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Abstract

Projective vector fields are the infinitesimal transformations whose local flow preserves geodesics up to reparametrisation. In 1882 Sophus Lie posed the problem of describing 2-dimensional metrics admitting a non-trivial projective vector field, which was solved in recent years. In the present paper, we solve the analog of Lie's problem in dimension 3, for Riemannian metrics and, more generally, for Levi-Civita metrics of arbitrary signature.

Résumé

Les champs vectoriels projectifs sont les transformations infinitésimales dont le flux local préserve les courbes géodésiques sans égard au paramétrage. En 1882, Sophus Lie posa le problème de décrire tous les métriques bi-dimensionnelles qui admettent des champs vectoriels projectifs non-banals. Ce problème a été résolu récemment. Dans l'article présent, on résolve l'analogue tri-dimensionnel du problème de Lie pour les métriques riemanniennes et, plus généralement, pour les métriques de Levi-Civita de signature arbitraire.

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1. Introduction

1.1. Basic definitions and description of the problem

Let M be a smooth manifold of dimension n. A *metric* on M is a symmetric, non-degenerate $(0,2)$-tensor field g (of arbitrary signature). In particular, if g is positive (resp. negative) definite, it is called a *Riemannian* (resp. *anti-Riemannian*) metric. If the signature of g is either $(+\cdots+-)$ or $(+-\cdots-)$, it is called *Lorentzian.* A metric is of *constant curvature* if its sectional curvatures coincide and are constant. A curve $\mathbb{R}\supseteq I\ni t\mapsto\gamma(t)\in M$ such that

$$\nabla_{\dot\gamma}\dot\gamma=f(t)\dot\gamma \tag{1}$$

for some function $f(t)$ is an *unparametrised geodesic of* g.

The present paper studies metrics that admit vector fields whose local flow preserves unparametrised geodesics.

Definition 1. A *projective transformation* is a (local) diffeomorphism of M that sends geodesics into geodesics (where geodesics are to be understood as unparametrised curves). A vector field on M is *projective* if its (local) flow acts by projective transformations. A projective vector field v is *homothetic* (for the metric g) if the Lie derivative of g along v satisfies $\mathcal{L}_vg=\lambda g$ for some constant $\lambda\in\mathbb{R}$. If $\lambda\neq0$, we say that v is *properly homothetic.* If v is not homothetic, we call it an *essential* projective vector field.

The projective vector fields of a given metric g form a Lie algebra [1], which we denote by $\mathfrak{p}(g)$. It is easy to confirm that the following definition indeed gives rise to an equivalence relation.

Definition 2. We say that two metrics (on the same manifold) are *projectively equivalent* if they share the same geodesics (as unparametrised curves). The set of all metrics projectively equivalent to a given metric g is called the *projective class* of g.

In the 1880s, Sophus Lie posed the following problem for 2-dimensional surfaces [2, 1]: *Find the metrics that describe surfaces whose geodesic curves admit an infinitesimal transformation*[1]. This problem has been solved during the recent years in [3, 4, 5]. Related work can be found in [6, 7, 8, 9, 10, 11].

[1] Original German wording in [2]: *"Es wird verlangt, die Form des Bogenelementes einer jeden Fläche zu bestimmen, deren geodätische Curven eine infinitesimale Transformation gestatten."*

The present paper is concerned with the analogous problem for manifolds of dimension 3, i.e., to find 3-dimensional metrics admitting a non-trivial projective vector field. We solve this problem for 3-dimensional Riemannian metrics and, more generally, for Levi-Civita metrics (thoroughly introduced in Section 4) of arbitrary signature.

1.2. State of the art

It is well-known that, given a 3-dimensional metric g that admits a homothetic vector field v, there exist, around a non-singular point of v, local coordinates (x, y, z) such that $v = \partial_x$ and

$$g = e^{\lambda x} h(y,z)\,, \quad \lambda \in \mathbb{R}\,, \tag{2}$$

where $h(y, z)$ is a non-degenerate metric depending on the coordinates y and z only. It is easy to confirm that, in the case when all metrics in a projective class $[g]$ are proportional (by a real constant), any projective vector field is homothetic and thus the metric g locally can be written in the form (2). For this reason, we focus our attention to metrics whose projective class contains a pair of non-proportional metrics.

It follows from [12, 13] that a Riemannian metric g of non-constant curvature and belonging to the aforementioned class can be taken to be locally either in the form

$$\pm (X_1 - X_2)(X_1 - X_3)\,(dx^1)^2 \pm (X_2 - X_1)(X_2 - X_3)\,(dx^2)^2 \\ \pm (X_3 - X_1)(X_3 - X_2)\,(dx^3)^2\,,\ X_i = X_i(x^i) \tag{3}$$

or in the form

$$\zeta(x^3)\,(h \pm (dx^3)^2)\,, \tag{4}$$

where $h = \sum_{i,j=1}^2 h_{ij} dx^i dx^j$ is a 2-dimensional metric (note however, that not all metrics (3) and (4) are Riemannian). The metrics (3) and (4) are the so-called *Levi-Civita metrics*; this class of metrics will be thoroughly introduced in Section 4. Our aim will be to find the functions $X_i(x^i)$ and $\zeta(x^3)$, and the metrics h, such that a projective vector field exists for g.

Projective vector fields for Levi-Civita metrics, particularly those of the form (3), are studied in [14], under the assumption of Riemannian signature. The reference then finds explicit ordinary differential equations (ODEs) for the functions X_i and the projective symmetries, both essential and homothetic ones. Generalisations of some of these ODEs can be found in recent works concerned with splitting-gluing constructions of projectively equivalent metrics [15] and c-projective vector fields [16] and will be discussed in detail later.

However, explicit descriptions of 3-dimensional metrics with projective vector fields do not exist to date except for some special cases. The current paper closes this gap by providing explicit descriptions of 3-dimensional metrics with projective vector fields.

1.3. A first look at the results

We explicitly describe all metrics (3) and (4) with non-zero projective vector fields. The explicit metrics can be found in Sections 5 and 6, see Theorems 2 and 3, respectively. The following theorem is a first look at the results we obtain, stated for 3-dimensional Riemannian metrics. If the metrics in Theorem 1 are of non-Riemannian signature, they still admit the projective vector fields indicated. We proceed to make this precise in the main body of the paper. A final remark concerns the case when the metric assumes the form (2) in suitable local coordinates. A description of the projective algebras of these metrics will be given in this paper, in Theorems 2 and 3, along with Corollary 3 and Proposition 14, but is omitted here for brevity.

Theorem 1. *Let g be a 3-dimensional Riemannian metric with a non-vanishing projective vector field. Then either g is locally of the form (2) or it is among the following:*

1. *The metric, for $\beta \neq 0, k_i \neq 0$,*

$$g = k_1(\tanh(x) - \tanh(y))(\tanh(x) - \tanh(z))\, e^{\frac{2x}{\beta}}\, dx^2 + k_2(\tanh(y) - \tanh(x))(\tanh(y) - \tanh(z))\, e^{\frac{2y}{\beta}}\, dy^2 + k_3(\tanh(z) - \tanh(x))(\tanh(z) - \tanh(y))\, e^{\frac{2z}{\beta}}\, dz^2$$

   with the essential projective vector field $v = \partial_x + \partial_y + \partial_z$.

2. *The metric, for $k_i \neq 0$,*

$$g = k_1 \left(\tfrac{1}{x} - \tfrac{1}{y}\right)\left(\tfrac{1}{x} - \tfrac{1}{z}\right) e^{2x}\, dx^2 + k_2 \left(\tfrac{1}{y} - \tfrac{1}{x}\right)\left(\tfrac{1}{y} - \tfrac{1}{z}\right) e^{2y}\, dy^2 + k_3 \left(\tfrac{1}{z} - \tfrac{1}{x}\right)\left(\tfrac{1}{z} - \tfrac{1}{y}\right) e^{2z}\, dz^2$$

   with the essential projective vector field $v = \partial_x + \partial_y + \partial_z$.

3. *The metric, for* $\beta \neq 0, k_i \neq 0$,

$$
\begin{aligned}
g = & k_1(\tan(x)-\tan(y))(\tan(x)-\tan(z))\, e^{\frac{2x}{\beta}}\, dx^2 \\
& + k_2(\tan(y)-\tan(x))(\tan(y)-\tan(z))\, e^{\frac{2y}{\beta}}\, dy^2 \\
& + k_3(\tan(z)-\tan(x))(\tan(z)-\tan(y))\, e^{\frac{2z}{\beta}}\, dz^2
\end{aligned}
$$

with the essential projective vector field $v = \partial_x + \partial_y + \partial_z$.

4. *The metric, for* $k_i \neq 0$,

$$
\begin{aligned}
g = & k_1(\tan(x)-\tan(y))(\tan(x)-\tan(z))\, dx^2 \\
& + k_2(\tan(y)-\tan(x))(\tan(y)-\tan(z))\, dy^2 \\
& + k_3(\tan(z)-\tan(x))(\tan(z)-\tan(y))\, dz^2
\end{aligned}
$$

with the essential projective vector field $v = \partial_x + \partial_y + \partial_z$.

5. *The metric, for* $k_i \neq 0$,

$$
\begin{aligned}
g = & k_1(\tanh(x)-\tanh(y))(\tanh(x)-\tanh(z))\, dx^2 \\
& + k_2(\tanh(y)-\tanh(x))(\tanh(y)-\tanh(z))\, dy^2 \\
& + k_3(\tanh(z)-\tanh(x))(\tanh(z)-\tanh(y))\, dz^2
\end{aligned}
$$

with the essential projective vector field $v = \partial_x + \partial_y + \partial_z$.

6. *The metric*

$$
g = \frac{\beta}{z^2}\,(h + dz^2)
$$

with the essential projective vector field $v = \frac{1}{z}\,\partial_z$, *where* $h = h_{11}dx^2 + 2h_{12}dxdy + h_{22}dy^2$ *does not admit any homothetic vector field.*

7. *The metric*

$$
g = \beta\,(1 + \tan^2(z))(h + dz^2)
$$

with the essential projective vector field $v = \tan(z)\,\partial_z$, *where* $h = h_{11}dx^2 + 2h_{12}dxdy + h_{22}dy^2$ *does not admit any Killing vector field.*

8. *The metric*

$$
g = \beta\,(1 - \tanh^2(z))(h + dz^2)
$$

with the essential projective vector field $v = \tanh(z)\,\partial_z$, *where* $h = h_{11}dx^2 + 2h_{12}dxdy + h_{22}dy^2$ *does not admit any Killing vector field.*

1.4. Structure and strategy of the paper

The basic theory of metrisable projective connections, of projectively equivalent metrics, and of projective vector fields (from the angle of Lie theory of symmetries of differential equations) is reviewed in Section 2. An important object are certain (1,1)-tensors associated to a pair of projectively equivalent metrics, called Benenti tensors. Their eigenvalues are the functions X_i and ζ that determine, respectively, the metrics (3) and (4).

As we explain later the projective class of these metrics is described by a 2-dimensional linear space, cf. Definition 4 and Proposition 5. This space is endowed with the action of the projective symmetry algebra, studied in Section 3. Its matrix description defines a polynomial (called Solodovnikov's polynomial) which gives rise to differential equations for the eigenvalues of the Benenti tensors, see Lemma 6. Together with the differential equations arising from the projective symmetry action on the metric, see Lemma 7, this allows us to obtain the explicit metrics by solving certain systems of ODEs. The details of the procedure will be elaborated in Sections 5 and 6.

In Section 4 we formally introduce Levi-Civita metrics in dimension n, which in the specific case $n = 3$ lead to (3) and (4). The remaining two sections contain the main outcomes of the paper: In Section 5, we obtain local normal forms for (3) (see Theorem 2) and in Section 6 those for (4) (see Theorem 3).

The proof of the main results obtained in this paper are based on a combination of the methods from [14] with the more recent techniques from the classical Lie problem [3, 4], from c-projective geometry [16], and from the splitting-gluing theory of projectively equivalent metrics [15, 17]. The latter, in particular, allows us to reduce projective vector fields to lower-dimensional homothetic vector fields, see Propositions 8 and 12 as well as 13 for details.

Note the following notation that we apply from now on: We use a comma to denote usual derivatives, e.g. "$,a$" denotes differentiation w.r.t. a-th coordinate direction. The comma will be omitted when derivatives are interpreted as coordinates on certain jet spaces. Unless otherwise clarified, Einstein's summation convention applies to repeated indices. We omit the comma and use a simple subscript to refer to coordinates on the jet space, see Section 2.4 for more details.

2. Preliminaries

2.1. Metrisability of projective connections

Let us now consider an n-dimensional metric g, given in terms of an explicit system of coordinates

$$(x^1, x^2, \dots, x^n) = (x, y^2, \dots, y^n)\,.$$

It gives rise, via its Levi-Civita connection, to a system of second order ODEs

$$y^k_{xx} = -\Gamma^k_{11} - \sum_{i=2}^{n} (2\Gamma^k_{1i} - \delta^k_i \Gamma^1_{11}) y^i_x - \sum_{i,j=2}^{n} (\Gamma^k_{ij} - 2\delta^k_i \Gamma^1_{1j}) y^i_x y^j_x + \sum_{i,j=2}^{n} \Gamma^1_{ij}\, y^i_x y^j_x y^k_x\,, \tag{5}$$

$(k = 2, \dots, n)$ where $y^k = y^k(x)$ and where

$$\Gamma^k_{ij} = \frac{1}{2} g^{kh} (g_{jh,i} + g_{ih,j} - g_{ij,h}) \tag{6}$$

are the Christoffel symbols of the Levi-Civita connection of g. System (5) is called *the projective connection associated to* g. The name is justified by the fact that, for a solution $y^k(x)$ to (5), the curve $(x, y^2(x), \dots, y^n(x))$ is a geodesic of g up to reparametrisation. In fact, System (5) can be achieved by eliminating the external parameter from the classical geodesic equation (1).

Remark 1. In the context of the theory of symmetries of (systems of) differential equations, local diffeomorphisms

$$(x^1, \dots, x^n) \to \big(\tilde{x}^1(x^1, \dots, x^n), \dots, \tilde{x}^n(x^1, \dots, x^n)\big)$$

preserving (5) are called *finite point symmetries*. These send solutions of (5) to solutions and are projective transformations of g, because solutions to (5) are unparametrised geodesics of g by construction. Infinitesimal point symmetries of (5) are in 1-to-1 correspondence with projective vector fields of g.

Example 1. For $n = 2$, in particular, and with $(x, y^2) = (x, y)$, System (5) reduces to the classical 2-dimensional projective connection associated to a 2-dimensional metric [18]

$$y_{xx} = -\Gamma^2_{11} + (\Gamma^1_{11} - 2\Gamma^2_{12})\, y_x - (\Gamma^2_{22} - 2\Gamma^1_{12})\, y_x^2 + \Gamma^1_{22}\, y_x^3\,. \tag{7}$$

This equation has been extensively studied in [3].

A general (n-dimensional) projective connection has the following form:

$$y^k_{xx} = f^k_{11} + \sum_{i=2}^{n} f^k_{1m} y^i_x + \sum_{i,j=2}^{n} f^k_{ij} y^i_x y^j_x + \sum_{i,j=2}^{n} f_{ij}\, y^i_x y^j_x y^k_x\,, \quad k = 2, \dots, n\ , \tag{8}$$

where w.l.o.g. we can suppose f^k_{ij} and $f^1_{ij} := f_{ij}$ symmetric in the lower indices. A natural question is whether or not the projective connection (8) is metrisable, i.e. if there exists an n-dimensional metric g such that (5) is equal to (8). This is equivalent to the existence of a solution to the system

$$-\Gamma^k_{11} = f^k_{11}\,,\ \ -(2\Gamma^k_{1i} - \delta^k_i \Gamma^1_{11}) = f^k_{1i}\,,\ \ -(\Gamma^k_{ij} - \delta^k_i \Gamma^1_{1j} - \delta^k_j \Gamma^1_{1i}) = f^k_{ij}\,,\ \ \Gamma^1_{ij} = f_{ij} \tag{9}$$

where Γ^k_{ij} is given by (6).

Definition 3. The projective connection (8) is *metrisable* if there exists a Levi-Civita connection ∇ such that (9) is satisfied.

If in (9) we perform the substitution

$$\sigma^{ij} = |\det(g)|^{\frac{1}{n+1}} g^{ij} \in S^2(M) \otimes (\Lambda^n(M))^{\frac{2}{n+1}}\,, \tag{10}$$

where g^{ij} are the entries of the inverse metric of g_{ij}, we obtain a *linear* system in the unknowns σ^{ij} (see [19]). The inverse transformation is

$$g^{ij} = |\det(\sigma)|\sigma^{ij}\,. \tag{11}$$

More precisely we have the following proposition.

Proposition 1 ([19])**.** *A metric g on an n-dimensional manifold lies in the projective class of a given connection ∇ if and only if σ^{ij} defined by (10) is a solution of the linear system*

$$\nabla_a \sigma^{bc} - \frac{1}{n+1}(\delta^c_a \nabla_i \sigma^{ib} + \delta^b_a \nabla_i \sigma^{ic}) = 0 \tag{12}$$

with

$$\nabla_a \sigma^{bc} = \sigma^{bc}_{,a} + \Gamma^b_{ad}\sigma^{dc} + \Gamma^c_{ad}\sigma^{db} - \frac{2}{n+1}\Gamma^d_{da}\sigma^{bc}\,,$$

where Γ^k_{ij} are the Christoffel symbols of the Levi-Civita connection of g.

The metrisability of projective connections can also be given in terms of differential invariants, see [11] and [20] for 2-dimensional and 3-dimensional projective structures, respectively. A characterisation of metrisability in terms of pseudo-holomorphic curves can be found in [21].

We proceed with two definitions that follow the conventions in [22].

Definition 4. The linear space of solutions of (12) is denoted by

$$\mathfrak{A} = \mathfrak{A}(g) := \left\{ \sigma \in S^2(M) \otimes (\Lambda^n(M))^{\frac{2}{n+1}} \mid \sigma = \sigma^{ij} \text{ is a solution to (12)} \right\}.$$

We refer to $\mathfrak{A}$ as the *metrisation space*. Its dimension is called the *degree of mobility* of g. If $\sigma \in \mathfrak{A}$ is represented by a matrix of rank r, we say that σ is a *rank-r solution*. If $r < n$, the solution is also called *degenerate* or a *lower-rank solution*. If $r = n$, the solution is called *full-rank solution*.

Proposition 2. *Let g and $\bar{g}$ be two n-dimensional projectively equivalent metrics on the same n-dimensional manifold. Then, if defined for $t_1, t_2 \in \mathbb{R}$,*

$$\frac{\left(t_1 \det(g)^{\frac{1}{n+1}} g^{-1} + t_2 \det(\bar{g})^{\frac{1}{n+1}} \bar{g}^{-1}\right)^{-1}}{\det\left(t_1 \det(g)^{\frac{1}{n+1}} g^{-1} + t_2 \det(\bar{g})^{\frac{1}{n+1}} \bar{g}^{-1}\right)} \tag{13}$$

is a metric that is projectively equivalent to g and $\bar{g}$.

Proof. To the metrics $g, \bar{g}$ correspond, respectively, the tensors $\sigma, \bar{\sigma}$ via formula (10). Then the claim follows taking into account the linearity of System (12). □

2.2. Benenti tensors and projective equivalence

Definition 5. Let $g, \bar{g}$ be two projectively equivalent metrics on the same n-dimensional manifold M. The $(1,1)$-tensor field

$$L(g, \bar{g}) = \left|\frac{\det(\bar{g})}{\det(g)}\right|^{\frac{1}{n+1}} \bar{g}^{-1} g \tag{14}$$

on M is called the *Benenti tensor* associated with $(g, \bar{g})$. In view of Formula (11), we also set

$$L(\sigma, \bar{\sigma}) = \bar{\sigma}\sigma^{-1}. \tag{15}$$

Proposition 3. *Let $g, \bar{g}$ be two projectively equivalent metrics on the same n-dimensional manifold M. Then $L = L(g, \bar{g})$ is self-adjoint w.r.t. g. In particular, if g is a Riemannian metric, then $L(g, \bar{g})$ is diagonalisable.*

Proof. It will be sufficient to prove that the matrix $(gL)_{ik} = g_{ij}L^j{}_k$ is symmetric. Indeed, setting $f = \left|\frac{\det(\bar g)}{\det(g)}\right|^{\frac{1}{n+1}}$ and denoting by A^T the transpose of matrix A, we have that

$$(gL)^T = (fg\bar g^{-1}g)^T = fg^T(\bar g^1)^T g^T = fg\bar g^{-1}g = gL\,. \tag{16}$$

□

Remark 2. Using (16), gL^k is symmetric for any $k \in \mathbb{N}$ by induction. Indeed, supposing gL^{k-1} symmetric, we have that

$$(gL^k)^T = (gL^{k-1}L)^T = L^T gL^{k-1} = (gL)^T L^{k-1} = (gL)L^{k-1} = gL^k\,.$$

If g is of mixed signature, $L(g, \bar g)$ is not necessarily diagonalisable. For instance, consider

$$g = 2(y^2 + x)dxdy \quad \text{and} \quad \bar g = -2\frac{y^2+x}{y^3}dxdy + \frac{(y^2+x)^2}{y^4}dy^2\,.$$

These Lorentzian metrics are studied in [4]. The metrics g and $\bar g$ are projectively equivalent as their projective connection is (see (7))

$$y_{xx} = \frac{1}{y^2+x}y_x - \frac{2y}{y^2+x}y_x^2\,,$$

and $L(g, \bar g)$ is not diagonalisable,

$$L(g, \bar g) = -\begin{pmatrix} y & y^2+x \\ 0 & y \end{pmatrix}.$$

We quote the following proposition, which follows from a more general statement in the reference.

Proposition 4 ([15], Theorem 1)**.** *Let g and $\bar g$ be projectively equivalent metrics. Let us assume that $L(g, \bar g)$ is diagonalisable. Then the eigendistributions of $L(g, \bar g)$ are integrable.*

2.3. Metrisability and the degree of mobility in dimension 3

Proposition 5 ([23], Corollary 3, p. 413)**.** *The degree of mobility of a 3-dimensional metric g of non-constant curvature is at most 2.*

Remark 3. In case the degree of mobility is 1, any metric $\bar{g}$ projectively equivalent to g is a constant multiple of g, i.e. $\bar{g} = c\, g$ with $c \neq 0$. Thus, for such g, any projective vector field $w \neq 0$ is homothetic. In this case, in a sufficiently small neighborhood around a non-singular point of w, we may thus choose coordinates (x, y, z) such that $w = \partial_x$, and this choice implies that the metric g can locally be written as in (2).

The following lemma is useful for the understanding eigendistributions of Benenti tensors $L(g, \bar{g})$ for 3-dimensional metrics.

Lemma 1. *Let g be an n-dimensional metric of non-constant curvature with degree of mobility 2. Let $\bar{g}$ be a metric projectively equivalent to g such that $\bar{g} \neq cg$, $\forall\, c \in \mathbb{R}$. Then the number and multiplicity of distinct eigenvalues of the Benenti tensor $L = L(g, \bar{g})$ does not depend on the choice of $\bar{g}$.*

Proof. The sought functions are solutions of $\det(L - s\ \mathrm{Id}) = 0$. Set $\sigma^{ij} = |\det(g)|^{\frac{1}{n+1}} g^{ij}$ and $\bar{\sigma}$ analogously (cf. (10)). We now replace the metric $\bar{g}$, using $\hat{g} = |\det(\hat{\sigma})|^{-1}\hat{\sigma}^{-1}$ instead. Since we assume that the degree of mobility is equal to 2, there exist two real constants c_1, c_2 such that

$$\hat{\sigma} = c_1\sigma + c_2\bar{\sigma}\,, \quad \text{with } c_2 \neq 0\,.$$

As a consequence, we replace $L = L(g, \bar{g})$ by $\hat{L} = L(g, \hat{g}) = \hat{\sigma}\sigma^{-1} = c_1 \mathrm{Id} + c_2 L$. Therefore, a solution t of $\det(\hat{L} - t\ \mathrm{Id}) = 0$ is in one-to-one correspondence with a solution of $\det(L - s\ \mathrm{Id}) = 0$. Indeed,

$$0 = \det(\hat{L} - t\ \mathrm{Id}) = c_2^n\ \det\left(L + \frac{1}{c_2}(c_1 - t)\,\mathrm{Id}\right)\,,$$

and thus we identify $s = \frac{c_1}{c_2} - \frac{t}{c_2}$. Note that the eigenvalues themselves change, but their number and multiplicity is preserved. □

Corollary 1. *Let g be a metric on a 3-dimensional manifold. Let $\bar{g}$ and $\hat{g}$ be metrics on the same manifold and projectively equivalent with g. Assume g, $\bar{g}$ and $\hat{g}$ are pairwise non-proportional. Then the number and multiplicity of eigenvalues of the Benenti tensor $L(g, \bar{g})$ corresponds with those of $L(g, \hat{g})$.*

Proof. This follows from Lemma 1 in combination with Proposition 5. □

Lemma 2. *There is a metrisable 3-dimensional projective connection such that $\dim \mathfrak{A} = 2$ if and only if there is a rank-3 basis $(\sigma, \bar{\sigma})$ of $\mathfrak{A}$.*

Proof. The backwards direction is trivial. Thus let us assume $\dim \mathfrak{A} = 2$ and the existence of a rank-3 solution σ (otherwise the projective connection would not be metrisable). Then, since (12) is linear, all multiples $k\sigma$ with $k \neq 0$ are rank-3 solutions. Now, since the condition $\mathrm{rank}(\sigma) < 3$ is a closed condition, the set $\mathfrak{A} \setminus \{\sigma | \det \sigma = 0\}$ is open. Thus, there exists another rank-3 solution, say $\bar{\sigma}$, that is not proportional to σ. Since $\dim \mathfrak{A} = 2$ the pair $(\sigma, \bar{\sigma})$ forms a rank-3 basis of $\mathfrak{A}$. □

2.4. Projective connections and projective vector fields in dimension 3

In coordinates $(x^1, x^2, x^3) = (x, y, z)$, in view of (8), a 3-dimensional projective connection is given by

$$\begin{cases} y_{xx} & = f^2_{11} + f^2_{12} y_x + f^2_{13} z_x + f^2_{22} y_x^2 + 2 f^2_{23} y_x z_x \\ & \qquad + f^2_{33} z_x^2 + f_{11} y_x^3 + 2 f_{12} y_x^2 z_x + f_{22} y_x z_x^2 \\ & =: F^2(x, y, z, y_x, z_x) \\ \\ z_{xx} & = f^3_{11} + f^3_{12} y_x + f^3_{13} z_x + f^3_{22} y_x^2 + 2 f^3_{23} y_x z_x \\ & \qquad + f^3_{33} z_x^2 + f_{11} y_x^2 z_x + 2 f_{12} y_x z_x^2 + f_{22} z_x^3 \\ & =: F^3(x, y, z, y_x, z_x) \end{cases} \tag{17}$$

Taking into account Remark 1, below we write the system of differential equations that the components of a vector field

$$v = v^1 \partial_x + v^2 \partial_y + v^3 \partial_z \,, \quad v^i = v^i(x, y, z) \tag{18}$$

have to satisfy to be a projective vector field for the projective connection (17). For this purpose, it is useful to recall some basic facts concerning the theory of infinitesimal symmetries of a system of type (17). More details can be found, for instance, in [24]. Consider the second jet space $J^2(1, 2)$ with an independent variable x and two dependent variables y, z (since we are working locally, we may think of $J^2(1, 2)$ as $\mathbb{R}^7$ with coordinates $(x, y, z, y_x, z_x, y_{xx}, z_{xx})$). We define the total derivative operator restricted to system (17):

$$D_x = \partial_x + y_x \partial_y + z_x \partial_z + F^2 \partial_{y_x} + F^3 \partial_{z_x} \,.$$

The prolongation of vector field (18) to the second jet $J^2(1,2)$ is a vector field given by

$$
\begin{aligned}
v^{(2)} = v^1\partial_x + v^2\partial_y + v^3\partial_z \\
+ \left(D_x(v^2) - y_x D_x(v^1)\right)\partial_{y_x} + \left(D_x(v^3) - z_x D_x(v^1)\right)\partial_{z_x} \\
+ \left(D_x^2(v^2) - y_x D_x^2(v^1) - 2y_{xx}D_x(v^1)\right)\partial_{y_{xx}} \\
+ \left(D_x^2(v^3) - z_x D_x^2(v^1) - 2z_{xx}D_x(v^1)\right)\partial_{z_{xx}}
\end{aligned}
\tag{19}
$$

where $D_x^2 = D_x \circ D_x$. The above vector field is determined by the local one-parametric group of transformations $\phi_t^{(2)}$ given by the prolongation to the second jet space $J^2(1,2)$ of the local flow ϕ_t of (18) ([24, Sec. 2.3]).

The vector field (18) is an infinitesimal symmetry of system (17) (i.e. in other words, a projective vector field of the projective connection (17), see again Remark 1) if and only if

$$
\left.\begin{array}{l} v^{(2)}(y_{xx} - F^2) = 0 \\ v^{(2)}(z_{xx} - F^3) = 0 \end{array}\right\} \text{ when restricted to } \{y_{xx} = F^2\,, z_{xx} = F^3\}\,. \tag{20}
$$

A straightforward computation shows that conditions (20) give a second order system of 18 partial differential equations (PDEs) in the unknown functions (v^1, v^2, v^3).

3. Action of the projective algebra on the metrisation space $\mathfrak{A}$.

The action Φ of a projective vector field $v \in \mathfrak{p}$ on $\mathfrak{A}$ is

$$
\Phi : \mathfrak{p} \times \mathfrak{A} \to \mathfrak{A}\,, \quad \Phi(v,\sigma) = \mathcal{L}_v\sigma\,, \tag{21}
$$

where $\mathcal{L}_v$ is the Lie derivative along v.

Remark 4. Note that $\mathfrak{A}$ is invariant under Φ [4, 19]. This can be proven in an elementary way by choosing local coordinates $(x, y^2, \ldots, y^n)$ such that $v = \partial_x$. Then in (5) the coefficients depend on $y^2, \ldots, y^n$ only and the Lie derivative acts by usual derivatives. Therefore $\mathcal{L}_v\sigma \in \mathfrak{A}$ [19].

Since $\mathfrak{A} \sim \mathbb{R}^2$, action (21), for a specific $v \in \mathfrak{p}$, is encoded in a 2×2 matrix A, see [4]:

$$
\mathcal{L}_v \begin{pmatrix} \sigma \\ \bar\sigma \end{pmatrix} := \begin{pmatrix} \mathcal{L}_v\sigma \\ \mathcal{L}_v\bar\sigma \end{pmatrix} = A \begin{pmatrix} \sigma \\ \bar\sigma \end{pmatrix}\,. \tag{22}
$$

where

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad a, b, c, d \in \mathbb{R}\,. \tag{23}$$

The elements $\sigma, \bar{\sigma} \in \mathfrak{A}$ can correspond to metrics ($\sigma, \bar{\sigma}$ are full-rank solutions), or they can be of lower rank.
Concerning (22) we also remark the following:

- For a tensor section σ of (projective) *weight* w (compare [19]), the Lie derivative is related to the covariant derivative by

$$\mathcal{L}_v\sigma^{ij} = v^k\nabla_k\sigma^{ij} - \nabla_k v^i\,\sigma^{kj} - \nabla_k v^j\,\sigma^{ik} + w\,\nabla_k v^k\,\sigma^{ij}\,.$$

- A non-degenerate, weighted tensor section $\sigma \in \mathfrak{A}$ has weight -2. Let us recall that g and σ are related by (10) and (11). Then their Lie derivatives are related by

$$\mathcal{L}_v\sigma^{ij} = |\det(g)|^{\frac{1}{n+1}}\left(\frac{1}{n+1}\,\mathrm{trace}(g^{-1}\mathcal{L}_v g)g^{ij} + \mathcal{L}_v g^{ij}\right) \tag{24}$$

  and

$$\mathcal{L}_v g^{ij} = |\det(\sigma)|\big(\mathcal{L}_v\sigma^{ij} + \mathrm{trace}(\sigma_{ac}\mathcal{L}_v\sigma^{cb})\sigma^{ij}\big) \tag{25}$$

- In the present paper we follow the notation in [4]. The matrix A in (22) therefore is not the matrix representing, in the basis $(\sigma, \bar{\sigma})$, the Lie derivative by the usual conventions, but its transpose.

The following lemma is based on similar statements in [25] and [14].

Lemma 3. *Let g be a metric of degree of mobility 2 admitting a projective vector field v. Let σ be as in (10). Then there exist a constant $k \neq 0$ and a solution $\bar{\sigma} \neq 0$ of (12) such that*

$$\mathcal{L}_{kv}\sigma = a\sigma + b\bar{\sigma}.$$

where $\bar{\sigma}$ and σ are not proportional and where

1. *$a = b = 0$ if v is a Killing vector field for g.*
2. *$b = 0, a = 1$ if v is homothetic for g, and not a Killing vector field for g.*

3. $a = 0, b = 1$ if v is an essential projective vector field for g.

The lemma normalises the constants a, b in (23). Note however that in the lemma, for essential v, the solution $\bar{\sigma}$ is not necessarily of full rank.

Proof. If v is Killing, then $\mathcal{L}_v g = 0$ and thus $\mathcal{L}_v \sigma = 0$. We set $k = 1$. Let $\bar{g}$ be projectively equivalent to g, but not proportional to g. Then we obtain a solution $\bar{\sigma}$ of (12) from $\bar{g}$ using (10). For any such $\bar{\sigma}$, we find $a = b = 0$. If v is properly homothetic, we have $\mathcal{L}_v \sigma = \mu\sigma$ for some constant $\mu \neq 0$. We decree $k = \mu^{-1}$, and obtain $\mathcal{L}_{kv}\sigma = \sigma$. We conclude $a = 1, b = 0$ analogously to the previous case. Finally, if v is an essential projective vector field, we let $\bar{\sigma} := \mathcal{L}_v \sigma$ and $k = 1$. Then $a = 0, b = 1$. □

We have the following lemma, found in [15], see also [16]. It generalises equations in [14].

Lemma 4 ([15])**.** *Let $L = L(\sigma, \bar{\sigma})$ with σ of full rank. Then, for a projective vector field v,*

$$\mathcal{L}_v L = -bL^2 + (d-a)L + c\,\mathrm{Id} \tag{26}$$

Proof. This follows from a direct computation, using the formula $\mathcal{L}_v(\sigma^{-1}) = -\sigma^{-1}(\mathcal{L}_v\sigma)\sigma^{-1}$ that holds if σ is not of lower rank. Indeed, on account of (22),

$$\begin{aligned}\mathcal{L}_v L &= \mathcal{L}_v(\bar{\sigma}\sigma^{-1}) = \mathcal{L}_v(\bar{\sigma})\sigma^{-1} + \bar{\sigma}\mathcal{L}_v\sigma^{-1} = \mathcal{L}_v(\bar{\sigma})\sigma^{-1} - \bar{\sigma}\sigma^{-1}(\mathcal{L}_v\sigma)\sigma^{-1} \\ &= (c\,\mathrm{Id} + dL) - L(a\,\mathrm{Id} + bL) = -bL^2 + (d-a)L + c\,\mathrm{Id}\ .\end{aligned}$$

□

Lemma 5. *Let $L = L(g, \bar{g})$ with g and $\bar{g}$ projectively equivalent. Let v be a projective vector field of g. Then $(\mathcal{L}_v g)L$ is symmetric. In particular $(\mathcal{L}_v g)\big(L(u), w\big) = (\mathcal{L}_v g)\big(u, L(w)\big)$.*

Proof. We have that

$$(\mathcal{L}_v g)L = \mathcal{L}_v(gL) - g(\mathcal{L}_v L)\,.$$

Since gL is symmetric (see the proof of Proposition 3), also $\mathcal{L}_v(gL)$ is symmetric. So, in order to prove the statement of the lemma, in view of the above formula, it would be enough to prove that $g(\mathcal{L}_v L)$ is symmetric. Since $\mathcal{L}_v L$ is given by (26), this follows by Remark 2. □

Definition 6. Let v be a projective vector field. Let $\mathcal{L}_v$ be described by a matrix A as in (22). We call the formal polynomial

$$\mathrm{S}_A(t) = -bt^2 + (d-a)t + c \tag{27}$$

Solodovnikov's polynomial.

Lemma 6. *Let g be a 3-dimensional metric with degree of mobility 2 admitting projective vector field v. Let $\bar{g}$ be projectively equivalent, but non-proportional, to g. Let λ be an eigenvalue of $L = L(g, \bar{g})$. Then*

$$\mathrm{S}_A(\lambda) = v(\lambda)\,, \tag{28}$$

with S_A given by (27). In particular, if λ is constant, then it is a root of S_A.

Proof. Let w be an eigenvector of L with the eigenvalue λ, i.e. $Lw = \lambda w$. Because of the identity

$$\mathcal{L}_v(\lambda w) = \mathcal{L}_v(Lw) = (\mathcal{L}_v L)w + L\mathcal{L}_v w = (\mathcal{L}_v L)w + L[v,w]\,,$$

and due to (26), we have

$$0 = \mathrm{S}_A(L)w - (\mathcal{L}_v L)w = \mathrm{S}_A(\lambda)w - v(\lambda)w + (L - \lambda\,\mathrm{Id})\,[v,w]\,. \tag{29}$$

This implies

$$\big(-\mathrm{S}_A(\lambda) + v(\lambda)\big)g(w,w) = g\big((L-\lambda\,\mathrm{Id})\,[v,w],w\big) = 0\,, \tag{30}$$

where the second equality holds because of

$$g\big((L-\lambda\,\mathrm{Id})\,[v,w],w\big) \overset{(*)}{=} g\big([v,w],(L-\lambda\,\mathrm{Id})w\big) = g\big([v,w],0\big) = 0\,.$$

The equality $(*)$ above holds true as $gL - \lambda g$ is symmetric, see the proof of Proposition 3.
If $g(w,w) \neq 0$, Equation (30) immediately yields (28).[2] Thus let $g(w,w) = 0$. If $[v,w]$ is proportional to w, i.e. $[v,w] = \mu w$ for some function μ, then $(L-\lambda\,\mathrm{Id})\,[v,w] = \mu\big(Lw - \lambda w\big) = 0$ and, because of (29), we obtain (28). It

[2]Note that this proves the claim for any non-null eigenvector w, and in particular for Riemannian metrics, even in arbitrary dimension.

remains to prove the claim for $g(w,w)=0$ if $u:=[v,w]$ is not proportional to w. By virtue of (29), in this case we have

$$Lu = v(\lambda)w + \lambda u - \mathrm{S}_A(\lambda)w\,. \tag{31}$$

Hence Equation (26) yields the first equality in

$$\left(-bL^2 + (d-a)L + c\,\mathrm{Id}\right)u = (\mathcal{L}_v L)u = \mathcal{L}_v(Lu) - L[v,u]\,, \tag{32}$$

where the second equality follows from the Leibniz rule. In order to compute the left hand side of this identity, we use (31) to obtain

$$L^2 u = 2v(\lambda)\lambda w + \lambda^2 u - 2\mathrm{S}_A(\lambda)\lambda w \tag{33}$$

After resubstituting (31) and (33) into (32), we have

$$2(\mathrm{S}_A(\lambda) - v(\lambda))u + Bw = (\lambda\,\mathrm{Id} - L)[v,u] \tag{34}$$

for some expression B. To finish the proof, we study (34) for the two possible cases: First, in the case when $g(u,w)\neq 0$, Equation (34) together with Proposition 3 implies

$$2(\mathrm{S}_A(\lambda) - v(\lambda))g(u,w) = g\big([v,u],(\lambda\,\mathrm{Id} - L)w\big) = 0\,.$$

We obtain (28). Second, in the case when $g(u,w)=0$, we have

$$(\mathcal{L}_v g)(u,w) + g([v,u],w) + g(u,u) = v(g(u,w)) = 0\,, \tag{35}$$

along with

$$(\mathcal{L}_v g)(w,w) = v(g(w,w)) = 0\,, \tag{36}$$

which follows already from $g(w,w)=0$. Therefore, using (34) and then Lemma 5, we conclude

$$2(\mathrm{S}_A(\lambda) - v(\lambda))(\mathcal{L}_v g)(u,w) = (\mathcal{L}_v g)\big([v,u],(\lambda\,\mathrm{Id} - L)w\big) = 0\,. \tag{37}$$

Note that the conditions $g(u,w)=0=g(w,w)$ imply, together with the assumption that u and w are non-proportional, that

$$g(u,u)\neq 0\,, \tag{38}$$

as otherwise g would be degenerate. Taking the scalar product w.r.t. g of (34) and u, we arrive at

$$
\begin{aligned}
&2(\mathrm{S}_A(\lambda) - v(\lambda))g(u,u) \\
&\qquad \overset{\text{Prop. 3}}{=} g\big([v,u], (\lambda \operatorname{Id} - L)u\big) \overset{(31)}{=} g\big([v,u], \mathrm{S}_A(\lambda)w - v(\lambda)w\big) \\
&\qquad\qquad \overset{(35)}{=} -(\mathrm{S}_A(\lambda) - v(\lambda))\big((\mathcal{L}_v g)(u,w) + g(u,u)\big) \\
&\qquad\qquad\qquad \overset{(37)}{=} -(\mathrm{S}_A(\lambda) - v(\lambda))g(u,u)
\end{aligned}
$$

implying (28) in view of (38). This concludes the proof. □

In view of (28) and (29), we obtain the following corollary.

Corollary 2. *Let the hypotheses be as in Lemma 6. Then $[v,w]$ is an eigenvector of L with eigenvalue λ.*

Lemma 7. *Let g and $\bar{g}$ be a pair of projectively equivalent, non-proportional n-dimensional metrics with non-constant curvature and degree of mobility 2. Let g (and $\bar{g}$) admit the projective vector field v. Then*

$$\mathcal{L}_v g_{ij} = -(n+1)a\, g_{ij} - b\left(\operatorname{tr}(L)\, g_{ij} + g_{ik}L^k{}_j\right) \tag{39a}$$

$$\mathcal{L}_v \bar{g}_{ij} = -(n+1)d\, \bar{g}_{ij} - c\left(\operatorname{tr}(\bar{L})\, \bar{g}_{ij} + \bar{g}_{ik}\bar{L}^k{}_j\right) \tag{39b}$$

where $L = L(g,\bar{g})$ is the Benenti tensor of $(g,\bar{g})$. By $\bar{L}$ we denote the inverse of L, $\bar{L} = L^{-1}$.

Proof. Recall the formula $0 = \mathcal{L}_v(g^{-1}g) = \mathcal{L}_v(g^{-1})g + g^{-1}\mathcal{L}_v g$, whereby $\mathcal{L}_v g = -g\mathcal{L}_v(g^{-1})g$. By a direct computation using formula (10), and the identities (22) and (25), it then follows that

$$
\begin{aligned}
\mathcal{L}_v g &= -\det(\sigma)\left[g(a\sigma + b\bar{\sigma})g + \operatorname{tr}\left(\sigma^{-1}(a\sigma + b\bar{\sigma})\right) g\sigma g\right] \\
&= -\big((n+1)ag + bgL + b\operatorname{tr}(L)g\big)\,.
\end{aligned}
$$

This proves Equation (39a). Equation (39b) is obtained analogously. □

4. Levi-Civita metrics

Definition 7. A metric g is said to be a *Levi-Civita metric* if there exits a metric $\bar{g}$ on the same manifold M such that

- g and $\bar{g}$ are projectively equivalent metrics,
- $\bar{g}$ is not proportional to g, and
- $L(g, \bar{g})$ is diagonalisable.

Remark 5. Let g be a 3-dimensional Levi-Civita metric. Then it is either of constant curvature or its degree of mobility is exactly 2. The reason is that by Proposition 5 the metric is either of constant curvature or the degree of mobility is *at most* 2. But if the degree of mobility is equal to one, then any metric $\bar{g}$ projectively equivalent to g is proportional to g, contradicting the fact that g is a Levi-Civita metric.

Proposition 6. *Let us assume that the degree of mobility of a metric g is 2. Then g is a Levi-Civita metric if and only if $L(g, \bar{g})$ is diagonalisable for all metrics $\bar{g}$ on the same manifold which are projectively equivalent to g.*

Proof. The "$\Leftarrow$" implication follows immediately from Definition 7. For the "$\Rightarrow$" implication we assume that the degree of mobility of g is 2. By definition of Levi-Civita metrics, there exists $\bar{g}$ non proportional to g and such that $L = L(g, \bar{g})$ is diagonalisable. Let σ and $\bar{\sigma}$ be the corresponding solutions for $g, \bar{g}$, respectively, according to (10). Then σ and $\bar{\sigma}$ are linearly independent. Since the degree of mobility is 2, we obtain that any other solution $\hat{\sigma}$ of (12) is a linear combination of σ and $\bar{\sigma}$, with $s \neq 0 \neq t$,

$$\hat{\sigma} = s\sigma + t\bar{\sigma} \,.$$

Let $\hat{L} = L(\sigma, \hat{\sigma})$. The characteristic polynomial of $\hat{L}$ is

$$\begin{aligned}\det(\hat{L} - \lambda\,\mathrm{Id}) &= \det(\hat{\sigma}\sigma^{-1} - \lambda\,\mathrm{Id}) \\ &= \det((s\sigma + t\bar{\sigma})\sigma^{-1} - \lambda\,\mathrm{Id}) = \det(tL - (\lambda - s)\,\mathrm{Id})\end{aligned}$$

Therefore, any eigenvalue λ of $\hat{L}$ corresponds with an eigenvalue $\lambda' = \frac{\lambda - s}{t}$ of L, and vice versa. This confirms that the algebraic multiplicity is the same, c.f. Lemma 1. We now verify that also the geometric multiplicities coincide. For a specific eigenvalue λ of $\hat{L}$, the eigenspace is (note $t \neq 0$)

$$\ker(\hat{L} - \lambda\,\mathrm{Id}) = \ker(tL + s\,\mathrm{Id} - \lambda\,\mathrm{Id}) = \ker(tL - t\lambda'\,\mathrm{Id}) = \ker(L - \lambda'\,\mathrm{Id}) \,.$$

It follows that $\hat{L}$ is diagonalisable, since L is diagonalisable. ∎

In order to have a local description of Levi-Civita metrics of non-constant curvature and with degree of mobility equal to 2, we can proceed taking into account the following considerations:

1) Since such metrics have degree of mobility equal to 2, in view of Proposition 2, it will be enough to have a local description of a *pair* of metrics g and $\bar{g}$ in the considered projective class in a common system of coordinates;

2) This is facilitated by the diagonalisability of $L(g, \bar{g})$ (see Proposition 6) and by the integrability of its eigendistributions (see Proposition 4).

Concerning the second point, let us assume to have m integrable eigendistributions $\mathcal{D}_1, \dots, \mathcal{D}_m$ of $L(g, \bar{g})$, with $\dim(\mathcal{D}_i) = k_i$. We can then choose a system of coordinates $(x^1, \dots, x^n)$ on M

$$(x^1, \dots, x^n) = \left(x_1^1, \dots, x_1^{k_1}, x_2^1, \dots, x_2^{k_2}, \dots, x_m^1, \dots, x_m^{k_m}\right), \quad \sum_{i=1}^{m} k_i = n,$$

such that $\mathcal{D}_i = \langle \partial_{x_i^1}, \dots, \partial_{x_i^{k_i}} \rangle$. For cosmetic reasons, w.l.o.g., we assume that the coordinates corresponding to 1-dimensional eigendistributions are the first ones, i.e. the above system of coordinates is rearranged as follows:

$$\begin{aligned} (x^1, \dots, x^n) = \left(x_1^1, x_2^1, \dots, x_r^1, x_{r+1}^1, \dots, x_{r+1}^{k_{r+1}}, \dots, x_m^1, \dots, x_m^{k_m}\right), \\ \sum_{i=r+1}^{m} k_i = n - r, \ k_i \geq 2. \end{aligned} \tag{40}$$

Proposition 7 ([15, 17, 13])**.** *Let g be a Levi-Civita metric. Then there exists a metric $\bar{g}$ that is projectively equivalent and non-proportional to g, and (almost everywhere, in a sufficiently small neighborhood) local coordinates (40) such that $g, \bar{g}$ assume the form*

$$g = \sum_{i=1}^{r} P_i (dx_i^1)^2 + \sum_{i=r+1}^{m} \left[P_i \sum_{\alpha_i, \beta_i}^{k_i} (h_i(x_i))_{\alpha_i, \beta_i} dx_i^{\alpha_i} dx_i^{\beta_i} \right] \tag{41a}$$

$$\bar{g} = \sum_{i=1}^{r} P_i \rho_i (dx_i^1)^2 + \sum_{i=r+1}^{m} \left[P_i \rho_i \sum_{\alpha_i, \beta_i}^{k_i} (h_i(x_i))_{\alpha_i, \beta_i} dx_i^{\alpha_i} dx_i^{\beta_i} \right] \tag{41b}$$

where (x_i) *stands for the set of coordinates* $(x_i^1, \dots, x_i^{k_i})$ *and where*

$$P_i = \pm \prod (X_i - X_j) \qquad \text{and} \qquad \rho_i = \frac{1}{X_i \prod_\alpha X_\alpha^{k_\alpha}}\,,$$

with X_i *denoting the eigenvalue of* $L(g, \bar{g})$ *for the eigendistribution* $\mathcal{D}_i$*. Here, the* k_i *are numbers larger or equal to 2.*

Remark 6. The metrics in (41) are local. According to [15] they can be achieved after a coordinate change, in a sufficiently small neighborhood of a point where $\mathrm{image}(X_i) \cap \mathrm{image}(X_j) = \emptyset$, $\forall\, i \neq j$. We remark that one may assume $X_i > X_{i+1}$ for $1 \leq i \leq r$ and, respectively, for $r + 1 \leq i \leq m$.

Due to [17, Theorem 3], the X_i are constants for $i \geq r + 1$, i.e. for the building blocks with geometric multiplicity $k_i \geq 2$, see also [26, 12] and [27, Lemma 6]. For $i \leq r$, in the coordinates of (41) the $X_i = X_i(x_i^1)$ are univariate functions.

4.1. Levi-Civita metrics of dimension 3

The current paper concerns itself with Levi-Civita metrics in dimension 3. Due to Proposition 7 we therefore have to study two distinct cases:

[1-1-1] metrics In this case $r = m = 3$, and (41) gives

$$\begin{aligned} g &= \pm(X_1 - X_2)(X_1 - X_3)\,(dx^1)^2 \pm (X_2 - X_1)(X_2 - X_3)\,(dx^2)^2 \\ &\quad \pm (X_3 - X_1)(X_3 - X_2)\,(dx^3)^2 \end{aligned} \tag{42a}$$

$$\begin{aligned} \bar{g} &= \pm\frac{(X_1 - X_2)(X_1 - X_3)}{X_1^2 X_2 X_3}\,(dx^1)^2 \pm \frac{(X_2 - X_1)(X_2 - X_3)}{X_1 X_2^2 X_3}\,(dx^2)^2 \\ &\quad \pm \frac{(X_3 - X_1)(X_3 - X_2)}{X_1 X_2 X_3^2}\,(dx^3)^2 \end{aligned} \tag{42b}$$

and without loss of generality it may be assumed that $X_3(x^3) > X_2(x^2) > X_1(x^1)$. Indeed, the eigenvalues of $L(g, \bar{g})$ are not allowed to coincide as otherwise already the following case would occur.

[2-1] metrics In this case $r = 1$, $m = 2$ and (41) gives

$$g = \pm(X_1 - X_2)\,\left[(dx^1)^2 \mp h\right] \tag{43a}$$

$$\bar{g} = \pm\frac{X_1 - X_2}{X_1 X_2^2}\,\left[\frac{(dx^1)^2}{X_1} \mp \frac{h}{X_2}\right] \tag{43b}$$

where $h = \sum_{i,j=2}^{3} h_{ij} dx^i dx^j$ is any 2-dimensional metric, $X_1 = X_1(x^1)$ and X_2 is a constant (cf. Remark 6). For cosmetic reasons we are going to rename the coordinates such that the manifold underlying h has coordinates x and y and such that the manifold underlying one-dimensional distribution has the coordinate z. We thus have

$$g = \zeta(z)\,(h \pm dz^2) \tag{44a}$$

$$\bar{g} = \frac{\zeta(z)}{Z(z)\,\rho^2}\left(\frac{h}{\rho} \pm \frac{dz^2}{Z(z)}\right) \tag{44b}$$

where $h = h_{11}dx^2 + 2h_{12}dxdy + h_{22}dy^2$, $h_{ij} = h_{ij}(x,y)$, and

$$\zeta(z) = Z(z) - \rho\,, \quad \rho \in \mathbb{R}\,.$$

The manifold M underlying g and $\bar{g}$ is a product of a 2-dimensional manifold M_2 with coordinates x, y and a 1-dimensional manifold M_1 with coordinate z.

Note that the case $r = 0$, $m = 1$ cannot appear. Indeed, if g were such a Levi-Civita metric then there would exist a metric $\bar{g}$ projectively equivalent to g but not proportional to it. The tensor (14) thus has one eigenvalue of multiplicity three, i.e. it is a multiple of the identity. Consequently g and $\bar{g}$ must be already proportional, contradicting the hypothesis to be of Levi-Civita type.

Example 2. For a Riemannian 3-dimensional Levi-Civita [1-1-1] metric g we find local coordinates $(x_1^1, x_2^1, x_3^1) = (x^1, x^2, x^3)$ such that

$$g = |X_1 - X_2|\,|X_1 - X_3|\,(dx^1)^2 + |X_2 - X_1|\,|X_2 - X_3|\,(dx^2)^2 + |X_3 - X_1|\,|X_3 - X_2|\,(dx^3)^2$$

where $X_i = X_i(x^i)$ are univariate functions, which are also the eigenvalues of (14). Reordering the coordinates, we may w.l.o.g. assume $X_1 > X_2 > X_3$. The above metric g appears for the first time in [12]. Some differential projective aspects of such metric is discussed in [14, 13].

For Riemannian Levi-Civita metrics in arbitrary dimension $n \geq 3$ that admit a projective vector field v, [14] provides a collection of defining ODEs under the requirement that the metric consists of solely 1-dimensional blocks, i.e. the metrics of Example 2 and their higher-dimensional counterparts.

4.2. Projective vector fields of Levi-Civita metrics

The following lemma can be obtained from [15, 16, 14].

Lemma 8. *Let g be an n-dimensional Levi-Civita metric of the form (41a) and not of constant curvature. Let v be a projective vector field for g. Then the components of v, in the system of coordinates (40), are*

$$\big(v_1(x_1^1), \dots, v_r(x_r^1), v_{r+1}(x_{r+1}), \dots, v_m(x_m)\big)$$

where, for $1 \leq i \leq r$, v_i are univariate functions of variable x_i^1 and, for $i > r$, $v_i = (v_i^1, \dots, v_i^{k_i})$ depend on the coordinates $x_i = (x_i^1, \dots, x_i^{k_i})$.

Proof. Let $\bar{g}$ be the metric (41b) and let $L = L(g, \bar{g})$. Since g is a Levi-Civita metric, L is diagonal. Let i and j be indices such that x^i and x^j are coordinates on different eigendistributions, i.e. such that $\frac{\partial}{\partial x^i}$ and $\frac{\partial}{\partial x^j}$ belong to different eigenspaces of L. We denote the associated eigenvalues by λ_k, λ_ℓ, respectively, with $\lambda_k \neq \lambda_\ell$. Because L is diagonal, we infer $L^i{}_j = 0$. For the entry on the i-th row and j-th column of Equation (26), we therefore find:

$$0 = \mathcal{L}_v L^i{}_j = v^c L^i{}_{j,c} - v^i{}_{,c}\, L^c{}_j + v^c{}_{,j}\, L^i{}_c = (\lambda_k - \lambda_\ell)\, v^i{}_{,j}\,,$$

where we use that L is diagonal. Since $\lambda_k - \lambda_\ell \neq 0$, we conclude $v^i{}_{,j} = 0$. Repeating this for all pairs (i, j) as above, we confirm the assertion. □

Example 3. Let us consider the metric (42a) of non-constant curvature. According to Lemma 8, a projective vector field v for this metric has the following components:

$$\big(v_1(x_1^1), v_2(x_2^1), v_3(x_3^1)\big) = \big(v^1(x), v^2(y), v^3(z)\big)\,.$$

For the metric (43a), of non-constant curvature, a projective vector field v has the following components:

$$\big(v_1(x_1^1), v_2(x_2^1, x_2^2)\big)\,.$$

In view of the consideration after equations (43a), for the metric (44a), of non-constant curvature, a projective vector field v has the following components:

$$\big(v^1(x, y), v^2(x, y), v^3(z)\big)\,.$$

5. Levi-Civita metrics of type [1-1-1]

We begin our discussion with the case of Levi-Civita metrics of type [1-1-1], i.e. metrics that may be put into the form (42a) at least in a small neighborhood around almost every point. This case includes Riemannian metrics of the form (42a). In the current section, we admit Riemannian as well as Lorentzian signature. The partner metric for (42a) is (42b) such that the eigenvalues of $L = L(g, \bar{g})$ are given by three univariate functions, more precisely

$$L = X_1(x^1)\, \partial_{x^1} \otimes dx^1 + X_2(x^2)\, \partial_{x^2} \otimes dx^2 + X_3(x^3)\, \partial_{x^3} \otimes dx^3 \tag{45}$$

Remark 7. The form of the metric (42a) is preserved under any simultaneous affine transformation of the coordinates,

$$x^i_{\text{new}} = \mu x^i_{\text{old}} + \kappa_i\,, \quad \mu, \kappa_i \in \mathbb{R}\,,\ \mu \neq 0\,,$$

if we simultaneously redefine

$$X_i^{\text{new}}(x^i_{\text{new}}) = \frac{1}{|\mu|}\ X_i^{\text{old}}\left(\frac{x^i_{\text{new}} - \kappa_i}{\mu}\right) + t\,,$$

with $t \in \mathbb{R}$. Note that, in particular, we may therefore transform $L_{\text{new}} = L_{\text{old}} - t\,\mathrm{Id}$, even without any modification of the coordinate system. Indeed, such a transformation corresponds with choosing a different metric $\bar{g}$ (respectively $\bar{\sigma}$ via (10)) projectively equivalent, but non-proportional, to g. This even allows us to transform, without changing the metric g,

$$L_{\text{new}} = \kappa L_{\text{old}} - t\,\mathrm{Id}$$

for $\kappa, t \in \mathbb{R}$, $\kappa \neq 0$.

Equation (28), in the [1-1-1] case, provide a set of ODEs for the eigenvalues of (45), i.e. for X_1, X_2 and X_3. More precisely, we have the following proposition.

Lemma 9. *Let g be a Levi-Civita metric of type [1-1-1] admitting a projective vector field v. Then $v = v^i \partial_{x^i}$ where $v^i = v^i(x^i)$ are univariate functions. Moreover the following system of ODEs is satisfied:*

$$v^i \frac{dX_i}{dx^i} = -b\, X_i^2 + (d-a)\, X_i + c \tag{46a}$$

$$\frac{dv^i}{dx^i} = -(a+d) \tag{46b}$$

Proof. In view of the first part of Example 3, the components v^i of the projective vector field v are such that $v^i = v^i(x^i)$. Then (46a) is nothing but (28) with $\lambda = X_i(x^i)$. Resubstituting (46a) into (39), we can solve the obtained system w.r.t. the first derivatives of the component v^i of the projective vector field v, thus obtaining (46b). □

The following proposition shows that all projective symmetries Levi-Civita metrics of type [1-1-1] arise from homotheties of 1-dimensional metrics.

Proposition 8.
(i) Let g be a Levi-Civita metric of type [1-1-1] admitting a projective vector field $v = v^i\partial_{x^i}$, $v^i = v^i(x^i)$. Then v is a homothetic vector field for the 1-dimensional metric $(dx^i)^2$.
(ii) For $i \in \{1,2,3\}$, let h_i be 1-dimensional metrics. Assume $u^i\partial_{x^i} \neq 0$ (no summation), $u^i = u^i(x^i)$, is a homothetic vector field for h_i, $\mathcal{L}_{u^i}h_i = -2Bh_i$ for a common constant $B \in \mathbb{R}$. Let X_1, X_2, X_3 be three functions satisfying

$$(-2Bx^i + k_i)\frac{dX_i}{dx^i} = -b\,X_i^2 + \mu\,X_i + c$$

for $k_i \in \mathbb{R}$ and common constants $b, c, \mu \in \mathbb{R}$. Then the X_i define a [1-1-1]-type Levi-Civita metric (42) with projective vector field $v = u^i\partial_{x^i}$.

Proof. For part (i), consider $h = (dx^i)^2$ and $u = f(x^i)\partial_{x^i}$. Then u is homothetic for h if

$$\mathcal{L}_u h = -2f'(x^i)h = -2Bh$$

for constant $B \in \mathbb{R}$. The claim then follows from Equation (46b).

Part (ii) follows immediately From Lemma 9. □

In [14] formulas (46a) and (46b) are obtained in the Riemannian case. We compare the results. In the reference, two branches are distinguished:

1. If v is an *essential projective vector field*, then $b \neq 0$. We then rescale v by $-b$ and find the projective vector field $w = -\frac{v}{b}$. Now, define $f_i = X_i + \frac{a}{b}$. We then obtain from (46) (no summation convention applies)

$$w^i\,\frac{df_i}{dx^i} = f_i^2 + \alpha_1 f_i + \alpha_0 \tag{47a}$$

$$\frac{dw^i}{dx^i} = -\alpha_1\,, \tag{47b}$$

where we define

$$\alpha_1 = -\frac{a+d}{b}\,, \quad \alpha_0 = \frac{ad-cb}{b^2}\,.$$

This system corresponds with Equations (5.5) of [14].

2. If v is a *homothetic projective vector field*, then $b=0$. In order to compare (46) with the reference, we introduce $\eta \in \mathbb{R}$ such that $\mathcal{L}_v g = 2\eta\, g$. Using Equation (24), we then find $\mathcal{L}_v\sigma = -\frac{\eta}{2}\sigma$ and conclude $\eta = -2a$. Equations (46) thus take the form (no summation convention)

$$v^i\,\frac{dX_i}{dx^i} = (d-a)\,X_i + c \tag{48a}$$

$$\frac{dv^i}{dx^i} = -(a+d)\,. \tag{48b}$$

If v is Killing, we have $a=0$. If v is properly homothetic, the have $a\neq 0$ and by rescaling v we can fix a to be any non-zero constant. Therefore, the number of free parameters is two. Note that (48) correspond with Equations (5.8) of [14].

Remark 8. We emphasise the following difference between the two cases: In the homothetic case, we may immediately take a solution to (48) and use it to write down the metrics (42a) and (42b). In the essential case, for solutions to (47), this is only possible if $\mathcal{L}_v\sigma$ is invertible. Otherwise, one of the f_i is going to be zero. Indeed, if $\det(a\sigma+b\bar{\sigma})=0$, then $-\frac{a}{b}$ is an eigenvalue of L, and so $f_i = X_i + \frac{a}{b} = -\frac{a}{b} + \frac{a}{b} = 0$ for some i (see Remark 7). Note that $g = \sum_i dx_i^2 \prod_{j\neq i}(X_i - X_j) = \sum_i dx_i^2 \prod_{j\neq i}(f_i - f_j)$. However, (42b) cannot be computed for f_i. Instead, we obtain the metrics projectively equivalent to g by computing

$$\bar{g} = \mu\Bigg(\pm \tfrac{(f_1-f_2)(f_1-f_3)}{(f_1+\ell)^2(f_2+\ell)(f_3+\ell)}\,(dx^1)^2 \pm \tfrac{(f_2-f_1)(f_2-f_3)}{(f_1+\ell)(f_2+\ell)^2(f_3+\ell)}\,(dx^2)^2 \pm \tfrac{(f_3-f_1)(f_3-f_2)}{(f_1+\ell)(f_2+\ell)(f_3+\ell)^2}\,(dx^3)^2\Bigg)$$

wherever defined (here ℓ and $\mu\neq 0$ are real constants). This formula can also be found in [12, Eq. (26′)]

The following two lemmas are the foundation for a full description of [1-1-1]-type Levi-Civita metrics with projective vector fields. We begin with the solution of the system (47), which yields essential projective vector fields[3]

Lemma 10. *The general solution to system* (47)*, up to a translation of* x^i*, is described below.*

1. $\alpha_1 \neq 0$

   (a) $\alpha_1^2 - 4\alpha_0 > 0$

$$
\begin{aligned}
& w^i = -\alpha_1 x^i \,, \\
& f_i = \frac{1}{2}\tanh\left(\frac{\ln(c_i|x^i|)}{2\alpha_1}\sqrt{\alpha_1^2 - 4\alpha_0}\right)\sqrt{\alpha_1^2 - 4\alpha_0} - \frac{1}{2}\alpha_1 \,, \\
& c_i \in \mathbb{R}, c_i > 0 \,.
\end{aligned} \tag{49}
$$

   (b) $\alpha_1^2 - 4\alpha_0 = 0$

$$
w^i = -\alpha_1 x^i \,, \quad f_i = \alpha_1\left(\frac{1}{\ln(c_i|x^i|)} - \frac{1}{2}\right) \,, \quad c_i \in \mathbb{R}, c_i > 0 \,. \tag{50}
$$

   (c) $\alpha_1^2 - 4\alpha_0 < 0$

$$
\begin{aligned}
& w^i = -\alpha_1 x^i \,, \\
& f_i = -\frac{1}{2}\tan\left(\frac{\ln(c_i|x^i|)}{2\alpha_1}\sqrt{-\alpha_1^2 + 4\alpha_0}\right)\sqrt{-\alpha_1^2 + 4\alpha_0} - \frac{1}{2}\alpha_1 \,, \\
& c_i \in \mathbb{R}, c_i > 0 \,.
\end{aligned} \tag{51}
$$

2. $\alpha_1 = 0$

   (a) $\alpha_0 > 0$*. In this case*

$$
w^i = c_i \,, \quad f_i = \sqrt{\alpha_0}\tan\left(\frac{\sqrt{\alpha_0}\, x^i}{c_i}\right) \,, \quad c_i \in \mathbb{R} \,, \; c_i \neq 0 \,.
$$

[3] We remark that, in the statement of the lemma, the condition $\alpha_1^2 - 4\alpha_0 = 0$ is equivalent to requiring that the eigenvalues of (23) be coincident. Indeed $b^2(\alpha_1^2 - 4\alpha_0) = (\kappa_1 - \kappa_2)^2$, where κ_i are the eigenvalues of (23).

(b) $\alpha_0 = 0$. *In this case*

$$w^i = c_i\,,\quad f_i = -\frac{c_i}{x^i}\,,\quad c_i \in \mathbb{R}\,.$$

(c) $\alpha_0 < 0$. *In this case, either*

$$w^i = c_i\,,\quad f_i = -\sqrt{-\alpha_0}\tanh\left(\tfrac{\sqrt{-\alpha_0}\,x^i}{c_i}\right),\quad c_i \in \mathbb{R}\,,\ c_i \neq 0$$

or

$$w^i = 0\,,\quad f_i = \mp\sqrt{-\alpha_0}$$

Proof. All the involved ODEs are of Riccati type, which can be straightforwardly solved. □

The following lemma covers homothetic vector fields of [1-1-1]-type Levi-Civita metrics.

Lemma 11. *The general solution to system* (48)*, up to a translation of* x^i*, is described below.*

1. $a + d \neq 0$.

   (a) $a - d \neq 0$.
   In this case (48b) *gives*

   $$v^i = -(a+d)x^i\,.$$

   By substituting it into (48a) *we obtain an ODE whose general solution is*

   $$X_i = \frac{c}{a-d} + k_i|x^i|^{\frac{a-d}{a+d}}\,,\ k_i \in \mathbb{R}\,.$$

   (b) $a - d = 0$, $a \neq 0$.
   In this case (48b) *gives*

   $$v^i = -2ax^i\,.$$

   By substituting it into (48a) *we obtain an ODE whose general solution is*

   $$X_i = -\frac{c\ln(|x^i|)}{2a} + k_i\,,\quad k_i \in \mathbb{R}$$

2. $a + d = 0$.

(a) $a \neq 0$.

In this case, either

$$v^i = 0\,, \quad X_i = \frac{c}{2a}$$

or

$$v^i = k_i\,, \quad X_i = \frac{c}{2a} + h_i e^{-\frac{2a}{k_i}x^i}\,, \quad k_i \in \mathbb{R}\,,\ k_i \neq 0\,, \quad h_i \in \{-1, 0, 1\}\,.$$

(b) $a = 0 = d$, $c = 0$. *Then either*

$$v^i = 0\,,\ \ X_i = X_i(x^i) \text{ is an arbitrary non-constant function of } x^i$$

or

$$v^i = k_i\,,\ X_i = h_i\,,\ k_i \in \mathbb{R}\,, k_i \neq 0\,,\ h_i \in \mathbb{R}$$

(c) $a = 0 = d$, $c \neq 0$

Then

$$v^i = k_i\,, \quad X_i = \frac{c}{k_i}x^i\,, \quad k_i \in \mathbb{R}\,,\ k_i \neq 0$$

Proof. The claim is straightforwardly obtained in analogy to Lemma 10. □

Recall that if an eigenvalue of $L = L(g, \bar{g})$ of the form (45) is constant, i.e. $X_i(x^i) = \text{constant}$ for some i, then ∂_{x^i} is a Killing vector field as the metric (42) does not depend on x^i in that case. Moreover, note that by definition, for a metric of [1-1-1] type, no eigenvalue of L can have multiplicity greater than one. Finally, if all three eigenvalues of L are constant, the metric (42) has constant curvature, and we are therefore going to assume in the following that at least one eigenvalue of L is non-constant.

Note that if $L = L(\sigma, \bar{\sigma})$ has a constant eigenvalue, we may w.l.o.g. change $\bar{\sigma}$ such that $X_1 = 0$. Also, for a metric (42), L cannot have repeated eigenvalues. If L has two constant eigenvalues, we may in this case w.l.o.g. assume $X_1 = 0, X_2 = \rho \neq 0$ by reordering the coordinates.

For the following proposition, also recall the coordinate transformations outlined in Remark 7.

Proposition 9. *Consider a metric (42) of non-constant curvature such that $L(g, \bar{g})$ has two constant eigenvalues. Then the projective algebra is $\mathfrak{p}(g) = \langle \partial_{x^1}, \partial_{x^2} \rangle$ and coincides with the Killing algebra.*

Proof. In Lemma 11 only the case 2b contains solutions compatible with L having two constant eigenvalues. We observe that this case we have the two trivial Killing vector fields ∂_{x^1} and ∂_{x^2}. The metric in case 2c of Lemma 10 is a special case of the aforementioned, and by virtue of the transformation in Remark 7 we obtain, in new coordinates (x, y, z), the metric $(k \neq 0)$

$$g = \pm 2\left(1 - \tanh\left(\frac{z}{k}\right)\right) dx^2 \pm 2\left(1 + \tanh\left(\frac{z}{k}\right)\right) dy^2 \\ \pm \left(1 - \tanh\left(\frac{z}{k}\right)\right)\left(1 + \tanh\left(\frac{z}{k}\right)\right) dz^2 \,.$$

It is easily checked that this metric has constant curvature. The claim then follows straightforwardly. □

The proof of the following proposition is analogous to the previous one.

Proposition 10. *Consider a metric (42) of non-constant curvature such that $L(g, \bar{g})$ has exactly one constant eigenvalue. Then $\mathfrak{p}(g) = \langle \partial_{x^1} \rangle$, or there is a coordinate transformation to new coordinates (x, y, z) that identifies g as one of the following:*

1. *The metric*

$$g = \pm k_2 k_3\, y^h z^h\, dx^2 \pm k_2 y^h (k_2 y^h - k_3 z^h)\, dy^2 \pm k_3 z^h (k_3 z^h - k_2 y^h)\, dz^2$$

where $h \notin \{-1, 0, 1\}, k_i \in \mathbb{R}, k_i \neq 0$. Its projective algebra is generated by

$$\partial_x \quad \text{and} \quad x\partial_x + y\partial_y + z\partial_z \,.$$

2. *The metric*

$$g = \pm k_2 k_3\, y^{-1} z^{-1}\, dx^2 \pm k_2 y^{-1} (k_2 y^{-1} - k_3 z^{-1})\, dy^2 \pm k_3 z^{-1} (k_3 z^{-1} - k_2 y^{-1})\, dz^2$$

($k_i \in \mathbb{R}$, $k_i \neq 0$) whose projective algebra is generated by

$$\partial_x \,, \quad k_2 \partial_y + k_3 \partial_z \quad \text{and} \quad x\partial_x + y\partial_y + z\partial_z \,.$$

3. *The metric*

$$g = \pm h_2 h_3 e^{\frac{y}{k_2}} e^{\frac{z}{k_3}}\, dx^2 \pm h_2 e^{\frac{y}{k_2}} \left(h_2 e^{\frac{y}{k_2}} - h_3 e^{\frac{z}{k_3}}\right) dy^2 \\ \pm h_3 e^{\frac{z}{k_3}} \left(h_3 e^{\frac{z}{k_3}} - h_2 e^{\frac{y}{k_2}}\right) dz^2$$

where $h_i \in \{\pm 1\}$, $k_i \in \mathbb{R}$, $k_i \neq 0$ and $k_2 \neq \pm k_3$. Its projective algebra is generated by

$$\partial_x \quad \textit{and} \quad k_2\partial_y + k_3\partial_z\,.$$

4. *The metric*

$$\begin{aligned} g = &\pm \left(\varepsilon - \tanh\left(\tfrac{y}{k_2}\right)\right)\left(\varepsilon - \tanh\left(\tfrac{z}{k_3}\right)\right)\,dx^2 \\ &\pm \left(\tanh\left(\tfrac{y}{k_2}\right) - \varepsilon\right)\left(\tanh\left(\tfrac{y}{k_2}\right) - \tanh\left(\tfrac{z}{k_3}\right)\right)\,dy^2 \\ &\pm \left(\tanh\left(\tfrac{z}{k_3}\right) - \varepsilon\right)\left(\tanh\left(\tfrac{z}{k_3}\right) - \tanh\left(\tfrac{y}{k_2}\right)\right)\,dz^2 \end{aligned}$$

with $k_i \in \mathbb{R}$, $k_i \neq 0$, $k_3 \neq \pm k_2$, $\varepsilon \in \{\pm 1\}$. Its projective algebra is generated by

$$\partial_x \quad \textit{and} \quad k_2\partial_y + k_3\partial_z\,.$$

Proof. In Lemma 10 only the cases 2b and 2c can lead to metrics g with $L(g,\bar{g})$ having exactly one constant eigenvalue. In Lemma 11 only the cases 1a, 2b and 2c can be of this kind.

All these cases are special cases of 2b of Lemma 11, which is the generic case. Generically, according to Lemma 11, we have the Killing vector field ∂_{x^i}. Let us now consider the remaining, non-generic cases. The first case is case 1a from Lemma 11, assuming $h \neq 1$. Note that for $h = 1$, this metric coincides with case 2c of the same Lemma. In this case we therefore have a larger projective algebra. The remaining cases follow in an analogous manner. $\square$

Proposition 11. *Consider a metric (42) of non-constant curvature such that $L(g,\bar{g})$ has no constant eigenvalue. Assume that g admits a non-vanishing projective vector field. Then, after a change to new coordinates (x,y,z), the metric is one of the following.*

1. *The metric*

$$\begin{aligned} g = &\pm(k_1|x|^h - k_2|y|^h)(k_1|x|^h - k_3|z|^h)\,dx^2 \\ &\pm (k_2|y|^h - k_1|x|^h)(k_2|y|^h - k_3|z|^h)\,dy^2 \\ &\pm (k_3|z|^h - k_1|x|^h)(k_3|z|^h - k_2|y|^h)\,dz^2 \end{aligned}$$

where $h \notin \{-1,0,1\}$, $k_i \in \mathbb{R}$, $k_i \neq 0$. Its projective algebra is generated by

$$x\partial_x + y\partial_y + z\partial_z\,.$$

2. *The metric*

$$g = \pm(k_1x - k_2y)(k_1x - k_3z)\,dx^2 \pm (k_2y - k_1x)(k_2y - k_3z)\,dy^2 \pm (k_3z - k_1x)(k_3z - k_2y)\,dz^2$$

($k_i \in \mathbb{R}$, $k_i \neq 0$) whose projective algebra is generated by

$$\frac{1}{k_1}\,\partial_x + \frac{1}{k_2}\,\partial_y + \frac{1}{k_3}\,\partial_z \quad \textit{and} \quad x\partial_x + y\partial_y + z\partial_z\,.$$

3. *The metric*

$$g = \pm\left(\frac{k_1}{x} - \frac{k_2}{y}\right)\left(\frac{k_1}{x} - \frac{k_3}{z}\right)\,dx^2 \pm \left(\frac{k_2}{y} - \frac{k_1}{x}\right)\left(\frac{k_2}{y} - \frac{k_3}{z}\right)\,dy^2 \pm \left(\frac{k_3}{z} - \frac{k_1}{x}\right)\left(\frac{k_3}{z} - \frac{k_2}{y}\right)\,dz^2$$

($k_i \in \mathbb{R}$, $k_i \neq 0$) whose projective algebra is generated by

$$k_1\,\partial_x + k_2\,\partial_y + k_3\,\partial_z \quad \textit{and} \quad x\partial_x + y\partial_y + z\partial_z\,.$$

4. *The metric*

$$g = \pm\ln\left(\tfrac{k_1|x|}{k_2|y|}\right)\ln\left(\tfrac{k_1|x|}{k_3|z|}\right)\,dx^2 \pm \ln\left(\tfrac{k_2|y|}{k_1|x|}\right)\ln\left(\tfrac{k_2|y|}{k_3|z|}\right)\,dy^2 \pm \ln\left(\tfrac{k_3|z|}{k_1|x|}\right)\ln\left(\tfrac{k_3|z|}{k_2|y|}\right)\,dz^2$$

($k_i \in \mathbb{R}$, $k_i > 0$) whose projective algebra is generated by

$$x\partial_x + y\partial_y + z\partial_z\,.$$

5. *The metric*

$$g = \pm\left(h_1e^{\frac{x}{k_1}} - h_2e^{\frac{y}{k_2}}\right)\left(h_1e^{\frac{x}{k_1}} - h_3e^{\frac{z}{k_3}}\right)\,dx^2 \pm \left(h_2e^{\frac{y}{k_2}} - h_1e^{\frac{x}{k_1}}\right)\left(h_2e^{\frac{y}{k_2}} - h_3e^{\frac{z}{k_3}}\right)\,dy^2 \pm \left(h_3e^{\frac{z}{k_3}} - h_1e^{\frac{x}{k_1}}\right)\left(h_3e^{\frac{z}{k_3}} - h_2e^{\frac{y}{k_2}}\right)\,dz^2$$

($h_i \in \{-1, 1\}$, $k_i \in \mathbb{R}$, $k_i \neq 0$) whose projective algebra is generated by

$$k_1\partial_x + k_2\partial_y + k_3\partial_z\,.$$

6. *The metric*

$$\begin{aligned} g = \pm &\left(\tanh\left(\ln(k_1|x|^\beta)\right) - \tanh\left(\ln(k_2|y|^\beta)\right)\right) \\ &\cdot \left(\tanh\left(\ln(k_1|x|^\beta)\right) - \tanh\left(\ln(k_3|z|^\beta)\right)\right)\, dx^2 \\ \pm &\left(\tanh\left(\ln(k_2|y|^\beta)\right) - \tanh\left(\ln(k_1|x|^\beta)\right)\right) \\ &\cdot \left(\tanh\left(\ln(k_2|y|^\beta)\right) - \tanh\left(\ln(k_3|z|^\beta)\right)\right)\, dy^2 \\ \pm &\left(\tanh\left(\ln(k_3|z|^\beta)\right) - \tanh\left(\ln(k_1|x|^\beta)\right)\right) \\ &\cdot \left(\tanh\left(\ln(k_3|z|^\beta)\right) - \tanh\left(\ln(k_2|y|^\beta)\right)\right)\, dz^2 \end{aligned}$$

($k_i \in \mathbb{R}$, $k_i > 0$, $\beta \neq 0$) whose projective algebra is generated by

$$x\partial_x + y\partial_y + z\partial_z\,.$$

7. *The metric*

$$\begin{aligned} g = \pm &\left(\frac{1}{\ln(k_1|x|)} - \frac{1}{\ln(k_2|y|)}\right)\left(\frac{1}{\ln(k_1|x|)} - \frac{1}{\ln(k_3|z|)}\right) dx^2 \\ \pm &\left(\frac{1}{\ln(k_2|y|)} - \frac{1}{\ln(k_1|x|)}\right)\left(\frac{1}{\ln(k_2|y|)} - \frac{1}{\ln(k_3|z|)}\right) dy^2 \\ \pm &\left(\frac{1}{\ln(k_3|z|)} - \frac{1}{\ln(k_1|x|)}\right)\left(\frac{1}{\ln(k_3|z|)} - \frac{1}{\ln(k_2|y|)}\right) dz^2 \end{aligned}$$

($k_i \in \mathbb{R}$, $k_i > 0$) whose projective algebra is generated by

$$x\partial_x + y\partial_y + z\partial_z\,.$$

8. *The metric*

$$\begin{aligned} g = \pm &\left(\tan\left(\ln(k_1|x|^\beta)\right) - \tan\left(\ln(k_2|y|^\beta)\right)\right) \\ &\cdot \left(\tan\left(\ln(k_1|x|^\beta)\right) - \tan\left(\ln(k_3|z|^\beta)\right)\right)\, dx^2 \\ \pm &\left(\tan\left(\ln(k_2|y|^\beta)\right) - \tan\left(\ln(k_1|x|^\beta)\right)\right) \\ &\cdot \left(\tan\left(\ln(k_2|y|^\beta)\right) - \tan\left(\ln(k_3|z|^\beta)\right)\right)\, dy^2 \\ \pm &\left(\tan\left(\ln(k_3|z|^\beta)\right) - \tan\left(\ln(k_1|x|^\beta)\right)\right) \\ &\cdot \left(\tan\left(\ln(k_3|z|^\beta)\right) - \tan\left(\ln(k_2|y|^\beta)\right)\right)\, dz^2 \end{aligned}$$

($k_i \in \mathbb{R}$, $k_i > 0$, $\beta \neq 0$) whose projective algebra is generated by

$$x\partial_x + y\partial_y + z\partial_z\,.$$

9. The metric

$$g = \pm \left(\tan\left(\tfrac{x}{k_1}\right) - \tan\left(\tfrac{y}{k_2}\right)\right)\left(\tan\left(\tfrac{x}{k_1}\right) - \tan\left(\tfrac{z}{k_3}\right)\right)\, dx^2$$
$$\pm \left(\tan\left(\tfrac{y}{k_2}\right) - \tan\left(\tfrac{x}{k_1}\right)\right)\left(\tan\left(\tfrac{y}{k_2}\right) - \tan\left(\tfrac{z}{k_3}\right)\right)\, dy^2$$
$$\pm \left(\tan\left(\tfrac{z}{k_3}\right) - \tan\left(\tfrac{x}{k_1}\right)\right)\left(\tan\left(\tfrac{z}{k_3}\right) - \tan\left(\tfrac{y}{k_2}\right)\right)\, dz^2$$

($k_i \in \mathbb{R}$, $k_i \neq 0$) whose projective algebra is generated by

$$k_1\partial_x + k_2\partial_y + k_3\partial_z\,.$$

10. The metric

$$g = \pm \left(\tanh\left(\tfrac{x}{k_1}\right) - \tanh\left(\tfrac{y}{k_2}\right)\right)\left(\tanh\left(\tfrac{x}{k_1}\right) - \tanh\left(\tfrac{z}{k_3}\right)\right)\, dx^2$$
$$\pm \left(\tanh\left(\tfrac{y}{k_2}\right) - \tanh\left(\tfrac{x}{k_1}\right)\right)\left(\tanh\left(\tfrac{y}{k_2}\right) - \tanh\left(\tfrac{z}{k_3}\right)\right)\, dy^2$$
$$\pm \left(\tanh\left(\tfrac{z}{k_3}\right) - \tanh\left(\tfrac{x}{k_1}\right)\right)\left(\tanh\left(\tfrac{z}{k_3}\right) - \tanh\left(\tfrac{y}{k_2}\right)\right)\, dz^2$$

($k_i \in \mathbb{R}$, $k_i \neq 0$) whose projective algebra is generated by

$$k_1\partial_x + k_2\partial_y + k_3\partial_z\,.$$

Proof. In view of Remark 7, consider the solutions in Lemmas 10 and 11. Consider the first case of Lemma 11. Without changing the metric, we translate the solutions $X_i = \frac{c}{a-d} + k_i|x^i|^{\frac{a-d}{a+d}}$ by the common constant $\frac{c}{a-d}$ and rename $h = \frac{a-d}{a+d}$. We thus arrive at the claim for $h \neq 0$ (for $h = 0$ the metric would have constant curvature). Note that the projective vector field, up to multiplication by a constant, is given by the components $v^i = x^i$. Consider the second case in Lemma 11, $X_i = -\frac{c}{2a}\ln(|x^i|) + k_i$. By an affine transformation of the coordinates of the form $x^i \to \frac{2a}{c}x^i$, and using the standard identities for logarithms, we arrive at the claim. The remaining cases follow similarly, from the remaining cases of Lemma 11 and the solutions from Lemma 10. Recall that no two eigenvalues of the Benenti tensor $L(g, \bar{g})$ are allowed to coincide. □

The previous three propositions answer the question which Levi-Civita metrics of type [1-1-1] admit a projective symmetry algebra of at least dimension 1. The following theorem summarises these results.

Theorem 2. *Consider a metric (42) of non-constant curvature with a non-vanishing projective vector field. Then it is a metric in Proposition 9, 10 or 11. In particular:*

a) If the projective algebra is exactly 2-dimensional, it is as in Proposition 9, or as in 1, 3 or 4 of Proposition 10, or as in 2 or 3 of Proposition 11.

b) If the projective algebra is exactly 3-dimensional, it is as in 2 of Proposition 10.

If the projective algebra is at least 4*-dimensional, the metric is already of constant curvature.*

6. Levi-Civita metrics of type [2-1]

In order to conclude our study of 3-dimensional Levi-Civita metrics, we now focus on Levi-Civita metrics of type [2-1] of non-constant curvature that admit a projective vector field v. We consider solely the positive branch of (44a),

$$g = \zeta(z)(h + dz^2)\,, \quad \zeta(z) = Z(z) - \rho\,, \tag{52}$$

because $\zeta(z)(h - dz^2) = -\zeta(z)(-h + dz^2)$. For the projective vector field v, in view of Example 3, we have the following expression:

$$v = u + \alpha(z)\partial_z\,, \tag{53}$$

with

$$u = v^1(x,y)\partial_x + v^2(x,y)\partial_y\,. \tag{54}$$

Our aim is analogous to that in Section 5, where we have considered Levi-Civita metrics of type [1-1-1]: We shall determine the functions $\zeta(z)$ such that (52) admits non-vanishing projective symmetries.

Remark 9. The coordinate transformations that preserve the form of (52) are ($k_i \in \mathbb{R}$)

$$(x_{\text{new}}, y_{\text{new}}, z_{\text{new}}) = \big(x_{\text{new}}(x_{\text{old}}, y_{\text{old}}), y_{\text{new}}(x_{\text{old}}, y_{\text{old}}), z_{\text{new}}(z_{\text{old}}) = k_1 z_{\text{old}} + k_2\big)\,. \tag{55}$$

In addition we can simultaneously transform

$$\rho_{\text{new}} = \rho_{\text{old}} + t\,, \quad Z_{\text{new}} = Z_{\text{old}} + t$$

for constant $t \in \mathbb{R}$. This transformation does not change (52).

We now describe first all those metrics (52) that are of constant curvature and thus admit a 15-dimensional algebra of projective symmetries. For metrics (52) of non-constant curvature, we are then going to show, in Section 6.1, that projective vector fields arise from homothetic vector fields of h. We begin by showing that if (52) is of constant curvature, then also h is.

Lemma 12. *Let g be a metric (52) of constant scalar curvature. Then h has constant curvature.*

Proof. Let us denote the scalar curvature of g and h by, respectively, R_g and R_h. It is easily verified that

$$R_g(x,y,z) = \frac{R_h(x,y)}{\zeta(z)} + \frac{1}{2\zeta(z)^3}\left(3\left(\frac{d\zeta(z)}{dz}\right)^2 - 4\,\zeta(z)\frac{d^2\zeta(z)}{dz^2}\right)$$

In view of the above formula, since R_g is constant, we have that

$$\frac{\partial R_g}{\partial x} = \frac{\partial R_g}{\partial y} = 0 = \frac{\partial R_h}{\partial x} = \frac{\partial R_h}{\partial y}\,.$$

Thus, R_h is a constant. □

The following lemma allows us simplify (52) if h has constant curvature.

Lemma 13. *Let g be a metric (52) such that h has constant curvature equal to $\kappa \neq 0$. Then such a metric is isometric to*

$$\tilde{\zeta}(\tilde{z})(\tilde{h} + d\tilde{z}^2)\,, \quad \tilde{\zeta}(\tilde{z}) := \frac{1}{|\kappa|}\,\zeta\left(\frac{\tilde{z}}{\sqrt{|\kappa|}}\right) \tag{56}$$

with $\tilde{h}$ a 2-dimensional metric with constant curvature equal to $\frac{\kappa}{|\kappa|}$.

Proof. The metric h is locally isometric to $\frac{\tilde{h}}{|\kappa|}$ with $\tilde{h}$ having curvature equal to $\frac{\kappa}{|\kappa|}$. Then, by considering the transformation $z = \frac{\tilde{z}}{\sqrt{|\kappa|}}$, metric (52) assumes the form (56). □

We are now able to formulate precisely which functions ζ lead to constant curvature metrics.

Lemma 14. *Consider the metric (52). If it has constant curvature, then by a change of coordinate of type (55), locally we achieve one of the following:*

1. *g has zero curvature.*

   (a) $g = \varepsilon_0\,(\varepsilon_1 dx^2 + \varepsilon_2 dy^2 + dz^2)$.

   (b) $g = \frac{\varepsilon_0 k}{z^2}\,(\varepsilon_1\, dx^2 + \varepsilon_2\, dy^2 + dz^2)$.

2. *g has positive constant curvature.*

   (a) $g = \frac{k}{\cosh^2(z)}\,(dx^2 + \varepsilon_2 \sin^2(x) dy^2 + dz^2)$.

   (b) $g = -\frac{k}{\cos^2(z)}\,(-dx^2 + \varepsilon_2 \sin^2(x) dy^2 + dz^2)$.

   (c) $g = \frac{k}{\cosh^2(z)}\,(-dx^2 + \varepsilon_2 \sinh^2(x) dy^2 + dz^2)$.

   (d) $g = -\frac{k}{\cos^2(z)}\,(dx^2 + \varepsilon_2 \sinh^2(x) dy^2 + dz^2)$.

3. *g has negative constant curvature.*

   (a) $g = -\frac{k}{\cosh^2(z)}\,(dx^2 + \varepsilon_2 \sin^2(x) dy^2 + dz^2)$.

   (b) $g = \frac{k}{\cos^2(z)}\,(-dx^2 + \varepsilon_2 \sin^2(x) dy^2 + dz^2)$.

   (c) $g = -\frac{k}{\cosh^2(z)}\,(-dx^2 + \varepsilon_2 \sinh^2(x) dy^2 + dz^2)$.

   (d) $g = \frac{k}{\cos^2(z)}\,(dx^2 + \varepsilon_2 \sinh^2(x) dy^2 + dz^2)$.

where $\varepsilon_i \in \{\pm 1\}$ *and* $k \in \mathbb{R}$, $k > 0$.

Proof. We give the explicit proof for the case when h is Riemannian. For the other signatures the proof is analogous. So assume that h has signature $(++)$. Due to Lemma 12, we have that h is of constant curvature. In view of Lemma 13, w.l.o.g., we can suppose such curvature equal to 0, 1 or -1.

• If h has curvature equal to 0, by a change of coordinates (x, y) we have

$$g = \zeta(z)(dx^2 + dy^2 + dz^2)\,. \tag{57}$$

Now, let us suppose that (57) has constant curvature equal to κ. Thus (by cutting with the planes $\langle \partial_x\,, \partial_y \rangle$ and $\langle \partial_x\,, \partial_z \rangle$) we have that

$$-\frac{1}{4\zeta(z)^3}\left(\frac{d\zeta(z)}{dz}\right)^2 = \frac{\left(\frac{d\zeta(z)}{dz}\right)^2 - \zeta(z)\,\frac{d^2\zeta(z)}{dz^2}}{2\zeta(z)^3} = \kappa\,.$$

By solving the first equation we obtain

$$\zeta(z) = \frac{\varepsilon}{(c_1 z + c_2)^2}\,, \quad c_i \in \mathbb{R}\,,\ \varepsilon \in \{\pm 1\}\,. \tag{58}$$

On the other hand, we observe that a metric (57) with $\zeta(z)$ given by (58) has constant curvature equal to $\pm c_1^2$. If $c_1 = 0$ then we have $\zeta \in \mathbb{R}$ and we easily get the form 1a of the claim by rescaling the coordinates. If however $c_1 \neq 0$, then we achieve the form 1b.

• If h has curvature equal to 1, then by a change of coordinates (x, y) we have

$$g = \zeta(z)(dx^2 + \sin(x)^2 dy^2 + dz^2) \tag{59}$$

Similarly as in the above case, if the metric (59) has constant curvature κ then

$$-\frac{\left(\frac{d\zeta(z)}{dz}\right)^2 - 4\zeta(z)^2}{4\zeta(z)^3} = \frac{\left(\frac{d\zeta(z)}{dz}\right)^2 - \zeta(z)\,\frac{d^2\zeta(z)}{dz^2}}{2\zeta(z)^3} = \kappa$$

and by solving the first equation we obtain

$$\zeta(z) = \frac{\varepsilon}{(c_1 \cosh(z) + c_2 \sinh(z))^2}\,, \quad c_i \in \mathbb{R}\,,\ \varepsilon \in \{\pm 1\}\,.$$

The resulting metric (59) has constant curvature equal to $c_1^2 - c_2^2$. Defining $c = \sqrt{|c_1^2 - c_2^2|}$ and $\tanh(\gamma) = \frac{c_2}{c_1}$, we arrive at

$$\zeta(z) = \frac{\varepsilon}{c \cosh(z + \gamma)^2}\,,$$

and after a translation of z by γ we arrive at the claim.

• The case when h has curvature equal to -1 is very similar to the above case: we eventually arrive to the form 3d of the claim. We omit the details. □

6.1. A splitting-gluing result for projective vector fields of [2-1]-type Levi-Civita metrics

Let us now restrict to Levi-Civita metrics of [2-1] type of non-constant curvature that admit projective symmetries. Recall that a projective vector field v of non-constant curvature metrics (52) is necessarily of the form (53). It is easy to see that u is a projective vector field for h. Indeed, by computing the geodesic equations (17), where the f_{ij}^k's are given by (9), with Γ_{ij}^k the Christoffel symbols of the Levi-Civita connection of g, we realise that the first equation of system (17) coincides with the 2-dimensional projective connection associated to the metric h. The following proposition refines this statement further. In preparation of it, recall Lemma 6, which provides the following equations:

$$0 = b\rho^2 - (d-a)\rho - c \tag{60a}$$

$$\alpha \frac{dZ}{dz} = -bZ^2 + (d-a)Z + c \tag{60b}$$

Proposition 12.
(i) Let g be a 3-dimensional Levi-Civita metric (52) of non-constant curvature with projective vector field (53). Then (54) is homothetic for the metric h, with

$$\mathcal{L}_u h = -Ch \tag{61}$$

where

$$C = 2b\rho + 3a + d. \tag{62}$$

(ii) Conversely, let u be a homothetic vector field for a 2-dimensional metric h, such that (61) holds. Let g be the metric (52). Then the vector field (53) is a projective vector field for g if and only if

$$-2\alpha'(z) = C + b\zeta(z) \tag{63a}$$

$$\alpha(z)\zeta'(z) = -b\zeta^2(z) + (2B - C)\,\zeta(z) \tag{63b}$$

where

$$B = a + d \tag{64}$$

and a, b, d are as in (23).

Proof. We begin with part (i) by computing the projections of (39a) onto the 2-dimensional component M_2 and the 1-dimensional component M_1, respectively, see the discussion after (44). Since v is of the form (53), we infer,

$$\mathcal{L}_v g = \mathcal{L}_v(\zeta(h + dz^2)) = \zeta'\alpha\, h + \zeta\mathcal{L}_u h + \alpha\zeta' dz^2 + 2\zeta\alpha' dz^2\,. \tag{65}$$

Substituting this in (39a), and inserting $n = 3$ as well as $L = \rho(\partial_x \otimes dx + \partial_y \otimes dy) + (\zeta + \rho)\partial_z \otimes dz$, we arrive at

$$\mathcal{L}_u h = \left(-\frac{\zeta'}{\zeta}\,\alpha - 4a - b(4\rho + \zeta)\right) h \tag{66}$$

$$-2\alpha' = \frac{\zeta'}{\zeta}\,\alpha + 4a + 2b\rho\zeta + 4b\rho\,. \tag{67}$$

Combining the equations (60), we obtain (63b). Reinserting (63b) into (66), we find (61).

For part (ii) consider (65) and insert (61). Then (63a) holds due to (60b). Reinserting into (67), we find (63b). We compute $\mathcal{L}_v g = \mathcal{L}_v(\zeta(h + dz^2))$, but now for a given h with $\mathcal{L}_u h = -Ch$. We substitute this into (39a) and take the projection onto the 2-dimensional and the 1-dimensional component. This yields (63a). Equation (63b) is (60) rewritten in terms of ζ. □

Remark 10. In the hypotheses of Proposition 12, vector field $u = v^1(x, y)\partial_x + v^2(x, y)\partial_y$ is a homothety for the metric h. In the case when h is a Riemannian metric this implies that u is a holomorphic vector field in the following sense: by working in conformal coordinates (x, y), function $v^1(x, y)+iv^2(x, y)$ turns out to be holomorphic. In the case when h is of Lorentzian signature, u turns out to be a para-holomorphic vector field, i.e., by working in null coordinates (x, y) (i.e., $h = e^{f(x,y)}dxdy$ for some function f), $v^1 = v^1(x)$ and $v^2 = v^2(y)$, see [28] and [29] for more details.

Proposition 13. *Let $h = h_{11}dx^2 + 2h_{12}dxdy + h_{22}dy^2$, $h_{ij} = h_{ij}(x, y)$, be a 2-dimensional metric with homothetic vector field u that is nowhere vanishing such that $\mathcal{L}_u h = -Ch$, $C \in \mathbb{R}$. Then the vector field (53) is projective for the metric $g = \zeta(z)\,(h + dz^2)$ if and only if ζ and α satisfy the system*

$$\zeta(\alpha\zeta'' - \alpha'\zeta' - C\zeta') - \alpha{\zeta'}^2 = 0 \tag{68a}$$

$$\zeta(-2\alpha''\zeta + \alpha\zeta'' + \alpha'\zeta') - \alpha{\zeta'}^2 = 0 \tag{68b}$$

with ζ being a nowhere vanishing solution.

Proof. In an appropriate system of coordinates (x, y),

$$u = \partial_x\,, \quad h = e^{-Cx}\begin{pmatrix} E(y) & F(y) \\ F(y) & G(y) \end{pmatrix}, \quad C \in \mathbb{R}\,. \tag{69}$$

A direct computation shows that the 18 PDEs forming System (20) reduce to only 2 independent conditions, namely (68). □

Remark 11. For any h of the form (69), there exist non-zero functions $\zeta(z)$ and $\alpha(z)$ such that the vector field (53), with $\mathcal{L}_u h = -Ch$, is a projective vector field for the metric (52). For instance, if $C \neq 0$, the functions

$$\zeta = Cz\,, \quad \alpha = -\frac{1}{2}Cz$$

solve system (68). For the case $C = 0$, the functions $\zeta = \frac{1}{z^2}$, $\alpha = \frac{1}{z}$ solve system (68), for example.

Remark 12. Note that, in Proposition 13, Equations (68) are consequences of (63a) and (63b). Indeed (63a) gives

$$b = -\frac{2\alpha' + C}{\zeta}\,. \tag{70}$$

Then differentiate (70) once w.r.t. z to obtain the equation

$$2\zeta'\alpha' - 2\zeta\alpha'' + C\zeta' = 0\,. \tag{71}$$

Likewise, solve (63b) for $2B - C$, then differentiate w.r.t. z, to obtain:

$$\zeta(\alpha'\zeta' + \alpha\zeta'') + b\zeta'\zeta^2 = \alpha\zeta'^2\,. \tag{72}$$

Inserting (70) into the Equation (72), we obtain (68a). Equation (68b) then follows from (70),(71) and (72). Conversely, note that from (68) we obtain (71), which is equivalent to (70) since b an arbitrary constant.

The following lemma covers the most basic situation, namely constant $\zeta(z)$.

Corollary 3. *Let h be a 2-dimensional metric. Let $g = k\,(h + dz^2)$ for some non-zero constant k. If g is of non-constant curvature, the projective symmetries of g are the vector fields*

$$v = u + (k_1 z + k_0)\partial_z\,,$$

where u is a homothetic vector field of h, and where $k_0, k_1 \in \mathbb{R}$.

Proof. Substituting $\zeta = k$ into the system (68) of Proposition 13, we arrive at $-2k^2\alpha''(z) = 0$, i.e. $\alpha''(z) = 0$. This proves the claim. ☐

The following example illustrates the situation.

Example 4. Take the metric $h = dx^2 + \sin^2(x)dy^2$, which has the homothetic vector fields (in fact, these are all Killing vector fields)

$$u = k_0\partial_y + k_1\,(-\cos(y)\partial_x + \cot(x)\sin(y)\partial_y) + k_2\,(\sin(y)\partial_x + \cot(x)\cos(y)\partial_y)\,.$$

The metric $g = dx^2 + \sin^2(x)dy^2 + dz^2$, on the other hand, admits the projective symmetry algebra

$$\begin{aligned} u = k_0\partial_y + k_1\,&(-\cos(y)\partial_x + \cot(x)\sin(y)\partial_y) \\ &+ k_2\,(\sin(y)\partial_x + \cot(x)\cos(y)\partial_y) + k_3 z\partial_z + k_4\partial_z\,. \end{aligned}$$

Remark 13. A 3-dimensional Riemannian metric, which is not of constant curvature, has a projective algebra of dimension ≤ 5 [30]. The sub-maximal dimension 5 is realised in Example 4, and it is realised also for $g = dx^2 + \sinh^2(x)dy^2 + dz^2$.

Note that for a 3-dimensional metric of Lorentz signature the sub-maximal dimension of the projective algebra is 6 instead. According to [30], see also the references therein, this is realised for the metrics ($k \neq 0$)

$$\begin{aligned}
g_1 &= kdx^2 + 2(2-c)e^{cx}dxdy + e^{2x}dz^2\,, \quad c \notin \{1,2\} \\
g_2 &= kdx^2 + e^{2x}(2dxdy - dz^2) \\
g_3 &= kdx^2 + e^{x\sqrt{4-\omega^2}}\left(2dxdy - \tfrac{4}{\omega^2}\cos^2(\tfrac{\omega x}{2})dz^2\right)\,, \quad \omega \neq 0
\end{aligned}$$

None of these metrics is of Levi-Civita type.

The following example is based on a 2-dimensional metric of non-constant curvature.

Example 5. Consider the metric $h = e^{(\beta+2)x}dx^2 + e^{\beta x}dy^2$, which has the homothetic vector fields

$$u = k_0\,(\partial_x + y\partial_y) + k_1\,\partial_y\,.$$

The metric $g = h + dz^2$, on the other hand, admits the projective symmetry algebra

$$u = k_0\,(\partial_x + y\partial_y) + k_1\,\partial_y + k_2\,z\partial_z + k_3\,\partial_z\,.$$

6.2. Characterisation of $\zeta(z)$ for metrics (52) with projective vector fields

The purpose of the current section is to find the functions $\zeta(z)$ that can appear in (52), assuming the metric is of non-constant curvature and admits non-trivial projective vector fields. The main outcome of this section is an ODE for $\zeta(z)$ (see Equation (73) below), which holds under mild hypotheses. We start by considering the very special case when the polynomial S_A (see (27)) vanishes.

Lemma 15. *Let g be the metric (52) with $\zeta'(z) \neq 0$ and of non-constant curvature. Assume v is a projective vector field of g.*

1. *If $\mathrm{S}_A = 0$ (recall Definition 6), then $A = 0$.*

2. *If* $\mathrm{S}_A \neq 0$, *then* $\zeta(z)$ *satisfies the ODE*

$$\zeta'' = \frac{3b\zeta + C - 4B}{2\zeta\,(b\zeta + C - 2B)}\zeta'^2\,, \tag{73}$$

where C *is as in* (62) *and* B *as in* (64).

Proof. For the first part of the claim, $\mathrm{S}_A = 0$ implies that $b = 0$, $a = d$ and $c = 0$. Because of Equation (60), $\alpha(z) = 0$. Using (63a) we conclude $C = 0$, and thus $d = -3a$ because $b = 0$. Combining this with $a = d$, it follows that $a = d = 0$ and thus $A = 0$.

For the second part of the claim, since $\zeta'(z) \neq 0$, we can solve (63b) for $\alpha(z)$. Resubstituting this expression into (63a), and reorganising the terms, we obtain(73). Note that the denominator does not identically vanish because of (60b), keeping in mind that $\zeta(z)$ cannot be zero. □

We conclude the section with the following lemma.

Lemma 16. *Let* g *be a [2-1] Levi-Civita metric* (52) *that is not of constant curvature. Then* $v = \alpha(z)\partial_z$ *is a projective vector field for* g *if* $\alpha(z)$ *and* $\zeta(z)$ *satisfy the set of equations* (63a) *and* (63b).

Proof. Note that if $u = 0$ holds in (53), then (61) holds with $C = 0$. Consequently, (63a) and (63b) become

$$\alpha' = -\frac{b}{2}\,\zeta \quad \text{and, respectively,} \quad \alpha\zeta' = -b\zeta^2 + 2B\zeta\,.$$

These two equations are thus equivalent to (39a) given the conditions $u = 0$ and $C = 0$. □

6.3. Levi-Civita metrics of type [2-1] with projective symmetries

We now aim to describe all Levi-Civita metrics of type [2-1] that have projective symmetries and do not fall into the situation of Corollary 3, i.e. such that $\zeta'(z) \neq 0$.

Lemma 17. *Let* h *be a 2-dimensional metric and let* $g = \zeta(z)(h + dz^2)$ *where* $\zeta' \neq 0$. *Assume that the vector field* $v = \alpha(z)\partial_z$ *is a non-zero projective vector field of* g. *Then one of the following is realised after a coordinate transformation of the type* (55).

1. $\zeta(z) = \varepsilon e^{\beta z}$ *and* $\alpha = k$.

2. $\zeta(z) = \frac{\beta}{z^2}$ *and* $\alpha(z) = \frac{k}{z}$

3. $\zeta(z) = \beta(1 + \tan^2(\xi z))$ *and* $\alpha = k\, \tan(\xi z)$

4. $\zeta(z) = \beta(1 - \tanh^2(\xi z))$ *and* $\alpha = k\, \tanh(\xi z)$

where $\xi, \beta \neq 0$ *and* $k \in \mathbb{R}$ *and* $\varepsilon = \pm 1$*, at least after a constant translation of* z*.*

Proof. We integrate (63a) and (63b) under the hypotheses of the statement. Since $u = 0$ in (53), we have $C = 0$ and thus

$$\alpha'(z) = \mu\zeta \qquad \text{and} \qquad \alpha(z)\zeta'(z) = 2\mu\zeta^2(z) + \eta\zeta$$

with $\mu = -\frac{b}{2}$ and $\eta = 2B - C$.
First assume $\mu = 0$ (i.e., $b = 0$). Then $\alpha(z) = k \neq 0$ and so $\zeta' = \frac{\eta}{k}\zeta$. We infer $\eta \neq 0$ and $\zeta(z) = e^{\beta z}$ with $\beta = \frac{\eta}{k}$. This is case 1 of the claim. Next assume $\mu \neq 0$ (i.e., $b \neq 0$). Then, substituting $\zeta(z) = \mu^{-1}\,\alpha'(z)$,

$$\alpha(z)\alpha''(z) = 2\,\alpha'(z)^2 + \eta\alpha'(z)\,.$$

If $\eta = 0$, we have $\alpha(z) = \frac{k}{z}$ and obtain $\zeta \propto \frac{1}{z^2}$. This is case 2 of the claim. If $\eta \neq 0$, let $f(z) = \frac{\alpha(z)}{\eta}$ and obtain

$$f(z)f''(z) = 2\,f'(z)^2 + f'(z)\,.$$

which has the two solutions

$$f(z) = \frac{\tan\left(\frac{c_1 z}{2}\right)}{c_1} \qquad \text{and} \qquad f(z) = \frac{\tanh\left(\frac{c_1 z}{2}\right)}{c_1}\,.$$

These solutions yield the remaining two cases of the claim. We find

$$\zeta \propto 1+\tan(\xi z)^2\,, \quad \alpha \propto \tan(\xi z) \qquad \text{or} \qquad \zeta \propto 1-\tanh(\xi z)^2\,, \quad \alpha \propto \tanh(\xi z)$$

where $\xi \neq 0$. □

Since in Lemma 17 the metric h is fixed, the allowed transformation (55) are those with $k_1 \in \{\mp 1\}$.

Lemma 18. *Let* $g = \zeta(z)(h + dz^2)$ *where* $\zeta' \neq 0$*. Assume that* h *admits a homothetic algebra of dimension* ≤ 1*. Assume, too, that the projective algebra of* g *is at least 2-dimensional,* $\dim \mathfrak{p}(g) \geq 2$*. Then* h *admits a 1-dimensional algebra of homothetic vector fields, generated by a non-vanishing vector field* u*, and after a translation of* z *we obtain*

1. *if* $\zeta(z) = \varepsilon e^{\beta z}$ *and* u *is Killing, then the projective symmetries of* g *are* $v = k_0 u + k_1 \partial_z$.

2. *if* $\zeta(z) = \frac{\beta}{z^2}$ *and* u *is properly homothetic with* $\mathcal{L}_u h = -Ch$, $C \neq 0$, *then the projective symmetries of* g *are* $v = k_0 \left(u - \frac{C}{2} z\partial_z\right) + \frac{k_1}{z}\partial_z$.

3. *if* $\zeta(z) = \beta(1 + \tan^2(\xi z))$ *and* u *is Killing, then the projective symmetries of* g *are* $v = k_0 u + k_1 \tan(\xi z)$, $\xi \in \mathbb{R} \setminus \{0\}$.

4. *if* $\zeta(z) = \beta(1 - \tanh^2(\xi z))$ *and* u *is Killing, then the projective symmetries of* g *are* $v = k_0 u + k_1 \tanh(\xi z)$, $\xi \in \mathbb{R} \setminus \{0\}$.

for constants $\beta \neq 0$ *and* $\varepsilon \in \{\pm 1\}$ *and the constant* C *with* $\mathcal{L}_u h = -Ch$.

Proof. Let h have no projective symmetries, except for the trivial $u = 0$. Then g has a non-zero projective vector field only if ζ and α are as in Lemma 17. In these cases, the projective algebra is 1-dimensional. Otherwise, it is 0-dimensional. Thus let h admit a 1-dimensional homothetic algebra, generated by a vector field u with $\mathcal{L}_u h = -Ch$. We prove the assertion by contradiction. Assume that there is $\zeta(z) \neq 0$ that admits a higher dimensional algebra of projective symmetries. We conclude that there are two independent solutions $\alpha_1(z), \alpha_2(z)$, satisfying (68) for the same $\zeta(z)$ and λ. From the linearity of (68) w.r.t. α it follows that the function $\alpha(z) := \alpha_2(z) - \alpha_1(z) \neq 0$ satisfies

$$(\alpha\zeta'' - \alpha'\zeta')\,\zeta = \alpha\zeta'^2$$
$$(-2\zeta\alpha'' + \alpha\zeta'' + \alpha'\zeta')\,\zeta = -\alpha\zeta'^2$$

By Remark 12 it follows that

$$b\zeta(z) = -2\alpha'(z) \qquad \text{and} \qquad \alpha(z)\zeta'(z) = -b\zeta^2(z) + 2\eta\zeta(z)\,.$$

This is the system solved in Lemma 17, and so $\zeta(z)$ has to be one of the solutions in the list. In order to have a second projective vector field, independent of $v = u + \alpha(z)\partial_z$, we need to be able to find another projective vector field of the form $\bar{v} = u + \bar{\alpha}(z)\partial_z$, linearly independent of v. Indeed, this is not possible for all of the metrics of Lemma 17, yet for some examples it is. In order to find these metrics g, i.e. the functions ζ, such that desired $\bar{v}$ exists, we solve Equations (63a) and (63b), which in terms of $\bar{\alpha}(z)$ read

$$-2\bar{\alpha}' = C + b\zeta \tag{74a}$$
$$\bar{\alpha}\zeta' = -b\zeta^2 + \eta\zeta \tag{74b}$$

Note that in these equations $\zeta(z)$ is given explicitly, possibly involving parameters. We seek a solution $\bar{\alpha}$. Since $\zeta'(z) \neq 0$, we have $\bar{\alpha} = \frac{-b\zeta^2+\eta\zeta}{\zeta}$ due to (74b). Resubstituting into Equation (74a), we obtain a condition on C, b and η. Explicitly, for the cases of Lemma 17 we find

1. If $\zeta = \varepsilon e^{\beta z}$, we solve (74b) for $\bar{\alpha} = \frac{-b\varepsilon}{\beta} e^{\beta z} + \frac{\eta}{\beta}$. Next, from (74a), we obtain $\varepsilon b e^{\beta z} - C = 0$. We conclude that $C = 0$, and thus u is a Killing vector field. Moreover, we conclude $b = 0$, i.e. v is homothetic. Finally, we compute $\bar{\alpha} = \frac{\eta}{\beta}$.

2. If $\zeta = \frac{\beta}{z^2}$, we find $\bar{\alpha} = \frac{b\beta}{2z} - 2\eta z$ and the condition $C = 4\eta$. Thus, $\bar{\alpha} = \frac{b\beta}{2z} - \frac{Cz}{2}$.

3. If $\zeta = \beta(1 + \tan^2(\xi z))$, we find analogously that $\bar{\alpha} = -\frac{1}{2} \frac{b\beta \tan(\xi z)^2 + b\beta - \eta}{\xi \tan(\xi z)}$ and the condition $(b\beta + C - \eta) + \frac{b\beta - \eta}{\tan(\xi z)^2} = 0$ and so $\eta = C + b\beta$ and $\eta = b\beta$, implying $C = 0$. We conclude $\bar{\alpha} = -\frac{1}{2} \frac{b\beta}{\xi} \tan(\xi z)$.

4. The remaining case follows analogously.

□

We are now able to formulate an explicit description of Levi-Civita metrics of [2-1] type that admit projective vector fields. We begin with the case when the projective algebra is 1-dimensional.

Proposition 14. *Let $g = \zeta(z)(h + dz^2)$ be a Levi-Civita metric (52) where $\zeta(z)$ is not a constant and g is not of constant curvature. If g admits a projective algebra of dimension exactly 1, then around almost every point and locally up to a change of coordinates g falls into one of the following cases.*

(1) *$g = \zeta(z)\,(h + dz^2)$ where h has an exactly 1-dimensional Killing algebra generated by the vector field u and where*

$$\zeta(z) \notin \left\{ \eta e^{\beta(z+z_0)}, \frac{\eta}{(z+z_0)^2}, \eta(1 + \tan^2(kz + z_0)), \right.$$
$$\left. \eta(1 - \tanh^2(kz + z_0)) : \eta, \beta, k, z_0 \in \mathbb{R} \right\}$$

Then $\mathfrak{p}(g) = \langle u \rangle$ is Killing.

(2) $g = \pm e^{\beta z}\,(h + dz^2)$, $\beta \neq 0$, *where* h *has no Killing vector field. Then* $\mathfrak{p}(g) = \langle \partial_z \rangle$ *is homothetic.*

(3) $g = \frac{\eta}{z^2}\,(h + dz^2)$, $\eta \neq 0$, *where* h *has no homothetic vector field. Then* $\mathfrak{p}(g) = \left\langle \frac{1}{z}\partial_z \right\rangle$ *is essential.*

(4) $g = \eta(1 + \tan^2(z))\,(h + dz^2)$, $\eta \neq 0$, *where* h *has no Killing vector field. Then* $\mathfrak{p}(g) = \langle \tan(z)\partial_z \rangle$ *is essential.*

(5) $g = \eta(1 - \tanh^2(z))\,(h + dz^2)$, $\eta \neq 0$, *where* h *has no Killing vector field.*
Then $\mathfrak{p}(g) = \langle \tanh(z)\partial_z \rangle$ *is essential*

Proof. The claim follows directly from Lemmas 17 and 18. □

For Levi-Civita metrics of type [2-1] that admit a projective algebra of dimension at least 2, we then arrive at the following theorem.

Theorem 3. *Let* $g = \zeta(z)(h + dz^2)$ *be a Levi-Civita metric* (52) *where* $\zeta(z)$ *is not a constant and* g *is not of constant curvature. If* g *admits a projective algebra of dimension at least* 2*, then around almost every point and locally up to a change of coordinates* g *falls into one of the following cases.*

a) The metric h *has constant curvature.*

b) The homothetic algebra of metric h *is exactly of dimension 1. Then there exist local coordinates such that*

(1) $g = \varepsilon e^{\beta z}\,(h + dz^2)$, $\beta \neq 0$, *with projective vector fields*

$$v = k_0\partial_x + k_1\partial_z\,,$$

$k_i \in \mathbb{R}$, *where* ∂_x *is a Killing vector field of* h.

(2) $g = \frac{\eta}{z^2}\,(h + dz^2)$ *with projective vector fields*

$$v = k_0\left(\partial_x - \frac{C}{2}\,z\partial_z\right) + \frac{k_1}{z}\partial_z\,,$$

$k_i \in \mathbb{R}$, *where* ∂_x *is a properly homothetic vector field of* h *with* $\mathcal{L}_{\partial_x} h = h$.

(3) $g = \eta\,(1+\tan^2(z))\,(h+dz^2)$ *with projective vector fields*

$$v = k_0\partial_x + k_1\tan(z)\partial_z\,,$$

$k_i \in \mathbb{R}$, *where* ∂_x *is a Killing vector field of* h.

(4) $g = \eta\,(1-\tanh^2(z))\,(h+dz^2)$ *with projective vector fields*

$$v = k_0\partial_x + k_1\tanh(z)\partial_z\,,$$

$k_i \in \mathbb{R}$, *where* ∂_x *is a Killing vector field of* h.

c) *The homothetic algebra of metric* h *is exactly of dimension 2. Locally there exist coordinates such that* h *can be brought into the form*

$$h = \varepsilon_1 e^{(\beta+2)x}dx^2 + \varepsilon_2 e^{\beta x}dy^2 \tag{75}$$

with $\beta \in \mathbb{R}\setminus\{-2,0\}$ *and* $\varepsilon_i \in \{\pm 1\}$. *Then there exist local coordinates such that* g *assumes one of the following forms.*

(1) *For* $\eta \in \mathbb{R}, \eta \neq 0$, *the metric* $g = \frac{\eta}{z^2}\,(h+dz^2)$, *has the projective vector fields*

$$v = k_0\partial_y + k_1\,(2\partial_x + 2y\partial_y + (\beta+2)z\partial_z) + \tfrac{k_2}{z}\partial_z\,,$$

where $k_i \in \mathbb{R}$.

(2) *The metric* $g = \eta e^z(h+dz^2)$, $\eta \in \mathbb{R}, \eta \neq 0$, *has the projective vector fields*

$$v = k_0\partial_y + k_2\partial_z$$

which are homothetic, with $k_i \in \mathbb{R}$.

(3) *For* $k \neq -2$ *the metric* $g = \varepsilon\,|z|^k\,(h+dz^2)$, $\varepsilon \in \{\pm 1\}$ *admits the projective vector fields*

$$v = k_0\partial_y + k_1\,(2\partial_x + 2y\partial_y + (\beta+2)z\partial_z)\,,$$

which are homothetic, with $k_i \in \mathbb{R}$.

(4) *The metric* $g = \eta\,(1+\tan^2(z))\,(h+dz^2)$, $\eta \in \mathbb{R}, \eta \neq 0$, *has the projective vector fields*

$$v = k_0\partial_y + k_2\tan(z)\partial_z\,,$$

with $k_i \in \mathbb{R}$.

(5) *The metric* $g = \eta\,(1-\tanh^2(z))\,(h+dz^2)$, $\eta \in \mathbb{R}, \eta \neq 0$, *has the projective vector fields*

$$v = k_0\partial_y + k_2\tanh(z)\partial_z\,,$$

with $k_i \in \mathbb{R}$.

(6) *For* $k \in \mathbb{R}$, $k \neq -1$, *the metric* $g = \eta\psi'(z)\,(h+dz^2)$, $\eta \neq 0$, *where*

$$(\psi(z)-z)\psi''(z) = 2\psi'(z)\,(\psi'(z)-k),$$

has the projective vector fields

$$v = k_0\partial_y + k_1\,(2\partial_x + 2y\partial_y + (\beta+2)(z-\psi(z))\partial_z)$$

with $k_i \in \mathbb{R}$.

In Section 6.4 we are going to give explicit normal forms also for the case a), which are omitted here for conciseness.

6.4. Proof of Theorem 3

6.4.1. Proof of cases a) and b)

By hypothesis the metric g has a projective algebra of dimension ≥ 2. Therefore, due to Proposition 12 and keeping in mind the considerations within the proof of Lemma 18, we conclude that h has a homothetic algebra of at least dimension 1. This leaves us with three distinct situations: If h has a homothetic algebra of exactly dimension 1, it must be one of the metrics listed in Lemma 18. If it is of constant curvature, then it has a homothetic algebra of maximal dimension 4. Two-dimensional metrics with a projective algebra of dimension 2 or 3 are classified in [3, Theorem 1]. It is then easy to verify that only (75) admits a homothetic algebra of dimension $2 \leq n \leq 3$. This leaves us with the following distinct cases:

a) If h has constant curvature, then there is a coordinate transformation (55), see Lemma 13, such that (without loss of generality) exactly one of the following cases occurs:

   1) If h has zero curvature, it is locally flat, $h = dx^2 + dy^2$.
   2) If h has positive constant curvature, it is locally the round sphere, $h = dx^2 + \sin^2(x)dy^2$.

3) If h has negative constant curvature, it is locally of the form $h = dx^2 + \sinh^2(x)dy^2$.

b) If h has a homothetic algebra of dimension 1, then after a coordinate transformation it is either of the form

$$h = h_{11}(y)dx^2 + 2h_{12}(y)dxdy + h_{22}(y)dy^2$$

or of the form

$$h = e^x \left(h_{11}(y)dx^2 + 2h_{12}(y)dxdy + h_{22}(y)dy^2\right) .$$

In the first case ∂_x is Killing, in the latter it is properly homothetic with $\mathcal{L}_{\partial_x} h = h$.

c) If h belongs to neither of the previous cases, then it can be brought into the form (75).

For each of these cases, we need to integrate (68) for $\zeta(z)$ and $\alpha(z)$, where C is determined by the homothetic vector fields u of h. In fact, see Remark 12, this task is equivalent to integrating (63a) and (63b) given C and for suitable $b, B \in \mathbb{R}$. The parameter choices can be reduced using Lemma 3.

Cases a) and b) of Theorem 3 are therefore proven, after a suitable transformation (55) of the normal forms in case b). It remains to conclude the proof in the case of (75).

6.4.2. Proof of case c)

Note that (75) admits the homothetic algebra parametrised by

$$u = k_0\partial_y + k_1\left(\partial_x + y\partial_y\right), \quad k_i \in \mathbb{R}\,.$$

To proceed, we make use of Lemma 3, which allows us to find all projective symmetries of g by integrating the following cases. By a direct computation we find

$$C = -(\beta+2)k_1\,.$$

Lemma 19. *Let $g = \zeta(z)\left(h + dz^2\right)$ as in Theorem 3. Assume g admits a projective algebra larger than the one generated by ∂_y.*

1. *If g admits a Killing vector field in addition to ∂_y, then after a transformation (55) and multiplying g with a constant, we have the metric*

$$g = \frac{1}{z^2}(h + dz^2)\,,$$

   which admits the additional projective vector field $v = \partial_x + y\partial_y - \frac{\beta+2}{2}\,z\partial_z$.

2. *If g admits a properly homothetic vector field in addition to the Killing vector field ∂_y, then, after a transformation (55) and multiplying g with a constant, either we have the metric*

$$g = z^k\,(h + dz^2)$$

   ($k \neq 0$), which admits the additional projective vector field $v = \partial_x + y\partial_y - \frac{\beta+2}{2}\,z\partial_z$, or we have the metric

$$g = e^{kz}\,(h + dz^2)$$

   ($k \neq 0$), which admits the additional projective vector field $v = \partial_z$.

3. *If g admits an essential projective vector field v in addition to the Killing vector field ∂_y, then, after a transformation (55) and multiplying g with a constant, one of the following is attained*

   - $g = \frac{1}{z^2}\,(h + dz^2)$ *with projective vector field* $v = \frac{1}{z}\,\partial_z$.
   - $g = (1+\tan^2(z))\,(h+dz^2)$ *with projective vector field* $v = \tan(z)\,\partial_z$.
   - $g = (1 - \tanh^2(z))\,(h + dz^2)$ *with projective vector field*

$$v = \tanh(z)\,\partial_z.$$

   - $g = \psi'(z)\,(h + dz^2)$ *where*

$$(\psi(z) - z)\,\psi''(z) = 2\psi'(z)\,(\psi'(z) - k)\,, \qquad k \in \mathbb{R}\,, \tag{76}$$

     and with the essential projective vector field

$$v = \partial_x + y\partial_y + \frac{1}{2}\,(z - \psi(z))\,\partial_z.$$

Proof. We begin with part 1 of the claim and compute Killing vector fields v for g by setting $a = 0, b = 0$ (and thus $C = B = d$) in (63a) and (63b). We hence need to solve

$$
\begin{aligned}
-2\alpha'(z) &= C \\
\alpha(z)\zeta'(z) &= C\zeta(z)\,.
\end{aligned}
$$

We begin by assuming $C \neq 0$. After a suitable translation (55), we arrive at the first case. If $C = 0$, on the other hand, we immediately obtain $\alpha(z) = 0$, for any non-constant function ζ. We can omit this solution because its projective algebra is spanned by ∂_y and thus 1-dimensional.

We continue with part 2 of the claim. Properly homothetic vector fields v are obtained assuming $a = 1, b = 0$ ($C = 3 + d, B = 1 + d = C - 2$). We need to integrate

$$
\begin{aligned}
-2\alpha'(z) &= C \\
\alpha(z)\zeta'(z) &= (C-4)\zeta(z)\,.
\end{aligned}
$$

If $C \neq 0$, then after a suitable transformation (55),

$$
\alpha(z) = -\frac{C}{2}\,z\,, \qquad \zeta(z) = C_1 z^{2\left(\frac{4}{C}-1\right)}
$$

where $C_1 \in \mathbb{R}$. If $C = 0$, we find the solution

$$
\alpha(z) = C_0 \neq 0 \qquad \text{for} \qquad \zeta(z) = C_1\, e^{-\frac{4}{C_0}z}\,.
$$

Finally, we consider part 3 of the claim. Essential vector fields v are obtained assuming $a = 0, b = 1$ ($C = 2\rho + d$, $B = d$). We need to integrate

$$
\begin{aligned}
-2\alpha'(z) &= C + \zeta(z) \\
\alpha(z)\zeta'(z) &= -\zeta^2(z) + (2B - C)\zeta(z)
\end{aligned}
$$

We solve the first equation for ζ and substitute the result into the second condition, obtaining

$$
\zeta = -2\alpha' - C \qquad \text{where} \quad -2\alpha\alpha'' = -(2\alpha' + C)^2 - (2B - C)(2\alpha' + C) \tag{77}
$$

We begin with the case $C = 0$. The ODE becomes

$$
\alpha\alpha'' = 2(\alpha')^2 + 2B\alpha'
$$

If $B=0$, we have $\alpha = \frac{C_2}{C_1+z}$ and thus $\zeta = \frac{1}{z^2}$ up to a constant translation of z and a multiplication of ζ with a constant. Thus assume $B \neq 0$. Introducing $\psi = \frac{\alpha}{2B}$ the ODE translates into

$$\psi(z)\psi''(z) = 2\psi'(z)^2 + \psi'(z).$$

with the solutions

$$\psi(z) = \frac{1}{C_1}\tan\left(\frac{C_1 z}{2}\right) \quad \text{and} \quad \psi(z) = \frac{1}{C_1}\tanh\left(\frac{C_1 z}{2}\right).$$

We thus have

$$\alpha(z) = \frac{2B}{C_1}\tan(\tfrac{C_1 z}{2})$$

and, up to multiplication by a constant factor,

$$\zeta(z) = (1+\tan^2(C_1 z)) \quad \text{or} \quad \zeta(z) = (1-\tanh^2(C_1 z)),$$

from which the claim is easily obtained.

Finally, consider $C \neq 0$. In this case it is helpful to introduce

$$\psi(z) = \frac{2\alpha}{C} + z,$$

which turns (77) into the ODE (76) with $k = 1 - \frac{2B}{C}$. □

For the cases (1) to (5) of the claim in part c) of Theorem 3, the proof is a direct consequence of Lemma 19. Case (6) of part c) is also a direct consequence of Lemma 19, but we need to exclude all the solutions that are equivalent, under (55), to one of the previous cases. We can check this by assuming that there is, for some $k \in \mathbb{R}$, a solution ψ of (76), such that the metric has the form $\psi'(z)\,(h + dz^2)$, with h as in (75). For the metric in case (1) we have $\psi'(z) = \frac{1}{z^2}$ (note that the constant conformal factor η is irrelevant). We thus have $\psi(z) = \frac{1}{z} + s$ for some constant $s \in \mathbb{R}$. Resubstituting into (76) yields the condition

$$(k+1)z - s = 0,$$

which can only be realised if $k=-1$ and $s=0$. On the other hand, if $k=-1$, the ODE (76) has the solution

$$\psi(z) = -\frac{k_0 + k_1 z}{k_1 + z},$$

for some $k_0, k_1 \in \mathbb{R}$, from which we obtain

$$\psi'(z) = \frac{k_0 - k_1^2}{(k_1 + z)^2} \, .$$

This corresponds to the metric ($\mu \in \mathbb{R}, \mu \neq 0$)

$$g = \eta\psi'(z)\,(h + dz^2) = \frac{\mu}{(k_1 + z)^2}$$

and after a transformation (55) we obtain the metric from case (1) of Theorem 3. We therefore exclude the value $k = -1$ from case (6).

It is easily confirmed that none of the cases (2) to (5) can be realised as a special case of (6). For example, if $\psi'(z) = e^z$, then $\psi(z) = e^z + s$ for some $s \in \mathbb{R}$. Resubstituting into (76), we have

$$(s + 2k - z)e^z - e^{2z} = 0 \, ,$$

which cannot be realised for any s, k with generic z. The cases (3) to (5) are checked analogously.

6.4.3. Solutions of Equation (76)

The last case of Theorem 3 is given implicitly through solutions of an ODE for a function $\psi(z)$, from which both the metric and the projective symmetries are obtained. It is however possible to obtain solutions in more concrete form using the system (73). Consider case (6) of Theorem 3. A closer inspection of the proof, and a comparison with Lemma 15, shows that instead of solving (76), we can also solve (73). Since we need to determine $\zeta(z)$ only up to a constant factor, we can equivalently solve

$$\zeta''(z) = \frac{3\zeta(z) + 2k - 1}{2\zeta(z)\,(\zeta(z) + k)}\,\zeta'(z)^2$$

We claim that $\zeta''(z) \neq 0$. Indeed, if $\zeta(z) = c_0 z + c_1$, then

$$0 = \frac{3c_0 z + 3c_1 + 2k - 1}{2c_0 z + 3c_1\,(c_0 z + c_1 + k)}\,c_0^2 \, ,$$

implying $c_0 = 0$, which contradicts the hypothesis of Theorem 3. We conclude that $\zeta''(z) \neq 0$.

Example 6. Let us now consider the special case when $k = 0$. Since $\zeta''(z) \neq 0$, we introduce $f(\zeta(z)) = \zeta'(z)^2$. Thus we need to solve

$$f'(\zeta) = \frac{3\zeta - 1}{\zeta^2}\, f(\zeta)\,.$$

We obtain

$$\zeta'' = f(\zeta) = c_2 f(\zeta)^3 e^{\frac{1}{f(\zeta)}}\,.$$

This equation can be solved and we obtain, up to a transformation (55) and up to a constant factor,

$$\zeta(z) = \frac{1}{\operatorname{inverf}(z)}\,,$$

where inverf is the inverse error function.

The cases with $k \neq 0$ are generally less easy to solve. Introducing again $f(\zeta(z)) = \zeta'(z)^2$, we arrive at

$$f'(\zeta) = \frac{3\zeta + 2k - 1}{\zeta\,(\zeta + k)}\, f(\zeta)\,.$$

The computer algebra software *Wolfram Mathematica 12.3* [31] yields the solution

$$f(\zeta) = c_3 \zeta^{\frac{2k+1}{k}} (\zeta - k)^{\frac{k-1}{k}}\,.$$

Solving $f(\zeta(z)) = \zeta''(z)^2$, the software finds

$$\zeta(z) = F^{-1}(z)\,,$$

up to a transformation (55) and up to a constant factor, where F^{-1} denotes the inverse of the function

$$F : t \mapsto -\frac{2k\; t^{-\frac{1}{2k}}\,(t-k)^{\frac{1}{2k}}\,\left(\frac{k-t}{k}\right)^{\frac{k-1}{2k}}\;{}_2F_1\left[\frac{k-1}{2k}, -\frac{1}{2k}, \frac{2k-1}{2k}, \frac{t}{k}\right]}{\sqrt{k-t}}$$

for the hypergeometric special function ${}_2F_1$.

Example 7. Let us consider the special case when $k = 1$. The solution to (76) is

$$\psi(z) = z - \frac{\tanh(k_0 + k_1 z)}{k_1}\,,\ k_1 \neq 0 \quad \text{or} \quad \psi(z) = z\,,$$

so that

$$\psi'(z) = \tanh^2(k_0 + k_1 z)\,,\ k_1 \neq 0 \quad \text{or} \quad \psi'(z) = 1\,.$$

The latter solution is not allowed by the hypotheses of Theorem 3. Up to a transformation (55), and up to multiplication of the metric by a constant factor, this yields

$$g = \tanh^2(z)\,(h + dz^2)\,,$$

where h is given by (75). Its projective algebra is indeed parametrised by

$$v = k_0\partial_y + k_1\,(2\partial_x + 2y\partial_y + (\beta + 2)\tanh(z)\partial_z)\,,$$

which is an essential projective vector field for any $k_1 \neq 0$.

6.5. [2-1]-type Levi-Civita metrics with constant curvature metric h

In this section we consider case a) of Theorem 3. For brevity we shall consider only the case when h has Riemannian signature as for the other signatures the corresponding results can be obtained in complete analogy.

6.5.1. [2-1]-type Levi-Civita metrics with flat metric h

For for any $\zeta \neq 0$, the metric

$$g = \zeta(z)(dx^2 + dy^2 + dz^2) \tag{78}$$

admits the Killing vector fields

$$v = k_0(y\partial_x - x\partial_y) + k_1\partial_x + k_2\partial_y\,,$$

$k_i \in \mathbb{R}$, which straightforwardly arise from Killing vector fields of h for any $\zeta(z) \neq 0$. The metric h also admits the properly homothetic vector field

$$u = k_3\,(x\partial_x + y\partial_y)\,.$$

If g admits a larger projective algebra, then g assumes at least one of the following forms after a suitable coordinate transformation.

1. For $k(k-2) \neq 0$, the metric $g = \varepsilon|z|^{-k}\ (dx^2 + dy^2 + dz^2)$, $\varepsilon \in \{\pm 1\}$, has the projective vector fields

$$v = k_0(y\partial_x - x\partial_y) + k_1\partial_x + k_2\partial_y + k_3(x\partial_x + y\partial_y + z\partial_z)$$

2. The metric $g = \varepsilon e^z(dx^2 + dy^2 + dz^2)$, $\varepsilon \in \{\pm 1\}$, has the projective vector fields

$$v = k_0(y\partial_x - x\partial_y) + k_1\partial_x + k_2\partial_y + k_4\partial_z\,.$$

3. The metric $g = \eta(1+\tan^2(z))(dx^2+dy^2+dz^2)$, $\eta \neq 0$, has the projective vector fields

$$v = k_0(y\partial_x - x\partial_y) + k_1\partial_x + k_2\partial_y + k_4\tan(z)\partial_z\,.$$

4. The metric $g = \eta(1 - \tanh^2(z))(dx^2 + dy^2 + dz^2)$, $\eta \neq 0$, has the projective vector fields

$$v = k_0(y\partial_x - x\partial_y) + k_1\partial_x + k_2\partial_y + k_4\tanh(z)\partial_z\,.$$

5. For $k \in \mathbb{R}$, $k \neq -1$, the metric $g = \eta\psi'(z)\,(dx^2 + dy^2 + dz^2)$, $\eta \neq 0$, where
$$(\psi(z) - z)\psi''(z) = 2\psi'(z)\,(\psi' - k)\,,$$
has the projective vector fields

$$v = k_0(y\partial_x - x\partial_y) + k_1\partial_x + k_2\partial_y + k_3(x\partial_x + y\partial_y + (z - \psi(z))\partial_z)$$

Remark 14. The metric (78) is of constant curvature if and only if $\zeta(z) = \mp\frac{1}{(c_0+c_1z)^2}$ with $c_i \in \mathbb{R}$ such that $c_0^2 + c_1^2 \neq 0$. There are no metrics of type 5 with constant curvature.

In order to obtain the above list, we proceed analogously to the proof of Theorem 3. Generically the metric $h = dx^2 + dy^2$ admits the homothetic vector fields

$$u = k_0(y\partial_x - x\partial_y) + k_1\partial_x + k_2\partial_y + k_3(x\partial_x + y\partial_y)\,,$$

and we easily compute
$$C = -2k_3\,.$$
The admissible functions $\zeta(z)$, i.e. the functions that lead to new projective vector fields, can be found analogously to Lemma 19.

First, note that the metrics g with $\zeta \propto z^\mu$ are of constant curvature if and only if $\mu(\mu + 2)(\mu - 4) = 0$, and thus $\zeta \propto \frac{1}{z^2}$ as well as $\zeta \propto z^4$ must be omitted. The remaining metrics are of non-constant curvature. Hence we obtain the first case in the list, after a suitable rescaling of the coordinates. Similarly, the metrics with $\zeta \propto e^{-kz}$ are of constant curvature if and only if $k = 0$ and thus we find the second case in the list after a suitable affine transformation of the coordinates. Next, the metrics g with $\zeta \propto e^{-\frac{\mu}{z}}$ do not admit additional projective symmetries and can therefore be omitted from the list. For $\zeta \propto (1+\tan^2(C_1 z))$, we obtain metrics of non-constant curvature if and only if $C_1 \neq 0$, from which we infer the third case of the list, after a suitable rescaling of the coordinates. The last case follows analogously.

6.5.2. [2-1]-type Levi-Civita metrics with spherical metric h

With exactly the same strategy as in Section 6.5.1 we can also consider the case when h has positive constant curvature. According to Lemma 13, we can restrict to curvature $+1$.

For $\zeta \neq 0$, we consider the metric

$$g = \zeta(z)\,(dx^2 + \sin^2(x)dy^2 + dz^2)\,, \tag{79}$$

which admits the linearly independent Killing vector fields

$$\begin{aligned} u_0 &= \partial_y \\ u_1 &= \cos(y)\partial_x - \frac{\sin(y)}{\tan(x)}\partial_y \\ u_2 &= \sin(y)\partial_x + \frac{\cos(y)}{\tan(x)}\partial_y\,, \end{aligned}$$

that straightforwardly arise from the Killing vector fields of h. If g admits a larger projective algebra, then g assumes at least one of the following forms after a suitable coordinate transformation.

1. The metric $g = \frac{\eta}{z^2}\,(dx^2 + \sin^2(x)dy^2 + dz^2)$, $\eta \neq 0$, has the projective vector fields
$$v = k_0\,u_0 + k_1\,u_1 + k_2\,u_2 + \frac{k_3}{z}\partial_z$$
2. The metric $g = \varepsilon e^z(dx^2+\sin^2(x)dy^2+dz^2)$, $\varepsilon \in \{\pm1\}$, has the projective vector fields
$$v = k_0\,u_0 + k_1\,u_1 + k_2\,u_2 + k_3\partial_z\,.$$

3. The metric $g = \eta(1+\tan^2(z))(dx^2+\sin^2(x)dy^2+dz^2)$, $\eta \neq 0$, has the projective vector fields
$$v = k_0\, u_0 + k_1\, u_1 + k_2\, u_2 + k_3 \tan(z)\partial_z\,.$$

4. The metric $g = \eta(1-\tanh^2(z))(dx^2+\sin^2(x)dy^2+dz^2)$, $\eta \neq 0$, has the projective vector fields
$$v = k_0\, u_0 + k_1\, u_1 + k_2\, u_2 + k_3 \tanh(z)\partial_z\,.$$

Remark 15. The metric (79) is of constant curvature if and only if $\zeta(z) = \mp\frac{e^{-2z}}{(c_0+c_1e^{-2z})^2}$, with $c_i \in \mathbb{R}$ such that $c_0^2 + c_1^2 \neq 0$.

6.5.3. [2-1]-type Levi-Civita metrics with hyperbolic metric h

We conclude this section with the case when h has negative constant curvature, i.e., according to Lemma 13, curvature -1. The strategy is the same as in the previous two cases. For $\zeta \neq 0$, we consider the metric
$$g = \zeta(z)\,(dx^2+\sinh^2(x)dy^2+dz^2)\,, \tag{80}$$
which admits the Killing vector fields
$$\begin{aligned} u_0 &= \partial_y \\ u_1 &= \cos(y)\partial_x - \frac{\sin(y)}{\tanh(x)}\partial_y \\ u_2 &= \sin(y)\partial_x + \frac{\cos(y)}{\tanh(x)}\partial_y \end{aligned}$$
that straightforwardly arise from the Killing vector fields of h. If g admits a larger projective algebra, then g assumes at least one of the following forms after a suitable coordinate transformation.

1. The metric $g = \frac{\eta}{z^2}\,(dx^2+\sin^2(x)dy^2+dz^2)$, $\eta \neq 0$, has the projective symmetries
$$v = k_0\, u_0 + k_1\, u_1 + k_2\, u_2 + \frac{k_3}{z}\partial_z$$

2. The metric $g = \varepsilon e^z(dx^2+\sinh^2(x)dy^2+dz^2)$, $\varepsilon \in \{\pm 1\}$, has the projective symmetries
$$v = k_0\, u_0 + k_1\, u_1 + k_2\, u_2 + k_3\partial_z\,.$$

3. The metric $g = \eta(1+\tan^2(z))(dx^2+\sinh^2(x)dy^2+dz^2)$, $\eta \neq 0$, has the projective symmetries

$$v = k_0\, u_0 + k_1\, u_1 + k_2\, u_2 + k_3 \tan(z)\partial_z\,.$$

4. The metric $g = \eta(1-\tanh^2(z))(dx^2+\sinh^2(x)dy^2+dz^2)$, $\eta \neq 0$, has the projective symmetries

$$v = k_0\, u_0 + k_1\, u_1 + k_2\, u_2 + k_3 \tanh(z)\partial_z\,.$$

Remark 16. The metric (80) is of constant curvature if and only if $\zeta(z) = \mp\frac{1}{(c_1\sin(z)+c_2\cos(z))^2}$, with $c_i \in \mathbb{R}$ such that $c_1^2+c_2^2 \neq 0$.

7. Proof of Theorem 1

First let us recall that the metrics of Theorem 1 are of non-constant curvature, and by assumption they do not admit any homothetic vector field. The degree of mobility is therefore 2, and since the metric is of Riemannian signature, it follows that it is a Levi-Civita metric either of type [1-1-1] or of type [2-1]. Let us begin with type [1-1-1]. Reviewing Theorem 2, we observe that the metrics in Propositions 9 and 10 do admit a Killing vector field and therefore do not fall under the assumptions of Theorem 1. The desired metrics thus follow, maybe after an obvious change of coordinates, from a straightforward analysis of the metrics in Proposition 11. We arrive at the cases 1 to 5 of Theorem 1.

Hence let us proceed to Levi-Civita metrics of [2-1] type. Note that the conformal factor $\zeta(z)$ must be non-constant since otherwise there necessarily exists a Killing vector field contrary to the hypothesis. We are thus left with the metrics covered in Theorem 3 and Proposition 14. The metrics covered by Theorem 3 always admit a homothetic vector field, however, and thus we are left with those in Proposition 14. Among these metrics only three cases are compatible with the assumptions of Theorem 1. We arrive at cases 6 to 8 in the claim.

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