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Development of a New Method for Modeling a Physical System in a Rational Expression

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Abstract — The presented paper shows a new method for modeling of a physical system in a rational expression. The method is based on a hybrid combination of the conventional least square method (LSM) and a robust recursive procedure. In the first step of the method, a sampling function is employed to establish a system of linear equations. Then, the QR decomposition method and a very fast iterative technique are simultaneously used to determine the unknown parameters. Also, it is shown that by regarding a scalar gain parameter, the stability of the final model can be controlled. To verify the performance of the presented method, several theoretical and practical examples are examined and the obtained results are compared with the simulation and measurement data.

Index Terms — Least square method, rational approximation, recursive regression

I. INTRODUCTION

Most of the time, analysis of many practical problems in engineering is very difficult based on the first principle [1-2]. On the other hand, simulation of physical structures using the full-wave simulators is commonly time-consuming, due to the complexity of them [3]. Moreover, to study of a complex structure in the presence of nonlinear components, a hybrid combination of frequency and time domain are necessary [4]. Hence, a helpful intuitive concept for understanding the properties of a complicated structures is its equivalent circuit model. Commonly, a frequency domain response of these structures is available. So, it is desired to consider an equivalent circuit for them [5-6].

There are two types of modeling techniques for the theoretical and practical systems, including the white-box and black-box modeling. In a white-box modeling approach, some features of the under-studying system are known. Unlike the first one, in black-box modeling, only the input and output responses of the under-studying system are available. It is clear that the black-box modeling is more difficult. So far, several black-box modeling approaches of a physical system are introduced in literature. Some of them include brute force lumped segmentation modeling [7], the Loewner method [8, 9], passive reduced-order interconnect modeling algorithm (PRIMA) [10], vector fitting (VF) algorithm [11, 12], hybrid combination of data integration and least square method [13], matrix rational approximation (MRA) technique [14, 15], compact difference (CD) method [16], integral congruent transformation (ICT) [17, 18], etc. The efficiency, accuracy and complexity of each of these methods are different. Therefore, the pros and cons of the currently available techniques in the literature show that the modeling of a system is still a challengeable problem.

In this work, combination of the conventional least square method (LSM) and a robust recursive regression procedure is employed for modeling of a theoretical and physical system. For this purpose, using a sampling function, the data of a dynamic system is expressed into a system of linear equations. In the converting process, a scalar gain parameter is employed to guarantee the stability of the final model. Since, the equation system itself is biased and due to the sensitivity of the final model with respect to the computational errors, the accuracy of the conventional LSM technique is low. Therefore, the QR factorization method with a recursive regression technique is simultaneously employed to conquer the mentioned problems. Also, it is shown that the proposed method converges rapidly for a very small number of iterations. To assess the performance of the introduced method, several theoretical and practical cases are examined.

The organization of the presented work is as follows. In section II, the theoretical of the proposed method is mathematically described. In section III, several test cases are studied, and the acquired results are compared with those obtained by the simulation and measurement data. Finally, section IV concludes the paper.

II. MATHEMATIC DESCRIPTION

The frequency response of any physical system can be expressed as the following rational function.

$$H(s)=\sum_{n=1}^{N}\frac{r_n}{s-p_n}+r_0 \tag{1}$$

where N, r_n, p_n, r_0, $H(s)$ and s are the number of poles, residues, poles, constant term, response of the system and Laplace variable defined by $s=j\omega$, in which ω is the angular frequency. It is common to show $H(s)$ as the ratio of two polynomials as follows.

$$H(s)=\frac{a_0+a_1s+...+a_Ns^N}{1+b_1s+...+b_Ns^N} \tag{2}$$

In the present form, it is assumed that the degree of numerator and denominator polynomials are equal. For practical applications, the real parts of poles must be negative due to stability condition. Therefore, the following closed-loop system (Figure 1) can be considered for the under-studying system.

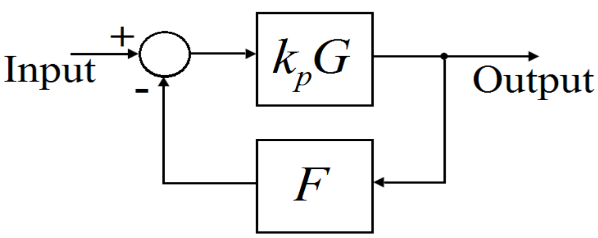

Fig. 1. Block diagram of a closed-loop system

The forward, backward and scalar gain of the system are shown with $G(s)$, $F(s)$ and $k_p\geq 1$, respectively. The closed-loop transfer function (TF) is as follows.

$$TF(s)=\frac{k_pG(s)}{1+k_pG(s)F(s)} \tag{3}$$

Typically, the location of transfer function's poles is changed by varying values of the scalar gain k_p [19]. Hence, the stability of the system can be controlled by the parameter k_p. The new expression of Eq. (2) can be obtained by comparing (2) with (3).

$$H(s)=\frac{N(s)}{D(s)}=\frac{k_pa_0+k_pa_1s+...+k_pa_Ns^N}{1+k_pb_1s+...+k_pb_Ns^N} \tag{4}$$

Multiplying both sides of Eq. (4) by $D(s)$, leading to the following expression.

$$\begin{aligned}&\left(k_pa_0+k_pa_1s+...+k_pa_Ns^N\right)-\\&H(s)\left(k_pb_1s+...+k_pb_Ns^N\right)=H(s)\end{aligned} \tag{5}$$

It is well-known from the sampling theory that the transfer function $H(s)$ can be reconstructed using only M samples [20].

$$H_r(s)=\sum_{m=1}^{M}H(s_m)S(s-s_m) \tag{6}$$

where $H_r(s)$ is the reconstructed transfer function, and $S(x)$ is a sampling function described by.

$$S(x)=\frac{\sin(Mx/2)}{M\ \sin(x/2)} \tag{7}$$

The samples of the transfer function are taken at M points. It is shown in [13] that the minimum value of M can be determined as.

$$M\geq\frac{8N}{\omega_{\max}-\omega_{\min}} \tag{8}$$

So, Eq. (5) can be expressed in matrix form as.

$$\mathbf{A}_{M\times(2N+1)}\mathbf{X}_{(2N+1)\times 1}=\mathbf{B}_{M\times 1} \tag{9}$$

where,

$$\mathbf{X}=\begin{bmatrix}a_0 & \cdots & a_N & b_1 & \cdots & b_N\end{bmatrix}^T \tag{10}$$

$$\mathbf{B}=\begin{bmatrix}H(s_1) & \cdots & H(s_M)\end{bmatrix}^T \tag{11}$$

$$\mathbf{A}_m=k_p\begin{bmatrix}\mathbf{A}_{m1} & \mathbf{A}_{m2}\end{bmatrix}^T \tag{12a}$$

$$\mathbf{A}_{m1}=\begin{bmatrix}1 & s_m & \cdots & s_m^N\end{bmatrix} \tag{12b}$$

$$\mathbf{A}_{m2}=\begin{bmatrix}-s_mH(s_m) & \cdots & -s_m^NH(s_m)\end{bmatrix} \tag{12c}$$

where m denotes the m-th data sample. The unknown vector $\mathbf{X}$ can be determined using the conventional least square method (LSM) as follows [21].

$$\mathbf{X}=\left(\mathbf{A}^T\mathbf{A}\right)^{-1}\mathbf{A}^T\mathbf{B} \tag{13}$$

In (13), it is supposed that the importance of each equation is the same. Since, $H(s)$ is multiplied with $D(s)$ in (4), the system equation (9) itself is biased [22]. Moreover, it is shown in [13] that the conventional LSM provides a low accuracy solution, due to the sensitivity of solution with respect to the numerical errors. To mitigate the mentioned problem, several techniques are proposed such as Iteratively Reweighted Least Squares (IRLS), the Sanathanan-Koerner (SK) method and the Noda (Noda) technique [22]. Although, these techniques have been able to solve this problem to some extent, but the high computational cost and sometimes low accuracy of the final solution have not been solved so far. On the other hand, these approaches does not prevent the ill-conditioning of the system [13]. Additionally, in these techniques, it is assumed that k_p=1, and these methods do not guarantee the stability of the final model. To overcome the mentioned challenges, first, the system equation (9) is rewritten as.

$$\mathbf{G}_{2M\times(2N+1)}\mathbf{X}_{(2N+1)\times 1}=\mathbf{C}_{2M\times 1} \tag{14}$$

where,

$$\mathbf{G}_{(2m-1)\times n} = \mathrm{real}\left(\mathbf{A}_{m\times n}\right); \quad \begin{matrix} n = 1,...,2N+1 \\ m = 1,...,2M \end{matrix} \tag{15a}$$

$$\mathbf{G}_{2m\times n} = \mathrm{imag}\left(\mathbf{A}_{m\times n}\right); \quad \begin{matrix} n = 1,...,2N+1 \\ m = 1,...,2M \end{matrix} \tag{15b}$$

$$\mathbf{C}_{2m-1} = \mathrm{real}\left(\mathbf{B}_{m}\right); \quad m = 1,...,2M \tag{16a}$$

$$\mathbf{C}_{2m} = \mathrm{imag}\left(\mathbf{B}_{m}\right); \quad m = 1,...,2M \tag{16b}$$

Also, in order to decrease the numerical errors, a QR decomposition is used as [23].

$$\mathbf{G} = \mathbf{QR} \tag{17}$$

where **Q**, **R** are the orthogonal and upper triangular matrix of the new coefficient matrix **G**, respectively. Using **Q** and **R**, the system equation (14) is converted to the new expression [23].

$$\mathbf{RX} = \mathbf{Q}^{T}\mathbf{C} \tag{18}$$

The objective function that should be minimized can be considered as.

$$F\left(\mathbf{X}\right) = \frac{1}{2}\left\|\mathbf{RX} - \mathbf{Q}^{T}\mathbf{C}\right\|_{2}^{2} \tag{19}$$

where $\|.\|_2$ is the norm-2. It is worth noting that $F(\mathbf{X})$ is a convex function [23]. So, the following equation is a sufficient condition for optimality.

$$\nabla F\left(\mathbf{X}\right) = 0 \rightarrow \mathbf{R}^{T}\mathbf{RX} - \mathbf{R}^{T}\mathbf{Q}^{T}\mathbf{C} = 0 \tag{20}$$

Since $\mathbf{R}^{\mathrm{T}}\mathbf{R}$ is a positive semi-definite matrix, so it has only positive eigenvalues [23]. As a result, the solution of (20) can be regarded as the following recursive expression.

$$\mathbf{X}\left(t+1\right) = \mathbf{X}\left(t\right) - \zeta\nabla F\left(\mathbf{X}\left(t\right)\right) \tag{21}$$

Substituting the gradient of $F(\mathbf{X}(t))$ into (21) gives

$$\mathbf{X}\left(t+1\right) = \mathbf{X}\left(t\right) - \zeta\left(\mathbf{RX}\left(t\right) - \mathbf{Q}^{T}\mathbf{C}\right) \\ t = 1,...,T \tag{22}$$

where t, T and $\zeta>0$ are the iteration index, total number of iterations and relation parameter, respectively. It is clear that Eq. (22) has the correct fixed points $\mathbf{X}(t+1)=\mathbf{X}(t)$, and $\mathbf{X}(t)$ is an approximate solution of (18). By defining the error $\mathbf{E}(t)=\mathbf{X}(t)-\mathbf{X}$, in which **X** is the exact solution, the recursive expression of the error can be found as follows.

$$\mathbf{E}\left(t+1\right) = \left(\mathbf{1} - \zeta\mathbf{R}\right)\mathbf{E}\left(t\right) \tag{23}$$

in which **1** is the identity matrix. The norm-2 of $\mathbf{E}(t+1)$ can be calculated as.

$$\left\|\mathbf{E}\left(t+1\right)\right\|_{2} = \\ \left\|\left(\mathbf{1} - \zeta\mathbf{R}\right)\mathbf{E}\left(t\right)\right\|_{2} \leq \left\|\mathbf{1} - \zeta\mathbf{R}\right\|_{2}\left\|\mathbf{E}\left(t\right)\right\|_{2} \tag{24}$$

It is obvious that if $\|\mathbf{1}-\zeta\mathbf{R}\|_2<1$, the method will be convergent. If **R** is a symmetric positive definite matrix, the error tends to zero if $|1-\zeta\lambda_i|<1$, in which λ_i's are the eigenvalues of **R** [24]. Nevertheless, in general case, it can be mathematically proved that the solution of (22) will be converges if and only if [24].

$$\zeta = \frac{2}{\lambda_{\min}\left(\mathbf{R}\right) + \lambda_{\max}\left(\mathbf{R}\right)} \tag{25}$$

where λ_{min} and λ_{max} are the minimum and maximum eigenvalues of **R**, respectively. For the cases that $\lambda_{min}(\mathbf{R})+\lambda_{max}(\mathbf{R})$ is zero, the best value of ζ is $1/\|\mathbf{R}\|_2$. Our studies show that for the practical applications, a high accuracy solution can be achieved by choosing $T\leq 3$. So, the proposed method is very fast.

As mentioned before, by changing the scalar gain k_p, the stability condition could be met. For this purpose, first, the poles are calculated by assuming $k_p=1$. Then, by plotting the root locus diagram, the acceptable value of it can be determined, in which guarantees the stability condition.

Our studies show that for most of the practical applications, when a stability condition is met, the passivity is also established. However, for other cases, the passivity condition can be acquired through the conventional two-step methods introduced in [22]. Moreover, the proposed method can be easily developed for Multi-Input-Multi-Output systems (MIMO) using the similar procedure introduced in [22].

III. RESULTS VERIFICATION AND DISCUSSION

In this section, several test cases are examined to verify the performance of the proposed method.

A. Noisy Data

In the first case, a synthetic transfer function $H(s)$ is considered. This function has 16 poles, which are reported in Table 1. The magnitude and phase of the reconstructed function $H_r(s)$ using the proposed method are compared with the prescribed function $H(s)$ in Figures (2) and (3), respectively. It is worth noting that $H_r(s)$ is only synthesized with 12 poles, and the accuracy of the obtained results is good over a wide frequency range 2GHz≤ f ≤30GHz.

Table 1. Poles and residues of TF of the first case

Poles (GHz)	Residues (GHz)
-0.6132±j3.4551	-0.9877±j0.0809
-0.3940±j7.3758	-0.2067±j0.0131
-0.0880±j14.3024	-0.1382±j0.0145
-0.4097±j17.7864	-0.1182±j0.0166
-0.2991±j28.4622	-0.2426±j0.0145
-0.6447±j35.2669	-0.4043±j0.0297
-1.0135±j37.9655	-0.6787±j0.1465
-0.5711±j57.4748	-0.2626±j0.1037
r_0=0.1	

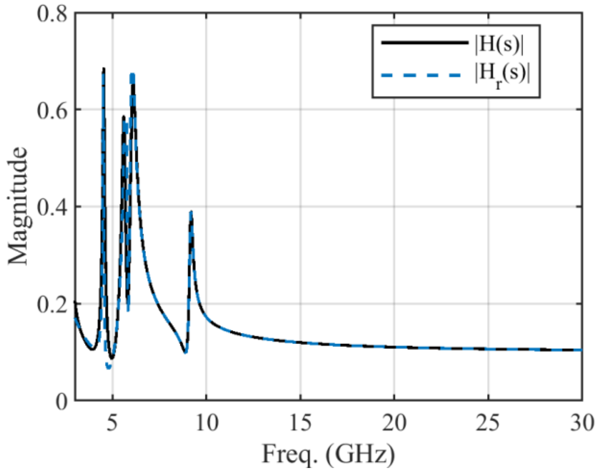

Fig. 2. The magnitude of synthesized TF with 12 poles.

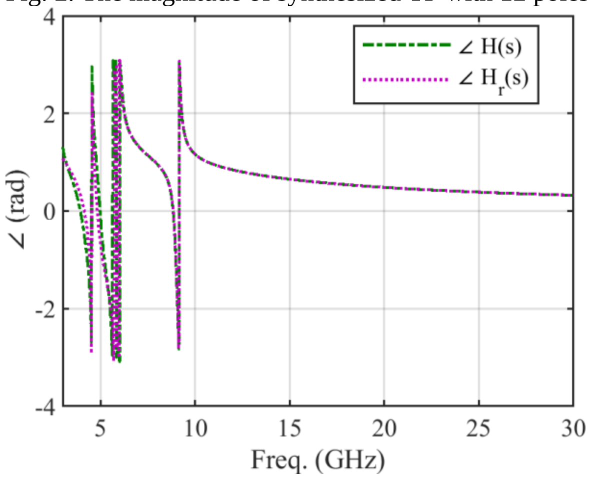

Fig. 3. The phase of synthesized TF with 12 poles.

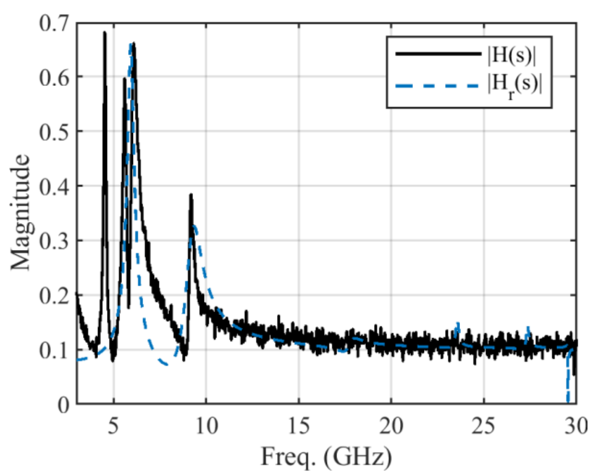

Fig. 4. The magnitude of synthesized TF in presence of noise.

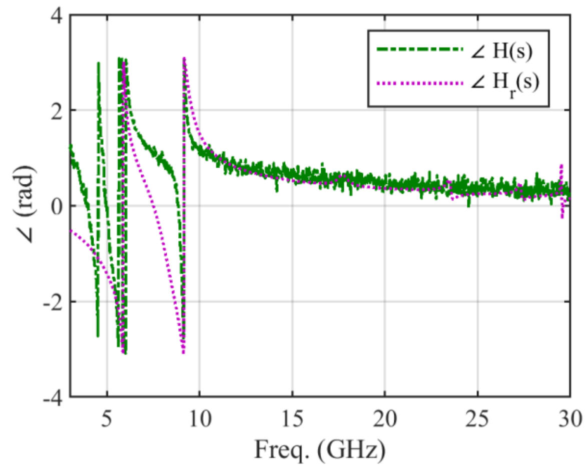

Fig. 5. The phase of synthesized TF in presence of noise.

Now, we want to investigate the performance of the method in the presence of noise. For this purpose, both real and imaginary parts of $H(s)$ are infected by white Gaussian noise with signal to noise ratio (SNR) about 20 dB. The proposed method is applied to the noise-infected data, and the obtained results are depicted in Figures (4), (5). Several fluctuations are seen in the lower frequency bands. For the proposed and other methods which are established based on the Least Square (LS) technique, the noisy data worsens the biasing effect of the LS solution. This is due to the fact that, noise perturbs the original position of the poles. To overcome this problem, it may require additional iterations to converge. But, additional iteration does not guarantee increased accuracy. Although, the accuracy of the method is low for the lower frequencies, but it is acceptable over the higher frequencies range. Hence, similar to the other techniques, the proposed method may be vulnerable in the presence of noise.

B. Smalls Spherical Helix Antenna

Electrically small antennas are widely used in electromagnetic compatibility (EMC) applications. The target of the second case is to equate an equivalent circuit model for a small spherical helix antenna over a ground plane. The number of arms, number of turns and total length of antenna are 3, 1.58 and 64.07cm, respectively. More details of the antenna can be found in [25]. The proposed method is applied to the measurement data of input impedance of the antenna. The synthesized results of real and imaginary parts of the under-studying antenna are depicted in Figure (6). The obtained model is synthesized using 6 poles. A very good agreement is seen between the measured and synthesized impedance of the antenna. After determining the equivalent model, return loss of the antenna can be calculated. In other words, after fitting the data of input impedance of the antenna in rational form, the equivalent circuit can be derived, and the other required parameters, such as return loss, can be easily determined. The synthesizing procedure can be done using well-known software like ADS or microwave office.

Figure (7) shows a comparison between the calculated return loss using the model and those obtained by measurement. A very small frequency shift is seen. However, the accuracy of the proposed model is very good.

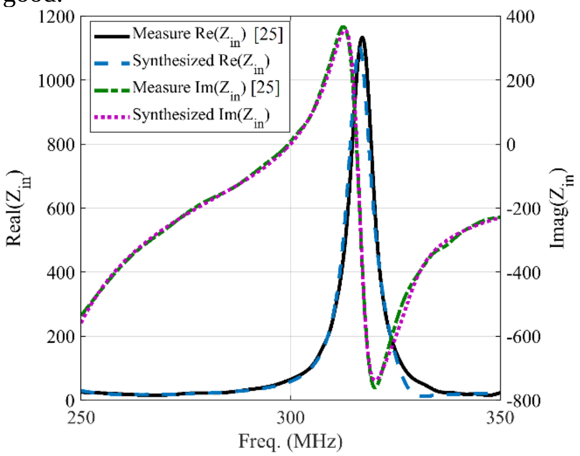

Fig. 6. The obtained real and imaginary parts of input impedance of the small antenna.

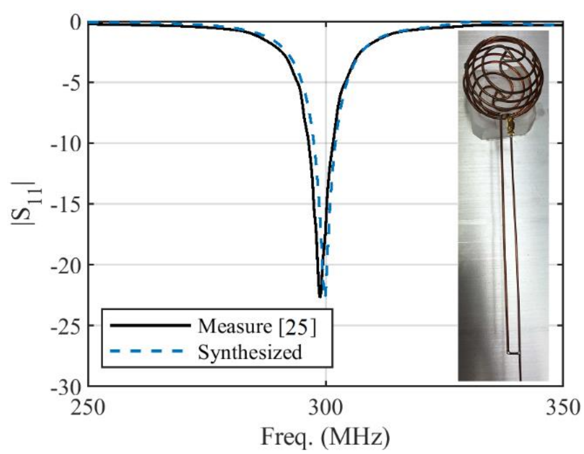

Fig. 7. The obtained return loss of the small antenna.

C. Tschebyscheff Band-Pass Filter

Filters are extensively used in the electronic and communications systems. In the third case, a band-pass Tschebyscheff filter over a frequency range $0 \leq f \leq 3$GHz is regarded. The frequency response of the filter is generated using Advanced Design System (ADS). The lower, upper passband corners frequency, and the lower, upper stopband frequency edges of the desired response are 1GHz, 2GHz, 0.5GHz and 2.5GHz, respectively. Also, passband and stopband attenuation are 3dB and 20 dB, respectively. After applying the proposed method, the synthesized results, including the input impedance and scattering parameters, are plotted in Figures (8), (9). The synthesizing procedure of the filter response is done using 6 poles. It is seen that the accuracy of the method is very well.

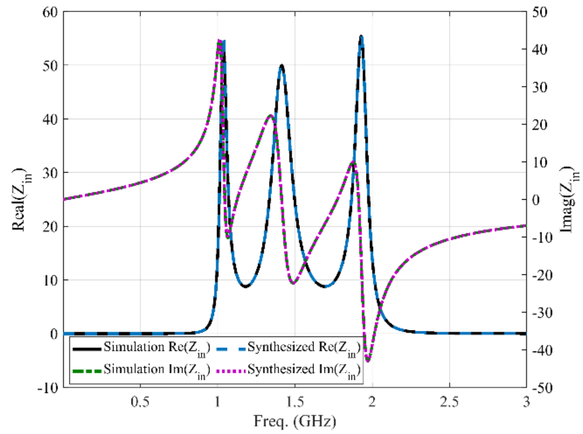

Fig. 8. The obtained real and imaginary parts of input impedance of the passband filter.

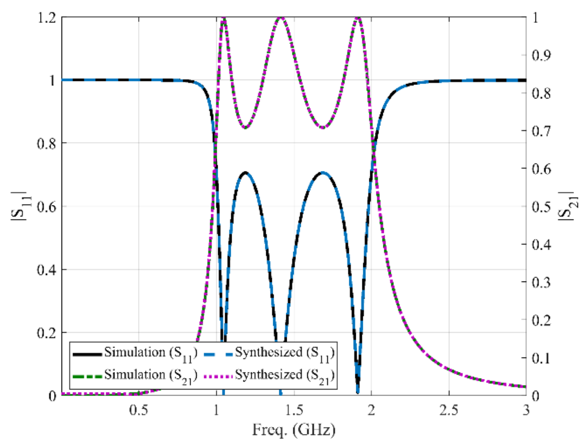

Fig. 9. The obtained frequency response of the passband filter.

D. Substrate Integrated Waveguide Modeling

Waveguides, including substrate integrated waveguide (SIW), are typically used in various applications in microwave engineering. Determining an equivalent circuit of a waveguide is complicated. In this example, the scattering responses of a non-uniform SIW over a high frequency range 14GHz$\leq f \leq$18GHz, including S_{11} and S_{21}, are considered as the input data. The dielectric permittivity, loss tangent, substrate thickness, length of substrate, load and source impedances, diameter of vias, distance between vias and the minimum and maximum width of SIW are 3.66, 0.0037, 0.254 mm, 44 mm, 50 Ω, 1mm, 2mm, 10mm and

20 mm, respectively. In Figure (10), the synthesized results using the introduced method and measured data reported in [26] are compared. The accuracy of S_{21} is very good. A deviation is seen in the magnitude of S_{11} in the minimal frequencies. The reflection coefficient shows two resonances around frequencies 14.8 GHz and 16.7 GHz. This is probably due to the fact that, the one or more zeros of the fitted $H(s)$ in Eq. (2) are not properly matched in the modeling procedure. To improve the accuracy, the order of the model (N) in Eq. (1) or (2) can be increased. However, it can be seen that the obtained results are in a reasonable agreement.

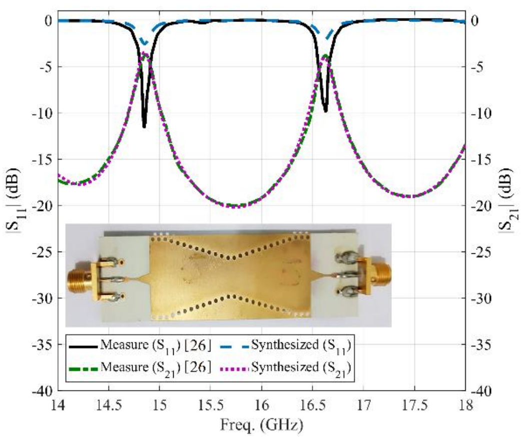

Fig. 10. The synthesized frequency response of the SIW.

Table 2. The important parameters of all studied cases

Case#	I		II	III	IV
	Noise-free	**Noisy**			
N	12	15	6	6	8
k_p	1	1	1	1	1
MSE	2.6e-3	2.4e-2	1.1e-3	4.5e-6	3.4e-4
τ	2.0e9	1.0e6	7.9e3	3.41e2	2.3e5
ζ	0.29	0.26	0.42	0.43	0.42
time (ms)	29	47	9.5	9.1	13.8
T	1	3	1	1	1

The important parameters of all studied cases, including the number of poles, scalar gain k_p, mean square error (MSE), condition number (τ), relation parameter (ζ), required time and the total number of iterations (T), are reported in Table 2. In all examples, the stability condition is met for k_p=1. For all studied cases, accuracy of the proposed method are very good. However, the error of the noisy data is a little high. It should be noted that the condition number shows the sensitivity of the proposed method with respect to the computational error (i.e., ill-conditioning). It is seen that for all examples, the condition number is an acceptable range in comparison to the conventional LSM [13]. The condition number required by the conventional LSM technique for the first to last example are 3.46e163, 2.34e63, 1.73e62 and 1.67e94 respectively. It is seen that the condition number of the proposed method is very lower than the conventional LSM technique. Since, the total number of iteration for all cases is low, the running time is also low. Hence, the method is very fast.

Table 3 contains a comparison of the proposed method and other introduced techniques in the literature, including Levy Method (LM), Weighted Least Square (WLS), Sanathanan Koerner (SK), Noda Iteration (NI), Levenberg-Marquardt method (LMM), and Data Integration (DI). The first five techniques are described in [22] and the last one is introduced in [13]. Comparison factors include the complexity, accuracy, efficiency, and noise resistance.

Table 3. The comparison of the introduced and other methods

	Complexity	Accuracy	Efficiency	Noise Resistance
This Work	middle	high	high	no
LM	low	middle	high	no
WLS	middle	middle	low	no
SK	high	middle	middle	no
NI	high	middle	middle	no
LMM	high	low	low	no
DI	middle	middle	high	yes

It is obvious that any iterative technique like WLS, SK, NI and LMM are complex. Although the proposed approach is an iterative technique, but the number of iterations is very low (lower than three iterations) in comparison to the other ones. The computational cost of the non-iterative methods or the iterative methods with low iterations like the proposed method, LM and DI techniques, are very low. So, the efficiency of these methods will be high. The accuracy of each mentioned methods in this table can be concluded by the presented examples of each technique [13], [22]. It is directly expressed in [13], [22] that only DI method can be used for modeling of the noise-infected data. The performance of the other ones is weak in the presence of noisy data.

IV. CONCLUSION

In the presented work, a new method is proposed for modeling of a prescribed data in a rational approximation. This method is developed based on a hybrid combination of the conventional LSM, QR factorization and a very fast recursive regression technique. In the modeling procedure, a scalar gain parameter is defined to control the stability of the final model. Several theoretical and practical examples are investigated. Comparison of the obtained results shows that the introduced method is efficient, which provide a reasonable model with a very good accuracy.

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