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Instantons and no wormholes in $AdS_3 \times S^3 \times CY_2$

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We study supergravity instantons sourced by axion (and saxion) fields in the Euclidean $AdS_3 \times S^3 \times CY_2$ vacua of IIB supergravity. Such instantons are described by geodesic curves on the moduli space; the timelike geodesics can describe Euclidean wormholes, the lightlike geodesics describe (generalizations of) D instantons, and spacelike geodesics are subextremal versions thereof. We perform a concrete classification of such geodesics and find that, despite earlier claims, the wormholes fail to be regular. A subclass of the lightlike geodesics is supersymmetric and, up to dualities, lifts to Euclidean strings wrapping 2-cycles in the CY_2. The dual of these instantons is expected to be worldsheet instantons of the D1–D5 conformal field theory.

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I. INTRODUCTION

The study of the Euclidean path integral for gravity has a long history with recent breakthroughs for low-dimensional gravity theories like Jackiw–Teitelboim gravity; see, for instance, Ref. [1]. One of the main lessons to be learned from low-dimensional theories is that it does make sense to sum over saddle points with different topologies and they tend to have a holographic description in terms of ensemble theories, akin to Coleman's α parameters and the associated absorbtion and emission of baby universes [2]. However, the rules of the game for actual Einstein-Hilbert gravity (coupled to matter) in dimensions 3 and higher remain somewhat unclear. It has even been suggested that it is vastly different from the lessons learned in low-dimensional gravity theories. The role Euclidean wormholes[1] can play is not a clear picture at this point in time. In fact, some well-motivated ideas on quantum gravity and the swampland [4,5] suggest that wormholes do not contribute in the ways envisaged in the early works [2,6,7] (see also Ref. [8]).

String theory provides a UV completion of quantum gravity, and therefore various ideas on the semiclassical formulation of gravity should be testable. Since saddle point expansions are nonperturbative, one is naturally led to consider holographic dual pairs in string theory as they provide a nonperturbative definition of string theory in certain anti-de Sitter (AdS) backgrounds. This topic was initiated in some early works [9–11] (see also Refs. [12–14]) with the hope that AdS/CFT should inform us about which saddle points contribute and how. In this regard, the most natural Euclidean wormholes to consider are wormholes sourced by axion fields [7], as axions provide a natural source of negative Euclidean energy momentum required to sustain a wormhole geometry. As emphasized in Refs. [4,10,15–17], such setups come with a bonus: axions in string theory pair up with saxions that have positive energy momentum instead, and one can find configurations which interpolate between negative (wormholes) through zero, toward positive (Euclidean) energy momentum (EM). An example of the zero EM solutions are D(-1)-branes, also known as D instantons [18,19]. They can be supersymmetric (SUSY), and their role in string theory is well understood. Similar to black holes and branes, one can think in terms of an extremality property:

(i) Negative energy momentum corresponds to wormholes as superextremal instantons
(ii) Zero energy momentum corresponds to extremal instantons
(iii) Positive energy momentum corresponds to subextremal solutions.

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[1]Reviewed nicely in Ref. [3].

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This picture can be made explicit by the c map (reduction over time) where the extremality properties of black holes reduce to the corresponding extremality properties of instantons [16,20–23]. Alternative (but related) viewpoints are obtained by computing the action to charge ratio [24–26] or using probe D-instanton actions to infer Euclidean "repulsion or attraction" [4,17].

There is, however, one caveat with the c-map picture: the wormholes obtained from reducing superextremal black holes have badly singular axion-saxion profiles which lift to the naked singularity of the superextremal "black hole." To obtain smooth wormholes, one needs a specific inequality to hold on the axion-saxion coupling [11], which was claimed to be possible in Euclidean $\mathrm{AdS}_3 \times S^3 \times \mathrm{CY}_2$ [11] and found more recently in Euclidean $\mathrm{AdS}_5 \times S^5/Z_k$ with $k > 1$ [27]. One of the results of this paper is that the regularity condition is in fact not satisfied in $\mathrm{AdS}_3 \times S^3 \times CY_2$, making the five-dimensional examples of Ref. [27] the so far unique embeddings of axion wormholes in AdS.

Using the latter explicit embedding and its dual $N = 2$ necklace quiver description [28–30], some properties of the instantons were able to be inferred in Refs. [31,32]: the extremal instantons were argued to map to specific SUSY and non-SUSY instantons of the gauge theory. The SUSY instantons have the same orientation of the (anti-)self-dual gauge fields at every gauge node, whereas the non-SUSY instantons had at least one gauge node with opposite orientation. The subextremal solutions remain unclear, and a speculative description in terms of non-self-dual gauge field configurations was given earlier in Ref. [10] for $\mathrm{AdS}_5 \times S^5$ and is readily extended to $\mathrm{AdS}_5 \times S^5/Z_k$. The superextremal solutions (the wormholes) are problematic since the holographic one-point functions violate a positivity bound, suggesting the wormholes are in fact unphysical. We interpret this as a manifestation of the recently discovered infinite number of perturbative instabilities (negative modes) of four-dimensional axion wormholes sourced by a single axion [27], correcting earlier contradicting claims in Refs. [33,34]. It is natural to expect that the instabilities also arise when multiple axions and saxions interact in some general sigma model [35]. If so, the macroscopic wormholes cannot contribute to the path integral, whereas a similar configuration of widely separated microscopically sized solutions with unit axion charge should be the dominant saddle points [4,17].[2] They are, however, outside of the supergravity regime and should not be interpreted as wormhole geometries.

In this paper, we continue our investigation of AdS moduli spaces, their geodesics, and the relation with supergravity and conformal field theory (CFT) instantons initiated in Refs. [27,31,32] and extend it to Euclidean $\mathrm{AdS}_3 \times S^3 \times CY_2$ with CY_2 either $\mathbb{T}^4$ or K_3. This holographic background is well studied, and its dual CFT, known as the D1–D5 CFT, has a Lagrangian description in the free orbifold limit [37–39]. Despite $\mathrm{AdS}_3 \times S^3 \times CY_2$ being one of the most well-known AdS/CFT backgrounds, there has been surprisingly little investigation of the instantons in these backgrounds up to two works we are aware of [11,40]. This is in rather stark contrast with the study of instantons in $\mathrm{AdS}_5 \times S^5$ [41–45], which constitutes one of the main early breakthroughs in our understanding of AdS/CFT. The aim of this paper is to carefully classify the instantons with O(3) symmetry sourced by the AdS moduli (axions and saxions), which boils down to explicitly construct and classify geodesic curves on the moduli space. We will find disagreement with the earlier investigations of Refs. [11,40]. A dual description of the extremal supersymmetric instantons in terms of instantons in the D1–D5 CFT is left for a follow-up work.

II. GENERAL SETUP

The strategy of Refs. [27,31,32] to embed Euclidean axion wormholes in AdS compactifications of 10D/11D supergravity is to truncate the compactification down to its moduli space of scalars such that the resulting Lagrangian after the truncation reads

$$e^{-1}\mathcal{L} = \mathcal{R} - \frac{1}{2}\mathcal{G}_{IJ}\partial\phi^I\partial\phi_I - \Lambda, \tag{2.1}$$

where the ϕ^I are the AdS moduli, $\mathcal{G}_{IJ}$ is the metric on moduli space, and Λ is the negative vacuum energy at the AdS critical point of the scalar potential. In this definition, moduli are not just massless, but they have no appearance whatsoever in the effective potential at the vacuum.[3] A holographic dual statement is that the dual operators are *exactly* marginal, and the moduli space is then dual to the conformal manifold describing a (continuous) set of CFTs labeled by the vevs (vacuum expectation values) of the moduli dual to the values of the coefficients in front of the exactly marginal operators in the CFT Lagrangian (if any).

In Euclidean supergravity, the moduli space metric is not necessarily positive definite. Even more, it seems that its signature is not uniquely fixed by supersymmetry since there are sign ambiguities in defining Euclidean [ten-dimensional (10D)] supergravity [46]. However, this ambiguity is resolved if one wishes to study instantons sourced by axions since these instantons are interpreted as axion charge transitions and then the boundary conditions in the path integral fix completely the sign; see, for instance, Refs. [10,47]: axions get flipped signs (consistently with

[2]Interesting recent work has reported on "brane-nucleation" type instabilities (different from negative modes) in general classes of Euclidean AdS wormholes embedded in string/M theory [36].

[3]That is, the potential, once expanded about the vacuum, features no cubic or higher order terms in these scalar fields.

them enjoying a shift symmetry), while the description of the other scalar fields remains unchanged.

To make this paper self-contained, we briefly review the general form of instanton solutions in $D > 2$ as presented in detail in Refs. [4,20,21]. Once a radially symmetric instanton ansatz is made,

$$ds^2 = f(\tau)^2 d\tau^2 + a(\tau)^2 d\Omega_{D-1}^2, \qquad \phi^I = \phi^I(\tau), \tag{2.2}$$

the scalar field equations of motion are purely geodesic, and with the gauge choice, $f = a^{D-1}$, the geodesics have an affine parametrization in terms of the harmonic function ρ on the Euclidean geometry, and consequently the geodesic velocity is a constant c:

$$\mathcal{G}_{IJ}\partial_\rho \phi^I \partial_\rho \phi^J = c. \tag{2.3}$$

As emphasized earlier, due to the presence of axions, c can have any sign, and the solution for the metric is independent of the sigma model details. A particularly simple expression that exists for the gauge $a = \tau$ can be found,

$$f(\tau)^2 = \left(1 + \frac{\tau^2}{\ell^2} + \frac{c}{2(D-2)(D-1)}\tau^{-2(D-2)}\right)^{-1}, \tag{2.4}$$

where $\Lambda = -(D-1)(D-2)/\ell^2$. For $c = 0$, this is the metric on Euclidean AdS; for $c > 0$, this is a singular solution whose metric coincides with the Euclidean version of Gubser's "dilaton-driven confinement" solution [48]; and for $c < 0$, the metric describes a wormhole.[4] The wormhole metric $c < 0$ is smooth in proper coordinates, but the scalars do not need to be. There can be unphysical kinklike singularities as, for instance, observed for $AdS_5 \times S^5$ [10]. A regularity condition, involving the length of timelike geodesics, was found in Ref. [11] and shown to be possible in $AdS_5 \times S^5/\mathbb{Z}_k$ when $k > 1$ [27]. In what follows, we investigate the wormhole regularity criterion for $AdS_3 \times S^3 \times CY_2$ in Euclidean IIB supergravity and further construct all the solutions and check whether supersymmetry can be preserved for $c = 0$.

To answer the questions laid out above, we need to know the moduli space at stake. Early work by Cecotti [49] on two-dimensional (2D) CFTs of the kind we expect to find suggests conformal manifolds for the Lorentzian CFT of the form

$$\frac{SO(4,n)}{SO(4) \times SO(n)}. \tag{2.5}$$

[4]Note that these particular coordinates only cover half the wormhole since there is a coordinate singularity at $\tau = \tau^*$ where $f(\tau^*) = \infty$, and there one can consistently glue a mirror copy to have the whole smooth wormhole metric. Other coordinates can make this more explicit but are not needed here and are described in the references quoted earlier.

This is confirmed by AdS/CFT since the moduli spaces of $AdS_3 \times S^3 \times \mathbb{T}_4$ should be the one with $n = 5$ [50,51] and the moduli space of $AdS_3 \times S^3 \times K_3$ should have $n = 20$ [52]. Below, we will construct the Wick-rotated version of this moduli space. But before we set up this general machinery of coset spaces and geodesic curves, we take a different, more 10D viewpoint to find some simple truncations of the moduli space and their corresponding solutions. The group theory in Sec. IV will then prove that the results obtained from this particularly simple truncation is extended to the whole moduli space. In other words, the truncation of the moduli space discussed in the next section contains the *seed* solutions that generate all other solutions of interest which lie within a general set of orbits with respect to the isometry (duality) group of the full moduli space. In the same section, we construct the generating solutions of all the geodesics in the moduli space.

The paper is organized as follows. In Sec. III, we restrict ourselves to a simple truncation of the moduli space and derive therein the geodesics which describe the supersymmetric configuration corresponding to two Euclidean D1-branes in the D1–D5 background. The uplift of these geodescis to $D = 10$ is performed, and their on-shell action is computed. We also prove that this simple truncation contains no regular wormhole solution. In Sec. IV, using the theory of Lie groups and Lie algebras, we discuss how general the results obtained in the previous section are. We rigorously define the Wick rotation and prove that the duality orbits of geodesics in the moduli space, both in the $\mathbb{T}_4$ and in the K_3 cases, have a representative in the smaller Wick-rotated version of (2.5) with $n = 4$. The classification of the extremal solutions will require tools borrowed from the theory of nilpotent orbits in classical Lie algebras. We shall prove, in Sec. IV C 1, that the Euclidean D1-brane solutions derived in Sec. III are generating all order-2 nilpotent orbits [with reference to a suitable representation of $\mathfrak{so}(4,4)$]. As for the remaining orbits, we give the explicit form of the generating extremal and nonextremal geodesics, in terms of the $D = 10$ fields, in Sec. IV B. Finally, in Sec. IV D, we elaborate on the existence of regular wormholes and argue that, in light of the criterion put forward in Ref. [11], the negative result of Sec. III C extends to the whole moduli space. We end with concluding remarks.

III. SIMPLE TRUNCATION AND ITS SOLUTIONS

A. Brane intersections

It is insightful to recall the brane intersection whose near horizon gives the vacuum. The 10D brane picture is given by

$$D1 \times \times - - - - - - - -$$
$$D5 \times \times - - - - \times \times \times \times.$$

Our notation is such that, upon taking the near horizon limit, the first three directions generate AdS_3, the next three

correspond to S^3, and the remaining four directions correspond to $\mathbb{T}^4$ or K_3. This picture naturally suggests the existence of SUSY instantons localized in AdS_3 from Euclidean D1 strings wrapping 2-cycles in the CY_2. In particular, for $CY_2 = \mathbb{T}^4$, we would have

$$\begin{aligned} &D1 \times \times - - - - - - - - \\ &D5 \times \times - - - - \times \times \times \times \\ &D1 - - - - - - - - \times \times \\ &D1 - - - - - - \times \times - -. \end{aligned} \tag{3.1}$$

The naive counting of supercharges works as follows: the D1–D5 intersection preserves eight supercharges but doubles to 16 upon taking the near horizon limit. If only one stack of Euclidean D1-branes is present, then SUSY is broken to eight supercharges. The presence of both stacks would further reduce it to a configuration preserving 4 of the original 32. An intersection diagram that contains an euclidean D3 branewrapping $\mathbb{T}^4$ suggests that this configuration breaks all the supersymmetry.

We will present a detailed analysis of the moduli space later in this paper, but we can already make some educated guesses as to where the potential axions in three dimensions can come from: integrating the NSNS (Neveu-Schwarz) and RR (Ramond-Ramond) 2-forms B_2, C_2 over the 2-cycles in CY_2, integrating C_4 over the whole CY_2 and then the RR axion C_0 itself. Note that the Kaluza Klein vectors from the reduction on $\mathbb{T}^4$, as well as the vectors from B_2, C_2 over the $\mathbb{T}^4$ 1-cycles, could in principle be dualized to axions in three dimensions, but we should not do so. They are "true vectors" and confine in three dimensions. This is related to the choice of boundary conditions for vectors in AdS_3 as explained in Ref. [53].

The full moduli space from the $\mathbb{T}^4$ reduction will deliver too many axions because some will be lifted by the F_3 flux on AdS_3 and S^3; one can demonstrate [37,39] that only 5 out of the 25 $\mathbb{T}^4$ moduli[5] in six dimensions get lifted by the fluxes. These are the a linear combination of torus volume and dilaton, a linear combination of C_0 and C_4, and three from the C_2 field (the self-dual ones). The remaining 20 moduli span the manifold

$$\frac{\mathrm{SO}(5,4)}{\mathrm{SO}(4)\times\mathrm{SO}(5)}. \tag{3.2}$$

B. Dimensional reduction and truncation

The above discussion inspires to find a simple consistent truncation for $\mathbb{T}^4$. We keep the torus volume, the volume of a 2-cycle (which determines the volume of the orthogonal 2-cycle). Then, we also keep two axions from C_2 reduced over these two 2-cycles and call them c_1 and c_2. The ansatz in the 10D Einstein frame is

$$\begin{aligned} ds_{10}^2 = {} & e^{2\alpha\varphi} ds_6^2 + e^{2\beta\varphi}(e^{2\gamma\psi} d\theta_1^2 + e^{2\gamma\psi} d\theta_2^2 \\ & + e^{-2\gamma\psi} d\theta_3^2 + e^{-2\gamma\psi} d\theta_4^2), \end{aligned} \tag{3.3}$$

$$\hat{C}_2 = C_2 + c_1 d\theta_1 \wedge d\theta_2 + c_2 d\theta_3 \wedge d\theta_4, \tag{3.4}$$

where $\alpha = 1/4 = -\beta$ and $\gamma^2 = 1/8$. The Lagrangian of our truncation in six dimensions is

$$\begin{aligned} e^{-1}\mathcal{L} = {} & \mathcal{R}_6 - \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}(\partial\varphi)^2 - \frac{1}{2}(\partial\psi)^2 \\ & - \frac{1}{2} e^{\phi+\varphi-4\gamma\psi}(\partial c_1)^2 - \frac{1}{2} e^{\phi+\varphi+4\gamma\psi}(\partial c_2)^2 \\ & - \frac{1}{23!} e^{\frac{1}{2}\varphi+\phi} F_3^2. \end{aligned} \tag{3.5}$$

Now, we reduce further down to three dimensions using electric and magnetic flux,

$$ds_6^2 = \mathrm{e}^{2\bar{\alpha}\bar{\varphi}} ds_3^2 + \mathrm{e}^{2\bar{\beta}\bar{\varphi}} \mathrm{d}\Omega_3^2, \tag{3.6}$$

$$F_3 = Q_1 e^{-\phi} e^{3(\bar{\alpha}-\bar{\beta})\bar{\varphi}} \epsilon_3 + Q_5 \tilde{\epsilon}_3, \tag{3.7}$$

where $\bar{\alpha}^2 = 3/8$ and $\bar{\alpha} = -3\bar{\beta}$. The forms ϵ_3 and $\tilde{\epsilon}_3$ are the volume forms of ds_3^2 and $\mathrm{d}\Omega_3^2$ respectively. We introduced $\bar{\varphi}$, the volume scalar of the S^3. We then find[6]

$$e^{-1}\mathcal{L} = \mathcal{R}_3 - \text{kinetic} - V(\phi,\psi,\varphi,\bar{\varphi}), \tag{3.8}$$

where

$$\begin{aligned} 2\times\text{kinetic} = {} & (\partial\phi)^2 + (\partial\varphi)^2 + (\partial\psi)^2 + (\partial\bar{\varphi})^2 \\ & + e^{\phi+\varphi}(e^{-4\gamma\psi}(\partial c_1)^2 + e^{4\gamma\psi}(\partial c_2)^2), \end{aligned} \tag{3.9}$$

$$V(\phi,\psi,\varphi,\bar{\varphi}) = \frac{1}{2} Q_1^2 e^{-(\phi-\varphi)+4\bar{\alpha}\bar{\varphi}} + \frac{1}{2} Q_5^2 e^{(\phi-\varphi)+4\bar{\alpha}\bar{\varphi}} - 6 e^{-8\bar{\beta}\bar{\varphi}}. \tag{3.10}$$

Note that the dependence of the axion kinetic term on $\bar{\varphi}$ canceled out. The potential stabilizes the scalar $\bar{\varphi}$ and the combination $(\phi - \varphi)$. Interestingly, it is exactly the orthogonal combination $(\phi + \varphi)$ that is appearing in the axion kinetic term, which will prove necessary for our truncation. So, let us call

$$\sqrt{2}\tilde{\phi} = \phi + \varphi. \tag{3.11}$$

We find that the three-dimensional (3D) action, in the vacuum, truncates to

[5]Ten from the metric, six from the B_2 field, six from the C_2 field, the string coupling, one from the C_4, and one from C_0.

[6]We normalize the curvature of the 3-sphere metric $\mathrm{d}\Omega_3^2$ to be 6.

$$e^{-1}\mathcal{L} = R - \frac{1}{2}(\partial\tilde{\phi})^2 - \frac{1}{2}(\partial\psi)^2 - \frac{1}{2}e^{\sqrt{2}\tilde{\phi}-4\gamma\psi}(\partial c_1)^2 - \frac{1}{2}e^{\sqrt{2}\tilde{\phi}+4\gamma\psi}(\partial c_2)^2 - \Lambda, \tag{3.12}$$

where the AdS vacuum lives at the following values for the scalars:

$$e^{\phi-\varphi} = \left|\frac{Q_1}{Q_5}\right|, \qquad e^{4\overline{\beta\varphi}} = \frac{|Q_1 Q_5|}{4}, \tag{3.13}$$

and

$$\Lambda = -\frac{32}{|Q_1 Q_5|^2}. \tag{3.14}$$

What we have described here is a consistent truncation of the bigger action (2.1) that is itself a truncation down to the moduli space of the AdS_3 vacuum. The truncation (3.12) is consistent and will be shown below to generate the solutions of interest by means of $SO(4,5)$ for $\mathbb{T}^4$ or $SO(4,20)$ for K_3.

Interestingly, the two dilaton vectors appearing in the axion kinetic terms of (3.12) are orthogonal since $2 - 16\gamma^2 = 0$. This means we have effectively two decoupled $\frac{SL(2,\mathbb{R})}{SO(2)}$ pairs in the truncation. To make this manifest, we define

$$\phi_1 \equiv \frac{1}{\sqrt{2}}(\tilde{\phi} - \psi), \qquad \phi_2 \equiv \frac{1}{\sqrt{2}}(\tilde{\phi} + \psi), \tag{3.15}$$

and then (3.12) becomes

$$e^{-1}\mathcal{L} = \mathcal{R}_3 - \frac{1}{2}(\partial\phi_1)^2 - \frac{1}{2}e^{2\phi_1}(\partial c_1)^2 - \frac{1}{2}(\partial\phi_2)^2 - \frac{1}{2}e^{2\phi_2}(\partial c_2)^2 - \Lambda. \tag{3.16}$$

Note that in Euclidean signature the kinetic terms of c_1 and c_2 are flipped.

C. No wormholes in the truncation

For an action in three Euclidean dimensions that consists of decoupled axion-saxion pairs as follows,

$$e^{-1}\mathcal{L} = \mathcal{R}_3 - \frac{1}{2}\sum_{i=1}^{2}((\partial\phi_i)^2 - e^{b_i\phi_i}(\partial c_i)^2) - \Lambda, \tag{3.17}$$

regular wormholes are possible once [11]

$$\sum_{i=1}^{2}\frac{1}{b_i^2} > 1. \tag{3.18}$$

This regularity condition, carefully derived in Ref. [11], roughly comes about as follows: the Einstein equations are blind to the details of the sigma model and only see the total geodesic velocity. Since the radial symmetric harmonic function is the affine parametrization of the geodesic, the Einstein equations predict a certain length for the timelike geodesic as it moves from one side of the wormhole to the other. Whether such geodesics can fit the length predicted by the explicit sigma model is not guaranteed. When it cannot, it represents itself as a singularity in the physical scalar fields. For example, consider the axiodilaton in IIB supergravity; they form the sigma model of the maximally supersymmetric moduli space of $AdS_5 \times S^5$. The dilaton coupling, however, is too large to allow regular wormholes [10], and this manifests itself by an expression for the dilaton of the following form,

$$e^{\phi} \sim \cos(H), \tag{3.19}$$

with H being a harmonic that ranges beyond the interval $(0,\pi)$, such that the dilaton becomes complex somewhere in the wormhole geometry.

In our case (3.16), we have instead $b_1 = b_2 = 2$, which *do not* satisfy the regularity condition. Hence, there are no regular wormholes in this truncation, but neither are there any in the full moduli space since we will later demonstrate that all timelike geodesics have lengths bounded by the ones in the truncation. This is inconsistent with the claim in Ref. [11], and we have traced this discrepancy back to a Wick rotation of "axion" fields in Ref. [11] that were not really independent axion-dilaton pairs in moduli space.[7]

D. Uplift to Euclidean D1 strings

The two axion-dilaton pairs in (3.12) exactly source the Euclidean D1-branes wrapping the torus 2-cycles as explained earlier. In here, we make this manifest by uplifting the 3D *extremal* instantons.

The 3D metric for the extremal solution is undeformed Euclidean AdS_3, and the expressions for the scalars are

$$e^{\phi_i(\rho)} = e^{k_i}(1 - a_i\rho), \quad c_i(\rho) = \pm\frac{e^{-k_i}a_i\rho}{1 - a_i\rho} + c_{i0}, \tag{3.20}$$

where $i = 1, 2$ and a_i, k_i are integration constants. The function $\rho(\tau)$ is a spherical harmonic function on AdS_3,

$$\nabla^2\rho = 0, \tag{3.21}$$

for the metric[8]

$$ds_3^2 = \frac{d\tau^2}{1 + \frac{\tau^2}{\ell^2}} + \tau^2 d\Omega_2^2, \tag{3.22}$$

[7] In a recent paper [36], regular wormholes were found in this setting, but they are not the axion wormholes we are considering here.

[8] Here, $\Lambda = -2/ell^2$.

whose details we do not need aside from fixing its shift such that the boundary of AdS_3 (the UV) lives at $\rho = 0$ and the IR lives at $\rho = -\infty$. An explicit expression can be found in the Appendix of Ref. [10].

Regularity requires $a_i > 0$. The axion charges are given by

$$q_i = \frac{1}{Vol(S^2)} \int_{S^2} (e^{2\phi_i} \partial_\rho c_i)_{\rho=0} = \pm e^{k_i} a_i \tag{3.23}$$

and should be properly quantized.

Now, we are ready to uplift the extremal instanton solutions in AdS_3. The uplift formula for the metric and the dilaton are[9]

$$2\phi = \phi_1 + \phi_2 + \ln(|Q_1 Q_5^{-1}|), \tag{3.24}$$

$$2\varphi = \phi_1 + \phi_2 - \ln(|Q_1 Q_5^{-1}|), \tag{3.25}$$

$$\sqrt{2}\psi = \phi_1 - \phi_1. \tag{3.26}$$

Such that the 10D metric in Einstein becomes

$$ds_{10}^2 = \left|\frac{Q_1}{Q_5}\right|^{\frac{1}{4}} \left(\left(\frac{h_2}{h_1^3}\right)^{\frac{1}{4}} [d\theta_1^2 + d\theta_2^2] + \left(\frac{h_1}{h_2^3}\right)^{\frac{1}{4}} [d\theta_3^2 + d\theta_4^2] \right) + (h_1 h_2)^{\frac{1}{4}} \left|\frac{Q_5}{Q_1}\right|^{\frac{1}{4}} ds_6^2, \tag{3.27}$$

where the h_i are the following harmonics on AdS_3:

$$h_i = e^{k_i}(1 - a_i \rho). \tag{3.28}$$

The dilaton is given by

$$e^{2\phi} = \left|\frac{Q_1}{Q_5}\right| h_1 h_2. \tag{3.29}$$

To compare this with the intersection of Euclidean D1 strings, we present the usual supergravity solution for such an intersection based on the harmonic superposition rule and partial smearing [54]. In the 10D Einstein frame, the solution is given by

$$ds_{10}^2 = \left(\frac{H_2}{H_1^3}\right)^{\frac{1}{4}} [d\tilde{\theta}_1^2 + d\tilde{\theta}_2^2] + \left(\frac{H_1}{H_2^3}\right)^{\frac{1}{4}} [d\tilde{\theta}_3^2 + d\tilde{\theta}_4^2] + (H_1 H_2)^{\frac{1}{4}} d\tilde{s}_6^2, \tag{3.30}$$

$$e^{2\phi} = e^t H_1 H_2, \tag{3.31}$$

where $H_{1,2}$ are the harmonics of the two Euclidean D1 strings smeared over the S^3 and the transversal $\mathbb{T}^2$. The e^t factor in the dilaton is an integration constant that exists in the case of p-branes in flat noncompact space. Here, the background is $AdS_3 \times S^3 \times \mathbb{T}^4$, and that factor is fixed. Its exact value depends on the normalization of the harmonic functions $H_{1,2}$, which we have not (yet) specified.

The tildes in the above metric are indicating they could be rescaled with respect to the previous normalization for the metrics on the 4-torus and the six-dimensional space. Indeed, we find a full match upon identifying $H_i = h_i$, fixing $e^t = |Q_1 Q_5^{-1}|$ and rescaling the metrics on the $\mathbb{T}^4$ and ds_2^6 by constants involving Q_1, Q_5.

E. On-shell actions

Related, one can demonstrate the on-shell 3D bulk supergravity action for the instantons equals the on-shell value for the probe Euclidean D1 action in ten dimensions. Let us briefly sketch this.

We first compute the probe action for Euclidean D1 strings in the $AdS_3 \times S_3 \times \mathbb{T}_4$ vacuum. In 10D string units, the Dirac-Born-Infeld action equals

$$S = \frac{n_{1,2}}{g_s} \int_{\Sigma_2^{1,2}} \sqrt{g_2}, \tag{3.32}$$

in the string frame, with $n_{1,2}$ the number of strings wrapping the two 2-cycles indexed by the labels 1,2. By moving to the Einstein frame, we obtain

$$S = n_{1,2} e^{-\frac{\phi_0}{2} \pm \frac{\psi_0}{\sqrt{2}} - \frac{\varphi_0}{2}}, \tag{3.33}$$

The sign choice for ψ_0 determines which of the two 2-cycles inside $\mathbb{T}^4$ we wrap the strings around. To compute the on-shell action from the backreacted instanton solutions (so, beyond the probe level), we rely on a well-known fact, reviewed in, for instance, Ref. [31], that the on-shell action, after holographic renormalization, is only provided by the total derivative term that one generated from the action in which the axions are dualized to forms. In our setup, ignoring overall normalizations,[10] this gives

$$S_{\text{on-shell}}^{\text{real}} \sim \int \partial_\rho (c_1 e^{2\phi_1} \partial_\rho c_1 + c_2 e^{2\phi_2} \partial_\rho c_2) \sim e^{-k_1} n_1 + e^{-k_2} n_2. \tag{3.34}$$

Upon using the uplift formula to rewrite the exponentials in terms of the vevs of the scalars at the boundary,[11] we find a match with the 10D probe actions.

Similarly, the imaginary part of the action in 3D should equal the WZ (Wess-Zumino) actions in 10D. In 10D it is clear that the instantons have an imaginary part in the action coming from the WZ terms of the Euclidean D1 strings

[9] Here, we have fixed $\gamma = 1/\sqrt{8}$.

[10] Which would be absorbed in the 3D Planck mass.

[11] For example, $e^{-k_1} = e^{-\phi_1 0} = e^{-\frac{\phi_0}{2} + \frac{\psi_0}{\sqrt{2}} - \frac{\varphi_0}{2}}$.

$$S_{WZ} = in_{1,2} \int_{\Sigma_2^{1,2}} C_2. \tag{3.35}$$

But also in the 3D supergravity, the backreacted solutions have an imaginary piece as, for instance, explained in the Appendix of Ref. [16]. To find the imaginary pieces, we need the quantized axion charges (3.23) $q_{1,2} \sim n_{1,2}$ since

$$S^{\text{imaginary}}_{\text{on-shell}} \sim (ic_1(0)n_1 + ic_2(0)n_2). \tag{3.36}$$

This matches the probe computation on the nose since the axion vevs are, by construction, the C_2 form vevs integrated over the internal 2-cycles.

IV. SPACE OF ALL SOLUTIONS USING GROUP THEORY

So far, we have discussed a simple set of solutions corresponding to two stacks of Euclidean D1-branes wrapping the two orthogonal 2-cycles in the internal 4-torus as depicted in (3.1). We then showed that these solutions neatly correspond to specific lightlike geodesics on

$$\left(\frac{SL(2,\mathbb{R})}{SO(1,1)}\right)^2, \tag{4.1}$$

which is a consistent truncation of a bigger space corresponding to the proper Wick rotation of

$$\frac{SO(4,m)}{\mathrm{SO}(m) \times \mathrm{SO}(4)}, \tag{4.2}$$

with $m = 5$ for an internal 4-torus or $m = 20$ for a K3 surface. The aim of this section is to clarify what the Wick-rotated moduli space of (4.2) is and what the general set of instanton solutions is, by classifying the geodesics on that moduli space, modulo the action of the isometry group $\mathrm{SO}(4,m)$. After defining the Wick rotation of the moduli space (4.2), we shall characterize the general class of geodesics which the solution described in Sec. III belongs to. This class is characterized by Noether charge matrices, which are nilpotent elements of order 3 in the defining representation of $\mathrm{SO}(4,m)$. Solutions of this kind belong to specific *nilpotent orbits* with respect to the action of the isometry group $\mathrm{SO}(4,m)$. In Sec. IV C, we shall prove that, for generic m, all nilpotent orbits in the coset spaces of the Wick-rotated manifolds always have a representative in the maximally spit universal submanifold defined by $m = 4$.

Therefore, when dealing with extremal solutions, we can restrict our analysis to the study of lightlike geodesics in the Wick-rotated version of (4.2) corresponding to the maximal split case $m = 4$.

Let us start with the generic case of

$$\mathscr{M} = \frac{\mathrm{SO}(n,n)}{\mathrm{SO}(n) \times \mathrm{SO}(n)}, \tag{4.3}$$

describing the classical string moduli space of an n-torus $\mathbb{T}^n$. It is spanned by the internal components G_{ij}, B_{ij}, $i,j = 1,\ldots,n$, of the metric and of the B-field. We can equally think of it as the S-dual moduli space using the C_2 field, which we use later on. The relation with the AdS moduli space is to be understood as follows. When reducing IIB on the 4-torus, we end up with the maximal ungauged six-dimensional supergravity in which the scalar manifold has the form (4.3) with $n = 5$. The moduli $e^{-\frac{\phi}{2}}G_{ij}, C_{ij}$, $i,j = 1,\ldots,4$ span a submanifold of the form (4.3) with $n = 4$, ϕ being the ten-dimensional dilaton and G_{ij} being the $\mathbb{T}^4$ metric moduli in the Einstein frame. These 16 moduli are not lifted on the solution of the theory of the form $AdS_3 \times S^3$, which describe the near horizon geometry of a D1–D5 system. Their moduli space is indeed a submanifold of the 20-dimensional moduli space of the solution

$$\frac{\mathrm{SO}(4,4)}{\mathrm{SO}(4) \times \mathrm{SO}(4)} \subset \frac{\mathrm{SO}(4,5)}{\mathrm{SO}(5) \times \mathrm{SO}(4)}, \tag{4.4}$$

where $\mathrm{SO}(4,5)$ is the stabilizer in $\mathrm{SO}(5,5)$ of the D1–D5 charge vector [50,51].

A. Wick-rotated $\mathscr{M}^*$

Let us denote the Riemannian (i.e., non Wick-rotated) scalar manifold by $\mathscr{M} = G/H$, where G is the isometry group of the form $\mathrm{SO}(p,q)$ and the isotropy group $H = \mathrm{SO}(p) \times \mathrm{SO}(q)$ is the maximal compact subgroup of G, and let us denote the Wick-rotated manifold by $\mathscr{M}^* = G/H^*$, where now H^* is a different (i.e., noncompact) real form of the complexification of H.

Let us define the Wick rotation which is relevant to the problem under consideration. The effect of this rotation is to change the sign of the metric on the manifold along the directions of the axion fields. These scalars can be characterized as parameters of a maximal Abelian subalgebra $\mathfrak{A}$, consisting of nilpotent generators, of the isometry one $\mathfrak{g} = \mathfrak{so}(p,q)$ [55,56]. Let θ denote the Cartan involution on $\mathfrak{g}$, which defines its Cartan decomposition,

$$\mathfrak{g} = \mathfrak{K} \oplus \mathfrak{H}, \tag{4.5}$$

into the space of the noncompact generators (i.e., Hermitian in a suitable basis) $\mathfrak{K}$ and the maximal compact subalgebra $\mathfrak{H} = \mathfrak{so}(p) \oplus \mathfrak{so}(q)$ ($\theta(\mathfrak{K}) = -\mathfrak{K}, \theta(\mathfrak{H}) = \mathfrak{H}$). The space $\mathfrak{K}_2 \equiv \mathfrak{A} - \theta(\mathfrak{A})$ is a subspace of $\mathfrak{K}$, while $\mathfrak{H}_2 \equiv \mathfrak{A} + \theta(\mathfrak{A})$ is contained in $\mathfrak{H}$. The grading properties defining $\mathfrak{A}$ imply that the spaces $\mathfrak{K}$, $\mathfrak{H}$ decompose as follows,

$$\mathfrak{K} = \mathfrak{K}_1 \oplus \mathfrak{K}_2, \qquad \mathfrak{H} = \mathfrak{H}_1 \oplus \mathfrak{H}_2, \tag{4.6}$$

where $\mathfrak{H}_1$ is a subalgebra of $\mathfrak{H}$ generating a subgroup $H_c \subset H$, $H_c = e^{\mathfrak{H}_1}$. Under the adjoint action of H_c, the space $\mathfrak{H}_2$ transforms in a representation $\mathcal{R}$. The Wick rotation is effected by interchanging in (4.6) the spaces $\mathfrak{K}_2$ and $\mathfrak{H}_2$ so as to define

$$\mathfrak{K}^* = \mathfrak{K}_1 \oplus \mathfrak{H}_2, \qquad \mathfrak{H}^* = \mathfrak{H}_1 \oplus \mathfrak{K}_2, \tag{4.7}$$

where now $\mathfrak{K}^*$ is the coset space of the Wick-rotated manifold $\mathscr{M}^*$, isomorphic to its tangent space at the origin, while the algebra $\mathfrak{H}^*$ generates its noncompact isotropy group H^*. The decomposition of $\mathfrak{g}$ into $\mathfrak{K}^*, \mathfrak{H}^*$,

$$\mathfrak{g} = \mathfrak{H}^* \oplus \mathfrak{K}^*, \tag{4.8}$$

is referred to as *pseudo-Cartan* decomposition. These two spaces are now eigenspaces of a new involution, θ^*: $\theta^*(\mathfrak{H}^*) = \mathfrak{H}^*, \theta^*(\mathfrak{K}^*) = -\mathfrak{K}^*$. The metric on the tangent space at the origin of $\mathscr{M}^*$ is defined by the restriction of the Cartan-Killing metric of $\mathfrak{g}$ to $\mathfrak{K}^*$ and thus has negative signature directions along a basis of $\mathfrak{H}_2$. These are the directions of the axionic fields since only the axionic isometry generators have components in $\mathfrak{H}_2$. In particular, the axion charges are defined as the components of the Noether charge matrix Q of a geodesic along the generators of $\mathfrak{H}_2$.

As far as the $\mathfrak{g} = \mathfrak{so}(p,q)$ algebra is concerned, there are two kinds of maximal Abelian subalgebras which are relevant to our discussion:

(i) A generic $\mathfrak{so}(p,q)$ algebra always has a $(p+q-2)$-dimensional maximal Abelian subalgebra defined by the decomposition

$$\mathfrak{so}(p,q) = \mathfrak{so}(1,1)_0 \oplus \mathfrak{so}(p-1,q-1)_0 \oplus (\mathbf{p+q-2})_{+1} \oplus \overline{(\mathbf{p+q-2})_{-1}}, \tag{4.9}$$

where the grading refers to the $\mathfrak{so}(1,1)$-generator. Since there are no generators with grading $+2$ or -2, the subspaces in the representations $(\mathbf{p+q-2})_{+1}$ and $\overline{(\mathbf{p+q-2})_{-1}}$ are separately Abelian subalgebras. In this case, we can choose $\mathfrak{A} = (\mathbf{p+q-2})_{+1}$. An example of this subalgebra is the one parametrized by the eight RR scalars $C_{ij}, C_{ijkl}, C_{(0)}$ within $\mathfrak{so}(5,5)$ in the maximal $D=6$ theory originating from type IIB superstring compactified on $\mathbb{T}^4$. Another instance of such Abelian subalgebra is the one parametrized by the 22 components C_I of the Type IIB R-R 2-form $C_{(2)}$ along the 2-cycles of an internal K_3. In this case, the isometry group of the moduli space is $\mathrm{SO}(4,20)$.

(ii) Only for $p=q=n$, we have a maximal Abelian subalgebra of dimension $n(n-1)/2$ defined by the following decomposition:

$$\mathfrak{so}(n,n) = \mathfrak{so}(1,1)_0 \oplus \mathfrak{gl}(n)_0 \oplus \left(\frac{\mathbf{n(n-1)}}{\mathbf{2}}\right)_{+1} \oplus \overline{\left(\frac{\mathbf{n(n-1)}}{\mathbf{2}}\right)_{-1}}, \tag{4.10}$$

The same grading argument used in case i implies that the subspaces of generators with gradings $+1$ and -1 are separately Abelian subalgebras. In this case, $\mathfrak{A} = (\frac{\mathbf{n(n-1)}}{\mathbf{2}})_{+1}$, and an explicit construction of its generators, as $2n \times 2n$ matrices in a suitable basis, is given below in Eq. (4.18). Instances of this subalgebra is the one parametrized by the moduli B_{ij} in the algebra $\mathfrak{so}(n,n)$ acting on the moduli G_{ij}, B_{ij} of type IIB supergravity compactified on $\mathbb{T}^n$ or by the moduli C_{ij} within the $\mathfrak{so}(n,n)$ acting, in the same $D=6$ theory, on the moduli G_{ij}, C_{ij}. When $n=4$, this maximal Abelian subalgebra is isomorphic to the one in case i, both having dimension 6. They are related by triality.

In case i, the Wick-rotated manifold is

$$\mathscr{M}^* = \frac{\mathrm{SO}(p,q)}{\mathrm{SO}(1,p-1)\times\mathrm{SO}(1,q-1)}, \tag{4.11}$$

the group H_c is $\mathrm{SO}(p-1)\times\mathrm{SO}(q-1)$, and the representation $\mathcal{R}$ in which $\mathfrak{K}_2$, $\mathfrak{H}_2$ transform under the adjoint action of H_c is the $(\mathbf{p-1},\mathbf{1}) \oplus (\mathbf{1},\mathbf{q-1})$. We can therefore view $\mathfrak{K}_2$ as the coset space of the following symmetric manifold:

$$\frac{\mathrm{SO}(1,p-1)}{\mathrm{SO}(p-1)} \times \frac{\mathrm{SO}(1,q-1)}{\mathrm{SO}(q-1)} = e^{\mathfrak{K}_2}. \tag{4.12}$$

In case ii, the Wick-rotated manifold is

$$\mathscr{M}^* = \frac{\mathrm{SO}(n,n)}{\mathrm{SO}(n,\mathbb{C})}, \tag{4.13}$$

$H_c = \mathrm{SO}(n)$, and $\mathcal{R} = \frac{\mathbf{n(n-1)}}{\mathbf{2}}$.

Below, we shall expand on case ii and study the geometry of the Wick-rotated manifold. The manifold is parametrized by the moduli $\tilde{G}_{ij} = e^{-\phi/2}G_{ij}, C_{ij}$, and the Wick rotation flips the sign of the kinetic terms of C_{ij}. We are interested in the $n=4$ case, for which $\mathrm{SO}(4,\mathbb{C}) \sim \mathrm{SL}(2,\mathbb{C})^2 \sim \mathrm{SO}(1,3)^2$.

Let us use, as an $\mathrm{SO}(n,n)$-invariant metric in the defining representation, the matrix

$$\eta = \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix} = \sigma_1 \otimes \mathbf{1}_n, \tag{4.14}$$

where $\mathbf{1}_n$ is the $n \times n$ identity matrix and $\sigma_1, \sigma_2, \sigma_3$ are the Pauli matrices. According to the Cartan decomposition

(4.5), the isometry algebra $\mathfrak{g} = \mathfrak{so}(n,n)$ splits into its maximal compact subalgebra where $\mathfrak{H} = \mathfrak{so}(n) \oplus \mathfrak{so}(n)$ and the space $\mathfrak{K}$ consisting of the Hermitian matrices in the algebra $\mathfrak{g}$. According to our discussion above, we can further split the subspaces $\mathfrak{H}$ and $\mathfrak{K}$ as in (4.6), where $\mathfrak{K}_2$, $\mathfrak{H}_2$ are spanned, respectively, by the Hermitian and anti-Hermitian components of the elements of the maximal Abelian subalgebra $\mathfrak{A}$. In the $SO(n,n)$ defining representation, the generic representatives of the above subspaces have the following form,

$$\mathfrak{H}_1 = \{\mathbf{1}_2 \otimes \mathbf{A}\}, \qquad \mathfrak{H}_2 = \{\sigma_1 \otimes \mathbf{A}'\},$$
$$\mathfrak{K}_1 = \{\sigma_3 \otimes \boldsymbol{\gamma}\}, \qquad \mathfrak{K}_2 = \{i\sigma_2 \otimes \mathbf{C}\}, \tag{4.15}$$

the matrices $\mathbf{A}, \mathbf{A}', \mathbf{C}$ being generic $n \times n$ antisymmetric matrices and $\boldsymbol{\gamma}$ being a generic symmetric matrix. The subspace $\mathfrak{K}_1$ is the coset space of the metric moduli G_{ij} of $\mathbb{T}^n$, suitably combined with the ten-dimensional dilaton ϕ, and it generates the submanifold $\frac{GL(n,\mathbb{R})}{SO(n)}$ and is spanned by $\boldsymbol{\gamma} = (\gamma_{ij}) = \boldsymbol{\gamma}^T$. As discussed above, the Wick rotation is effected by exchanging the roles of the spaces $\mathfrak{H}_2$ and $\mathfrak{K}_2$, so that the algebra $\mathfrak{g}$ decomposes according to the pseudo-Cartan decomposition (4.8), where $\mathfrak{H}^*, \mathfrak{K}^*$ are given in (4.7). Now, $\mathfrak{H}^*$ is the algebra $\mathfrak{so}(n,\mathbb{C})$, while $\mathfrak{K}^*$ has $n(n-1)/2$ negative signature directions corresponding to the compact generators in $\mathfrak{H}_2$. The pseudo-Cartan decomposition is defined by an involution θ^*, defined by the matrix $\eta' = \sigma_3 \otimes \mathbf{1}_n$ as follows:

$$\theta^*(\mathfrak{H}^*) = -\eta'(\mathfrak{H}^*)^T\eta' = \mathfrak{H}^*, \;\; \theta^*(\mathfrak{K}^*) = -\eta'(\mathfrak{K}^*)^T\eta' = -\mathfrak{K}^*. \tag{4.16}$$

We then have the following local isometric representation,

$$\frac{SO(n,n)}{SO(n,\mathbb{C})} \sim \left(\frac{GL(n,\mathbb{R})}{SO(n)}\right) \ltimes e^{\mathfrak{A}}, \tag{4.17}$$

where $\mathfrak{A}$ is the Abelian algebra generated by nilpotent matrices parametrized by $\mathbf{C} = (C_{ij})$ while $\frac{GL(n,\mathbb{R})}{SO(n)}$ is spanned by $\boldsymbol{\gamma}$, related to the metric moduli of the internal torus. We can use the following matrix representations,

$$\mathfrak{A} = \left\{\sigma_+ \otimes \mathbf{C} = \sigma_+ \otimes \frac{1}{2} t^{ij} C_{ij}\right\}, \tag{4.18}$$

where $\sigma_+ \equiv (\sigma_1 + i\sigma_2)/2$ satisfies the relation $[\sigma_3, \sigma_+] = 2\sigma_+$, while $(t^{ij})_{kl} = 2\delta^{ij}_{kl}$. According to (4.17), we define the coset representative as follows,

$$L = e^{\mathfrak{A}} L_G, \tag{4.19}$$

where $L_G \in e^{\mathfrak{K}_1}$ is the coset representative of $\frac{GL(n,\mathbb{R})}{SO(n)}$. The matrix $\mathcal{M}$ locally describing the coset is defined as follows,

$$\mathcal{M} \equiv L\eta' L^T = e^{\mathfrak{A}} L_G \eta' L_G^T (e^{\mathfrak{A}})^T = e^{\mathfrak{A}} \mathcal{M}_G \eta' (e^{\mathfrak{A}})^T, \tag{4.20}$$

where $\mathcal{M}_G \equiv L_G L_G^T$ and we have used the property that L_G commutes with η'.

The generic element of the group $e^{\mathfrak{A}}$ and $\mathcal{M}_G$ have the form

$$e^{\mathfrak{A}} = \begin{pmatrix} \mathbf{1} & \mathbf{C} \\ \mathbf{0} & \mathbf{1} \end{pmatrix}, \qquad \mathcal{M}_G = \begin{pmatrix} \tilde{\mathbf{G}} & \mathbf{0} \\ \mathbf{0} & \tilde{\mathbf{G}}^{-1} \end{pmatrix}, \tag{4.21}$$

where $\tilde{\mathbf{G}} = (\tilde{G}_{ij}) \equiv e^{2\boldsymbol{\gamma}}$.

The matrix $\mathcal{M}$ reads

$$\mathcal{M} = \begin{pmatrix} \mathbf{1} & \mathbf{C} \\ \mathbf{0} & \mathbf{1} \end{pmatrix} \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & -\mathbf{1} \end{pmatrix} \begin{pmatrix} \tilde{\mathbf{G}} & \mathbf{0} \\ \mathbf{0} & \tilde{\mathbf{G}}^{-1} \end{pmatrix} \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ -\mathbf{C} & \mathbf{1} \end{pmatrix}$$
$$= \begin{pmatrix} \tilde{\mathbf{G}} + \mathbf{C}\tilde{\mathbf{G}}^{-1}\mathbf{C} & -\mathbf{C}\tilde{\mathbf{G}}^{-1} \\ \tilde{\mathbf{G}}^{-1}\mathbf{C} & -\tilde{\mathbf{G}}^{-1} \end{pmatrix}. \tag{4.22}$$

From this, we can compute the metric on moduli space as

$$ds^2 = \frac{1}{4}\mathrm{Tr}[\mathcal{M}^{-1}d\mathcal{M}\mathcal{M}^{-1}d\mathcal{M}]$$
$$= \frac{1}{2}(\tilde{G}^{mp}\tilde{G}^{nq}d\tilde{G}_{mn}d\tilde{G}_{pq} - \tilde{G}^{mp}\tilde{G}^{nq}dC_{mn}dC_{pq}), \tag{4.23}$$

where, as mentioned earlier,

$$\tilde{G}_{ij} = e^{-\frac{\phi}{2}} G_{ij},$$

G_{ij} being the metric of the 4-torus in the Einstein frame.[12] The sigma-model Lagrangian density then reads

$$\mathcal{L}_{(\tilde{G},C)} = -\frac{1}{4}(\tilde{G}^{mp}\tilde{G}^{nq}\partial_\mu\tilde{G}_{mn}\partial^\mu\tilde{G}_{pq} - \tilde{G}^{mp}\tilde{G}^{nq}\partial_\mu C_{mn}\partial^\mu C_{pq}). \tag{4.24}$$

We see that indeed the axion scalars have the opposite sign of the kinetic term. In what follows, we use the exponential map to solve and classify the geodesics equations. In practice, this means that the above sigma model can trivially be solved for geodesics in terms of the symmetric coset matrix $\mathcal{M}$. Let us, for the sake of notational simplicity, collectively denote the moduli $\tilde{G}_{ij}, C_{ij}$ by ϕ^I. The geodesics on $\mathscr{M}^*$ can be classified in orbits with respect to the action of the isometry group G. More precisely, using transformations in G/H^*, the initial point at $\rho = 0$ can always be chosen to coincide with a given one $\phi_0 = (\phi_0^I)$. Once this point is fixed, we still have the freedom of changing the *initial velocity*, represented by the Noether charge matrix Q_0, within the tangent space

[12]The combination $e^{\phi}\det(G_{ij})^{\frac{1}{2}}$ is fixed in terms of the D1–D5 charges.

$T_{\phi_0}(\mathscr{M}^*)$ to the moduli space at ϕ_0, by means of the isotropy group $H^*_{\phi_0}$ of ϕ_0. For the sake of simplicity, we can start fixing the initial point to be the origin

$$\phi_0 = O \Leftrightarrow \tilde{G}_{ij}(\rho = 0) = \delta_{ij}, C_{ij}(\rho = 0) = 0,$$

so that $H^*_{\phi_0} = H^*$ and the geodesics are completely determined by the initial velocity Q, now an element of $\mathfrak{K}^*$. The geodesic is solution to the matrix equation:

$$\mathcal{M}(\phi(\rho)) = \mathcal{M}(\tilde{\mathbf{G}}(\rho), \mathbf{C}(\rho)) = \eta' \cdot e^{2Q\rho}. \tag{4.25}$$

As an element of $\mathfrak{K}^*$, the general form of Q is

$$Q = \sigma_3 \otimes \boldsymbol{\gamma} + \sigma_1 \otimes \mathbf{c}, \tag{4.26}$$

where $\boldsymbol{\gamma}^t = \boldsymbol{\gamma}$ and $\mathbf{c}^t = -\mathbf{c}$.

The geodesic $\phi(\rho, \phi_0)$ through a generic point ϕ_0 at $\rho = 0$ is then obtained from the one through the origin by solving the matrix equation,

$$\mathcal{M}(\phi(\rho, \phi_0)) = L(\phi_0)\mathcal{M}(\phi(\rho))L(\phi_0)^T = \mathcal{M}(\phi_0)e^{2\rho Q_0}, \tag{4.27}$$

where $Q_0 \equiv L(\phi_0)^{-1T} Q L(\phi_0)^T$ is an element of the tangent space to the moduli space at ϕ_0.

The backreaction on the spacetime of a geodesic, to be denoted by (ϕ_0, Q_0), through a point ϕ_0 at $\rho = 0$, with initial velocity Q_0, is described by the Einstein equation,

$$\mathcal{R}_{\mu\nu} = \frac{g_{\mu\nu}}{D-2}\Lambda + \frac{1}{2}\mathrm{Tr}(Q_0^2)\partial_\mu\rho\partial_\nu\rho, \tag{4.28}$$

where in our case $D = 6$. The geodesic velocity c defined in (2.3) reads

$$c = \mathrm{Tr}(Q_0^2). \tag{4.29}$$

B. Teneral solution for the geodesics

Let us now describe the general form of the geodesics in $\mathscr{M}^*$ generated by a Noether charge matrix $Q \in \mathfrak{K}^*$, through the origin. They belong to the three classes:

(i) *Extremal instantons*.—These are the lightlike geodesics: $c = 0$. From (4.29), it follows that $\mathrm{Tr}(Q_0^2) = 0$. Regularity of the solution then requires the Q-matrix to be *nilpotent*. The scalar energy-momentum tensor vanishes, and so does the backreaction of these solutions on spacetime; see Eq. (4.28). As we shall prove in Sec. IV C, the maximal degree of nilpotency of a nilpotent element Q of $\mathfrak{K}^*$, in the representation $\mathbf{8}_v$ of $\mathrm{so}(4,4)$, is 4: $Q^4 = \mathbf{0}$. The extremal solutions constructed in Sec. III are generated by an order-2 nilpotent matrix Q;

(ii) *Overextremal instantons*.—These are the timelike geodesics and correspond to wormholes, but they will not be regular in their scalar profiles as we explained before. Then, Q is semisimple with imaginary eigenvalues. As we are interested in evaluating the maximal length of timelike geodesics, we can take Q in $\mathfrak{H}_2$.

(iii) *Subextremal instantons*.—These are the spacelike geodesics with Q having real eigenvalues in $\mathfrak{K}_1$.

The regularity condition on the above solutions is

$$\infty > \tilde{G}_{ij} > 0. \tag{4.30}$$

1. Extremal solutions

Since, in the representation we are currently considering, the Noether charge matrix Q is nilpotent of order at most 4, we give the explicit form of the generic solution in the $Q^4 = \mathbf{0}$ orbit. The two matrices $\boldsymbol{\gamma}$, $\mathbf{c}$ satisfy the conditions:

$$(\boldsymbol{\gamma}^2 + \mathbf{c}^2)^2 - [\boldsymbol{\gamma}, \mathbf{c}]^2 = \mathbf{0},$$
$$(\boldsymbol{\gamma}^2 + \mathbf{c}^2) \cdot [\boldsymbol{\gamma}, \mathbf{c}] = -[\boldsymbol{\gamma}, \mathbf{c}] \cdot (\boldsymbol{\gamma}^2 + \mathbf{c}^2). \tag{4.31}$$

The general form of the geodesic is

$$\tilde{\mathbf{G}}(\rho) = (\tilde{G}_{ij}(\rho)) = \left(\mathbf{1} - 2\rho\boldsymbol{\gamma} + 2\rho^2(\boldsymbol{\gamma}^2 + \mathbf{c}^2) - \frac{4}{3}((\boldsymbol{\gamma}^2 + \mathbf{c}^2) \cdot \boldsymbol{\gamma} + [\boldsymbol{\gamma}, \boldsymbol{c}] \cdot \boldsymbol{c})\rho^3\right)^{-1},$$
$$\mathbf{C}(\rho) = -2\rho\tilde{G}(\rho) \cdot \left(\mathbf{c} - \rho[\boldsymbol{\gamma}, \mathbf{c}] + \frac{2}{3}\rho^2((\boldsymbol{\gamma}^2 + \mathbf{c}^2) \cdot \boldsymbol{c} - [\boldsymbol{\gamma}, \boldsymbol{c}] \cdot \boldsymbol{\gamma})\right). \tag{4.32}$$

The matrices $\boldsymbol{\gamma}$ and $\boldsymbol{c}$ are constrained by the regularity condition (4.30).

If Q belongs to the Q^3-orbit, the following conditions hold,

$$(\boldsymbol{\gamma}^2 + \mathbf{c}^2) \cdot \boldsymbol{\gamma} + [\boldsymbol{\gamma}, \mathbf{c}] \cdot \mathbf{c} = \mathbf{0}, \quad (\boldsymbol{\gamma}^2 + \mathbf{c}^2) \cdot \mathbf{c} - [\boldsymbol{\gamma}, \mathbf{c}] \cdot \boldsymbol{\gamma} = \mathbf{0}, \tag{4.33}$$

which set the ρ^3 terms in the solution (4.32) to zero. Finally, if $Q^2 = 0$, we have the stronger condition,

$$\boldsymbol{\gamma}^2 + \mathbf{c}^2 = \mathbf{0}, \qquad [\boldsymbol{\gamma}, \mathbf{c}] = \mathbf{0}, \tag{4.34}$$

and also the ρ^2 terms in (4.32) vanish. We shall discuss a normal form for a Q in this orbit in Sec. IV C 1.

2. Semisimple Q in $\mathfrak{K}_1$

Consider now Q semisimple in $\mathfrak{K}_1$. It has real eigenvalues. This is the case if we set $\mathbf{c} = \mathbf{0}$ so that $Q = \sigma_3 \otimes \boldsymbol{\gamma}$ in the coset space of $\mathrm{GL}(4,\mathbb{R})/\mathrm{SO}(4)$. The general geodesic has the form

$$\tilde{\mathbf{G}}(\rho) = \cosh(2\rho\boldsymbol{\gamma}) + \sinh(2\rho\boldsymbol{\gamma}), \qquad \mathbf{C}(\rho) = \mathbf{0}. \tag{4.35}$$

The inverse of $\tilde{G}$ is $\tilde{G}^{-1}(\rho) = \cosh(2\rho\boldsymbol{\gamma}) - \sinh(2\rho\boldsymbol{\gamma})$. This matrix can be diagonalized by an SO(4) rotation. We now denote by γ_i the eigenvalues of $\boldsymbol{\gamma}$. In the basis in which this matrix is diagonal also the metric is diagonal and reads

$$\tilde{G}_{ij}(\rho) = \delta_{ij}(\cosh(2\rho\gamma_i) + \sinh(2\rho\gamma_i)). \tag{4.36}$$

3. Semisimple Q in $\mathfrak{H}_2$

Consider now Q semisimple in $\mathfrak{H}_2$. It has imaginary eigenvalues. This is the case if we set $\boldsymbol{\gamma} = \mathbf{0}$ so that $Q = \sigma_1 \otimes \boldsymbol{c}$. The general geodesic has the form

$$\begin{aligned} \tilde{\mathbf{G}}(\rho) &= \cosh(2\rho\mathbf{c})^{-1}, \\ \mathbf{C}(\rho) &= -\cosh(2\rho\mathbf{c})^{-1} \cdot \sinh(2\rho\mathbf{c}). \end{aligned} \tag{4.37}$$

By means of an SO(4) rotation, $\mathbf{c}$ can be brought to a skew-diagonal form $\mathbf{c}_{\mathrm{SD}}$, with only nonvanishing entries $c_{12} = c_1$ and $c_{34} = c_2$:

$$\mathbf{c}_{\mathrm{SD}} = \begin{pmatrix} 0 & c_1 & 0 & 0 \\ -c_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & c_2 \\ 0 & 0 & -c_2 & 0 \end{pmatrix}. \tag{4.38}$$

In this basis, the solution is characterized by the following only nonvanishing components of $\tilde{G}(\rho)$ and $C(\rho)$:

$$\begin{aligned} \tilde{G}_{11}(\rho) = \tilde{G}_{22}(\rho) = \cos(2\rho c_1)^{-1}, \quad & \tilde{G}_{33}(\rho) = \tilde{G}_{44}(\rho) = \cos(2\rho c_2)^{-1}, \\ C_{12}(\rho) = -C_{21}(\rho) = -\tan(2\rho c_1), \quad & C_{34}(\rho) = -C_{43}(\rho) = -\tan(2\rho c_2). \end{aligned} \tag{4.39}$$

This solution generates the most general timelike geodesic. It belongs to the truncation considered in Sec. III and describes singular wormholes.

So far, we have been working with the $\mathbf{8}_v$ representation of SO(4, 4) which branches with respect to $\mathrm{GL}(4,\mathbb{R})$ as $\mathbf{8}_v \to \mathbf{4}_+ + \bar{\mathbf{4}}_-$. When embedding the defining representation of SO(4, 4) within $\mathrm{SO}(4,m)$, $m > 4$, we shall be working with the representation $\mathbf{8}_s$ instead, related to $\mathbf{8}_v$ by triality, which branches with respect to the same subgroup as $\mathbf{8}_s \to \mathbf{6}_0 + \mathbf{1}_- + \mathbf{1}_+$. The maximal Abelian subalgebra $\mathfrak{A}$ will then be of kind i instead of ii, and some of the allowed nilpotent orbits for Q will change accordingly.

C. Issue of nilpotent orbits

Extremal solutions are described by a nilpotent Noether charge matrix Q in $\mathfrak{K}^*$, which is then classified in orbits with respect to the adjoint action of H^*. To formalize this concept, we refer to the discussion in the paragraph below Eq. (4.24). Once a global symmetry transformation in G/H^*, on a geodesic solution, has been used to make its initial point ϕ_0 at $\rho = 0$ coincide with the origin O (this transformation always exists being the manifold homogeneous), we can still act on the geodesic by means of transformations in H^*, which is the symmetry group of the origin and thus acts on the initial velocity (i.e., Noether change matrix) Q in the tangent space $T_O(\mathscr{M}^*)$ at O. Two solutions with $\phi_0 = O$ are connected by a transformation in H^* if and only if the corresponding Noether charge matrices Q and Q' are related by the adjoint action of H^*, namely, iff:

$$\exists\, \mathbf{h} \in H^*: \qquad Q' = \mathbf{h}^{-1} \cdot Q \cdot \mathbf{h}. \tag{4.40}$$

We say that Q and Q' (and thus the corresponding solutions) belong to the same H^*-orbit and the two solutions have the same physical properties. In this sense, we classify the geodesic solutions in our model within H^*-orbits. Given two geodesic solutions (ϕ_0, Q_0), (ϕ_0', Q_0') with initial points ϕ_0, ϕ_0' and initial velocities $Q_0 \in T_{\phi_0}(\mathscr{M}^*)$ and $Q_0' \in T_{\phi_0'}(\mathscr{M}^*)$, respectively, let (O, Q), (O, Q') be the corresponding G/H^*-transformed solutions through the origin O at $\rho = 0$. The two geodesics (ϕ_0, Q_0), (ϕ_0', Q_0') are then related by a G-transformation iff Q and Q' belong to the same H^*-orbit.The G-transformation connecting (ϕ_0, Q_0) to (ϕ_0', Q_0') consists in $L(\phi_0)^{-1}$ mapping ϕ_0 into the origin, combined with the H^*-element $\mathbf{h}$ transforming Q into Q', combined, in turn, with $L(\phi_0')$, which maps the origin into ϕ_0'.

As pointed out earlier, extremal solutions are characterized by a nilpotent Noether change matrix $Q_0 \in T_{\phi_0}(\mathscr{M}^*)$. This implies that $Q \in T_O(\mathscr{M}^*)$ belongs to a nilpotent orbit with respect to the adjoint representation of H^*.

In the previous sections, we focused on the geodesic solutions in a moduli space of the form (4.2) with $m = 4$. Here, we discuss how general this choice is and prove that the nilpotent orbits of Q in the moduli space with $m = 5$ all

have a representative in the maximally split subspace with $m = 4$. We shall refrain from reviewing the theory of nilpotent orbits of a semisimple Lie group, for which we refer the reader to some useful reviews [57,58]. The nilpotent orbits in $\mathfrak{so}(p,q)$ were classified in Ref. [59]. As explained above, the general problem which is relevant to our analysis is that of studying the nilpotent orbits within $\mathfrak{K}^*$ with respect to the adjoint action of H^*. This problem is referred to, in the mathematical literature, as that of classifying the nilpotent orbits of the vector space $\mathfrak{K}^*$ associated with the real semisimple symmetric pair $(\mathfrak{g}, \mathfrak{H}^*)$. For the sake of concreteness, we shall consider $\mathfrak{g} = \mathfrak{so}(p,q)$. In the case $p = q = 4$ and $\mathfrak{H}^* = \mathfrak{sl}(2,\mathbb{R})^4$, the problem was solved in Refs. [60,61]. However, the real semisimple symmetric pair which is relevant to our present analysis is the one with $\mathfrak{g} = \mathfrak{so}(p,q)$ and $\mathfrak{H}^* = \mathfrak{so}(1,p-1) \oplus \mathfrak{so}(1,q-1)$, for the special values $p = 4, q = m$. Here, we shall limit ourselves to identifying, in the latter case, those G-nilpotent orbits which have a representative in $\mathfrak{K}^*$, without further splitting them with respect to the action of H^*.

According to the Jacobson-Morozov theorem [57], any nilpotent element e of a real Lie algebra $\mathfrak{g}$ can be thought of as part of a *standard triple* of $\mathfrak{sl}(2,\mathbb{R})$-generators $\{h, e, f\}$ satisfying the standard commutation relations

$$[h,e] = e, \qquad [h,f] = -f, \qquad [e,f] = h.$$

We are interested in nilpotent elements e which lie in the coset space $\mathfrak{K}^*$. Then, the standard triple can be chosen so that $f \in \mathfrak{K}^*$ and h are a noncompact generator in $\mathfrak{H}^*$ (see Sec. 4.2 of Ref. [60] and references therein[13]). It is known that the nilpotent orbits in the complexification $\mathfrak{g}^{\mathbb{C}}$ of $\mathfrak{g}$ with respect to $G^{\mathbb{C}} = \exp \mathfrak{g}^{\mathbb{C}}$ are defined by the inequivalent embeddings of the $\mathfrak{sl}(2,\mathbb{C}) = \mathrm{Span}(h,e,f)$ inside $\mathfrak{g}^{\mathbb{C}}$, which in turn are defined by the different decompositions of the defining representation of $G^{\mathbb{C}}$ with respect to the corresponding $\mathrm{SL}(2,\mathbb{C})$ group (with a certain multiplicity prescription). Each of these decompositions is characterized by a partition of the dimension of the $(\mathbf{p}+\mathbf{q})$ representation of $\mathfrak{g}^{\mathbb{C}} = \mathfrak{so}(p,q;\mathbb{C})$. If, with respect to $\mathrm{SL}(2,\mathbb{C})$, the $(\mathbf{p}+\mathbf{q})$ representation branches as follows,

$$(\mathbf{p}+\mathbf{q}) \to \bigoplus_{i=1}^{\ell} \mathrm{k}_i \times [\mathbf{s}_i], \tag{4.41}$$

where we have used the ordering $s_\ell \geq s_{\ell-1} \geq \ldots \geq s_1$, the partition is denoted by $[(2s_\ell+1)^{\mathrm{k}_\ell}, \ldots, (2s_1+1)^{\mathrm{k}_1}]$ and represented by a corresponding Young tableau. According to the general theory, only certain partitions can occur and with certain multiplicities. When we consider real nilpotent orbits, there is a finer structure, and each nilpotent $\mathrm{SO}(p,q)$-orbit in $\mathfrak{so}(p,q)$ is described by a *graded Young tableau* [57]. The order of nilpotency of the corresponding orbit in the defining representation is $2s_\ell + 1$ since the h-grading of the element e of the orbit is 1 and the minimal and maximal eigenvalues of h in the defining representation are $-s_\ell$ and s_ℓ, respectively. For $\mathfrak{so}(4,4;\mathbb{C})$, the partitions are

$$[1^8], [2^2,1^4], [3,1^5], [2^4]^I, [2^4]^{II}, [3,2^2,1], [3^2,1^2],$$
$$[5,1^3], [4^2]^I, [4^2]^{II}, [5,3], [7,1], \tag{4.42}$$

$[1^8]$ being the trivial orbit corresponding to the zero-matrix. The orbits $[3,1^5], [2^4]^I, [2^4]^{II}$ are related to one another by $\mathrm{SO}(4,4)$-triality, and so are the orbits $[5,1^3], [4^2]^I, [4^2]^{II}$. We choose the embedding $\mathrm{SO}(4,4)$ inside $\mathrm{SO}(4,m)$ to be such that the defining representation $\mathbf{4}+\mathbf{m}$ of the latter, when branched with respect to the former, contains the $\mathbf{8}_s$ representation instead of the $\mathbf{8}_v$. The difference is that, with respect to the $\mathrm{GL}(4,\mathbb{R})$ group acting on the metric moduli of the 4-torus, the two eight-dimensional representations branch differently: $\mathbf{8}_s \to \mathbf{6}_0 + \mathbf{1}_- + \mathbf{1}_+, \mathbf{8}_v \to \mathbf{4}_+ + \bar{\mathbf{4}}_-$. This choice of the embedding of $\mathrm{SO}(4,4)$ inside $\mathrm{SO}(4,m)$ is appropriate to the problem at hand since if we consider the chain of embeddings $\mathrm{SO}(4,4) \subset \mathrm{SO}(4,5) \subset \mathrm{SO}(5,5)$, $\mathrm{SO}(5,5)$ being the global symmetry group of the maximal six-dimensional supergravity, when branching the $\mathbf{10}$ of the latter, describing the 3-form field strengths, with respect to $\mathrm{SO}(4,4) \times \mathrm{SO}(1,1)$ we have $\mathbf{10} \to \mathbf{8}_{s0} + \mathbf{1}_+ + \mathbf{1}_-$, since the $\mathbf{8}_{s0}$ contains the six 3-forms $H_{ij\mu\nu\rho}$ in the $\mathbf{6}_0$ of $\mathrm{GL}(4,\mathbb{R})$. For the same reason, the branching of the adjoint representation of $\mathrm{SO}(5,5)$ with respect to $\mathrm{SO}(4,4)$ contains the $\mathbf{8}_s$ instead of the $\mathbf{8}_v$. In the previous sections, we have been working with the $\mathrm{SO}(4,4)$-generators in the $\mathbf{8}_v$. Now, we shall use the $\mathbf{8}_s$ representation of the same group instead. This will affect the orbit assignment of a nilpotent generator in $\mathfrak{so}(4,4)$: a generator in the orbits $[2^4]^I, [2^4]^{II}$ as a matrix in the representations $\mathbf{8}_v$ or $\mathbf{8}_c$, in the $\mathbf{8}_s$, will belong to the orbit $[3,1^5]$. Similarly, triality will map the orbits $[4^2]^I$ or $[4^2]^{II}$, when the nilpotent generator is in the $\mathbf{8}_v$ or $\mathbf{8}_c$, into the orbit $[5,1^3]$ when it is represented in the $\mathbf{8}_s$.

The main observation is that the neutral element h of the standard triple associated with a nilpotent generator $e \in \mathfrak{K}^*$ can always be chosen to lie in the subspace $\mathfrak{K}_2 \in \mathfrak{H}^*$. It then transforms under the adjoint action of $H_c = \mathrm{SO}(p-1) \times \mathrm{SO}(q-1) \subset H^*$ in the representation $(\mathbf{p-1},\mathbf{1}) \oplus (\mathbf{1},\mathbf{q-1})$. Restricting to $p = 4$ and $q = m$, by acting on the whole triple by means of the compact symmetry group H_c, h can always be rotated into a minimal two-dimensional subspace $\mathfrak{K}_2^{(N)} = \mathrm{Span}(\mathcal{J}_\ell)_{\ell=1,2}$ of $\mathfrak{K}_2$ which is contained in the subalgebra $\mathfrak{so}(4,4)$ of $\mathfrak{so}(4,m)$. This subspace defines the noncompact rank of the coset $\frac{\mathrm{SO}(1,3)}{\mathrm{SO}(3)} \times \frac{\mathrm{SO}(1,3)}{\mathrm{SO}(3)}$. The reason behind this is that any n-vector

[13]In Ref. [60], the spaces $\mathfrak{H}^*$ and $\mathfrak{K}^*$ were denoted by $\mathfrak{g}_0$ and $\mathfrak{g}_1$, respectively.

$\mathbf{v}$ in the defining representation of $SO(n)$ can be rotated, by means of this group, in the normal form: $\mathbf{v} = (\pm|\mathbf{v}|, 0, \ldots, 0)$. Thus, we can always rotate a generic h in the coset space $\mathfrak{K}_2$ of $\frac{SO(1,3)}{SO(3)} \times \frac{SO(1,m-1)}{SO(m-1)}$, using $H_c = SO(3) \times SO(m-1)$, in a two-dimensional universal subspace $\mathfrak{K}_2^{(N)}$ which is common to the all the coset spaces of $\frac{SO(1,3)}{SO(3)} \times \frac{SO(1,m)}{SO(m)}$, including the $m = 4$ case. This allows us to compute the nonvanishing eigenvalues of a generic $h \in \mathfrak{K}_2$ which are

$$\text{eigenvalues}(h) = \left\{\frac{\kappa_1+\kappa_2}{2}, -\frac{\kappa_1+\kappa_2}{2}, \frac{\kappa_1-\kappa_2}{2}, -\frac{\kappa_1-\kappa_2}{2}, \overbrace{0, \ldots, 0}^{m}\right\}, \tag{4.43}$$

where κ_ℓ are real parameters. The above eigenvalues are compatible with the only orbits [59],

$$[1^{4+m}], \quad [2^2, 1^m], \quad [3, 1^{m+1}], \quad [3^2, 1^{m-2}], \quad [5, 1^{m-1}], \tag{4.44}$$

which all have nontrivial intersection with the corresponding $SO(4,4)$-orbits in (4.42). This motivates our choice of restricting to the $m = 4$ manifold for the study of the extremal solutions.

The extremal solutions discussed in Sec. III belong, for generic values of a_1, a_2, to the orbit $[3, 1^5]$ of $SO(4,4)$, and thus the corresponding Noether matrix Q is nilpotent of order 3. If $a_1 a_2 = 0$, the orbit becomes $[2^2, 1^4]$, and the same generator is then nilpotent of order 2. Below, we shall expand on these two orbits of solutions, leaving a systematic study of solutions belonging to the orbits $[3^2, 1^2], [5, 1^3]$, and of their supersymmetry properties, to a future work.

We conclude that the generating solutions of all the lightlike geodesics lie within the manifold $SO(4,4)/SO(1,3)^2$. If we work in the $\mathbf{8}_v$ of $SO(4,4)$ instead of the $\mathbf{8}_s$, the orbits $[3, 1^5]$ and $[5, 1^3]$ are replaced by $[2^4]$ and $[4^2]$, respectively, so that the maximal order of nilpotency of an element of $\mathfrak{K}^*$ in this representation is 4. Using this property, in Sec. IV B, we gave the most general form of the extremal geodesic written in terms of the string moduli $\tilde{G}_{ij}, C_{ij}$, with boundary conditions $\tilde{G}_{ij}(\rho = 0) = \delta_{ij}, C_{ij}(\rho = 0) = 0$.

In the next subsection, we show that, if we only consider the orbits $[2^1, 1^4]$ and $[2^4]$ (in the $\mathbf{8}_v$), we can restrict ourselves to an even simpler characteristic submanifold $\mathscr{M}_{(N)} = [SL(2,\mathbb{R})/SO(1,1)]^2$.

1. $[SL(2,\mathbb{R})/SO(1,1)]^2$ subspace and normal forms for the orbits $[2^2,1^4],[2^4]$

In this section, we construct a characteristic submanifold $\mathscr{M}_{(N)}$ of the Wick-rotated moduli space $\mathscr{M}^*$ which contains representative geodesics of the $[2^2, 1^4]$ and the $[3, 1^4]$ ($[2^4]$ in the $\mathbf{8}_v$) orbits. In this way, we can relate the abstract and completely general coset construction to the simple Euclidean D1 solutions discussed in Sec. III. The logic presented here was first worked out in detail in Ref. [21] for geodesics on cosets that appear in timelike reductions of supergravity. The general idea is that one truncates the coset to the smallest subspace that generates all geodesics belonging to a certain characteristic subset of all the G-orbits, by means of the isometry group G. This subspace is often, but not always, a simple product of $[SL(2,\mathbb{R})/SO(1,1)]$ factors. In light of the discussion in the previous section, we shall restrict ourselves to the Wick-rotated moduli spaces with $m = 4$.

We write $\mathscr{M}^* = G/H^*$ where $G = SO(4,4) = \exp(\mathfrak{g})$ and $H^* = SO(1,3)^2 = \exp(\mathfrak{H}^*)$.

The isotropy group H^* contains a maximal compact subgroup $H_c = \exp(\mathfrak{H}_1) = SO(3)^2$, which can be used to simplify the generator $Q \in \mathfrak{K}^*$ of a geodesic through the origin. In particular, the compact generators in $\mathfrak{K}^*$, which define the axion charges, span the subspace $\mathfrak{H}_2$ of $\mathfrak{K}^*$, and transform, under the adjoint action of $H_c = SO(3)^2$, in the $(\mathbf{3},\mathbf{1}) \oplus (\mathbf{1},\mathbf{3})$. Similarly the noncompact generators of $\mathfrak{H}^*$ span the subspace $\mathfrak{K}_2$ transforming, under the adjoint action of H_c, in the same representation $(\mathbf{3},\mathbf{1}) \oplus (\mathbf{1},\mathbf{3})$ as $\mathfrak{H}_2$. It was shown in Sec. IV C that, using $SO(3)^2$ transformations, we can always rotate a generic element of $\mathfrak{K}_2$ in a two-dimensional subspace $\mathfrak{K}_2^{(N)}$ (*normal space* of $\mathfrak{K}_2$) generated by two commuting noncompact operators $\mathcal{J}_\ell$, $\ell = 1, 2$. By the same token, using H_c, it is possible to rotate a generic element of $\mathfrak{H}_2$ (describing, for instance, the compact component of the Noether charge matrix Q of a geodesic) in a two-dimensional normal subspace $\mathfrak{H}_2^{(N)}$ of $\mathfrak{H}_2$. Let us denote by $\mathcal{K}_\ell$, $\ell = 1, 2$, a suitable basis of $\mathfrak{H}_2^{(N)}$. As proven in general in Ref. [21] and as we shall show here by direct construction, we can choose $\mathfrak{H}_2^{(N)}$ and $\mathfrak{K}_2^{(N)}$ so that their generators $\mathcal{K}_\ell$ and $\mathcal{J}_\ell$, together with $\mathcal{H}_\ell \equiv [\mathcal{K}_\ell, \mathcal{J}_\ell]$, close a characteristic $SL(2,\mathbb{R})^2$ subgroup of G, and a submanifold

$$\mathscr{M}_{(N)} = \left(\frac{SL(2,\mathbb{R})}{SO(1,1)}\right)^2 \subset \mathscr{M}^*, \tag{4.45}$$

where the $\mathrm{SO}(1,1)^2$ at the denominator is generated by $\mathcal{J}_\ell$ and the coset space of $\mathscr{M}_{(N)}$, to be denoted by $\mathfrak{K}_{(N)}$, is generated by $\{\mathcal{H}_\ell, \mathcal{K}_\ell\}$. This coset space contains representatives of the $[2^2,1^4]$ and $[2^4]$ (in the $\mathbf{8}_v$) orbits, and the corresponding geodesics in $\mathscr{M}_{(N)}$ are easily constructed. Let us define the matrix form of those generators. In the basis of the $\mathbf{8}_v$ of $\mathrm{SO}(4,4)$ used in Sec. IV A, the generators read

$$\begin{aligned}
\mathcal{J}_1 &= \frac{1}{2}(\mathbf{e}_{1,6} - \mathbf{e}_{2,5} - \mathbf{e}_{5,2} + \mathbf{e}_{6,1}),\\
\mathcal{J}_2 &= \frac{1}{2}(\mathbf{e}_{3,8} - \mathbf{e}_{4,7} - \mathbf{e}_{7,4} + \mathbf{e}_{8,3}),\\
\mathcal{K}_1 &= \frac{1}{2}(\mathbf{e}_{1,6} - \mathbf{e}_{2,5} + \mathbf{e}_{5,2} - \mathbf{e}_{6,1}),\\
\mathcal{K}_2 &= \frac{1}{2}(\mathbf{e}_{3,8} - \mathbf{e}_{4,7} + \mathbf{e}_{7,4} - \mathbf{e}_{8,3}),\\
\mathcal{H}_1 &= \frac{1}{2}(\mathbf{e}_{1,1} + \mathbf{e}_{2,2} - \mathbf{e}_{5,5} - \mathbf{e}_{6,6}),\\
\mathcal{H}_2 &= \frac{1}{2}(\mathbf{e}_{3,3} + \mathbf{e}_{4,4} - \mathbf{e}_{7,7} - \mathbf{e}_{8,8}),
\end{aligned} \tag{4.46}$$

where $\mathbf{e}_{i,j}$ are matrices with 1 in the entry (i,j) and 0 elsewhere. Next, we define the nilpotent generators $\mathcal{N}^{(\pm)}_\ell$ as follows:

$$\mathcal{N}^{(\pm)}_\ell = \mathcal{H}_\ell \mp \mathcal{K}_\ell. \tag{4.47}$$

These matrices satisfy the relations

$$[\mathcal{J}_\ell, \mathcal{N}^{(\pm)}_{\ell'}] = \pm\delta_{\ell\ell'}\mathcal{N}^{(\pm)}_{\ell'}. \tag{4.48}$$

Note that the two sets $\{\mathcal{J}_\ell, \mathcal{N}^{(+)}_\ell/\sqrt{2}, \mathcal{N}^{(-)}_\ell/\sqrt{2}\}$ are standard triples $\{h_\ell, e_\ell, f_\ell\}$ with nilpotent element e_ℓ in the orbit $[2^1,1^4]$, as can be easily ascertained from the eigenvalues of the neutral elements $h_\ell = \mathcal{J}_\ell$. As shown in the previous section, the most general neutral element h of a standard triple $\{h,e,f\}$ with $e,f \in \mathfrak{K}^*$, modulo an $H_c = \mathrm{SO}(3)^2$ transformation, can be written as $h = \sum_{\ell=1}^2 \kappa_\ell h_\ell = \sum_{\ell=1}^2 \kappa_\ell \mathcal{J}_\ell$. The eigenvalues of h are

$$\text{eigenvalues}(h) = \left\{\pm\frac{\kappa_1}{2}, \pm\frac{\kappa_1}{2}, \pm\frac{\kappa_2}{2}, \pm\frac{\kappa_2}{2}\right\}. \tag{4.49}$$

Note the difference between these eigenvalues and those given in (4.43) for $m=4$, which are referred to the same generator in a different, triality-related, representation: the $\mathbf{8}_s$.

If we try to complete this h into a standard triple $\{h,e,f\}$, with e, f inside the smaller space $\mathfrak{K}_{(N)} = \mathrm{Span}(\mathcal{K}_\ell, \mathcal{H}_\ell)$, coset space of $\mathscr{M}_{(N)}$, we see that we only succeed if $\kappa_\ell = 0, 1, -1$, corresponding to a nilpotent element e in the orbits $[2^2,1^4]$ (for $\kappa_1\kappa_2 = 0$) and $[2^4]$ ($\kappa_1\kappa_2 \neq 0$).[14] In both cases, this generator would have order of nilpotency 2.[15] Therefore, acting by means of G on the lightlike geodesics unfolding in $\mathscr{M}_{(N)}$, one can construct the most general geodesic within the orbits $[2^2,1^4], [2^4]$.

The generic nilpotent generator in the coset space $\mathfrak{K}_{(N)}$ has the form

$$Q = \sum_{\ell=1}^{2} \kappa^{(\pm)}_\ell \mathcal{N}^{(\pm)}_\ell \tag{4.50}$$

and has order of nilpotency 2 in the $\mathbf{8}_v$. A representative of the orbit $[2^4]$ is obtained when $\kappa^{(\pm)}_1\kappa^{(\pm)}_2 \neq 0$. Let us illustrate how this orbit splits into suborbits with respect to H^*. Using H^*-transformations generated by h_1, h_2, we can rescale $\kappa^{(\pm)}_1, \kappa^{(\pm)}_2$ by a positive factor, so that we can always set $|\kappa^{(\pm)}_\ell| = 1$. The inequivalent nilpotent elements in $\mathfrak{K}_{(N)}$ belonging to different H^*-orbits can then be reduced to the following four,

$$\begin{aligned}
&\mathcal{N}^{(+)}_1 + \mathcal{N}^{(+)}_2, \qquad \mathcal{N}^{(+)}_1 + \mathcal{N}^{(-)}_2,\\
&\mathcal{N}^{(+)}_1 - \mathcal{N}^{(+)}_2, \qquad \mathcal{N}^{(+)}_1 - \mathcal{N}^{(-)}_2,
\end{aligned} \tag{4.51}$$

and the $\mathrm{SO}(4,4;\mathbb{C})$-orbit $[2^4]$ split into four H^*-orbits as shown in Refs. [60,61]. The signs of $\kappa^{(\pm)}_\ell$ are indeed affected by a transformation of the form $e^{i\pi\mathcal{J}_\ell}$, which is in the complexification of H^*, while the grading $\pm$ of $\mathcal{N}^{(\pm)}_\ell$ is affected by a transformation of the form $e^{\pi\mathcal{K}_\ell}$. Both these transformations are not in H^*, and thus different signs of $\kappa^{(\pm)}_\ell$ and different gradings of $\mathcal{N}^{(\pm)}_\ell$ define different H^*-orbits

The components of Q along the compact generators $\mathcal{K}_\ell$ define the axion charges. Therefore, the grading $\pm$ of $\mathcal{N}^{(\pm)}_\ell$ is the sign of the corresponding axion charge.

Let us compute the most general lightlike geodesic in $\mathscr{M}_{(N)}$ passing through the origin. To this end, we define the coset representative in $\mathcal{M}_{(N)}$ in the solvable parametrization; that is, we describe the manifold as locally isometric to the solvable group exp(Solv), where the solvable Lie algebra Solv is generated by the matrices $\{\mathcal{H}_\ell, \mathcal{T}_\ell\}$, having defined

$$\mathcal{T}_\ell = (\mathcal{K}_\ell + \mathcal{J}_\ell). \tag{4.52}$$

The coset representative is then defined as follows:

[14] We neglect the trivial case $\kappa_1 = \kappa_2 = 0$.

[15] If we were working in the $\mathbf{8}_s$, we would have the orbit $[3,1^5]$ instead of the $[2^4]$, as explained in the previous section. The corresponding order of nilpotency would then be 3.

$$L = e^{\sum_\ell c_\ell T_\ell} \cdot e^{-\sum_\ell \phi_\ell \mathcal{H}_\ell}. \tag{4.53}$$

Next, we define the matrix $\mathcal{M}$:

$$\mathcal{M}(\phi) = L(\phi)\eta' L(\phi)^T. \tag{4.54}$$

From Eq. (4.22), we can extract from this matrix the matrices $\tilde{G}_{ij} = e^{-\frac{\phi}{2}} G_{ij}$ and C_{ij}, G_{ij} being the metric of the internal torus in the Einstein frame,

$$e^{-\frac{\phi}{2}} G_{ij} = \mathrm{diag}(e^{-\phi_1}, e^{-\phi_1}, e^{-\phi_2}, e^{-\phi_2}),$$
$$C_{ij} = \begin{pmatrix} 0 & c_1 & 0 & 0 \\ -c_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & c_2 \\ 0 & 0 & -c_2 & 0 \end{pmatrix}, \tag{4.55}$$

where, using the notation of Sec. III,

$$\phi_1 = \frac{\phi + \varphi}{2} - \frac{\psi}{\sqrt{2}}, \qquad \phi_2 = \frac{\phi + \varphi}{2} + \frac{\psi}{\sqrt{2}}. \tag{4.56}$$

The geodesic $\phi(\rho) = \{\phi_\ell(\rho), \chi_\ell(\rho)\}$ generated by Q, though the origin, is a solution to the matrix equation:

$$\mathcal{M}(\phi(\rho)) = \mathcal{M}(\phi_0) e^{2\rho Q} = \eta' e^{2\rho Q}. \tag{4.57}$$

Solving Eq. (4.57), we find

$$c_\ell = \pm \frac{\kappa_\ell^{(\pm)} \rho}{H_\ell}, \qquad e^{\phi_\ell} = H_\ell, \tag{4.58}$$

where

$$H_\ell \equiv 1 - \kappa_\ell^{(\pm)} \rho$$

are harmonic functions. If $\kappa_\ell^{(\pm)} \geq 0$, H_ℓ have no poles for $\rho \leq 0$, and the solution is regular. The above solution coincides with the one in (3.20) setting $k_\ell = c_{\ell 0} = 0$ and $\kappa_\ell^{(\pm)} = a_\ell$. Thus, the regularity condition selects two out of the four H^*-orbits within the complex orbit $[2^4]$. The grading of the two nilpotent generators is in turn related to the corresponding axion charge, i.e., to the charges of the Euclidean D1-branes:

$$q_\ell = \pm \kappa_\ell^{(\pm)}. \tag{4.59}$$

Only one choice, that with $q_\ell > 0$, defines a supersymmetric configuration. The other, defined by $\kappa_1^{(+)} > 0, \kappa_1^{(-)} > 0$, corresponds to an extremal, nonsupersymmetric, regular solution, in which the two D1-branes have opposite charges.

D. Remark on the regularity condition for wormholes

Our proof of the nonexistence of Euclidean wormholes can be summarized as follows:

(i) The initial velocity of a timelike geodesic is a compact generator in $\mathfrak{K}^*$ (i.e., an element of $\mathfrak{H}_2$). As discussed in Sec. IV C 1, using H_c, we can always rotate a generic element of $\mathfrak{H}_2$ into $\mathfrak{H}^{(N)}$, to be tangent to the normal submanifold $\mathscr{M}_{(N)}$, formally defined in Sec. IV C 1 and discussed in Sec. III;

(ii) In Sec. III C, it is proven that the condition on the maximal length for timelike geodesics in this truncation, for the existence of regular wormhole solutions, is not met.

This can also be verified by computing the maximal length $\ell_{\max}$ on the general timelike geodesic given in Sec. IV B. This value turns out to be $\ell_{\max} = \sqrt{2}\pi$, while regularity of wormhole solutions requires, in three-dimensions, $\ell_{\max} > 2\pi$.[16]

We wish here to briefly elaborate on the computation of $\ell_{\max}$ by considering all the inequivalent, totally geodesic $\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(1,1)$ submanifolds of $\mathscr{M}^*$ and the regularity condition (3.18) for the existence of regular wormholes. The latter condition follows from the requirement that the maximal length $\ell_{\max}$ of timelike geodesics is larger than the actual length of the same curve describing the wormhole solution. The former quantity $\ell_{\max}$ is referred to the arc of geodesic comprised between the boundaries of the *physical coordinate patch*, where the scalar fields become singular. The physical coordinate patch is selected by the dimensional reduction of string theory and is defined by the conditions

$$0 < G_{ij} < \infty, \qquad 0 < e^{\phi} < \infty.$$

The notion of maximal length is clearly dependent on the coordinate patch, and one can find other local coordinate patches in which the maximal length of a geodesic is larger than in the physical one. A same wormhole solution described in this patch can be regular while being singular when described in terms of the physical fields (coordinates of the physical patch). For example, we can consider inequivalent standard triples $\{e, f, h\}$ in $\mathfrak{so}(4,4)$, with $\{e, f\} \subset \mathfrak{K}^*$. The two-dimensional space $\{e, f\}$ generates a totally geodesic $\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(1,1)$ submanifold of $\mathscr{M}^*$. Restricting to this submanifold and describing the timelike geodesic generated by $e - f$ in the corresponding solvable patch,[17] one finds $\ell_{\max} = 2\pi/b = \pi\sqrt{d_h}$, where

[16] The corresponding condition in D dimensions is $\ell_{\max} > 2\pi\sqrt{\frac{D-1}{2(D-2)}}$. In the truncation discussed in Sec. III, $\ell_{\max}^2 = 4\pi^2 \sum_{i=1}^2 \frac{1}{b_i^2} = 2\pi^2$.

[17] The solvable coordinate patch is spanned by a dilatonic scalar and an axionic one, parametrizing the generators $\tilde{h} = \frac{e+f}{\sqrt{2}}$, $\tilde{e} = \frac{1}{\sqrt{2}}(h - \frac{e-f}{\sqrt{2}})$, respectively, with $[\tilde{h}, \tilde{e}] = \tilde{e}$. These generators close a solvable Lie algebra.

$$d_h = \mathrm{Tr}(h \cdot h) = \sum_{i=1}^{\ell} \mathrm{k}_i \sum_{m=-s_i}^{s_i} m^2$$

is characteristic of the nilpotent orbit of e. If this coordinate patch on the $\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(1,1)$ submanifold were contained in the physical one, the regularity condition would be satisfied for the $[4^2]$ or the $[5,1^3]$ orbit. However, this is not the case, and along the geodesic within this patch, G_{ij} fails to be positive definite. Only the subspaces defined by the triples corresponding to the orbits $[2^2,1^4]$ and $[2^4]$ (or $[3,1^5]$) have their solvable patches embedded in the physical patch on $\mathscr{M}^*$. Both these spaces can be realized within the truncation $\mathscr{M}_{(N)} = [\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(1,1)]^2$ considered in Secs. III and IV C 1. However, for these triples, $d_h = 1$ (for $[2^2,1^4]$) or $d_h = 2$ (for $[2^4]$ or $[3,1^5]$), and the regularity condition is not met. Indeed, the associated values of $b = 2/\sqrt{d_h}$ are 2 and $\sqrt{2}$, respectively, and the maximal length of timelike geodesics is realized in the latter $\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(1,1)$ submanifold and is $2\pi/b = \sqrt{2}\pi$. This is the same value computed on the general timelike geodesic given in Sec. IV B.

In summary, considering all inequivalent $\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(1,1)$ (totally geodesic) subspaces of $\mathscr{M}^*$ whose (solvable) coordinate patch is contained in the physical patch of the latter is a valuable approach for assessing the maximal length of timelike geodesics. Each of these two-dimensional subspaces is defined by a standard triple and is characterized by a value of the b-parameter. In the model under consideration, only two inequivalent $\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(1,1)$ subspaces satisfy the above requirement and correspond to the partitions $[2^2,1^4]$ and $[2^4]$ (or $[3,1^5]$). As pointed out above, both of them are also subspaces of $\mathscr{M}_{(N)}$.

A similar analysis was implicitly applied to the models considered in Refs. [27,31,32] in which two inequivalent such subspaces exist within the Wick-rotated universal hypermultiplet $\mathrm{SL}(3,\mathbb{R})/\mathrm{GL}(2,\mathbb{R})$, one with $b = 2$ and the other with $b = 1$. The latter contains the timelike geodesic of maximal length, defining, in that model, a regular wormhole. In this paper, we have mathematically formalized and generalized this approach.

V. CONCLUSIONS

Let us summarize the main points of this paper.

We have classified the instantons in Euclidean $\mathrm{AdS}_3 \times S^3 \times \mathbb{T}^4$ that are carried by the AdS moduli fields dual to the marginal operators of maximal supersymmetry in the dual CFT. On the supergravity side, this corresponds nicely to a classification of geodesics in the moduli space of the Euclidean theory, which we argued boiled down to studying the truncated moduli space

$$\frac{SO(4,4)}{SO(3,1)\times SO(3,1)} = \frac{SO(4,4)}{SO(4,\mathbb{C})}. \tag{5.1}$$

We constructed all geodesics and put particular emphasis on the null and timelike cases; see Table I. The null geodesics contain the subgroup of SUSY instantons that lift to Euclidean D1-branes wrapping 2-cycles inside the $\mathbb{T}^4$. It would be interesting to lift all extremal geodesics to 10D and understand their supersymmetry properties.

The timelike geodesics have metrics corresponding to the Giddings-Strominger wormholes [7], but they are not regular in their scalar profile, and hence there are no Giddings-Strominger wormholes in this setup, in constrast to the claim in Ref. [11].

We plan to investigate the meaning of the extremal instantons in the dual CFT. The dual CFT is thought to be a two-dimensional CFT with (4,4) supersymmetries and a central charge proportional to the product $Q_1 Q_5$. In the free orbifold point, the CFT target space is a large product of $\mathbb{T}^4$ factors divided out by the permutation group [62,63].

TABLE I. The general form of the geodesics on $\mathscr{M}^*$ defined by by a Noether charge matrix $Q = \sigma_3 \otimes \boldsymbol{\gamma} + \sigma_1 \otimes \boldsymbol{c}$, where $\boldsymbol{\gamma} = \boldsymbol{\gamma}^T, \boldsymbol{c} = -\boldsymbol{c}^T$. The elements of the matrix $\tilde{\mathbf{G}}$ are $e^{-\frac{\phi}{2}} G_{ij}$, ϕ being the $D = 10$ dilaton and G_{ij} being the metric moduli of $\mathbb{T}^4$ in the Einstein frame. The matrix elements of $\mathbf{C}$ are the components of the RR 2-form along the directions of the 4-torus.

Orbit	Moduli	Case
$Q^4 = 0$	$\tilde{\mathbf{G}}(\rho) = (\mathbf{1} - 2\rho\boldsymbol{\gamma} + 2\rho^2(\boldsymbol{\gamma}^2 + \mathbf{c}^2) - \frac{4}{3}((\boldsymbol{\gamma}^2 + \mathbf{c}^2)\cdot\boldsymbol{\gamma} + [\boldsymbol{\gamma},\boldsymbol{c}]\cdot\boldsymbol{c})\rho^3)^{-1}$, $\mathbf{C}(\rho) = -2\rho\tilde{G}(\rho)\cdot(\mathbf{c} - \rho[\boldsymbol{\gamma},\mathbf{c}] + \frac{2}{3}\rho^2((\boldsymbol{\gamma}^2 + \mathbf{c}^2)\cdot\boldsymbol{c} - [\boldsymbol{\gamma},\boldsymbol{c}]\cdot\boldsymbol{\gamma}))$	Extremal
$Q^3 = 0$	$\tilde{\mathbf{G}}(\rho) = (\mathbf{1} - 2\rho\boldsymbol{\gamma} + 2\rho^2(\boldsymbol{\gamma}^2 + \mathbf{c}^2))^{-1}$, $\mathbf{C}(\rho) = -2\rho\tilde{\mathbf{G}}(\rho)\cdot(\mathbf{c} - \rho[\boldsymbol{\gamma},\mathbf{c}])$	Extremal
$Q^2 = 0$	$\tilde{\mathbf{G}}(\rho) = (\mathbf{1} - 2\rho\boldsymbol{\gamma})^{-1}$, $\mathbf{C}(\rho) = -2\rho\tilde{\mathbf{G}}(\rho)\cdot\mathbf{c}$	Extremal
$Q = \sigma_3 \otimes \boldsymbol{\gamma}$, $\mathbf{c} = \mathbf{0}$	$\tilde{\mathbf{G}}(\rho) = \cosh(2\rho\boldsymbol{\gamma}) + \sinh(2\rho\boldsymbol{\gamma})$, $\mathbf{C}(\rho) = \mathbf{0}$	Subextremal
$Q = \sigma_1 \otimes \boldsymbol{c}$, $\boldsymbol{\gamma} = \mathbf{0}$	$\tilde{\mathbf{G}}(\rho) = \cosh(2\rho\mathbf{c})^{-1}$, $\mathbf{C}(\rho) = -\sinh(2\rho\mathbf{c})\cdot\cosh(2\rho\mathbf{c})^{-1}$	Overextremal

Following the procedure of Ref. [64], one could construct the corresponding worldsheet instantons by gauging the sigma-model. To find a correspondence with the supergravity solutions, one would hope to find a match between the on-shell actions and the charges. The charges should correspond to charges of the marginal operators dual to the axions. The latter operators are 2-forms,

$$dX^i \wedge dX^j, \tag{5.2}$$

with X a single copy of the CFT scalars carrying a vector $SO(4)$-index i under the $SO(4)$-symmetries generated by the internal $\mathbb{T}^4$ torus of the compactification in IIB. These are closed 2-forms that allow for nontrivial topological charges by integration. These should then correspond to the axion charges.

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Correction: The omission of a word from the title has been fixed.