A Transport Theory Route to the Dirac Equation

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Abstract
Starting from the similarity between the spherical harmonics approximation of order one to the linear transport equation (usually referred as P_1 approximation) and the Klein-Gordon equation of the quantum physics, an extended set of equations is introduced, which is proved to be equivalent to the Dirac equation with imaginary mass. Conversely, when a real mass is restored into the extended P_1 system, a new equation is obtained, whose solutions are superposition of the spinors for a $\frac{1}{2}-$spin particle and the corresponding antiparticle.

KEYWORDS
linear transport theory, P_1 approximation, Dirac equation

1. Introduction

Analogies in mathematical models are often found between fields of physics widely different. A "likeness" can be just curiosity, but sometimes it brings a new point of view on a phenomenon. For example, Poisson brackets of classical mechanics have clear analogies with commutators of linear operators, and the discovery of this analogy brought deep progress in the study of quantum mechanics. In a completely different field, the Author, about 40 years ago, collaborated to study a technique to solve the linear transport equation, which was based on the analogy with the multi-group diffusion equation (the A_N method, developed together with Prof. P. Ravetto (Coppa and Ravetto (1982))).
The present paper studies the analogy between the spherical harmonics approximation of order one to the linear transport equation and the famous Dirac equation for the electron. The spherical harmonics expansion of the angular dependence of the phase-space density (usually referred as P_N method) is a well-known technique employed for the solution of the linear transport equation (Davison, Sykes, and Cohen (1957)). According to this method, the angular flux $\varphi\left(\boldsymbol{x},\boldsymbol{v},t\right)$ is written as:

$$\varphi\left(\boldsymbol{x},\boldsymbol{v},t\right)=\sum_{n=0}^{N}\sum_{\beta=-n}^{n}\varphi_n^{\beta}\left(\boldsymbol{x},|\boldsymbol{v}|,t\right)Y_n^{\beta}\left(\boldsymbol{\Omega}\right), \tag{1}$$

being $\boldsymbol{\Omega}=\boldsymbol{v}/|\boldsymbol{v}|$. The order of the approximation is fixed by N. At the lowest order of practical interest, for $N=1$, the angular flux is approximated as:

$$\varphi\left(\boldsymbol{x},|\boldsymbol{v}|,\boldsymbol{\Omega},t\right)=\frac{1}{4\pi}\left\{\phi\left(\boldsymbol{x},|\boldsymbol{v}|,t\right)+3\boldsymbol{\Omega}\cdot\boldsymbol{J}(\boldsymbol{x},|\boldsymbol{v}|,t)\right\} \tag{2}$$

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where $\phi = \int \varphi \mathrm{d}\Omega$ and $\boldsymbol{J} = \int \boldsymbol{\Omega} \varphi \mathrm{d}\Omega$ represent the particle flux and current, respectively. In this work, the monokinetic problem, in which the particles undergo absorption and isotropic scattering with no change in the norm of the velocity, is considered, and the equations for the P_1 approximation are simply (Meghreblian and Holmes (1960)):

$$\begin{cases} \frac{1}{v}\partial_t \phi & = & -\mathrm{div}\boldsymbol{J} - \Sigma_a \phi + S, \\ \frac{1}{v}\partial_t \boldsymbol{J} & = & -\frac{1}{3}\nabla\phi - (\Sigma_a + \Sigma_\mathrm{s})\,\boldsymbol{J}. \end{cases} \tag{3}$$

where Σ_a and Σ_s are the absorption and the scattering cross sections, and S is the (isotropic) source. As it is well known, the P_1 system (3) is connected to the telegrapher's equation for transmission lines. In particular, if one takes the one-dimensional version of Eqs. (3) with $S = 0$, after defining the partial fluxes $\varphi_\pm$ as

$$\varphi_\pm(x,t) = \frac{\phi\,(x,t) \pm \sqrt{3} J_x(x,t)}{2}, \tag{4}$$

one obtains readily:

$$\begin{cases} \frac{1}{v}\partial_t \varphi_+ + \frac{1}{\sqrt{3}}\partial_x \varphi_+ & = & -\left(\Sigma_a + \frac{\Sigma_s}{2}\right)\varphi_+ + \frac{\Sigma_s}{2}\varphi_-, \\ \frac{1}{v}\partial_t \varphi_- - \frac{1}{\sqrt{3}}\partial_x \varphi_- & = & -\left(\Sigma_a + \frac{\Sigma_s}{2}\right)\varphi_- + \frac{\Sigma_s}{2}\varphi_+. \end{cases} \tag{5}$$

Equations (5) describe the transport of particles moving in x direction with velocity $\pm v/\sqrt{3}$, in presence of absorption and scattering, whose effect is reverting the direction of the motion. A similar phenomenon happens in a transmission line, in which waves can give origin to signals traveling in the opposite direction due to the resistance of the line. Instead, if the three-dimensional equations are considered, by eliminating $\boldsymbol{J}$ from Eqs.(3) one obtains the classic telegrapher's (Heizler (2010)):

$$\frac{1}{v^2}\partial_t^2\phi - \frac{1}{3}\nabla^2\phi + \frac{2}{v}\left(\Sigma_a + \frac{1}{2}\Sigma_s\right)\partial_t\phi + (\Sigma_a + \Sigma_s)\,\Sigma_a\phi = 0. \tag{6}$$

As it was already noticed (Chatzarakis, Livieratos, and Miliaras (2012)), the telegrapher's equation is related to the Klein-Gordon equation of the relativistic quantum mechanics (Cottingham and Greenwood (2007)). Moreover, the Klein-Gordon equation is connected with the Dirac equation for $\frac{1}{2}$–spin particles (Dirac (1988)); however, the original P_1 equations (3) are not completely equivalent to the Dirac equation, and a suitable extension of P_1 equations, as the one presented in Sect. 2, is needed. In Sect. 3 it is proved that the extension of the P_1 equation is equivalent to a Dirac-like equation with imaginary mass. Finally, by restoring the real mass into the extended P_1 system, in Sect. 4 a new equation is derived, which can be decoupled into two Dirac equations for particles having same mass and opposite charge (Sect. 5).

2. The extended P_1 system

We start by rewriting Eqs.(3) for $S = 0$ in an equivalent form, which is more suitable for the purposes of the work. To do this, after multiplying both Eqs.(3) by $\mathrm{e}^{(\Sigma_a+\Sigma_s/2)t}$, one readily obtains:

$$\begin{cases} \frac{1}{v}\partial_t\left[\phi\mathrm{e}^{(\Sigma_a+\Sigma_s/2)t}\right] & = & -\mathrm{div}\left[\boldsymbol{J}\mathrm{e}^{(\Sigma_a+\Sigma_s/2)t}\right] + \frac{\Sigma_s}{2}\phi\mathrm{e}^{(\Sigma_a+\Sigma_s/2)t} \\ \frac{1}{v}\partial_t\left[\boldsymbol{J}^{(\Sigma_a+\Sigma_s/2)t}\right] & = & -\frac{1}{3}\nabla\left[\boldsymbol{J}\mathrm{e}^{(\Sigma_a+\Sigma_s/2)t}\right] - \frac{\Sigma_s}{2}\boldsymbol{J}\mathrm{e}^{(\Sigma_a+\Sigma_s/2)t} \end{cases} \tag{7}$$

or:

$$\begin{cases} \dfrac{\partial\widetilde{\phi}}{\partial\widetilde{t}} &= -\mathrm{div}\widetilde{\boldsymbol{J}} + \mu\widetilde{\phi}, \\ \dfrac{\partial\widetilde{\boldsymbol{J}}}{\partial\widetilde{t}} &= -\nabla\widetilde{\phi} - \mu\widetilde{\boldsymbol{J}}, \end{cases} \tag{8}$$

being $\mu = \frac{\sqrt{3}}{2}\Sigma_s$, $\widetilde{t} = \frac{v}{\sqrt{3}}t$, $\widetilde{\phi} = \phi\mathrm{e}^{(\Sigma_a+\Sigma_s/2)t}$ and $\widetilde{\boldsymbol{J}} = \sqrt{3}\boldsymbol{J}\mathrm{e}^{(\Sigma_a+\Sigma_s/2)t}$. In the following, the tilde is dropped everywhere for simplicity of notation. Taking the time derivative of the first equation (8), one obtains:

$$\begin{aligned} \partial_t^2\phi &= -\mathrm{div}\left(\partial_t\boldsymbol{J}\right) + \mu\partial_t\phi \\ &= -\mathrm{div}\left(-\nabla\phi - \mu\boldsymbol{J}\right) + \mu\left(-\mathrm{div}\boldsymbol{J} + \mu\phi\right), \end{aligned} \tag{9}$$

and eventually

$$\Box\phi = \mu^2\phi. \tag{10}$$

As this equation is similar to the Klein-Gordon equation, $\left(\Box + m^2\right)\phi = 0$, the only difference being a sign, one can wonder if the fields ϕ, J_x, J_y, J_z can be represented as the four component of a Dirac-like equation (Dirac (1988)), in which the mass is imaginary ($m = \mathrm{i}\mu$). If this were true, also $\boldsymbol{J}$ should satisfy Eq. (10); in reality, taking the time derivative of the second equation (8), one has:

$$\begin{aligned} \partial_t^2\boldsymbol{J} &= -\nabla\left(\partial_t\phi\right) - \mu\partial_t\boldsymbol{J} \\ &= -\nabla\left(-\mathrm{div}\boldsymbol{J} + \mu\phi\right) - \mu\left(-\nabla\phi - \mu\boldsymbol{J}\right) \\ &= \nabla\mathrm{div}\boldsymbol{J} + \mu^2\boldsymbol{J} = \nabla^2\boldsymbol{J} + \mu^2\boldsymbol{J} + \nabla\times\nabla\times\boldsymbol{J}, \end{aligned} \tag{11}$$

and finally

$$\Box\boldsymbol{J} = \mu^2\boldsymbol{J} + \nabla\times\nabla\times\boldsymbol{J}. \tag{12}$$

Equation (12) is similar to the Klein-Gordon equation, but it contains an extra term. In order to obtain Eq. (10) also for $\boldsymbol{J}$, the system (8) must be suitably modified, e.g., by adding a term, $-\nabla\times\boldsymbol{A}$, to the second equation:

$$\partial_t\boldsymbol{J} = -\nabla\phi - \mu\boldsymbol{J} - \nabla\times\boldsymbol{A}; \tag{13}$$

in this way, the expression for $\mathrm{div}\left(\partial_t\boldsymbol{J}\right)$ is unchanged and Eqs.(9, 10) are still valid. If the calculation of $\partial_t^2\boldsymbol{J}$ is repeated, now one has:

$$\begin{aligned} \partial_t^2\boldsymbol{J} &= -\nabla\partial_t\phi - \mu\partial_t\boldsymbol{J} - \nabla\times(\partial_t\boldsymbol{A}) \\ &= -\nabla\left(-\mathrm{div}\boldsymbol{J} + \mu\phi\right) - \mu\left(-\nabla\phi - \mu\boldsymbol{J} - \nabla\times\boldsymbol{A}\right) - \nabla\times(\partial_t\boldsymbol{A}) \\ &= \nabla^2\boldsymbol{J} + \nabla\times\nabla\times\boldsymbol{J} + \mu^2\boldsymbol{J} + \mu\nabla\times\boldsymbol{A} - \nabla\times(\partial_t\boldsymbol{A}) \\ &= \nabla^2\boldsymbol{J} + \mu^2\boldsymbol{J} + \nabla\times(\nabla\times\boldsymbol{J} + \mu\boldsymbol{A} - \partial_t\boldsymbol{A}). \end{aligned} \tag{14}$$

Thus, if $\nabla\times\boldsymbol{J} + \mu\boldsymbol{A} - \partial_t\boldsymbol{A}$ is the gradient of a scalar field, $-W$, also $\boldsymbol{J}$ will satisfy the equation $\Box\boldsymbol{J} = \mu^2\boldsymbol{J}$. Does $\boldsymbol{A}$ satisfies the same equation? Starting from the equation for the time evolution of $\boldsymbol{A}$,

$$\partial_t\boldsymbol{A} = \nabla\times\boldsymbol{J} + \mu\boldsymbol{A} + \nabla W, \tag{15}$$

and taking its time derivative, one obtains:

$$\begin{aligned}
\partial_t^2 \boldsymbol{A} &= \nabla\times(\partial_t \boldsymbol{J}) + \mu\partial_t\boldsymbol{A} + \nabla\left(\partial_t W\right) \\
&= \nabla\times(-\nabla\phi - \mu\boldsymbol{J} - \nabla\times\boldsymbol{A}) + \mu\left(\nabla\times\boldsymbol{J} + \mu\boldsymbol{A} + \nabla W\right) + \nabla\left(\partial_t W\right) \\
&= -\nabla\times\nabla\times\boldsymbol{A} + \mu\boldsymbol{A} + \mu\nabla W + \nabla\left(\partial_t W\right),
\end{aligned} \tag{16}$$

or, equivalently:

$$\Box\boldsymbol{A} = \mu^2\boldsymbol{A} - \nabla\left(\mathrm{div}\boldsymbol{A} - \mu W - \partial_t W\right). \tag{17}$$

As W is arbitrary, the condition $\partial_t W = \mathrm{div}\boldsymbol{A} - \mu W$ can be imposed. In this way, also $\boldsymbol{A}$ satisfies the equation $\Box\boldsymbol{A} = \mu^2\boldsymbol{A}$. Finally, by taking the time derivative of the last condition, one has:

$$\begin{aligned}
\partial_t^2 W &= \mathrm{div}\left(\partial_t\boldsymbol{A}\right) - \mu\partial_t W \\
&= \mathrm{div}\left(\nabla\times\boldsymbol{J} + \mu\boldsymbol{A} + \nabla W\right) - \mu\left(\mathrm{div}\boldsymbol{A} - \mu W\right) \\
&= \nabla^2 W + \mu^2 W,
\end{aligned} \tag{18}$$

or, again, $\Box W = \mu^2 W$. In summary, starting from the original P_1 system for ϕ and $\boldsymbol{J}$, a new extended system:

$$\left\{\begin{array}{lcllll}
\partial_t\phi &=& \mu\phi & -\mathrm{div}\boldsymbol{J}, & & \\
\partial_t\boldsymbol{J} &=& -\nabla\phi & -\mu\boldsymbol{J} & -\nabla\times\boldsymbol{A}, & \\
\partial_t\boldsymbol{A} &=& & \nabla\times\boldsymbol{J} & +\mu\boldsymbol{A} & +\nabla W, \\
\partial_t W &=& & & \mathrm{div}\boldsymbol{A} & -\mu W,
\end{array}\right. \tag{19}$$

for $\phi, \boldsymbol{J}, \boldsymbol{A}, W$ has been introduced. The system can be written in a more concise way as:

$$\left\{\begin{array}{lcl}
\partial_t\begin{pmatrix}\phi\\ \boldsymbol{A}\end{pmatrix} &=& \hat{\mathbb{G}}\begin{pmatrix}W\\ \boldsymbol{J}\end{pmatrix} + \mu\begin{pmatrix}\phi\\ \boldsymbol{A}\end{pmatrix}, \\
\partial_t\begin{pmatrix}W\\ \boldsymbol{J}\end{pmatrix} &=& -\hat{\mathbb{G}}\begin{pmatrix}\phi\\ \boldsymbol{A}\end{pmatrix} - \mu\begin{pmatrix}W\\ \boldsymbol{J}\end{pmatrix},
\end{array}\right. \tag{20}$$

having definined the operator $\hat{\mathbb{G}}$ as:

$$\hat{\mathbb{G}} = \begin{pmatrix}0 & -\mathrm{div}\\ \nabla & \nabla\times\end{pmatrix}, \tag{21}$$

such that

$$\hat{\mathbb{G}}\begin{pmatrix}v_0\\ \boldsymbol{v}\end{pmatrix} = \begin{pmatrix}-\mathrm{div}\,\boldsymbol{v}\\ \nabla v_0 + \nabla\times\boldsymbol{v}\end{pmatrix}. \tag{22}$$

As can be readily verified, $\hat{\mathbb{G}}^2 = -\nabla^2\cdot\mathbb{I}_{4x4}$, so:

$$\begin{aligned}
\partial_t^2\begin{pmatrix}\phi\\ \boldsymbol{A}\end{pmatrix} &= \hat{\mathbb{G}}\left\{-\hat{\mathbb{G}}\begin{pmatrix}\phi\\ \boldsymbol{A}\end{pmatrix} - \mu\begin{pmatrix}W\\ \boldsymbol{J}\end{pmatrix}\right\} + \mu\left\{\hat{\mathbb{G}}\begin{pmatrix}W\\ \boldsymbol{J}\end{pmatrix} + \mu\begin{pmatrix}\phi\\ \boldsymbol{A}\end{pmatrix}\right\} \\
&= \left(\nabla^2 + \mu^2\right)\begin{pmatrix}\phi\\ \boldsymbol{A}\end{pmatrix},
\end{aligned} \tag{23}$$

and similarly for $\begin{pmatrix} W \\ \boldsymbol{J} \end{pmatrix}$. In summary, each unknown $(\phi, \boldsymbol{J}, \boldsymbol{A}, W)$ of the extended system (19) satisfies the modified Klein-Gordon equation, Eq. (10).
Before investigating the connection between system (19) and the Dirac equation, it is interesting to consider its relation with the original P_1 equations. By taking the divergence of the second and the third equations (19), one obtains two uncoupled sets of equations:

$$\left\{ \begin{array}{rcl} \partial_t \phi & = & \mu\phi - \mathrm{div}\boldsymbol{J}, \\ \partial_t \left(\mathrm{div}\boldsymbol{J}\right) & = & -\nabla^2\phi - \mu\,\mathrm{div}\boldsymbol{J}, \end{array} \right. \tag{24}$$

and

$$\left\{ \begin{array}{rcl} \partial_t W & = & \mathrm{div}\boldsymbol{A} - \mu W, \\ \partial_t \left(\mathrm{div}\boldsymbol{A}\right) & = & \nabla^2 W + \mu\,\mathrm{div}\boldsymbol{A}, \end{array} \right. \tag{25}$$

for $(\phi, \mathrm{div}\boldsymbol{J})$ and $(W, \mathrm{div}\boldsymbol{A})$ respectively. Equations (24) exactly correspond to system (8), ie., the original P_1 system. Instead, Equations (25) present "wrong" signs. Actually, by defining $\bar{t} = -t$, they assume exactly the form (24):

$$\left\{ \begin{array}{rcl} \dfrac{\partial W}{\partial \bar{t}} & = & \mu W - \mathrm{div}\boldsymbol{A}, \\ \dfrac{\partial}{\partial \bar{t}} \left(\mathrm{div}\boldsymbol{A}\right) & = & -\nabla^2 W - \mu\,\mathrm{div}\boldsymbol{A}. \end{array} \right. \tag{26}$$

In other words, W and $\boldsymbol{A}$ can be interpreted as a flux and a current, but they evolve backwards in time. As an alternative, by defining $U = 2\mu W - \mathrm{div}\boldsymbol{A}$, we readily find that the couple (W, U) satisfy the equations:

$$\left\{ \begin{array}{rcl} \partial_t W & = & \mu W - U, \\ \partial_t U & = & -\nabla^2 W - \mu U, \end{array} \right. \tag{27}$$

having the same form of Eqs.(24) for $(\phi, \mathrm{div}A)$. Going back to Eqs.(19), if the curl of the second and the third equations is evaluated, one obtains a new system o equations for $\mathscr{B} = \mathrm{rot}\boldsymbol{J}$ and $\mathscr{E} = \mathrm{rot}\boldsymbol{A}$:

$$\left\{ \begin{array}{rcl} \partial_t \mathscr{B} & = & -\mathrm{rot}\mathscr{E} - \mu\mathscr{B}, \\ \partial_t \mathscr{E} & = & \mathrm{rot}\mathscr{B} + \mu\mathscr{E}, \end{array} \right. \tag{28}$$

whose form recalls the Maxwell's equations of the electromagnetism (in which electric and magnetic current densities are present, $\boldsymbol{j}_e = -\mu\mathscr{E}$ and $\boldsymbol{j}_m = \mu\mathscr{B}$).

3. The Dirac equation with imaginary mass

It is well known (Dirac (1988)) that from the Dirac equation for a free particle[1]

$$\mathrm{i}\,\partial_t\psi = \left(\sum_{k=1}^{3} \alpha_k p^k + \alpha_m \cdot m \right)\psi, \quad p^k = -\mathrm{i}\,\partial_k, \tag{29}$$

[1] The Dirac matrices are defined as: $\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}$, $i = 1, 2, 3$, where σ_i are the Pauli matrices, and $\alpha_m = \begin{pmatrix} I_{2\times 2} & 0 \\ 0 & -I_{2\times 2} \end{pmatrix}$.

one deduces that each component of the spinor

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} \tag{30}$$

satisfies the Klein-Gordon equation

$$\left(\Box + m^2\right) \psi_a = 0, \quad a = 1, 2, 3, 4. \tag{31}$$

In order to find an equivalence between the Dirac equation and the extended P_1 system, the mass m must be replaced by an imaginary term, $\mathrm{i}\mu$, so obtaining the new equation:

$$\partial_t \psi + \sum_{k=1}^{3} \alpha_k \partial_k \psi = \alpha_m \mu \psi. \tag{32}$$

It must be noticed that the ψ_a's in Eq.(32) are still complex quantities, as α_2 has imaginary elements.
After splitting ψ into the two vectors $\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$ and $\begin{pmatrix} \psi_3 \\ \psi_4 \end{pmatrix}$, one obtains:

$$\begin{cases} \partial_t \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} + (\sigma{\cdot}\nabla) \begin{pmatrix} \psi_3 \\ \psi_4 \end{pmatrix} & = & \mu \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \\ \\ \partial_t \begin{pmatrix} \psi_3 \\ \psi_4 \end{pmatrix} + (\sigma{\cdot}\nabla) \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} & = & -\mu \begin{pmatrix} \psi_3 \\ \psi_4 \end{pmatrix}, \end{cases} \tag{33}$$

being (Itzykson and Zuber (1980)):

$$\sigma \cdot \nabla = \begin{pmatrix} \partial_z & \partial_x - \mathrm{i}\partial_y \\ \partial_x + \mathrm{i}\partial_y & -\partial_z \end{pmatrix}. \tag{34}$$

By comparing with system (20) for $\begin{pmatrix} \phi \\ \boldsymbol{A} \end{pmatrix}$ and $\begin{pmatrix} W \\ \boldsymbol{J} \end{pmatrix}$, the presence of the coefficients "$+\mu$" and "$-\mu$" suggests that ψ_1, ψ_2 are linear combinations of ϕ and $\boldsymbol{A}$, while ψ_3, ψ_4 are linear combinations of W and $\boldsymbol{J}$.
The next step is finding the relationship between the operators $\hat{\mathbb{G}}$ and $\boldsymbol{\sigma} \cdot \nabla$. To do this, from a generic 4-component real vector $\mathscr{V} = \begin{pmatrix} v_0 \\ \boldsymbol{v} \end{pmatrix}$, one can define a 2-component complex vector, $\mathscr{V}_{\mathbb{C}}$, as:

$$\mathscr{V}_{\mathbb{C}} = \begin{pmatrix} v_z + \mathrm{i}v_0 \\ v_x + \mathrm{i}v_y \end{pmatrix}. \tag{35}$$

In the following, this operation is referred as "$\mathbb{C}$-transform", and it is invertible. In particular, one has:

$$\left(\hat{\mathbb{G}}\mathscr{V}\right)_{\mathbb{C}} = \begin{pmatrix} -\operatorname{div} \boldsymbol{v} \\ \\ \nabla{\times}\boldsymbol{v} + \nabla v_0 \end{pmatrix}_{\mathbb{C}} = \begin{bmatrix} (\nabla{\times}\boldsymbol{v} + \nabla v_0)_z - \mathrm{i}\operatorname{div} \boldsymbol{v} \\ \\ (\nabla{\times}\boldsymbol{v} + \nabla v_0)_x + \mathrm{i}\,(\nabla{\times}\boldsymbol{v} - \nabla v_0)_y \end{bmatrix} \tag{36}$$

Moreover, it can be readily verified that:

$$
\begin{aligned}
(\sigma\cdot\nabla)\mathscr{V}_{\mathbb{C}} &= (\sigma\cdot\nabla)\begin{pmatrix} v_z + \mathrm{i}v_0 \\ v_x + \mathrm{i}v_y \end{pmatrix} \\
&= \begin{bmatrix} \operatorname{div}\boldsymbol{v} + \mathrm{i}\left(\nabla\times\boldsymbol{v} + \nabla v_0\right)_z \\ -\left(\nabla\times\boldsymbol{v} + \nabla v_0\right)_y + \mathrm{i}\left(\nabla\times\boldsymbol{v} + \nabla v_0\right)_x \end{bmatrix} \\
&= \mathrm{i}\left(\hat{\mathbb{G}}\mathscr{V}\right)_{\mathbb{C}}.
\end{aligned} \tag{37}
$$

Now, by $\mathbb{C}-$transforming the first equation (20), one obtains:

$$
\partial_t\left[-\mathrm{i}\begin{pmatrix}\phi \\ \boldsymbol{A}\end{pmatrix}_{\mathbb{C}}\right] = -(\sigma\cdot\nabla)\begin{pmatrix} W \\ \boldsymbol{J}\end{pmatrix}_{\mathbb{C}} + \mu\left[-\mathrm{i}\begin{pmatrix}\phi \\ \boldsymbol{A}\end{pmatrix}_{\mathbb{C}}\right], \tag{38}
$$

or

$$
\partial_t\begin{pmatrix}\psi_1 \\ \psi_2\end{pmatrix} + (\sigma\cdot\nabla)\begin{pmatrix}\psi_3 \\ \psi_4\end{pmatrix} = \mu\begin{pmatrix}\psi_1 \\ \psi_2\end{pmatrix}, \tag{39}
$$

having defined the vectors:

$$
\begin{pmatrix}\psi_1 \\ \psi_2\end{pmatrix} = -\mathrm{i}\begin{pmatrix}\phi \\ \boldsymbol{A}\end{pmatrix}_{\mathbb{C}} = \begin{pmatrix}\phi - \mathrm{i}A_z \\ A_y - \mathrm{i}A_x\end{pmatrix}, \tag{40}
$$

and

$$
\begin{pmatrix}\psi_3 \\ \psi_4\end{pmatrix} = \begin{pmatrix} W \\ \boldsymbol{J}\end{pmatrix}_{\mathbb{C}} = \begin{pmatrix} J_z + \mathrm{i}W \\ J_x + \mathrm{i}J_y\end{pmatrix}. \tag{41}
$$

Similarly, from the $\mathbb{C}$-transform of the second Eq. (20), noticing that:

$$
\left[\hat{\mathbb{G}}\begin{pmatrix}\phi \\ \boldsymbol{A}\end{pmatrix}\right]_{\mathbb{C}} = -\mathrm{i}\,(\sigma\cdot\nabla)\begin{pmatrix}\phi \\ \boldsymbol{A}\end{pmatrix}_{\mathbb{C}} = \sigma\cdot\nabla\begin{pmatrix}\psi_1 \\ \psi_2\end{pmatrix}, \tag{42}
$$

one finally obtains:

$$
\partial_t\begin{pmatrix}\psi_3 \\ \psi_4\end{pmatrix} + (\sigma\cdot\nabla)\begin{pmatrix}\psi_1 \\ \psi_2\end{pmatrix} = -\mu\begin{pmatrix}\psi_3 \\ \psi_4\end{pmatrix}. \tag{43}
$$

In summary, by defining:

$$
\psi = \begin{pmatrix}\psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4\end{pmatrix} = \begin{pmatrix}\phi - \mathrm{i}A_z \\ A_y - \mathrm{i}A_x \\ J_z + \mathrm{i}W \\ J_x + \mathrm{i}J_y\end{pmatrix}, \tag{44}
$$

it has been proved that the extended P_1 system is equivalent to the Dirac equation with imaginary mass, Eq.(32).

4. From the extended P_1 system to the original Dirac equation

In the previous section, it has been demonstrated that the system (20) can be written in the form (32), using the correspondence provided by Eq.(29) between $\{\phi, \boldsymbol{A}, W, \boldsymbol{J}\}$

and ψ. If μ is replaced by $-\mathrm{i}m$, the original Dirac equation is reobtained, and, in some way, it should be equivalent to the system:

$$\begin{cases} \partial_t \begin{pmatrix} \phi \\ \boldsymbol{A} \end{pmatrix} = \hat{\mathbb{G}} \begin{pmatrix} W \\ \boldsymbol{J} \end{pmatrix} - \mathrm{i}\, m \begin{pmatrix} \phi \\ \boldsymbol{A} \end{pmatrix}, \\ \partial_t \begin{pmatrix} W \\ \boldsymbol{J} \end{pmatrix} = -\hat{\mathbb{G}} \begin{pmatrix} \phi \\ \boldsymbol{A} \end{pmatrix} + \mathrm{i}\, m \begin{pmatrix} W \\ \boldsymbol{J} \end{pmatrix}, \end{cases} \tag{45}$$

in which now all the variables $\phi, \boldsymbol{A}, W, \boldsymbol{J}$ are complex quantities.
By writing $\phi = \phi' + \mathrm{i}\phi''$ $(\phi', \phi'' \in \mathbb{R})$ and similarly for all variables, one has:

$$\begin{cases} \partial_t \begin{pmatrix} \phi \\ \boldsymbol{A} \end{pmatrix}' = \hat{\mathbb{G}} \begin{pmatrix} W \\ \boldsymbol{J} \end{pmatrix}' + m \begin{pmatrix} \phi \\ \boldsymbol{A} \end{pmatrix}'', \\ \partial_t \begin{pmatrix} \phi \\ \boldsymbol{A} \end{pmatrix}'' = \hat{\mathbb{G}} \begin{pmatrix} W \\ \boldsymbol{J} \end{pmatrix}'' - m \begin{pmatrix} \phi \\ \boldsymbol{A} \end{pmatrix}', \\ \partial_t \begin{pmatrix} W \\ \boldsymbol{J} \end{pmatrix}' = -\hat{\mathbb{G}} \begin{pmatrix} \phi \\ \boldsymbol{A} \end{pmatrix}' + m \begin{pmatrix} W \\ \boldsymbol{J} \end{pmatrix}'', \\ \partial_t \begin{pmatrix} W \\ \boldsymbol{J} \end{pmatrix}'' = -\hat{\mathbb{G}} \begin{pmatrix} \phi \\ \boldsymbol{A} \end{pmatrix}'' - m \begin{pmatrix} W \\ \boldsymbol{J} \end{pmatrix}'. \end{cases} \tag{46}$$

The new system contains sixteen real unknowns, while the original Dirac equation has only eight. Thus, system (46) must contain more information with respect to the Dirac equation. Through $\mathbb{C}$-transformation, system (46) becomes

$$\begin{cases} \partial_t \begin{pmatrix} \phi' \\ \boldsymbol{A}' \end{pmatrix}_{\mathbb{C}} = -\mathrm{i}\,(\sigma\cdot\nabla) \begin{pmatrix} W' \\ \boldsymbol{J}' \end{pmatrix}_{\mathbb{C}} + m \begin{pmatrix} \phi'' \\ \boldsymbol{A}'' \end{pmatrix}_{\mathbb{C}}, \\ \partial_t \begin{pmatrix} \phi'' \\ \boldsymbol{A}'' \end{pmatrix}_{\mathbb{C}} = -\mathrm{i}\,(\sigma\cdot\nabla) \begin{pmatrix} W'' \\ \boldsymbol{J}'' \end{pmatrix}_{\mathbb{C}} - m \begin{pmatrix} \phi' \\ \boldsymbol{A}' \end{pmatrix}_{\mathbb{C}}, \\ \partial_t \begin{pmatrix} W' \\ \boldsymbol{J}' \end{pmatrix}_{\mathbb{C}} = +\mathrm{i}\,(\sigma\cdot\nabla) \begin{pmatrix} \phi' \\ \boldsymbol{A}' \end{pmatrix}_{\mathbb{C}} - m \begin{pmatrix} W'' \\ \boldsymbol{J}'' \end{pmatrix}_{\mathbb{C}}, \\ \partial_t \begin{pmatrix} W'' \\ \boldsymbol{J}'' \end{pmatrix}_{\mathbb{C}} = +\mathrm{i}\,(\sigma\cdot\nabla) \begin{pmatrix} \phi'' \\ \boldsymbol{A}'' \end{pmatrix}_{\mathbb{C}} + m \begin{pmatrix} W' \\ \boldsymbol{J}' \end{pmatrix}_{\mathbb{C}}. \end{cases} \tag{47}$$

By defining four vectors $\xi_{\mathrm{I}}, \xi_{\mathrm{II}}, \eta_{\mathrm{I}}, \eta_{\mathrm{II}} \in \mathbb{C}^2$ as:

$$\begin{array}{llll} \begin{pmatrix} \phi' \\ \boldsymbol{A}' \end{pmatrix}_{\mathbb{C}} = \mathrm{i}\xi_{\mathrm{I}}, & & \begin{pmatrix} \phi'' \\ \boldsymbol{A}'' \end{pmatrix}_{\mathbb{C}} = \mathrm{i}\xi_{\mathrm{II}}, \\ \begin{pmatrix} W' \\ \boldsymbol{J}' \end{pmatrix}_{\mathbb{C}} = \eta_{\mathrm{I}}, & & \begin{pmatrix} W'' \\ \boldsymbol{J}'' \end{pmatrix}_{\mathbb{C}} = \eta_{\mathrm{II}}, \end{array} \tag{48}$$

system (47) assumes the form:

$$\begin{cases} \partial_t \xi_{\mathrm{I}} + (\sigma \cdot \nabla)\, \eta_{\mathrm{I}} &= m\, \xi_{\mathrm{II}}, \\ \partial_t \eta_{\mathrm{I}} + (\sigma \cdot \nabla)\, \xi_{\mathrm{I}} &= -m\, \eta_{\mathrm{II}}, \\ \partial_t \xi_{\mathrm{II}} + (\sigma \cdot \nabla)\, \eta_{\mathrm{II}} &= -m\, \xi_{\mathrm{I}}, \\ \partial_t \eta_{\mathrm{II}} + (\sigma \cdot \nabla)\, \xi_{\mathrm{II}} &= m\, \eta_{\mathrm{I}}, \end{cases} \tag{49}$$

or

$$\begin{cases} \partial_t \psi_{\mathrm{I}} + \sum_k \alpha_k \partial_k \psi_{\mathrm{I}} = m\, \alpha_m \psi_{\mathrm{II}}, \\ \partial_t \psi_{\mathrm{II}} + \sum_k \alpha_k \partial_k \psi_{\mathrm{II}} = -m\, \alpha_m \psi_{\mathrm{I}}, \end{cases} \tag{50}$$

with

$$\psi_{\mathrm{I}} = \begin{pmatrix} \xi_{\mathrm{I}} \\ \eta_{\mathrm{I}} \end{pmatrix}, \quad \psi_{\mathrm{II}} = \begin{pmatrix} \xi_{\mathrm{II}} \\ \eta_{\mathrm{II}} \end{pmatrix}. \tag{51}$$

Finally, by defining the vector $\Psi \in \mathbb{C}^8$ as:

$$\Psi = \begin{pmatrix} \psi_{\mathrm{I}} \\ \psi_{\mathrm{II}} \end{pmatrix} = \begin{pmatrix} \xi_{\mathrm{I}} \\ \eta_{\mathrm{I}} \\ \xi_{\mathrm{II}} \\ \eta_{\mathrm{II}} \end{pmatrix}, \tag{52}$$

and the 8x8 matrices:

$$\mathscr{O}_k = \begin{pmatrix} \alpha_k & 0 \\ 0 & \alpha_k \end{pmatrix}, k = 1, 2, 3, \quad \mathscr{O}_m = \begin{pmatrix} 0 & \mathrm{i}\alpha_m \\ -\mathrm{i}\alpha_m & 0 \end{pmatrix}, \tag{53}$$

the system can be written as:

$$\mathrm{i}\partial_t \Psi = \sum_{k=1}^{3} \mathscr{O}_k\, (-\mathrm{i}\partial_k)\Psi + \mathscr{O}_m\, m\Psi. \tag{54}$$

It can be readily proved that the $\mathscr{O}$'s and Dirac's α matrices have similar properties. In particular:

$$\mathscr{O}_a^\dagger = \mathscr{O}_a, \quad \{\mathscr{O}_a, \mathscr{O}_b\} = 2\delta_{ab} I_{8\mathrm{x}8} \quad a, b, = 1, 2, 3, m. \tag{55}$$

Therefore, Eq.(54) is a sort of Dirac equation, in which the 4×4 Dirac matrices are replaced by the $8 \times 8\, \mathscr{O}$'s.

5. Discussion and concluding remarks

What is the meaning of Eq. (54) and how does it differ from the Dirac equation? A possible answer can be obtained by adding the electromagnetic field to the equations. When the electromagnetic interaction is included into the Dirac equation, the derivatives ∂^α are replaced by $\partial^\alpha + \mathrm{i}e\mathscr{A}^\alpha$, where $\left(\mathscr{A}^0, \boldsymbol{\mathscr{A}}\right)$ is the 4-potential for the field (Itzykson and Zuber (1980)). In particular, ∂_t and ∇ become $\partial_t + \mathrm{i}\mathrm{e}\mathscr{A}^0$ and $\nabla - \mathrm{i}\mathrm{e}\boldsymbol{\mathscr{A}}$, respectively. By operating in this way on the system (45), one obtains:

$$\left\{\begin{array}{rcl} \left(\partial_t + \mathrm{i}\,\mathrm{e}\,\mathscr{A}^0\right)\begin{pmatrix}\phi\\ \boldsymbol{A}\end{pmatrix} & = & \left(\hat{\mathbb{G}} - \mathrm{i}\,\mathrm{e}\,\mathbb{A}\right)\begin{pmatrix}W\\ \boldsymbol{J}\end{pmatrix} - \mathrm{i}\,m\begin{pmatrix}\phi\\ \boldsymbol{A}\end{pmatrix}, \\ \left(\partial_t + \mathrm{i}\,\mathrm{e}\,\mathscr{A}^0\right)\begin{pmatrix}W\\ \boldsymbol{J}\end{pmatrix} & = & \left(-\hat{\mathbb{G}} + \mathrm{i}\,\mathrm{e}\,\mathbb{A}\right)\begin{pmatrix}\phi\\ \boldsymbol{A}\end{pmatrix} + \mathrm{i}\,m\begin{pmatrix}W\\ \boldsymbol{J}\end{pmatrix}, \end{array}\right. \tag{56}$$

in which the 4×4 matrix $\mathbb{A}$ has been defined as

$$\mathbb{A} = \begin{pmatrix} 0 & \boldsymbol{\mathscr{A}}\cdot \\ \boldsymbol{\mathscr{A}} & \boldsymbol{\mathscr{A}}\times \end{pmatrix}. \tag{57}$$

Now, by splitting real and imaginary part of system (56), then $\mathbb{C}-$transforming the resulting equations, the new system

$$\left\{\begin{array}{rcl} \partial_t \xi_{\mathrm{I}} & = & -\left(\sigma{\cdot}\nabla\right)\eta_{\mathrm{I}} - \mathrm{e}\left(\sigma{\cdot}\boldsymbol{\mathscr{A}}\right)\eta_{\mathrm{II}} + \mathrm{e}\mathscr{A}^0\xi_{\mathrm{II}} + m\,\xi_{\mathrm{II}}, \\ \partial_t \xi_{\mathrm{II}} & = & -\left(\sigma{\cdot}\nabla\right)\eta_{\mathrm{II}} + \mathrm{e}\left(\sigma{\cdot}\boldsymbol{\mathscr{A}}\right)\eta_{\mathrm{I}} - \mathrm{e}\mathscr{A}^0\xi_{\mathrm{I}} - m\,\xi_{\mathrm{I}}, \\ \partial_t \eta_{\mathrm{I}} & = & -\left(\sigma{\cdot}\nabla\right)\xi_{\mathrm{I}} - \mathrm{e}\left(\sigma{\cdot}\boldsymbol{\mathscr{A}}\right)\xi_{\mathrm{II}} + \mathrm{e}\mathscr{A}^0\eta_{\mathrm{II}} - m\,\eta_{\mathrm{II}}, \\ \partial_t \eta_{\mathrm{II}} & = & -\left(\sigma{\cdot}\nabla\right)\xi_{\mathrm{II}} + \mathrm{e}\left(\sigma{\cdot}\boldsymbol{\mathscr{A}}\right)\xi_{\mathrm{I}} - \mathrm{e}\mathscr{A}^0\eta_{\mathrm{I}} + m\,\eta_{\mathrm{I}}, \end{array}\right. \tag{58}$$

is obtained. In system (58) all the unknowns are coupled, but in reality it can be split into two independent subsystems. If the time derivatives of $\xi_{\mathrm{I}} + \mathrm{i}\,\xi_{\mathrm{II}}$ and of $\eta_{\mathrm{I}} + \mathrm{i}\eta_{\mathrm{II}}$ are considered, one has:

$$\partial_t\left(\xi_{\mathrm{I}} + \mathrm{i}\xi_{\mathrm{II}}\right) = -\left(\sigma{\cdot}\nabla\right)(\eta_{\mathrm{I}} + \mathrm{i}\eta_{\mathrm{II}}) + \mathrm{e}\left(\sigma{\cdot}\boldsymbol{\mathscr{A}}\right)(\mathrm{i}\eta_{\mathrm{I}} - \eta_{\mathrm{II}}) + \left(\mathrm{e}\mathscr{A}^0 + m\right)\left(-\mathrm{i}\xi_{\mathrm{I}} + \xi_{\mathrm{II}}\right), \tag{59}$$

and

$$\partial_t\left(\eta_{\mathrm{I}} + \mathrm{i}\eta_{\mathrm{II}}\right) = -\left(\sigma{\cdot}\nabla\right)(\xi_{\mathrm{I}} + \mathrm{i}\xi_{\mathrm{II}}) + \mathrm{e}\left(\sigma{\cdot}\boldsymbol{\mathscr{A}}\right)(\mathrm{i}\xi_{\mathrm{I}} - \xi_{\mathrm{II}}) + \left(\mathrm{e}\mathscr{A}^0 - m\right)\left(-\mathrm{i}\eta_{\mathrm{I}} + \eta_{\mathrm{II}}\right), \tag{60}$$

or, equivalently,

$$\left\{\begin{array}{rcl} \mathrm{i}\partial_t\psi_a & = & -\mathrm{i}\sigma{\cdot}(\nabla - \mathrm{i}\mathrm{e}\boldsymbol{\mathscr{A}})\,\psi_b + \mathrm{e}\mathscr{A}^0\psi_a + m\,\psi_a, \\ \mathrm{i}\partial_t\psi_b & = & -\mathrm{i}\sigma{\cdot}(\nabla - \mathrm{i}\mathrm{e}\boldsymbol{\mathscr{A}})\,\psi_a + \mathrm{e}\mathscr{A}^0\psi_b - m\,\psi_b, \end{array}\right. \tag{61}$$

with $\psi_a = \xi_{\mathrm{I}} + \mathrm{i}\,\xi_{\mathrm{II}}$, $\psi_b = \eta_{\mathrm{I}} + \mathrm{i}\,\eta_{\mathrm{II}}$. Therefore, $\begin{pmatrix}\psi_a\\ \psi_b\end{pmatrix}$ is the (4-component) spinor associated to a particle of charge e, according to the classic Dirac equation. Moreover, by taking the time derivative of $\eta_{\mathrm{II}} + \mathrm{i}\,\eta_{\mathrm{I}}$ and of $\xi_{\mathrm{II}} + \mathrm{i}\,\xi_{\mathrm{I}}$, the new equations

$$\partial_t\left(\eta_{\mathrm{II}} + \mathrm{i}\eta_{\mathrm{I}}\right) = -\left(\sigma{\cdot}\nabla\right)(\xi_{\mathrm{II}} + \mathrm{i}\xi_{\mathrm{I}}) + \mathrm{e}\left(\sigma{\cdot}\boldsymbol{\mathscr{A}}\right)(\xi_{\mathrm{I}} - \mathrm{i}\xi_{\mathrm{II}}) + \left(\mathrm{e}\mathscr{A}^0 - m\right)\left(-\eta_{\mathrm{I}} + \mathrm{i}\eta_{\mathrm{II}}\right), \tag{62}$$

and

$$\partial_t\left(\xi_{\mathrm{II}} + \mathrm{i}\xi_{\mathrm{I}}\right) = -\left(\sigma{\cdot}\nabla\right)(\eta_{\mathrm{II}} + \mathrm{i}\eta_{\mathrm{I}}) + \mathrm{e}\left(\sigma{\cdot}\boldsymbol{\mathscr{A}}\right)(\eta_{\mathrm{I}} - \mathrm{i}\eta_{\mathrm{II}}) + \left(\mathrm{e}\mathscr{A}^0 + m\right)\left(-\xi_{\mathrm{I}} + \mathrm{i}\xi_{\mathrm{II}}\right) \tag{63}$$

are deduced. After defining $\psi_c = \eta_{\mathrm{II}} + \mathrm{i}\,\eta_{\mathrm{I}}$, $\psi_d = \xi_{\mathrm{II}} + \mathrm{i}\,\xi_{\mathrm{I}}$, one finally obtains:

$$\left\{\begin{array}{rcl} \mathrm{i}\partial_t\psi_c & = & -\mathrm{i}\sigma{\cdot}(\nabla + \mathrm{i}\mathrm{e}\boldsymbol{\mathscr{A}})\,\psi_d - \mathrm{e}\mathscr{A}^0\psi_c + m\,\psi_c, \\ \mathrm{i}\partial_t\psi_d & = & -\mathrm{i}\sigma{\cdot}(\nabla + \mathrm{i}\mathrm{e}\boldsymbol{\mathscr{A}})\,\psi_c - \mathrm{e}\mathscr{A}^0\psi_d - m\,\psi_d, \end{array}\right. \tag{64}$$

which is the Dirac equation for a particle of charge −e. In summary, the solution of the system (58) can be expressed as a suitable superposition of the solutions of two Dirac equations, one for a particle of mass m and charge e, the other for the corresponding antiparticle. The work shows that the Dirac equation with imaginary mass can be written in terms of scalar and vector fields (in particular, ϕ is a real scalar, $\boldsymbol{J}$ is a polar vector, W a pseudo-scalar, and $\boldsymbol{A}$ an axial vector), and the equations they satisfy [Eqs. (19) or (20)] involve the usual differential operators (∇, div and $\nabla\times$) in a curious mixture between diffusion and Maxwell's equations. Instead, by restoring the real mass [Eqs. (45)] the form of the equations is unchanged, but they correspond to an extended Dirac equation [Eq. (54)] rather than to the original one, for reason that are not easy to understand (at least to the Author). However, there is a limit situation, when $\mu = 0$, in which the system (19) is perfectly equivalent to the Dirac equation for a massless fermion. The study of these equations will be the object of a forthcoming paper.

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