration by parts in the last term and setting $\nu = u_r$, we deduce

$$\int_{B_R} V(r)|\Delta_{\mathbb{H}^N} u|^2 \, \mathrm{d}v_{\mathbb{H}^N} = \int_{B_R} V(r)|\nabla_{\mathbb{H}^N} \nu|^2 \, \mathrm{d}v_{\mathbb{H}^N}$$
$$+ (N-1) \int_{B_R} \left(\frac{V(r)}{\sinh^2 r} - \frac{V_r(r)\cosh r}{\sinh r} \right) |\nu|^2 \, \mathrm{d}v_{\mathbb{H}^N}. \tag{5.1}$$

On the other hand, from Lemma 3.1 for the function ν we have

$$\int_{B_R} V(r)|\nabla_{\mathbb{H}^N} \nu|^2 \, \mathrm{d}v_{\mathbb{H}^N} = \int_{B_R} W(r)|\nu|^2 \, \mathrm{d}v_{\mathbb{H}^N} + \int_{B_R} V(r)(f(r))^2 \left|\nabla_{\mathbb{H}^N}\left(\frac{\nu}{f(r)}\right)\right|^2 \mathrm{d}v_{\mathbb{H}^N}$$
$$- (N-1) \int_{B_R} V(r) \frac{f'}{f} \left(\coth r - \frac{1}{r} \right) |\nu|^2 \, \mathrm{d}v_{\mathbb{H}^N}.$$

By using this identity into (5.1) and writing back in terms of u we deduce (3.3). □

Spherical harmonics.
Before going to prove Corollary 3.1 and Theorem 3.2, we shall mention some useful facts about spherical harmonics, see [28, Lemma 2.1] and [33, Ch. 4].

Let $u(x) = u(r, \Theta) \in C_c^\infty(\mathbb{H}^N)$, $r \in (0, \infty)$ and $\Theta \in \mathbb{S}^{N-1}$, we can write

$$u(r, \Theta) = \sum_{n=0}^{\infty} a_n(r) P_n(\Theta) \tag{5.2}$$

in $L^2(\mathbb{H}^N)$, where $\{P_n\}$ is an orthonormal system of spherical harmonics and

$$a_n(r) = \int_{\mathbb{S}^{N-1}} u(r, \Theta) P_n(\Theta) \, \mathrm{d}\Theta.$$

A spherical harmonic P_n of order n is the restriction to $\mathbb{S}^{N-1}$ of a homogeneous harmonic polynomial of degree n. Moreover, it satisfies

$$-\Delta_{\mathbb{S}^{N-1}} P_n = \lambda_n P_n$$

for all $n \in \mathbb{N} \cup \{0\}$, where $\lambda_n = (n^2 + (N-2)n)$ are the eigenvalues of Laplace Beltrami operator $-\Delta_{\mathbb{S}^{N-1}}$ on $\mathbb{S}^{N-1}$ with corresponding eigenspace dimension c_n. We note that $\lambda_n \geq N-1$ for $n \geq 1$, $\lambda_0 = 0$, $c_0 = 1$, $c_1 = N$ and for $n \geq 2$

$$c_n = \binom{N+n-1}{n} - \binom{N+n-3}{n-2}.$$

In a continuation let us also describe the Gradient and Laplace Beltrami operator in this setting. Now onward, to shorten the notations, we will use the notation $\psi(r) = \sinh r$. The following identities hold:

$$|\nabla_{\mathbb{H}^N} u|^2 = \sum_{n=0}^{\infty} {a'_n}^2 P_n^2 + \frac{a_n^2}{\psi^2} |\nabla_{\mathbb{S}^{N-1}} P_n|^2$$

and

$$(\Delta_{\mathbb{H}^N} u)^2 = \sum_{n=0}^{\infty} \left(a''_n + (N-1)\frac{\psi'}{\psi} a'_n \right)^2 P_n^2 + \sum_{n=0}^{\infty} \frac{a_n^2}{\psi^4} (\Delta_{\mathbb{S}^{N-1}} P_n)^2 \tag{5.3}$$
$$+ 2\sum_{n=0}^{\infty} \left(a''_n + (N-1)\frac{\psi'}{\psi} a'_n \right) \frac{a_n}{\psi^2} (\Delta_{\mathbb{S}^{N-1}} P_n) P_n.$$

Along with this the radial contribution of the operators will be:

$$|\nabla_{r,\mathbb{H}^N} u|^2 = \sum_{n=0}^{\infty} {a'_n}^2 P_n^2$$

and

$$(\Delta_{r,\mathbb{H}^N} u)^2 = \sum_{n=0}^{\infty} \left(a''_n + (N-1)\frac{\psi'}{\psi} a'_n \right)^2 P_n^2 \, .$$

Proof of Corollary 3.1. By spherical harmonics, we decompose u as in (5.2). Now, exploiting Theorem 3.1 for each a_n, we deduce

$$\int_{\mathbb{H}^N} V(r) |\Delta_{r,\mathbb{H}^N} u|^2 \, \mathrm{d}v_{\mathbb{H}^N} = \sum_{n=0}^{\infty} \int_0^{\infty} V(r) \left(a''_n + (N-1)\frac{\psi'}{\psi} a'_n \right)^2 \psi^{N-1} \, \mathrm{d}r$$
$$= \sum_{n=0}^{\infty} \Bigg[\int_0^{\infty} W {a'_n}^2 \psi^{N-1} \, \mathrm{d}r + \int_0^{\infty} V f^2 \left[\left(\frac{a'_n}{f} \right)' \right]^2 \psi^{N-1} \, \mathrm{d}r$$
$$- (N-1) \int_0^{\infty} V \frac{f'}{f} \left(\coth r - \frac{1}{r} \right) (a'_n)^2 \psi^{N-1} \, \mathrm{d}r$$
$$+ (N-1) \int_0^{\infty} V {a'_n}^2 \psi^{N-3} \, \mathrm{d}r - (N-1) \int_0^{\infty} V_r \psi' (a'_n)^2 \psi^{N-2} \, \mathrm{d}r \Bigg]$$
$$= \int_{\mathbb{H}^N} W(r) |\nabla_{r,\mathbb{H}^N} u|^2 \, \mathrm{d}v_{\mathbb{H}^N} + \int_{\mathbb{H}^N} V(r) (f(r))^2 \left| \nabla_{r,\mathbb{H}^N} \left(\frac{u_r}{f(r)} \right) \right|^2 \mathrm{d}v_{\mathbb{H}^N}$$

$$- (N-1)\int_{\mathbb{H}^N} V(r)\frac{f'}{f}\left(\coth r - \frac{1}{r}\right)|\nabla_{r,\mathbb{H}^N} u|^2 \,\mathrm{d}v_{\mathbb{H}^N}$$
$$+ (N-1)\int_{\mathbb{H}^N} \left(\frac{V(r)}{\sinh^2 r} - \frac{V_r(r)\cosh r}{\sinh r}\right)|\nabla_{r,\mathbb{H}^N} u|^2 \,\mathrm{d}v_{\mathbb{H}^N} .$$

This completes the proof. □

Proof of Theorem 3.2. Again, by spherical decomposition we can write u as in (5.2). Having defined $\psi(r) = \sinh r$, the following identities hold:

$$\frac{\psi'^2(r)}{\psi^2(r)} = 1 + \frac{1}{\psi^2(r)} \quad \text{and} \quad \psi'(r)^2 = 1 + \psi^2(r) \quad \text{for all } r > 0; \tag{5.4}$$

we shall use them in the proof frequently.

Step 1. In this step we decompose the l.h.s. of (3.5) and, using (5.3), we get:

$$\int_{\mathbb{H}^N} V(r)|\Delta_{\mathbb{H}^N} u|^2 \,\mathrm{d}v_{\mathbb{H}^N} = \sum_{n=0}^{\infty}\Bigg[\int_0^\infty V(r)\left(a_n'' + (N-1)\frac{\psi'}{\psi}a_n'\right)^2 \psi^{N-1}\,\mathrm{d}r$$
$$+ \lambda_n^2\int_0^\infty V(r)\frac{a_n^2}{\psi^4}\psi^{N-1}\,\mathrm{d}r - 2\,\lambda_n\int_0^\infty V(r)\left(a_n'' + (N-1)\frac{\psi'}{\psi}a_n'\right)\frac{a_n}{\psi^2}\psi^{N-1}\,\mathrm{d}r\Bigg].$$

On the other hand, exploiting Corollary 3.1 for each a_n, we deduce

$$\int_{\mathbb{H}^N} V(r)|\Delta_{\mathbb{H}^N} a_n|^2 \,\mathrm{d}v_{\mathbb{H}^N} = N\omega_N\int_0^\infty V(r)\left(a_n'' + (N-1)\frac{\psi'}{\psi}a_n'\right)^2\psi^{N-1}\,\mathrm{d}r$$
$$= N\omega_N\Bigg[\int_0^\infty W {a_n'}^2\psi^{N-1}\,\mathrm{d}r + \int_0^\infty V f^2\left[\left(\frac{a_n'}{f}\right)'\right]^2\psi^{N-1}\,\mathrm{d}r$$
$$- (N-1)\int_0^\infty V\frac{f'}{f}\left(\coth r - \frac{1}{r}\right)(a_n')^2\psi^{N-1}\,\mathrm{d}r$$
$$+ (N-1)\int_0^\infty V{a_n'}^2\psi^{N-3}\,\mathrm{d}r - (N-1)\int_0^\infty V_r\psi'(a_n')^2\psi^{N-2}\,\mathrm{d}r\Bigg].$$

Step 2. In this step we compute the r.h.s of inequality (3.5):

$$\int_{\mathbb{H}^N} W(r)|\nabla_{\mathbb{H}^N} u|^2 \,\mathrm{d}v_{\mathbb{H}^N} + \int_{\mathbb{H}^N} V(r)(f(r))^2\left|\nabla_{\mathbb{H}^N}\left(\frac{u_r}{f(r)}\right)\right|^2 \mathrm{d}v_{\mathbb{H}^N}$$
$$- (N-1)\int_{\mathbb{H}^N} V(r)\frac{f'}{f}\left(\coth r - \frac{1}{r}\right)|\nabla_{\mathbb{H}^N} u|^2 \,\mathrm{d}v_{\mathbb{H}^N}$$
$$+ (N-1)\int_{\mathbb{H}^N}\left(\frac{V(r)}{\sinh^2 r} - \frac{V_r(r)\cosh r}{\sinh r}\right)|\nabla_{\mathbb{H}^N} u|^2 \,\mathrm{d}v_{\mathbb{H}^N}$$
$$= \sum_{n=0}^{\infty}\Bigg[\int_0^\infty W{a_n'}^2\psi^{N-1}\,\mathrm{d}r + \lambda_n\int_0^\infty W a_n^2\psi^{N-3}\,\mathrm{d}r + \int_0^\infty V f^2\left[\left(\frac{a_n'}{f}\right)'\right]^2\psi^{N-1}\,\mathrm{d}r$$
$$+ \lambda_n\int_0^\infty V{a_n'}^2\psi^{N-3}\,\mathrm{d}r - (N-1)\int_0^\infty V\frac{f'}{f}\left(\coth r - \frac{1}{r}\right){a_n'}^2\psi^{N-1}\,\mathrm{d}r$$
$$- (N-1)\lambda_n\int_0^\infty V\frac{f'}{f}\left(\coth r - \frac{1}{r}\right)a_n^2\psi^{N-3}\,\mathrm{d}r$$

$$+ (N-1)\int_0^\infty \left(\frac{V(r)}{\psi^2} - \frac{\psi'}{\psi}V_r(r)\right)(a_n')^2\psi^{N-1}\,\mathrm{d}r$$
$$\left. + (N-1)\lambda_n\int_0^\infty \left(\frac{V(r)}{\psi^2} - \frac{\psi'}{\psi}V_r(r)\right)\frac{a_n^2}{\psi^2}\psi^{N-1}\,\mathrm{d}r\right].$$

Step 3. Subtracting the r.h.s. of the identities obtained in Step 1 and Step 2 we obtain the expression below that we denote by $\mathcal{B}$ the following quantity:

(5.5)
$$\mathcal{B} := \sum_{n=0}^\infty \left[\lambda_n^2\int_0^\infty V(r)\frac{a_n^2}{\psi^4}\psi^{N-1}\,\mathrm{d}r - 2\,\lambda_n\int_0^\infty V(r)\left(a_n'' + (N-1)\frac{\psi'}{\psi}a_n'\right)\frac{a_n}{\psi^2}\psi^{N-1}\,\mathrm{d}r\right.$$
$$- \lambda_n\int_0^\infty W(r)\frac{a_n^2}{\psi^2}\psi^{N-1}\,\mathrm{d}r - (N-1)\lambda_n\int_0^\infty \left(\frac{V(r)}{\psi^2} - \frac{\psi'}{\psi}V_r(r)\right)\frac{a_n^2}{\psi^2}\psi^{N-1}\,\mathrm{d}r$$
$$\left. - \lambda_n\int_0^\infty V(a_n')^2\psi^{N-3}\,\mathrm{d}r + (N-1)\lambda_n\int_0^\infty V\frac{f'}{f}\left(\coth r - \frac{1}{r}\right)a_n^2\psi^{N-3}\,\mathrm{d}r\right].$$

In the steps below we shall show that $\mathcal{B}$ is non-negative, this will prove inequality (3.5). To this aim, we establish some preliminary identities.

Step 4. Set

$$\mathcal{I}_1 := \int_0^\infty V{a_n'}^2\psi^{N-3}\,\mathrm{d}r$$

and define $b_n(r) := \frac{a_n(r)}{\psi(r)}$, by Leibniz rule we have $a_n' = b_n'\psi + b_n\psi'$. Using this and by parts formula, we obtain

$$\mathcal{I}_1 = \int_0^\infty V{b_n'}^2\psi^{N-1}\,\mathrm{d}r - (N-3)\int_0^\infty Vb_n^2\psi^{N-3}\,\mathrm{d}r \tag{5.6}$$
$$- \int_0^\infty V_r b_n^2\psi'\psi^{N-2}\,\mathrm{d}r - (N-2)\int_0^\infty Vb_n^2\psi^{N-1}\,\mathrm{d}r.$$

Then applying Lemma 3.1 for b_n, we deduce

$$\int_0^\infty V{b_n'}^2\psi^{N-1}\,\mathrm{d}r = \int_0^\infty Wb_n^2\psi^{N-1}\,\mathrm{d}r + \int_0^\infty Vf^2\left[\left(\frac{b_n}{f}\right)'\right]^2\psi^{N-1}\,\mathrm{d}r$$
$$- (N-1)\int_0^\infty V\frac{f'}{f}\left(\coth r - \frac{1}{r}\right)b_n^2\psi^{N-1}\,\mathrm{d}r.$$

Using this estimate into (5.6) and writing b_n in terms of a_n, we have

$$\mathcal{I}_1 = \int_0^\infty Wa_n^2\psi^{N-3}\,\mathrm{d}r + \int_0^\infty Vf^2\left[\left(\frac{a_n}{f\psi}\right)'\right]^2\psi^{N-1}\,\mathrm{d}r - (N-1)\mathcal{I} \tag{5.7}$$
$$- (N-3)\int_0^\infty Va_n^2\psi^{N-5}\,\mathrm{d}r - \int_0^\infty V_r a_n^2\psi'\psi^{N-4}\,\mathrm{d}r - (N-2)\int_0^\infty Va_n^2\psi^{N-3}\,\mathrm{d}r,$$

where $\mathcal{I} = \int_0^\infty V\frac{f'}{f}\big(\coth r - \frac{1}{r}\big)a_n^2\psi^{N-3}\,\mathrm{d}r$.

Step 5. In this step we evaluate the terms

$$\mathcal{I}_2 := \int_0^\infty Va_n''a_n\psi^{N-3}\,\mathrm{d}r \quad \text{and} \quad \mathcal{I}_3 := \int_0^\infty Va_n'a_n\psi'\psi^{N-4}\,\mathrm{d}r$$

by means of integration by parts formula. Recalling (5.7), a computation provides

$$\mathcal{I}_2 = \frac{1}{2}\int_0^\infty V_{rr} a_n^2 \psi^{N-3}\, \mathrm{d}r + \frac{(N-3)}{2}\int_0^\infty V_r a_n^2 \psi' \psi^{N-4}\, \mathrm{d}r - \mathcal{I}_1 - (N-3)\mathcal{I}_3, \tag{5.8}$$

furthermore we have

$$\mathcal{I}_3 = -\frac{1}{2}\int_0^\infty V_r a_n^2 \psi' \psi^{N-4}\, \mathrm{d}r - \frac{(N-4)}{2}\int_0^\infty V a_n^2 \psi^{N-5}\, \mathrm{d}r - \frac{(N-3)}{2}\int_0^\infty V a_n^2 \psi^{N-3}\, \mathrm{d}r. \tag{5.9}$$

Step 6. Next using (5.8) into (5.5) we rewrite $\mathcal{B}$ as follows:

$$\begin{aligned}
\mathcal{B} &= \sum_{n=0}^{\infty}\Bigg[\lambda_n^2 \int_0^\infty V a_n^2 \psi^{N-5}\, \mathrm{d}r - 2\lambda_n \mathcal{I}_2 - 2(N-1)\lambda_n \mathcal{I}_3 - \lambda_n \int_0^\infty W a_n^2 \psi^{N-3}\, \mathrm{d}r \\
&- (N-1)\lambda_n \int_0^\infty V a_n^2 \psi^{N-5}\, \mathrm{d}r + (N-1)\lambda_n \int_0^\infty V_r a_n^2 \psi' \psi^{N-4}\, \mathrm{d}r - \lambda_n \mathcal{I}_1 + (N-1)\lambda_n \mathcal{I}\Bigg] \\
&= \sum_{n=0}^{\infty}\Bigg[\lambda_n^2 \int_0^\infty V a_n^2 \psi^{N-5}\, \mathrm{d}r - 2\lambda_n \Bigg\{\frac{1}{2}\int_0^\infty V_{rr} a_n^2 \psi^{N-3}\, \mathrm{d}r \\
&+ \frac{(N-3)}{2}\int_0^\infty V_r a_n^2 \psi' \psi^{N-4}\, \mathrm{d}r - \mathcal{I}_1 - (N-3)\mathcal{I}_3\Bigg\} - 2(N-1)\lambda_n \mathcal{I}_3 \\
&- \lambda_n \int_0^\infty W a_n^2 \psi^{N-3}\, \mathrm{d}r - (N-1)\lambda_n \int_0^\infty V a_n^2 \psi^{N-5}\, \mathrm{d}r \\
&+ (N-1)\lambda_n \int_0^\infty V_r a_n^2 \psi' \psi^{N-4}\, \mathrm{d}r - \lambda_n \mathcal{I}_1 + (N-1)\lambda_n \mathcal{I}\Bigg].
\end{aligned}$$

Simplifying the identity obtained above and recalling (5.9), we get

$$\begin{aligned}
\mathcal{B} &= \sum_{n=0}^{\infty}\Bigg[\lambda_n^2 \int_0^\infty V a_n^2 \psi^{N-5}\, \mathrm{d}r - \lambda_n \int_0^\infty V_{rr} a_n^2 \psi^{N-3}\, \mathrm{d}r + 2\lambda_n \int_0^\infty V_r a_n^2 \psi' \psi^{N-4}\, \mathrm{d}r \\
&+ \lambda_n \mathcal{I}_1 - 4\lambda_n \mathcal{I}_3 - \lambda_n \int_0^\infty W a_n^2 \psi^{N-3}\, \mathrm{d}r - (N-1)\lambda_n \int_0^\infty V a_n^2 \psi^{N-5}\, \mathrm{d}r + (N-1)\lambda_n \mathcal{I}\Bigg] \\
&= \sum_{n=0}^{\infty}\Bigg[\lambda_n^2 \int_0^\infty V a_n^2 \psi^{N-5}\, \mathrm{d}r - \lambda_n \int_0^\infty V_{rr} a_n^2 \psi^{N-3}\, \mathrm{d}r + 2\lambda_n \int_0^\infty V_r a_n^2 \psi' \psi^{N-4}\, \mathrm{d}r \\
&+ \lambda_n \mathcal{I}_1 - 4\lambda_n \Bigg\{-\frac{1}{2}\int_0^\infty V_r a_n^2 \psi' \psi^{N-4}\, \mathrm{d}r - \frac{(N-4)}{2}\int_0^\infty V a_n^2 \psi^{N-5}\, \mathrm{d}r \\
&- \frac{(N-3)}{2}\int_0^\infty V a_n^2 \psi^{N-3}\, \mathrm{d}r\Bigg\} - \lambda_n \int_0^\infty W a_n^2 \psi^{N-3}\, \mathrm{d}r \\
&- (N-1)\lambda_n \int_0^\infty V a_n^2 \psi^{N-5}\, \mathrm{d}r + (N-1)\lambda_n \mathcal{I}\Bigg].
\end{aligned}$$

By means of a further simplification we obtain

$$\begin{aligned}
\mathcal{B} &= \sum_{n=0}^{\infty}\Bigg[\lambda_n(\lambda_n + N - 7)\int_0^\infty V a_n^2 \psi^{N-5}\, \mathrm{d}r - \lambda_n \int_0^\infty V_{rr} a_n^2 \psi^{N-3}\, \mathrm{d}r \\
&+ 4\lambda_n \int_0^\infty V_r a_n^2 \psi' \psi^{N-4}\, \mathrm{d}r + 2\lambda_n (N-3)\int_0^\infty V a_n^2 \psi^{N-3}\, \mathrm{d}r
\end{aligned}$$

$$
\begin{aligned}
&- \lambda_n \int_0^\infty W a_n^2 \psi^{N-3}\,\mathrm{d}r + \lambda_n \mathcal{I}_1 + (N-1)\lambda_n \mathcal{I}\Bigg] \\
&= \sum_{n=0}^\infty \Bigg[\lambda_n(\lambda_n + N - 7) \int_0^\infty V a_n^2 \psi^{N-5}\,\mathrm{d}r - \lambda_n \int_0^\infty V_{rr} a_n^2 \psi^{N-3}\,\mathrm{d}r \\
&+ 4\lambda_n \int_0^\infty V_r a_n^2 \psi' \psi^{N-4}\,\mathrm{d}r + 2\lambda_n (N-3) \int_0^\infty V a_n^2 \psi^{N-3}\,\mathrm{d}r - \lambda_n \int_0^\infty W a_n^2 \psi^{N-3}\,\mathrm{d}r \\
&+ \lambda_n \Bigg\{ \int_0^\infty W a_n^2 \psi^{N-3}\,\mathrm{d}r + \int_0^\infty V f^2 \left[\left(\frac{a_n}{f\psi}\right)'\right]^2 \psi^{N-1}\,\mathrm{d}r - (N-3) \int_0^\infty V a_n^2 \psi^{N-5}\,\mathrm{d}r \\
&- \int_0^\infty V_r a_n^2 \psi' \psi^{N-4}\,\mathrm{d}r - (N-2) \int_0^\infty V a_n^2 \psi^{N-3}\,\mathrm{d}r \Bigg\}\Bigg],
\end{aligned}
$$

where in the last line we have exploited the definitions of $\mathcal{I}$ and $\mathcal{I}_1$.

Step 7. We conclude the proof by estimating $\mathcal{B}$:

$$
\begin{aligned}
\mathcal{B} &= \sum_{n=0}^\infty \Bigg[\lambda_n(\lambda_n - 4) \int_0^\infty V a_n^2 \psi^{N-5}\,\mathrm{d}r - \lambda_n \int_0^\infty V_{rr} a_n^2 \psi^{N-3}\,\mathrm{d}r \\
&+ 3\lambda_n \int_0^\infty V_r a_n^2 \psi' \psi^{N-4}\,\mathrm{d}r + \lambda_n (N-4) \int_0^\infty V a_n^2 \psi^{N-3}\,\mathrm{d}r \\
&+ \lambda_n \int_0^\infty V f^2 \left[\left(\frac{a_n}{f\psi}\right)'\right]^2 \psi^{N-1}\,\mathrm{d}r\Bigg] \\
&= \sum_{n=0}^\infty \lambda_n \Bigg[(\lambda_n - 4) \int_0^\infty V a_n^2 \psi^{N-5}\,\mathrm{d}r + \int_0^\infty \left\{\frac{3V_r\psi'}{\psi} - V_{rr}\right\} a_n^2 \psi^{N-3}\,\mathrm{d}r \\
&+ (N-4) \int_0^\infty V a_n^2 \psi^{N-3}\,\mathrm{d}r + \int_0^\infty V f^2 \left[\left(\frac{a_n}{f\psi}\right)'\right]^2 \psi^{N-1}\,\mathrm{d}r\Bigg] \\
&\geq \sum_{n=0}^\infty \lambda_n \Bigg[\int_0^\infty \left\{(N-5)\frac{V}{\psi^2} + 3\frac{V_r\psi'}{\psi} - V_{rr} + (N-4)V\right\} a_n^2 \psi^{N-3}\,\mathrm{d}r \\
&+ \int_0^\infty V f^2 \left[\left(\frac{a_n}{f\psi}\right)'\right]^2 \psi^{N-1}\,\mathrm{d}r\Bigg],
\end{aligned}
$$

where in the last line we have used $\lambda_n \geq N-1$ for all $n \geq 1$. Hence, $\mathcal{B}$ eventually turns out to be non-negative due to the hypothesis (3.4) and the non negativity of the last term. This concludes the proof. □

Proof of Theorem 3.3. We give the proof in the general case, the proof for radial operators follows with the same argument but by exploiting Corollary 3.1 instead of Theorem 3.2. First, under the assumptions of Theorem 3.3, from Theorem 3.2 we deduce that for all $u \in C_c^\infty(\mathbb{H}^N \setminus \{x_0\})$ there holds

$$\int_{\mathbb{H}^N} V(r)|\Delta_{\mathbb{H}^N} u|^2\,\mathrm{d}v_{\mathbb{H}^N} \geq \int_{\mathbb{H}^N} \tilde{W}(r)|\nabla_{\mathbb{H}^N} u|^2\,\mathrm{d}v_{\mathbb{H}^N}\,.$$

Finally, we use Hölder inequality and the above inequality to get:

$$\left(\int_{\mathbb{H}^N} |\nabla_{\mathbb{H}^N} u|^2\,\mathrm{d}v_{\mathbb{H}^N}\right)^2 = \left(\int_{\mathbb{H}^N} \sqrt{\tilde{W}(r)}|\nabla_{\mathbb{H}^N} u| \frac{|\nabla_{\mathbb{H}^N} u|}{\sqrt{\tilde{W}(r)}}\,\mathrm{d}v_{\mathbb{H}^N}\right)^2$$

$$\leq \left(\int_{\mathbb{H}^N} \frac{|\nabla_{\mathbb{H}^N} u|^2}{\tilde{W}(r)} \, \mathrm{d}v_{\mathbb{H}^N} \right) \left(\int_{\mathbb{H}^N} \tilde{W}(r) |\nabla_{\mathbb{H}^N} u|^2 \, \mathrm{d}v_{\mathbb{H}^N} \right)$$
$$\leq \left(\int_{\mathbb{H}^N} \frac{|\nabla_{\mathbb{H}^N} u|^2}{\tilde{W}(r)} \, \mathrm{d}v_{\mathbb{H}^N} \right) \left(\int_{\mathbb{H}^N} V(r) |\Delta_{\mathbb{H}^N} u|^2 \, \mathrm{d}v_{\mathbb{H}^N} \right)$$

which is the thesis. □

6. Concluding remarks

In this section we briefly discuss possibile extensions of our proofs and results to more general manifolds under appropriate curvature bounds.

The methods exploited in this article are in principle applicable to obtain Hardy-Rellich and Poincaré type identities, and inequalities on *Riemannian models*. An N-dimensional Riemannian model (M, g) is an N-dimensional Riemannian manifold admitting a pole $o \in M$ and whose metric g is given in spherical coordinates around o by

$$\mathrm{d}s^2 = \mathrm{d}r^2 + \psi^2(r) \, \mathrm{d}\omega^2,$$

where $d\omega^2$ denotes the canonical metric on the unit sphere $\mathbb{S}^{N-1}$ and ψ satisfies:

$$\begin{aligned} &\psi \text{ is a } C^\infty \text{ nonnegative function on } [0, +\infty), \text{ positive on } (0, +\infty) \\ &\text{such that } \psi'(0) = 1 \text{ and } \psi^{(2k)}(0) = 0 \text{ for all } k \geq 0\,. \end{aligned}$$

These conditions on ψ ensure that the manifold is smooth and the metric at the pole o is given by the euclidean metric. The coordinate r, by construction, represents the Riemannian distance from the pole o. In particular, all the assumptions above are satisfied by $\psi(r) = r$ and by $\psi(r) = \sinh(r)$: in the first case M coincides with the euclidean space $\mathbb{R}^N$, in the latter with the hyperbolic space $\mathbb{H}^N$.

We stress that our arguments relies on the careful analysis of the radial part of the Laplace-Beltrami operator on the hyperbolic space and exploiting the spectral analysis of $-\Delta_{\mathbb{S}^n}$ along with the notion of Bessel pair. In fact the Laplace-Beltrami operator on Riemannian models is given by:

$$\Delta_g = \underbrace{\frac{\partial^2}{\partial r^2} + (N-1) \frac{\psi'(r)}{\psi(r)} \frac{\partial}{\partial r}}_{\text{Radial part of the Laplacian}} + \frac{1}{\psi^2} \Delta_{\mathbb{S}^N}$$

which coincides with (2.1) for $\psi(r) = \sinh(r)$. Therefore, one can handle the radial part of the Laplace-Beltrami operator on M as done in the previous sections for $\mathbb{H}^N$, taking into account appropriately the terms involving the radial functions ψ, ψ'. Clearly, if $M \neq \mathbb{H}^N$, one cannot take advantage of the fundamental identities (5.4) which hold for $\psi(r) = \sinh(r)$; in fact, we expect some of the improved terms in the resulting inequalities (identities) would involve *curvature* terms depending on the functions ψ and ψ', as it happens for the analogous of (1.3) on more general manifolds, including Riemannian models as particular cases, see [4, Theorem 2.5]. Although, it's worth noticing that the passage from models to more general manifolds is not obvious in this higher order setting due to the lack of comparison principles.

Appendix: a family of improved Hardy-Poincaré equalities

In this appendix we present a family of improved Hardy-Poincaré equalities which follows as a corollary from [17, Theorem 3.2], i.e. Lemma 3.1 above, by exploiting the family of Bessel pairs $(r^{N-1}, r^{N-1}W_\lambda)$ introduced in Section 4 for all $0 \le \lambda \le \lambda_1(\mathbb{H}^N)$. If $\lambda = \lambda_1(\mathbb{H}^N)$ the identity we got is already known from [17, Theorem 3.2] while for $0 \le \lambda < \lambda_1(\mathbb{H}^N)$ it is new and improves (1.2), i.e. [5, Theorem 2.1], with the presence of an exact remainder term. The precise statement of the result reads as follows:

Theorem 6.1. *Let $N \ge 2$. For all $0 \le \lambda \le \lambda_1(\mathbb{H}^N) = \left(\frac{N-1}{2}\right)^2$ and for all $u \in C_c^\infty(\mathbb{H}^N \setminus \{x_0\})$ there holds*

$$\int_{\mathbb{H}^N} |\nabla_{\mathbb{H}^N} u|^2 \, dv_{\mathbb{H}^N} = \lambda \int_{\mathbb{H}^N} u^2 \, dv_{\mathbb{H}^N} + h_N^2(\lambda) \int_{\mathbb{H}^N} \frac{u^2}{r^2} \, dv_{\mathbb{H}^N}$$
$$+ \left[\frac{(N-2)^2}{4} - h_N^2(\lambda)\right] \int_{\mathbb{H}^N} \frac{u^2}{\sinh^2 r} \, dv_{\mathbb{H}^N} + \gamma_N(\lambda) h_N(\lambda) \int_{\mathbb{H}^N} \frac{r\coth r - 1}{r^2} u^2 \, dv_{\mathbb{H}^N}$$
$$+ \int_{\mathbb{H}^N} (\Psi_\lambda(r))^2 \left|\nabla_{\mathbb{H}^N}\left(\frac{u}{\Psi_\lambda(r)}\right)\right|^2 dv_{\mathbb{H}^N}$$

and for the radial operator we have

$$\int_{\mathbb{H}^N} |\nabla_{r,\mathbb{H}^N} u|^2 \, dv_{\mathbb{H}^N} = \lambda \int_{\mathbb{H}^N} u^2 \, dv_{\mathbb{H}^N} + h_N^2(\lambda) \int_{\mathbb{H}^N} \frac{u^2}{r^2} \, dv_{\mathbb{H}^N}$$
$$+ \left[\frac{(N-2)^2}{4} - h_N^2(\lambda)\right] \int_{\mathbb{H}^N} \frac{u^2}{\sinh^2 r} \, dv_{\mathbb{H}^N} + \gamma_N(\lambda) h_N(\lambda) \int_{\mathbb{H}^N} \frac{r\coth r - 1}{r^2} u^2 \, dv_{\mathbb{H}^N}$$
$$+ \int_{\mathbb{H}^N} (\Psi_\lambda(r))^2 \left|\nabla_{r,\mathbb{H}^N}\left(\frac{u}{\Psi_\lambda(r)}\right)\right|^2 dv_{\mathbb{H}^N} \, ,$$

where $\gamma_N(\lambda) := \sqrt{(N-1)^2 - 4\lambda}$, $h_N(\lambda) := \frac{\gamma_N(\lambda)+1}{2}$ and $\Psi_\lambda(r) := r^{-\frac{N-2}{2}} \left(\frac{\sinh r}{r}\right)^{-\frac{N-1+\gamma_N(\lambda)}{2}}$.

Proof. The proof follows by applying [17, Theorem 3.2], i.e. Lemma 3.1 above, with the Bessel pairs $(r^{N-1}, r^{N-1}W_\lambda)$, where W_λ is as given in (4.1). □

In particular, for $\lambda = N-2$ Theorem 6.1 yields the Hardy identity below which improves (1.4):

Corollary 6.1. *Let $N \ge 3$. For all $u \in C_c^\infty(\mathbb{H}^N \setminus \{x_0\})$ there holds*

$$\int_{\mathbb{H}^N} |\nabla_{\mathbb{H}^N} u|^2 \, dv_{\mathbb{H}^N} = \left(\frac{N-2}{2}\right)^2 \int_{\mathbb{H}^N} \frac{u^2}{r^2} \, dv_{\mathbb{H}^N} + (N-2) \int_{\mathbb{H}^N} u^2 \, dv_{\mathbb{H}^N}$$
$$+ \frac{(N-2)(N-3)}{2} \int_{\mathbb{H}^N} \frac{r\coth r - 1}{r^2} u^2 \, dv_{\mathbb{H}^N}$$
$$+ \int_{\mathbb{H}^N} \left(\frac{r^{1/2}}{\sinh r}\right)^{2(N-2)} \left|\nabla_{\mathbb{H}^N}\left(\frac{(\sinh r)^{N-2} u}{r^{(N-2)/2}}\right)\right|^2 dv_{\mathbb{H}^N}.$$

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