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The Faber–Krahn inequality for the short-time Fourier transform

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Abstract In this paper we solve an open problem concerning the characterization of those measurable sets $\Omega \subset \mathbb{R}^{2d}$ that, among all sets having a prescribed Lebesgue measure, can trap the largest possible energy fraction in time-frequency space, where the energy density of a generic function $f \in L^2(\mathbb{R}^d)$ is defined in terms of its Short-time Fourier transform (STFT) $\mathcal{V}f(x,\omega)$, with Gaussian window. More precisely, given a measurable set $\Omega \subset \mathbb{R}^{2d}$ having measure $s > 0$, we prove that the quantity

$$\Phi_\Omega = \max\left\{ \int_\Omega |\mathcal{V}f(x,\omega)|^2\, dx d\omega : f \in L^2(\mathbb{R}^d),\ \|f\|_{L^2} = 1\right\},$$

is largest possible if and only if Ω is equivalent, up to a negligible set, to a ball of measure s, and in this case we characterize all functions f that achieve equality. This result leads to a sharp uncertainty principle for the "essential support" of the STFT (when $d = 1$, this can be summarized by the optimal bound $\Phi_\Omega \le 1 - e^{-|\Omega|}$, with equality if and only if Ω is a ball). Our approach, using techniques from measure theory after suitably rephrasing the problem in the Fock space, also leads to a local version of Lieb's uncertainty inequality for the STFT in L^p when $p \in [2,\infty)$, as well as to L^p-concentration estimates

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when $p \in [1, \infty)$, thus proving a related conjecture. In all cases we identify the corresponding extremals.

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1 Introduction

The notion of energy concentration for a function $f \in L^2(\mathbb{R})$ in the time-frequency plane is an issue of great theoretical and practical interest and can be formalised in terms of time-frequency distributions such as the so-called Short-time Fourier transform (STFT), defined as

$$\mathcal{V} f(x, \omega) = \int_{\mathbb{R}} e^{-2\pi i y \omega} f(y) \varphi(x - y) dy, \qquad x, \omega \in \mathbb{R},$$

where φ is the "Gaussian window"

$$\varphi(x) = 2^{1/4} e^{-\pi x^2}, \quad x \in \mathbb{R}, \tag{1.1}$$

normalized in such way that $\|\varphi\|_{L^2} = 1$. It is well known that $\mathcal{V} f$ is a complex-valued, real analytic, bounded function and $\mathcal{V} : L^2(\mathbb{R}) \to L^2(\mathbb{R}^2)$ is an isometry (see [28,34]).

The STFT represents a versatile tool in several areas of pure and applied mathematics; it is also called *Bargmann transform* in PDEs [46,54] (cf. also the closely related FBI transform [38]), *coherent state transform* in quantum physics [43], *Gabor transform* or *windowed Fourier transform* in signal processing [44]. The basic idea of localizing a function before taking its Fourier transform turned out to be crucial in the study of the singularities of distributions [38] and, in more quantitative terms, to detect the blobs of energy of an L^2-function. This provides parsimonious representations for classes of functions, which is a fundamental paradigm of modern signal processing [44].

More precisely, it is customary to interpret $|\mathcal{V} f(x, \omega)|^2$ as the time-frequency energy density of f (see [34,44]). Consequently, the fraction of energy captured by a measurable subset $\Omega \subseteq \mathbb{R}^2$ of a function $f \in L^2(\mathbb{R})\backslash\{0\}$ will be given by the Rayleigh quotient (see [2,3,18,45])

$$\Phi_\Omega(f) := \frac{\int_\Omega |\mathcal{V} f(x, \omega)|^2 \, dx d\omega}{\int_{\mathbb{R}^2} |\mathcal{V} f(x, \omega)|^2 \, dx d\omega} = \frac{\langle \mathcal{V}^* \mathbb{1}_\Omega \mathcal{V} f, f \rangle}{\|f\|^2_{L^2}}. \tag{1.2}$$

The bounded, nonnegative and self-adjoint operator $\mathcal{V}^* \mathbb{1}_\Omega \mathcal{V}$ on $L^2(\mathbb{R})$ is known in the literature under several names, e.g. localization, concentration, Anti-Wick operator, as well as time-frequency or time-varying filter. Since its first appearance in the works by Berezin [8] and Daubechies [18], the applications of such operators have been manifold and the related literature is enormous: we refer to the books [9,55] and the survey [16], and the references therein, for an account of the main results.

Now, when Ω has finite measure, $\mathcal{V}^*\mathbb{1}_\Omega\mathcal{V}$ is a compact (in fact, trace class) operator. Its norm $\|\mathcal{V}^*\mathbb{1}_\Omega\mathcal{V}\|_{\mathcal{L}(L^2)}$, given by the quantity

$$\Phi_\Omega := \max_{f\in L^2(\mathbb{R})\setminus\{0\}} \Phi_\Omega(f) = \max_{f\in L^2(\mathbb{R})\setminus\{0\}} \frac{\langle \mathcal{V}^*\mathbb{1}_\Omega\mathcal{V}f, f\rangle}{\|f\|_{L^2}^2},$$

represents the maximum fraction of energy that can in principle be trapped by Ω for any signal $f \in L^2(\mathbb{R})$, and explicit upper bounds for Φ_Ω are of considerable interest. Indeed, the analysis of the spectrum of $\mathcal{V}^*\mathbb{1}_\Omega\mathcal{V}$ was initiated in the seminal paper [18] for radially symmetric Ω, in which case the operator is diagonal in the basis of Hermite functions –and conversely [1] if an Hermite function is an eigenfunction and Ω is simply connected then Ω is a ball centered at 0– and the asymptotics of the eigenvalues (Weyl's law), in connection with the measure of Ω, has been studied by many authors; again the literature is very large and we address the interested reader to the contributions [2,3,20,45,47] and the references therein.

The study of the time-frequency concentration of functions, in relation to uncertainty principles and under certain additional constraints (e.g. on subsets of prescribed measure in phase space, or under limited bandwidth etc.) has a long history which, as recognized by Landau and Pollak [41], dates back at least to Fuchs [30], and its relevance both to theory and applications has been well known since the seminal works by Landau-Pollack-Slepian, see e.g. [29,40,52], and other relevant contributions such as those of Cowling and Price [17], Donoho and Stark [23], and Daubechies [18].

However, in spite of the abundance of deep and unexpected results related to this circle of ideas (see e.g. the visionary work by Fefferman [26]) the question of characterizing the subsets $\Omega \subset \mathbb{R}^2$ of prescribed measure, which allow for the maximum concentration, is still open. In this paper we provide a complete solution to this problem proving that the optimal sets are balls in phase space, and, in dimension one, our result can be stated as follows (see Theorem 4.1 for the same result in arbitrary dimension).

Theorem 1.1 (Faber–Krahn inequality for the STFT) *Among all measurable subsets $\Omega \subset \mathbb{R}^2$ having a prescribed (finite, non zero) measure, the quantity*

$$\Phi_\Omega := \max_{f \in L^2(\mathbb{R})\setminus\{0\}} \frac{\int_\Omega |\mathcal{V} f(x,\omega)|^2 \, dx d\omega}{\int_{\mathbb{R}^2} |\mathcal{V} f(x,\omega)|^2 \, dx d\omega} = \max_{f \in L^2(\mathbb{R})\setminus\{0\}} \frac{\langle \mathcal{V}^* \mathbb{1}_\Omega \mathcal{V} f, f\rangle}{\|f\|^2_{L^2}} \tag{1.3}$$

achieves its maximum if and only if Ω is equivalent, up to a set of measure zero, to a ball.

Moreover, when Ω is a ball of center (x_0, ω_0), the only functions f that achieve the maximum in (1.3) are the functions of the kind

$$f(x) = c\, e^{2\pi i \omega_0 x} \varphi(x - x_0), \qquad c \in \mathbb{C}\setminus\{0\}, \tag{1.4}$$

that is, the scalar multiples of the Gaussian window φ defined in (1.1), translated and modulated according to (x_0, ω_0).

This "Faber–Krahn inequality" (see Remark 1.3 at the end of this section) proves, in the L^2-case, a conjecture by Abreu and Speckbacher [5] (the full conjecture is proved in Theorem 5.3), and confirms the distinguished role played by the Gaussian (1.4), as the first eigenfunction of the operator $\mathcal{V}^* \mathbb{1}_\Omega \mathcal{V}$ when Ω has radial symmetry (see [18]; see also [23] for a related conjecture on band-limited functions, and [17, page 162] for further insight).

When Ω is a ball of radius r, one can see that $\Phi_\Omega = 1 - e^{-\pi r^2}$ (this follows from the results in [18], and will also follow from our proof of Theorem 1.1). Hence we deduce a more explicit form of our result, which leads to a sharp form of the uncertainty principle for the STFT.

Theorem 1.2 (Sharp uncertainty principle for the STFT) *For every subset $\Omega \subset \mathbb{R}^2$ whose Lebesgue measure $|\Omega|$ is finite we have*

$$\Phi_\Omega \le 1 - e^{-|\Omega|} \tag{1.5}$$

and, if $|\Omega| > 0$, equality occurs if and only if Ω is a ball.

As a consequence, if for some $\epsilon \in (0,1)$, some function $f \in L^2(\mathbb{R})\setminus\{0\}$ and some $\Omega \subset \mathbb{R}^2$ we have $\Phi_\Omega(f) \ge 1 - \epsilon$, then necessarily

$$|\Omega| \ge \log(1/\epsilon), \tag{1.6}$$

with equality if and only if Ω is a ball and f has the form (1.4), where (x_0, ω_0) is the center of the ball.

Theorem 1.2 solves the long–standing problem of the optimal lower bound for the measure of the "essential support" of the STFT with Gaussian window. The best result so far in this direction was obtained by Gröchenig (see [34, Theorem 3.3.3]) as a consequence of Lieb's uncertainly inequality [42] for the STFT, and consists of the following (rougher, but valid for any window) lower bound

$$|\Omega| \geq \sup_{p>2} (1-\epsilon)^{p/(p-2)} (p/2)^{2/(p-2)} \tag{1.7}$$

(see Sect. 5 for a discussion in dimension d). Notice that the sup in (1.7) is a bounded function of $\epsilon \in (0, 1)$, as opposite to the optimal bound in (1.6) (see Fig. 1 in the Appendix for a graphical comparison).

We point out that, although in this introduction the discussion of our results is confined (for ease of notation and exposition) to the one dimensional case, our results are valid in arbitrary space dimension, as discussed in Sect. 4 (Theorem 4.1 and Corollary 4.2).

While addressing the reader to [12,29,35] for a review of the numerous uncertainty principles available for the STFT (see also [10,19,21,27]), we observe that inequality (1.5) is nontrivial even when Ω has radial symmetry: in this particular case it was proved in [33], exploiting the already mentioned diagonal representation in the Hermite basis.

Some concentration–type estimates were recently provided in [5] as an application of the Donoho-Logan large sieve principle [22] and the Selberg-Bombieri inequality [11]. However, though this machinery certainly has a broad applicability, as observed in [5] it does not seem to give sharp bounds for the problem above. For interesting applications to signal recovery we refer to [4,48,49,53] and the references therein.

One major difficulty in proving Theorem 1.1 is that, although optimal sets are balls and optimal functions are radial, symmetrization and rearrangement techniques cannot be used, since in general the rearrangement of a transform $\mathcal{V}f$ is not the transform of any function in $L^2(\mathbb{R})$. Therefore, even though some properties of rearrangements are used as auxiliary tools in some computations, the proofs call for new ideas and a new approach (based on a differential inequality) is introduced, after the problem has been reformulated as an equivalent statement in the Fock space. In order to present our strategy in a clear way and to better highlight the main ideas, we devote Sect. 3 to the proof of our main results in dimension one, while results in arbitrary dimension (which are more involved) are stated and proved in Sect. 4.

In Sect. 5 we discuss some extensions of the above results in different directions, such as a local version of Lieb's uncertainty inequality [25] for the STFT in L^p when $p \in [2, \infty)$ (Theorem 5.2), and L^p-concentration estimates for the STFT when $p \in [1, \infty)$ (Theorem 5.3, which proves [5, Conjecture

1]), identifying in all cases the extremals f and Ω, as above. We also study the effect of changing the window φ by a dilation or, more generally, by a metaplectic operator.

We believe that the techniques used in this paper could also shed new light on the Donoho-Stark uncertainty principle [23] and the corresponding conjecture [23, Conjecture 1], and that also the stability of (1.5) (via a quantitative version when the inequality is strict) can be investigated. We will address these issues in a subsequent work, together with applications to signal recovery.

Remark 1.3 The maximization of Φ_Ω among all sets Ω of prescribed measure can be regarded as a *shape optimization* problem (see [14]) and, in this respect, Theorem 1.1 shares many analogies with the celebrated Faber-Krahn inequality (beyond the fact that both problems have the ball as a solution). The latter states that, among all (quasi) open sets Ω of given measure, the ball minimizes the first Dirichlet eigenvalue

$$\lambda_\Omega := \min_{u\in H^1_0(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u(z)|^2\,dz}{\int_\Omega u(z)^2\,dz}.$$

On the other hand, if $T_\Omega : H^1_0(\Omega) \to H^1_0(\Omega)$ is the linear operator that associates with every (real-valued) $u \in H^1_0(\Omega)$ the weak solution $T_\Omega u \in H^1_0(\Omega)$ of the problem $-\Delta(T_\Omega u) = u$ in Ω, integrating by parts we have

$$\int_\Omega u^2\,dz = -\int_\Omega u\Delta(T_\Omega u)\,dz = \int_\Omega \nabla u\cdot\nabla(T_\Omega u)\,dz = \langle T_\Omega u, u\rangle_{H^1_0},$$

so that Faber-Krahn can be rephrased by claiming that

$$\lambda_\Omega^{-1} := \max_{u\in H^1_0(\Omega)\setminus\{0\}} \frac{\int_\Omega u(z)^2\,dz}{\int_\Omega |\nabla u(z)|^2\,dz} = \max_{u\in H^1_0(\Omega)\setminus\{0\}} \frac{\langle T_\Omega u, u\rangle_{H^1_0}}{\|u\|^2_{H^1_0}}$$

is maximized (among all open sets of given measure) by the ball. Hence the statement of Theorem 1.1 can be regarded as a Faber-Krahn inequality for the operator $\mathcal{V}^*\mathbb{1}_\Omega\mathcal{V}$.

2 Rephrasing the problem in the Fock space

It turns out that the optimization problems discussed in the introduction can be conveniently rephrased in terms of functions in the Fock space on $\mathbb{C}$. We address the reader to [34, Section 3.4] and [56] for more details on the relevant results that we are going to review, in a self-contained form, in this section.

The Bargmann transform of a function $f \in L^2(\mathbb{R})$ is defined as

$$\mathcal{B}f(z) := 2^{1/4} \int_{\mathbb{R}} f(y) e^{2\pi yz - \pi y^2 - \frac{\pi}{2} z^2} \, dy, \qquad z \in \mathbb{C}.$$

It turns out that $\mathcal{B}f(z)$ is an entire holomorphic function and $\mathcal{B}$ is a unitary operator from $L^2(\mathbb{R})$ to the Fock space $\mathcal{F}^2(\mathbb{C})$ of all holomorphic functions $F : \mathbb{C} \to \mathbb{C}$ such that

$$\|f\|_{\mathcal{F}^2} := \left(\int_{\mathbb{C}} |F(z)|^2 e^{-\pi|z|^2} dm_2(z) \right)^{1/2} < \infty, \tag{2.1}$$

where $dm_2(z)$ is the Lebesgue measure on $\mathbb{C}$. In fact, $\mathcal{B}$ maps the orthonormal basis of Hermite functions in $\mathbb{R}$ into the orthonormal basis of $\mathcal{F}^2(\mathbb{C})$ given by the monomials

$$e_k(z) := \left(\frac{\pi^k}{k!} \right)^{1/2} z^k, \qquad k = 0, 1, 2, \ldots; \quad z \in \mathbb{C}. \tag{2.2}$$

In particular, for the first Hermite function $\varphi(x) = 2^{1/4} e^{-\pi x^2}$, that is, the window in (1.1), we have $\mathcal{B}\varphi(z) = e_0(z) = 1$.

The connection with the STFT is based on the following crucial formula (see e.g. [34, Formula (3.30)]):

$$\mathcal{V}f(x, -\omega) = e^{\pi i x \omega} \mathcal{B}f(z) e^{-\pi|z|^2/2}, \qquad z = x + i\omega, \tag{2.3}$$

which allows one to rephrase the functionals in (1.2) as

$$\Phi_\Omega(f) = \frac{\int_\Omega |\mathcal{V}f(x,\omega)|^2 \, dx d\omega}{\|f\|_{L^2}^2} = \frac{\int_{\Omega'} |\mathcal{B}f(z)|^2 e^{-\pi|z|^2} \, dm_2(z)}{\|\mathcal{B}f\|_{\mathcal{F}^2}^2}$$

where $\Omega' = \{(x, \omega) : (x, -\omega) \in \Omega\}$. Since $\mathcal{B} : L^2(\mathbb{R}) \to \mathcal{F}^2(\mathbb{C})$ is a unitary operator, we can safely transfer the optimization problem in Theorem 1.1 directly on $\mathcal{F}^2(\mathbb{C})$, observing that

$$\Phi_\Omega = \max_{F \in \mathcal{F}^2(\mathbb{C}) \setminus \{0\}} \frac{\int_\Omega |F(z)|^2 e^{-\pi|z|^2} \, dm_2(z)}{\|F\|_{\mathcal{F}^2}^2}. \tag{2.4}$$

We will adopt this point of view in Theorem 3.1 below. In the meantime, two remarks are in order. First, we claim that the maximum in (2.4) is invariant under translations of the set Ω. To see this, consider for any $z_0 \in \mathbb{C}$, the operator U_{z_0} defined as

$$U_{z_0}F(z) = e^{-\pi|z_0|^2/2}e^{\pi z\overline{z_0}}F(z - z_0). \tag{2.5}$$

The map $z \mapsto U_z$ turns out to be a projective unitary representation of $\mathbb{C}$ on $\mathcal{F}^2(\mathbb{C})$, satisfying

$$|F(z - z_0)|^2 e^{-\pi|z-z_0|^2} = |U_{z_0}F(z)|^2 e^{-\pi|z|^2}, \tag{2.6}$$

which proves our claim. Invariance under rotations in the plane is also immediate.

Secondly, we observe that the Bargmann transform intertwines the action of the representation U_z with the so-called "time-frequency shifts":

$$\mathcal{B}M_{-\omega}T_x f = e^{-\pi i x\omega}U_z\mathcal{B}f, \qquad z = x + i\omega$$

for every $f \in L^2(\mathbb{R})$, where $T_x f(y) := f(y-x)$ and $M_\omega f(y) := e^{2\pi i y\omega}f(y)$ are the translation and modulation operators. This allows us to write down easily the Bargmann transform of the maximizers appearing in Theorem 1.1, namely $cU_{z_0}e_0$, $c \in \mathbb{C}\backslash\{0\}$, $z_0 \in \mathbb{C}$. For future reference, we explicitly set

$$F_{z_0}(z) := U_{z_0}e_0(z) = e^{-\frac{\pi}{2}|z_0|^2}e^{\pi z\overline{z_0}}, \quad z, z_0 \in \mathbb{C}. \tag{2.7}$$

The following result shows the distinguished role played by the functions F_{z_0} in connection with extremal problems. A proof can be found in [56, Theorem 2.7]. For the sake of completeness we present a short and elementary proof which generalises in higher dimension.

Proposition 2.1 *Let $F \in \mathcal{F}^2(\mathbb{C})$. Then*

$$|F(z)|^2 e^{-\pi|z|^2} \le \|F\|^2_{\mathcal{F}^2} \qquad \forall z \in \mathbb{C}, \tag{2.8}$$

and $|F(z)|^2 e^{-\pi|z|^2}$ vanishes at infinity. Moreover the equality in (2.8) occurs at some point $z_0 \in \mathbb{C}$ if and only if $F = cF_{z_0}$ for some $c \in \mathbb{C}$.

Proof By homogeneity we can suppose $\|F\|_{\mathcal{F}^2} = 1$, hence $F = \sum_{k\ge 0} c_k e_k$ (cf. (2.2)), with $\sum_{k\ge 0}|c_k|^2 = 1$. By the Cauchy-Schwarz inequality we obtain

$$|F(z)|^2 \le \sum_{k\ge 0}|e_k(z)|^2 = \sum_{k\ge 0}\frac{\pi^k}{k!}|z|^{2k} = e^{\pi|z|^2} \quad \forall z \in \mathbb{C}.$$

Equality in this estimate occurs at some point $z_0 \in \mathbb{C}$ if and only if $c_k = ce^{-\pi|z_0|^2/2}\overline{e_k(z_0)}$, for some $c \in \mathbb{C}$, $|c| = 1$, which gives

$$F(z) = ce^{-\pi|z_0|^2/2} \sum_{k\geq 0} \frac{\pi^k}{k!}(z\overline{z_0})^k = cF_{z_0}(z).$$

Finally, the fact that $|F(z)|^2 e^{-\pi|z|^2}$ vanishes at infinity is clearly true if $F(z) = z^k$, $k \geq 0$, and therefore holds for every $F \in \mathcal{F}^2(\mathbb{C})$ by density, because of (2.8). □

3 Proof of the main results in dimension 1

In this section we prove Theorems 1.1 and 1.2. In fact, by the discussion in Sect. 2, cf. (2.4), these will follow (without further reference) from the following result, which will be proved at the end of this section, after a few preliminary results have been established.

Theorem 3.1 *For every $F \in \mathcal{F}^2(\mathbb{C})\setminus\{0\}$ and every measurable set $\Omega \subset \mathbb{R}^2$ of finite measure, we have*

$$\frac{\int_\Omega |F(z)|^2 e^{-\pi|z|^2}\, dm_2(z)}{\|F\|^2_{\mathcal{F}^2}} \leq 1 - e^{-|\Omega|}. \tag{3.1}$$

Moreover, recalling (2.7), equality occurs (for some F and for some Ω such that $0 < |\Omega| < \infty$) if and only if $F = cF_{z_0}$ (for some $z_0 \in \mathbb{C}$ and some nonzero $c \in \mathbb{C}$) and Ω is equivalent, up to a set of measure zero, to a ball centered at z_0.

Throughout the rest of this section, in view of proving (3.1), given an arbitrary function $F \in \mathcal{F}^2(\mathbb{C})\setminus\{0\}$ we shall investigate several properties of the function

$$u(z) := |F(z)|^2 e^{-\pi|z|^2}, \tag{3.2}$$

in connection with its super-level sets

$$A_t := \{u > t\} = \left\{z \in \mathbb{R}^2 \;:\; u(z) > t\right\}, \tag{3.3}$$

its *distribution function*

$$\mu(t) := |A_t|, \qquad 0 \leq t \leq \max_{\mathbb{C}} u \tag{3.4}$$

(note that u is bounded due to (2.8)), and the *decreasing rearrangement* of u, i.e. the function

$$u^*(s) := \sup\{t \geq 0 \;:\; \mu(t) > s\} \qquad \text{for } s \geq 0 \tag{3.5}$$

(for more details on rearrangements, we refer to [7]). Since $F(z)$ in (3.2) is entire holomorphic, u (which letting $z = x + i\omega$ can be regarded as a real-valued function $u(x, \omega)$ on $\mathbb{R}^2$) has several nice properties which will simplify our analysis. In particular, u is *real analytic* and hence, since u is not a constant, *every* level set of u has zero measure (see e.g. [39]), i.e.

$$|\{u = t\}| = 0 \quad \forall t \geq 0 \tag{3.6}$$

and, similarly, the set of all critical points of u has zero measure, i.e.

$$|\{|\nabla u| = 0\}| = 0. \tag{3.7}$$

Moreover, since by Proposition 2.1 $u(z) \to 0$ as $|z| \to \infty$, by Sard's Lemma we see that for a.e. $t \in (0, \max u)$ the super-level set $\{u > t\}$ is a bounded open set in $\mathbb{R}^2$ with smooth boundary

$$\partial\{u > t\} = \{u = t\} \quad \text{for a.e. } t \in (0, \max u). \tag{3.8}$$

Since $u(z) > 0$ a.e. (in fact everywhere, except at most at isolated points),

$$\mu(0) = \lim_{t \to 0^+} \mu(t) = +\infty,$$

while the finiteness of $\mu(t)$ when $t \in (0, \max u]$ is entailed by the fact that $u \in L^1(\mathbb{R}^2)$, according to (3.2) and (2.1) (in particular $\mu(\max u) = 0$). Moreover, by (3.6) $\mu(t)$ is *continuous* (and not just right-continuous) at *every point* $t \in (0, \max u]$. Since μ is also strictly decreasing, we see that u^*, according to (3.5), is just the elementarily defined *inverse function* of μ (restricted to $(0, \max u]$), i.e.

$$u^*(s) = \mu^{-1}(s) \quad \text{for } s \geq 0, \tag{3.9}$$

which maps $[0, +\infty)$ decreasingly and continuously onto $(0, \max u]$.

In the following we will strongly rely on the following result.

Lemma 3.2 *The function μ is absolutely continuous on the compact subintervals of* $(0, \max u]$*, and*

$$-\mu'(t) = \int_{\{u=t\}} |\nabla u|^{-1} \, d\mathcal{H}^1 \qquad \text{for a.e. } t \in (0, \max u). \tag{3.10}$$

Similarly, the function u^* is absolutely continuous on the compact subintervals of $[0, +\infty)$*, and*

$$-(u^*)'(s) = \left(\int_{\{u=u^*(s)\}} |\nabla u|^{-1} \, d\mathcal{H}^1 \right)^{-1} \qquad \text{for a.e. } s \geq 0. \tag{3.11}$$

These properties of μ and u^* are essentially well known to the specialists in rearrangement theory, and follow e.g. from the general results of [6,13], which are valid within the framework of $W^{1,p}$ functions (see also [15] for the framework of BV functions, in particular Lemmas 3.1 and 3.2). We point out, however, that of these properties only the absolute continuity of u^* is valid in general, while the others strongly depend on (3.7) which, in the terminology of [6], implies that u is *coarea regular* in a very strong sense, since it rules out the possibility of a singular part in the (negative) Radon measure $\mu'(t)$ and, at the same time, it guarantees that the density of the absolutely continuous part is given (only) by the right-hand side of (3.10). As clearly explained in the excellent Introduction to [6], there are several subtleties related to the structure of the distributional derivative of $\mu(t)$ (which ultimately make the validity of (3.11) highly nontrivial), and in fact the seminal paper [13] was motivated by a subtle error in a previous work, whose fixing since [13] has stimulated a lot of original and deep research (see e.g. [15,32] and references therein).

However, since unfortunately we were not able to find a ready-to-use reference for (3.11) (and, moreover, our u is very smooth but strictly speaking it does not belong to $W^{1,1}(\mathbb{R}^2)$, which would require to fix a lot of details when referring to the general results from [6,13,15]), here we present an elementary and self-contained proof of this lemma, specializing to our case a general argument from [13] based on the coarea formula.

Proof of Lemma 3.2 The fact that u is locally Lipschitz guarantees the validity of the coarea formula (see e.g. [13,24]), that is, for every Borel function $h : \mathbb{R}^2 \to [0, +\infty]$ we have

$$\int_{\mathbb{R}^2} h(z) |\nabla u(z)| \, dm_2(z) = \int_0^{\max u} \left(\int_{\{u=\tau\}} h \, d\mathcal{H}^1 \right) d\tau,$$

where $\mathcal{H}^1$ denotes the one-dimensional Hausdorff measure (and with the usual convention that $0 \cdot \infty = 0$ in the first integral). In particular, when $h(z) = \chi_{A_t}(z) |\nabla u(z)|^{-1}$ (where $|\nabla u(z)|^{-1}$ is meant as $+\infty$ if z is a critical point of u), by virtue of (3.7) the function $h(z)|\nabla u(z)|$ coincides with $\chi_{A_t}(z)$ a.e., and recalling (3.4) one obtains

$$\mu(t) = \int_t^{\max u} \left(\int_{\{u=\tau\}} |\nabla u|^{-1} \, d\mathcal{H}^1 \right) d\tau \qquad \forall t \in [0, \max u]; \tag{3.12}$$

therefore we see that $\mu(t)$ is *absolutely continuous* on the compact subintervals of $(0, \max u]$, and (3.10) follows.

Now let $D \subseteq (0, \max u)$ denote the set where $\mu'(t)$ exists and coincides with the integral in (3.10), and let $D_0 = (0, \max u]\setminus D$. By (3.10) and the absolute continuity of μ, and since the integral in (3.10) is strictly positive for *every* $t \in (0, \max u)$ (note that $\mathcal{H}^1(\{u = t\}) > 0$ for every $t \in (0, \max u)$, otherwise we would have that $|\{u > t\}| = 0$ by the isoperimetric inequality), we infer that $|D_0| = 0$, so that letting $\widehat{D} = \mu(D)$ and $\widehat{D_0} = \mu(D_0)$, one has $|\widehat{D_0}| = 0$ by the absolute continuity of μ, and $\widehat{D} = [0, +\infty)\setminus\widehat{D_0}$ since μ is invertible. On the other hand, by (3.9) and elementary calculus, we see that $(u^*)'(s)$ exists for *every* $s \in \widehat{D}$ and

$$-(u^*)'(s) = \frac{-1}{\mu'(\mu^{-1}(s))} = \left(\int_{\{u=u^*(s)\}} |\nabla u|^{-1}\, d\mathcal{H}^1\right)^{-1} \qquad \forall s \in \widehat{D},$$

which implies (3.11) since $|\widehat{D_0}| = 0$. Finally, since u^* is differentiable *everywhere* on $\widehat{D}$, it is well known that u^* maps every negligible set $N \subset \widehat{D}$ into a negligible set. Since $\widehat{D} \cup \widehat{D_0} = [0, +\infty)$, and moreover $u^*(\widehat{D_0}) = D_0$ where $|D_0| = 0$, we see that u^* maps negligible sets into negligible sets, hence it is absolutely continuous on every compact interval $[0, a]$. □

The following estimate for the integral in (3.11), which can be of some interest in itself, will be the main ingredient in the proof of Theorem 3.1.

Proposition 3.3 *We have*

$$\left(\int_{\{u=u^*(s)\}} |\nabla u|^{-1}\, d\mathcal{H}^1\right)^{-1} \le u^*(s) \qquad \text{for a.e. } s > 0, \tag{3.13}$$

and hence

$$(u^*)'(s) + u^*(s) \ge 0 \quad \text{for a.e. } s \ge 0. \tag{3.14}$$

Proof Letting for simplicity $t = u^*(s)$ and recalling that, for a.e. $t \in (0, \max u)$ (or, equivalently, for a.e. $s > 0$, since u^* and its inverse μ are absolutely continuous on compact sets) the super-level set A_t in (3.3) has a smooth boundary as in (3.8), we can combine the Cauchy-Schwarz inequality

$$\mathcal{H}^1(\{u = t\})^2 \le \left(\int_{\{u=t\}} |\nabla u|^{-1}\, d\mathcal{H}^1\right) \int_{\{u=t\}} |\nabla u|\, d\mathcal{H}^1 \tag{3.15}$$

with the isoperimetric inequality in the plane

$$4\pi\, |\{u > t\}| \le \mathcal{H}^1(\{u = t\})^2 \tag{3.16}$$

to obtain, after division by t,

$$t^{-1}\left(\int_{\{u=t\}} |\nabla u|^{-1}\, d\mathcal{H}^1\right)^{-1} \le \frac{\int_{\{u=t\}} \frac{|\nabla u|}{t}\, d\mathcal{H}^1}{4\pi\, |\{u>t\}|}. \tag{3.17}$$

The reason for dividing by t is that, in this form, the right-hand side turns out to be (quite surprisingly, at least to us) independent of t. Indeed, since along $\partial A_t = \{u = t\}$ we have $|\nabla u| = -\nabla u \cdot \nu$ where ν is the outer normal to ∂A_t, along $\{u = t\}$ we can interpret the quotient $|\nabla u|/t$ as $-(\nabla \log u)\cdot \nu$, and hence

$$\int_{\{u=t\}} \frac{|\nabla u|}{t}\, d\mathcal{H}^1 = -\int_{\partial A_t} (\nabla \log u)\cdot \nu\, d\mathcal{H}^1 = -\int_{A_t} \Delta \log u(z)\, dm_2(z).$$

But by (3.2), since $\log |F(z)|$ is a harmonic function, we obtain

$$\Delta(\log u(z)) = \Delta(\log |F(z)|^2 + \log e^{-\pi|z|^2}) = \Delta(-\pi |z|^2) = -4\pi, \tag{3.18}$$

so that the last integral equals $4\pi|A_t|$. Plugging this into (3.17), one obtains that the quotient on the right equals 1, and (3.13) follows. Finally, (3.14) follows on combining (3.11) with (3.13). □

The following lemma establishes a link between the integrals of u on its super-level sets (which will play a major role in our main argument) and the function u^*.

Lemma 3.4 *The function*

$$I(s) = \int_{\{u>u^*(s)\}} u(z)\, dm_2(z), \qquad s \in [0, +\infty), \tag{3.19}$$

i.e. the integral of u on its (unique) super-level set of measure s, is of class C^1 on $[0, +\infty)$*, and*

$$I'(s) = u^*(s) \quad \forall s \ge 0. \tag{3.20}$$

Moreover, I' is (locally) absolutely continuous, and

$$I''(s) + I'(s) \ge 0 \quad \text{for a.e. } s \ge 0. \tag{3.21}$$

Proof We have for every $h > 0$ and every $s \ge 0$

$$I(s+h) - I(s) = \int_{\{u^*(s+h)<u\le u^*(s)\}} u(z)\, dm_2(z)$$

and, since by (3.9) and (3.4) $|A_{u^*(\sigma)}| = \sigma$,

$$\left|\{u^*(s+h) < u \le u^*(s)\}\right| = |A_{u^*(s+h)}| - |A_{u^*(s)}| = (s+h) - s = h,$$

we obtain

$$u^*(s+h) \le \frac{I(s+h) - I(s)}{h} \le u^*(s).$$

Moreover, it is easy to see that the same inequality is true also when $h < 0$ (provided $s + h > 0$), now using the reverse set inclusion $A_{u^*(s+h)} \subset A_{u^*(s)}$ according to the fact that u^* is decreasing. Since u^* is continuous, (3.20) follows letting $h \to 0$ when $s > 0$, and letting $h \to 0^+$ when $s = 0$.

Finally, by Lemma 3.2, $I' = u^*$ is absolutely continuous on $[0, a]$ for every $a \ge 0$, $I'' = (u^*)'$, and (3.21) follows from (3.14). □

We are now in a position to prove Theorem 3.1.

Proof of Theorem 3.1 By homogeneity we can assume $\|F\|_{\mathcal{F}^2} = 1$ so that, defining u as in (3.2), (3.1) is equivalent to

$$\int_\Omega u(z)\, dm_2(z) \le 1 - e^{-s} \tag{3.22}$$

for every $s \ge 0$ and every $\Omega \subset \mathbb{R}^2$ such that $|\Omega| = s$. It is clear that, for any fixed measure $s \ge 0$, the integral on the left is maximized when Ω is the (unique by (3.6)) super-level set $A_t = \{u > t\}$ such that $|A_t| = s$ (i.e. $\mu(t) = s$), and by (3.9) we see that the proper cut level is given by $t = u^*(s)$. In other words, if $|\Omega| = s$ then

$$\int_\Omega u(z)\, dm_2(z) \le \int_{A_{u^*(s)}} u(z)\, dm_2(z), \tag{3.23}$$

with strict inequality unless Ω coincides –up to a negligible set– with $A_{u^*(s)}$ (to see this, it suffices to let $E := \Omega \cap A_{u^*(s)}$ and observe that, if $|\Omega \setminus E| > 0$, then the integral of u on $\Omega \setminus E$, where $u \le u^*(s)$, is strictly smaller than the integral of u on $A_{u^*(s)} \setminus E$, where $u > u^*(s)$). Thus, to prove (3.1) it suffices to prove (3.22) when $\Omega = A_{u^*(s)}$, that is, recalling (3.19), prove that

$$I(s) \le 1 - e^{-s} \qquad \forall s \ge 0 \tag{3.24}$$

or, equivalently, letting $s = -\log\sigma$, that

$$G(\sigma) := I(-\log\sigma) \le 1 - \sigma \qquad \forall \sigma \in (0, 1]. \tag{3.25}$$

Note that

$$G(1) = I(0) = \int_{\{u>u^*(0)\}} u(z)\, dm_2(z) = \int_{\{u>\max u\}} u(z)\, dm_2(z) = 0, \tag{3.26}$$

while by monotone convergence, since $\lim_{s\to+\infty} u^*(s) = 0$,

$$\begin{aligned}\lim_{\sigma\to 0^+} G(\sigma) = \lim_{s\to+\infty} I(s) &= \int_{\{u>0\}} u(z)\, dm_2(z) \\ &= \int_{\mathbb{R}^2} |F(z)|^2 e^{-\pi|z|^2}\, dm_2(z) = 1,\end{aligned} \tag{3.27}$$

because we assumed F is normalized. Thus, G extends to a continuous function on $[0, 1]$ that coincides with $1 - \sigma$ at the endpoints, and (3.25) will follow by proving that G is convex. Indeed, by (3.21), the function $e^s I'(s)$ is non decreasing, and since $G'(e^{-s}) = -e^s I'(s)$, this means that $G'(\sigma)$ is non decreasing as well, i.e. G is convex as claimed.

Summing up, via (3.23) and (3.24), we have proved that for every $s \geq 0$

$$\begin{aligned}&\int_\Omega |F(z)|^2 e^{-\pi|z|^2}\, dm_2(z) = \int_\Omega u(z)\, dm_2(z) \\ &\leq \int_{A_{u^*(s)}} u(z)\, dm_2(z) = I(s) \leq 1 - e^{-s}\end{aligned} \tag{3.28}$$

for every F such that $\|F\|_{\mathcal{F}^2} = 1$.

Now assume that equality occurs in (3.1), for some F (we may still assume $\|F\|_{\mathcal{F}^2} = 1$) and for some set Ω of measure $s_0 > 0$: then, when $s = s_0$, equality occurs everywhere in (3.28), i.e. in (3.23), whence Ω coincides with $A_{u^*(s_0)}$ up to a set of measure zero, and in (3.24), whence $I(s_0) = 1 - e^{-s_0}$. But then $G(\sigma_0) = 1 - \sigma_0$ in (3.25), where $\sigma_0 = e^{-s_0} \in (0, 1)$: since G is convex on $[0, 1]$, and coincides with $1 - \sigma$ at the endpoints, we infer that $G(\sigma) = 1 - \sigma$ for every $\sigma \in [0, 1]$, or, equivalently, that $I(s) = 1 - e^{-s}$ for *every* $s \geq 0$. In particular, $I'(0) = 1$; on the other hand, choosing $s = 0$ in (3.20) gives

$$I'(0) = u^*(0) = \max u,$$

so that $\max u = 1$. But then by (2.8)

$$1 = \max u = \max |F(z)|^2 e^{-\pi|z|^2} \leq \|F\|^2_{\mathcal{F}^2} = 1 \tag{3.29}$$

and, since equality is attained, by Proposition 2.1 we infer that $F = cF_{z_0}$ for some $z_0, c \in \mathbb{C}$. We have already proved that $\Omega = A_{u^*(s_0)}$ (up to a negligible set) and, since by (2.7)

$$u(z) = |cF_{z_0}(z)|^2 e^{-\pi|z|^2} = |c|^2 e^{-\pi|z_0|^2} e^{2\pi\, Re(z\overline{z_0})} e^{-\pi|z|^2} = |c|^2 e^{-\pi|z-z_0|^2} \tag{3.30}$$

has radial symmetry about z_0 and is radially decreasing, Ω is (equivalent to) a ball centered at z_0. This proves the "only if part" of the final claim being proved.

The "if part" follows by a direct computation. For, assume that $F = cF_{z_0}$ and Ω is equivalent to a ball of radius $r > 0$ centered at z_0. Then using (3.30) we can compute, using polar coordinates

$$\int_\Omega u(z)\, dm_2(z) = |c|^2 \int_{\{|z|<r\}} e^{-\pi|z|^2}\, dm_2(z) = 2\pi|c|^2 \int_0^\rho \rho e^{-\pi\rho^2}\, d\rho$$
$$= |c|^2(1 - e^{-\pi r^2}),$$

and equality occurs in (3.1) because $\|cF_{z_0}\|^2_{\mathcal{F}^2} = |c|^2$. □

Remark 3.5 The "only if part" in the final claim of Theorem 3.1, once one has established that $I(s) = 1 - e^{-s}$ for every $s \geq 0$, instead of using (3.29), can also be proved observing that there must be equality, for a.e. $t \in (0, \max u)$, both in (3.15) and in (3.16) (otherwise there would be a strict inequality in (3.14), hence also in (3.24), on a set of positive measure). But then, for at least one value (in fact, for infinitely many values) of t we would have that A_t is a ball $B(z_0, r)$ (by the equality in the isoperimetric estimate (3.16)) and that $|\nabla u|$ is constant along $\partial A_t = \{u = t\}$ (by the equality in (3.15)).

By applying the "translation" U_{z_0} (cf. (2.5) and (2.6)) we can suppose that the super-level set $A_t = B(z_0, r)$ is centred at the origin, i.e. that $z_0 = 0$, and in that case we have to prove that F is constant (so that, translating back to z_0, one obtains that the original F had the form cF_{z_0}). Since now both u and $e^{-|z|^2}$ are constant along $\partial A_t = \partial B(0, r)$, also $|F|$ is constant there (and does not vanish inside $\overline{B(0, r)}$, since $u \geq t > 0$ there). Hence $\log |F|$ is constant along $\partial B(0, r)$, and is harmonic inside $B(0, r)$ since F is holomorphic: therefore $\log |F|$ is constant in $B(0, r)$, which implies that F is constant over $\mathbb{C}$.

Note that the constancy of $|\nabla u|$ along ∂A_t has not been used. However, also this property alone (even ignoring that A_t is a ball) is enough to conclude. Letting $w = \log u$, one can use that both w and $|\nabla w|$ are constant along ∂A_t, and moreover $\Delta w = -4\pi$ as shown in (3.18): hence every connected component of A_t must be a ball, by a celebrated result of Serrin [50]. Then the previous argument can be applied to just one connected component of A_t, which is a ball, to conclude that F is constant.

4 The multidimensional case

In this section we prove Theorem 4.1, which extends Theorem 3.1 (hence Theorems 1.1 and 1.2) to arbitrary dimension $d \geq 1$. Comparing the right hand sides of (4.5) and (1.5), we see that the former is given in terms of the γ function defined in (4.3): of course, when $d = 1$ these two quantitites coincide, but the explicit form in (1.5) is certainly more pleasant.

The STFT of a function $f \in L^2(\mathbb{R}^d)$, with a given window $g \in L^2(\mathbb{R}^d)$, is defined as

$$\mathcal{V}_g f(x,\omega) := \int_{\mathbb{R}^d} e^{-2\pi i y\cdot\omega} f(y)\overline{g(y-x)}\, dy, \qquad x,\omega \in \mathbb{R}^d. \tag{4.1}$$

Consider now the Gaussian function

$$\varphi(x) = 2^{-d/4} e^{-\pi|x|^2} \qquad x \in \mathbb{R}^d, \tag{4.2}$$

and the corresponding STFT in (4.1) with window $g = \varphi$; let us write shortly $\mathcal{V} = \mathcal{V}_\varphi$. Let $\boldsymbol{\omega}_{2d}$ be the measure of the unit ball in $\mathbb{R}^{2d}$. Recall also the definition of the (lower) incomplete γ function as

$$\gamma(k,s) := \int_0^s \tau^{k-1} e^{-\tau}\, d\tau \tag{4.3}$$

where $k \geq 1$ is an integer and $s \geq 0$, so that

$$\frac{\gamma(k,s)}{(k-1)!} = 1 - e^{-s}\sum_{j=0}^{k-1}\frac{s^j}{j!}. \tag{4.4}$$

Theorem 4.1 (Faber–Krahn inequality for the STFT in dimension d) *For every measurable subset $\Omega \subset \mathbb{R}^{2d}$ of finite measure and for every $f \in L^2(\mathbb{R}^d)\setminus\{0\}$ there holds*

$$\frac{\int_\Omega |\mathcal{V} f(x,\omega)|^2\, dx d\omega}{\|f\|^2_{L^2}} \leq \frac{\gamma(d, c_\Omega)}{(d-1)!}, \tag{4.5}$$

where $c_\Omega := \pi(|\Omega|/\boldsymbol{\omega}_{2d})^{1/d}$ is the symplectic capacity of the ball in $\mathbb{R}^{2d}$ having the same volume as Ω.

Moreover, equality occurs (for some f and for some Ω such that $0 < |\Omega| < \infty$) if and only if Ω is equivalent, up to a set of measure zero, to a ball centered at some $(x_0, \omega_0) \in \mathbb{R}^{2d}$, and

$$f(x) = ce^{2\pi i x\cdot\omega_0}\varphi(x - x_0), \qquad c \in \mathbb{C}\setminus\{0\}, \tag{4.6}$$

where φ is the Gaussian in (4.2).

We recall that the symplectic capacity of a ball of radius r in phase space is πr^2 in every dimension and represents the natural measure of the size of the ball from the point of view of the symplectic geometry [19,36,37].

Proof of Theorem 4.1 The definition of the Fock space $\mathcal{F}^2(\mathbb{C})$ extends essentially verbatim to $\mathbb{C}^d$, with the monomials $(\pi^{|\alpha|}/\alpha!)^{1/2}z^\alpha$, $z \in \mathbb{C}^d$, $\alpha \in \mathbb{N}^d$ (multi-index notation) as orthonormal basis. The same holds for the definition of the functions F_{z_0} in (2.7), now with $z, z_0 \in \mathbb{C}^d$, and Proposition 2.1 extends in the obvious way too. Again one can rewrite the optimization problem in the Fock space $\mathcal{F}^2(\mathbb{C}^d)$, the formula (2.3) continuing to hold, with $x, \omega \in \mathbb{R}^d$. Hence we have to prove that

$$\frac{\int_\Omega |F(z)|^2 e^{-\pi|z|^2}\, dm_{2d}(z)}{\|F\|^2_{\mathcal{F}^2}} \le \frac{\gamma(d, c_\Omega)}{(d-1)!} \tag{4.7}$$

for $F \in \mathcal{F}^2(\mathbb{C}^d)\setminus\{0\}$ and $\Omega \subset \mathbb{C}^d$ of finite measure, where $dm_{2d}(z)$ denotes the Lebesgue measure on $\mathbb{C}^d$, and that equality occurs if and only if $F = cF_{z_0}$ and Ω is equivalent, up to a set of measure zero, to a ball centered at z_0.

To this end, for $F \in \mathcal{F}^2(\mathbb{C}^d)\setminus\{0\}$, $\|F\|_{\mathcal{F}^2} = 1$, we set $u(z) = |F(z)|^2 e^{-\pi|z|^2}$, $z \in \mathbb{C}^d$, exactly as in (3.2) when $d = 1$, and define A_t, $\mu(t)$ and $u^*(s)$ as in Sect. 3, replacing $\mathbb{R}^2$ with $\mathbb{R}^{2d}$ where necessary, now denoting by $|E|$ the $2d$-dimensional Lebesgue measure of a set $E \subset \mathbb{R}^{2d}$, in place of the 2-dimensional measure. Note that, now regarding u as a function of $2d$ real variables in $\mathbb{R}^{2d}$, properties (3.6), (3.7) etc. are still valid, as well as formulas (3.10), (3.11) etc., provided one replaces every occurrence of $\mathcal{H}^1$ with the $(2d-1)$-dimensional Hausdorff measure $\mathcal{H}^{2d-1}$. Following the same pattern as in Proposition 3.3, now using the isoperimetric inequality in $\mathbb{R}^{2d}$ (see e.g. [31] for an updated account)

$$\mathcal{H}^{2d-1}(\{u = t\})^2 \ge (2d)^2 \boldsymbol{\omega}_{2d}^{1/d} |\{u > t\}|^{(2d-1)/d}$$

and the fact that $\triangle \log u = -4\pi d$ on $\{u > 0\}$, we see that now u^* satisfies the inequality

$$\left(\int_{\{u=u^*(s)\}} |\nabla u|^{-1}\, d\mathcal{H}^{2d-1}\right)^{-1} \leq \pi d^{-1} \boldsymbol{\omega}_{2d}^{-1/d} s^{-1+1/d} u^*(s) \quad \text{for a.e. } s > 0$$

in place of (3.13), and hence (3.14) is to be replaced with

$$(u^*)'(s) + \pi d^{-1} \boldsymbol{\omega}_{2d}^{-1/d} s^{-1+1/d} u^*(s) \geq 0 \quad \text{for a.e. } s > 0.$$

Therefore, with the notation of Lemma 3.4, $I'(t)$ is locally absolutely continuous on $[0, +\infty)$ and now satisfies

$$I''(s) + \pi d^{-1} \boldsymbol{\omega}_{2d}^{-1/d} s^{-1+1/d} I'(s) \geq 0 \quad \text{for a.e. } s > 0.$$

This implies that the function $e^{\pi \boldsymbol{\omega}_{2d}^{-1/d} s^{1/d}} I'(s)$ is non decreasing on $[0, +\infty)$. Then, arguing as in the proof of Theorem 3.1, we are led to prove the inequality

$$I(s) \leq \frac{\gamma(d, \pi(s/\boldsymbol{\omega}_{2d})^{1/d})}{(d-1)!}, \qquad s \geq 0$$

in place of (3.24). This, with the substitution

$$\gamma(d, \pi(s/\boldsymbol{\omega}_{2d})^{1/d})/(d-1)! = 1 - \sigma, \qquad \sigma \in (0, 1]$$

(recall (4.4)), turns into

$$G(\sigma) := I(s) \leq 1 - \sigma \quad \forall \sigma \in (0, 1].$$

Again G extends to a continuous function on $[0, 1]$, with $G(0) = 1$, $G(1) = 0$. At this point one observes that, regarding σ as a function of s,

$$G'(\sigma(s)) = -d!\pi^{-d} \boldsymbol{\omega}_{2d} e^{\pi(s/\boldsymbol{\omega}_{2d})^{1/d}} I'(s).$$

Since the function $e^{\pi(s/\boldsymbol{\omega}_{2d})^{1/d}} I'(s)$ is non decreasing, we see that G' is non increasing on $(0, 1]$, hence G is convex on $[0, 1]$ and one concludes as in the proof of Theorem 3.1. Finally, the "if part" follows from a direct computation, similar to that at the end of the proof of Theorem 3.1, now integrating on a ball in dimension $2d$, and using (4.3) to evaluate the resulting integral. □

As a consequence of Theorem 4.1 we deduce a sharp form of the uncertainty principle for the STFT, which generalises Theorem 1.2 to arbitrary dimension. To replace the function $\log(1/\epsilon)$ in (1.6) (arising as the inverse function of e^{-s}

in the right-hand side of (1.5)), we now denote by $\psi_d(\epsilon)$, $0 < \epsilon \le 1$, the inverse function of

$$s \mapsto 1 - \frac{\gamma(d,s)}{(d-1)!} = e^{-s} \sum_{j=0}^{d-1} \frac{s^j}{j!}, \qquad s \ge 0$$

(cf. (4.4)).

Corollary 4.2 *If for some $\epsilon \in (0,1)$, some $f \in L^2(\mathbb{R}^d)\setminus\{0\}$, and some $\Omega \subset \mathbb{R}^{2d}$ we have $\int_\Omega |\mathcal{V} f(x,\omega)|^2\, dx d\omega \ge (1-\epsilon)\|f\|^2_{L^2}$, then*

$$|\Omega| \ge \boldsymbol{\omega}_{2d} \pi^{-d} \psi_d(\epsilon)^d, \tag{4.8}$$

with equality if and only if Ω is a ball and f has the form (4.6), where (x_0, ω_0) is the center of the ball.

So far, the state-of-the-art in this connection has been represented by the lower bound

$$|\Omega| \ge \sup_{p>2} (1-\epsilon)^{p/(p-2)} (p/2)^{2d/(p-2)} \tag{4.9}$$

(which reduces to (1.7) when $d = 1$, see [34, Theorem 3.3.3]). See Fig. 1 in the Appendix for a graphical comparison with (4.8) in dimension $d = 2$. Figure 2 in the Appendix illustrates Theorem 4.1 and Corollary 4.2.

Remark Notice that $\psi_1(\epsilon) = \log(1/\epsilon)$, and $\psi_d(\epsilon)$ is increasing with d. Moreover, it is easy to check that

$$\psi_d(\epsilon) \sim (d!)^{1/d} (1-\epsilon)^{1/d}, \quad \epsilon \to 1^-$$
$$\psi_d(\epsilon) \sim \log(1/\epsilon), \quad \epsilon \to 0^+.$$

On the contrary, the right-hand side of (4.9) is bounded by e^d; see Fig. 1 in the Appendix.

5 Local Lieb's uncertainty inequality and other generalizations

In this section we discuss some generalizations in several directions.

5.1 Local Lieb's uncertainty inequality for the STFT

An interesting variation on the theme is given by the optimization problem

$$\sup_{f\in L^2(\mathbb{R})\setminus\{0\}} \frac{\int_\Omega |\mathcal{V}f(x,\omega)|^p\, dx d\omega}{\|f\|_{L^2}^p}, \tag{5.1}$$

where $\Omega \subset \mathbb{R}^2$ is measurable subset of finite measure and $2 \le p < \infty$. Again, we look for the subsets Ω, of prescribed measure, which maximize the above supremum.

Observe, first of all, that by the Cauchy-Schwarz inequality, $\|\mathcal{V}f\|_{L^\infty} \le \|f\|_{L^2}$, so that the supremum in (5.1) is finite and, in fact, it is attained.

Proposition 5.1 *The supremum in* (5.1) *is attained.*

Proof The desired conclusion follows easily by the direct method of the calculus of variations. We first rewrite the problem in the complex domain via (2.3), as we did in Sect. 2, now ending up with the Rayleigh quotient

$$\frac{\int_\Omega |F(z)|^p e^{-p\pi|z|^2/2}\, dm_2(z)}{\|F\|_{\mathcal{F}^2}^p}$$

with $F \in \mathcal{F}^2(\mathbb{C})\setminus\{0\}$. It is easy to see that this expression attains a maximum at some $F \in \mathcal{F}^2(\mathbb{C})\setminus\{0\}$. In fact, let $F_n \in \mathcal{F}^2(\mathbb{C})$, $\|F_n\|_{\mathcal{F}^2} = 1$, be a maximizing sequence, and let $u_n(z) = |F_n(z)|^p e^{-p\pi|z|^2/2}$. Since $u_n(z) = (|F_n(z)|^2 e^{-\pi|z|^2})^{p/2} \le \|F_n\|_{\mathcal{F}^2}^p = 1$ by Proposition 2.1, we see that the sequence F_n is equibounded on the compact subsets of $\mathbb{C}$. Hence there is a subsequence, that we continue to call F_n, uniformly converging on the compact subsets to a holomorphic function F. By the Fatou theorem, $F \in \mathcal{F}^2(\mathbb{C})$ and $\|F\|_{\mathcal{F}^2} \le 1$. Now, since Ω has finite measure, for every $\epsilon > 0$ there exists a compact subset $K \subset \mathbb{C}$ such that $|\Omega\setminus K| < \epsilon$, so that $\int_{\Omega\setminus K} u_n(z)\, dm_2(z) < \epsilon$ and $\int_{\Omega\setminus K} |F(z)|^p e^{-p\pi|z|^2/2}\, dm_2(z) < \epsilon$. Together with the already mentioned convergence on the compact subsets, this implies that $\int_\Omega u_n(z)\, dm_2(z) \to \int_\Omega |F(z)|^p e^{-p\pi|z|^2/2}\, dm_2(z)$. As a consequence, $F \ne 0$ and, since $\|F\|_{\mathcal{F}^2} \le 1 = \|F_n\|_{\mathcal{F}^2}$,

$$\lim_{n\to\infty} \frac{\int_\Omega |F_n(z)|^p e^{-p\pi|z|^2/2}\, dm_2(z)}{\|F_n\|_{\mathcal{F}^2}^p} \le \frac{\int_\Omega |F(z)|^p e^{-p\pi|z|^2/2}\, dm_2(z)}{\|F\|_{\mathcal{F}^2}^p}.$$

The reverse inequality is obvious, because F_n is a maximizing sequence. □

Theorem 5.2 (Local Lieb's uncertainty inequality for the STFT) *Let* $2 \leq p < \infty$*. For every measurable subset* $\Omega \subset \mathbb{R}^2$ *of finite measure, and every* $f \in L^2(\mathbb{R})\backslash\{0\}$,

$$\frac{\int_\Omega |\mathcal{V} f(x,\omega)|^p\, dx d\omega}{\|f\|_{L^2}^p} \leq \frac{2}{p}\Big(1 - e^{-p|\Omega|/2}\Big). \tag{5.2}$$

Moreover, equality occurs (for some f *and for some* Ω *such that* $0 < |\Omega| < \infty$*) if and only if* Ω *is equivalent, up to a set of measure zero, to a ball centered at some* $(x_0, \omega_0) \in \mathbb{R}^2$*, and*

$$f(x) = c e^{2\pi i x \omega_0} \varphi(x - x_0), \qquad c \in \mathbb{C}\backslash\{0\},$$

where φ *is the Gaussian in* (1.1).

Observe that when $p = 2$ this result reduces to Theorem 1.1. Moreover, by monotone convergence, from (5.2) we obtain

$$\int_{\mathbb{R}^2} |\mathcal{V} f(x,\omega)|^p\, dx d\omega \leq \frac{2}{p}\|f\|_{L^2}^p, \quad f \in L^2(\mathbb{R}), \tag{5.3}$$

which is Lieb's inequality for the STFT with a Gaussian window [25] (see also [42] and [34, Theorem 3.3.2]). Actually, (5.3) will be an ingredient of the proof of Theorem 5.2.

Proof of Theorem 5.2 Transfering the problem in the Fock space $\mathcal{F}^2(\mathbb{C})$, it is equivalent to prove that

$$\frac{\int_\Omega |F(z)|^p e^{-p\pi|z|^2/2}\, dm_2(z)}{\|F\|_{\mathcal{F}^2}^p} \leq \frac{2}{p}\Big(1 - e^{-p|\Omega|/2}\Big) \tag{5.4}$$

for $F \in \mathcal{F}^2(\mathbb{C})\backslash\{0\}$, $0 < |\Omega| < \infty$, and that the extremals are given by the functions $F = cF_{z_0}$ in (2.7), together with the balls Ω of center z_0. We give only a sketch of the proof, since the argument is similar to the proof of Theorem 3.1. Assuming $\|F\|_{\mathcal{F}^2} = 1$ and setting $u(z) = |F(z)|^p e^{-p\pi|z|^2/2}$, arguing as in the proof of Proposition 3.3 we obtain that

$$\left(\int_{\{u=u^*(s)\}} |\nabla u|^{-1}\, d\mathcal{H}^1\right)^{-1} \leq \frac{p}{2} u^*(s) \qquad \text{for a.e. } s > 0,$$

which implies $(u^*)'(s) + \frac{p}{2}u^*(s) \geq 0$ for a.e. $s \geq 0$. With the notation of Lemma 3.4 we obtain $I''(s) + \frac{p}{2}I'(s) \geq 0$ for a.e. $s \geq 0$, i.e. $e^{sp/2}I'(s)$ is non decreasing on $[0, +\infty)$. Arguing as in the proof of Theorem 3.1 we reduce ourselves to study the inequality $I(s) \leq \frac{2}{p}(1-e^{-ps/2})$ or equivalently, changing variable $s = -\frac{2}{p}\log\sigma$, $\sigma \in (0,1]$,

$$G(\sigma) := I\Big(-\frac{2}{p}\log\sigma\Big) \leq \frac{2}{p}(1-\sigma) \qquad \forall\sigma \in (0,1]. \tag{5.5}$$

We can prove this inequality and discuss the case of strict inequality as in the proof of Theorem 3.1, the only difference being that now $G(0) := \lim_{\sigma\to 0^+} G(\sigma) = \int_{\mathbb{R}^2} u(z)\, dm_2(z) \leq 2/p$ by (5.3) (hence, at $\sigma = 0$ strict inequality may occur in (5.5), but this is enough) and, when in (5.5) the equality occurs for some (and hence for every) $\sigma \in [0,1]$, in place of (3.29) we will have

$$\begin{aligned} 1 = \max u = \max |F(z)|^p e^{-p\pi|z|^2/2} &= (\max |F(z)|^2 e^{-\pi|z|^2})^{p/2} \\ &\leq \|F\|^p_{\mathcal{F}^2} = 1. \end{aligned}$$

The "if part" follows by a direct computation. □

Similarly, arguing as in the proof of Theorem 4.1 one obtains, in dimension d and for $2 \leq p < \infty$,

$$\frac{\int_\Omega |\mathcal{V}f(x,\omega)|^p\, dx d\omega}{\|f\|^p_{L^2}} \leq \Big(\frac{2}{p}\Big)^d \frac{\gamma(d, pc_\Omega/2)}{(d-1)!},$$

where $\Omega \subset \mathbb{R}^{2d}$ and $c_\Omega := \pi(|\Omega|/\boldsymbol{\omega}_{2d})^{1/d}$ is the symplectic capacity of the ball in $\mathbb{R}^{2d}$ having the same volume as Ω (the function γ is defined in (4.3)).

We also observe that we do not expect similar results for $p < 2$. Indeed, on the one hand, we know from [42] that the global estimate (5.3) holds reversed, in that case. However, the local estimate (5.4) continues to hold as it is when $s = |\Omega|$ is small (depending on F), because the right-hand side is $\sim s$ as $s \to 0^+$, whereas if $\|F\|_{\mathcal{F}^2} = 1$ and Ω is the super-level set of measure s of $u(z) = |F(z)|^p e^{-p\pi|z|^2/2}$, the left-hand side is $\sim s \max_{z\in\mathbb{C}} u(z)$, by (3.20), and such a maximum is strictly less than 1 if F is not one of the functions in (2.7).

5.2 L^p-concentration estimates for the STFT

We consider now the problem of the concentration in the time-frequency plane in the L^p sense, which has interesting applications to signal recovery (see e.g. [4]). More precisely, Theorem 5.3 below proves a conjecture of Abreu and Speckbacher [5, Conjecture 1] (note that when $p = 2$ one obtains Theorem 1.1).

Let $\mathcal{S}'(\mathbb{R})$ be the space of temperate distributions and, for $p \geq 1$, consider the subspace (called *modulation space* in the literature [34])

$$M^p(\mathbb{R}) := \{f \in \mathcal{S}'(\mathbb{R}) : \|f\|_{M^p} := \|\mathcal{V}f\|_{L^p(\mathbb{R}^2)} < \infty\}.$$

In fact, the definition of STFT (with Gaussian or more generally Schwartz window) and the Bargmann transform $\mathcal{B}$ extend (in an obvious way) to injective operators on $\mathcal{S}'(\mathbb{R})$. It is clear that $\mathcal{V} : M^p(\mathbb{R}) \to L^p(\mathbb{R}^2)$ is an isometry, and it can be proved that $\mathcal{B}$ is a *surjective* isometry from $M^p(\mathbb{R})$ onto the space $\mathcal{F}^p(\mathbb{C})$ of holomorphic functions $F(z)$ satisfying

$$\|F\|_{\mathcal{F}^p} := \Big(\int_{\mathbb{C}} |F(z)|^p e^{-p\pi|z|^2/2}\, dm_2(z) \Big)^{1/p} < \infty,$$

see e.g. [51] for a comprehensive discussion. Moreover the formula (2.3) continues to hold for $f \in \mathcal{S}'(\mathbb{R})$.

Theorem 5.3 (L^p-concentration estimates for the STFT) *Let $1 \leq p < \infty$. For every measurable subset $\Omega \subset \mathbb{R}^2$ of finite measure and every $f \in M^p(\mathbb{R})\setminus\{0\}$,*

$$\frac{\int_\Omega |\mathcal{V}f(x,\omega)|^p\, dx d\omega}{\int_{\mathbb{R}^2} |\mathcal{V}f(x,\omega)|^p\, dx d\omega} \leq 1 - e^{-p|\Omega|/2}. \tag{5.6}$$

Moreover, equality occurs (for some f and for some Ω such that $0 < |\Omega| < \infty$) if and only if Ω is equivalent, up to a set of measure zero, to a ball centered at some $(x_0, \omega_0) \in \mathbb{R}^2$, and

$$f(x) = c e^{2\pi i x \omega_0} \varphi(x - x_0), \qquad c \in \mathbb{C}\setminus\{0\}, \tag{5.7}$$

where φ is the Gaussian in (1.1).

We omit the proof, that is very similar to that of Theorem 3.1. We just observe that (2.8) extends to any $p \in [1, \infty)$ as

$$|F(z)|^p e^{-p\pi|z|^2/2} \leq \frac{p}{2} \|F\|^p_{\mathcal{F}^p}$$

and again the equality occurs at some point $z_0 \in \mathbb{C}$ if and only if $F = cF_{z_0}$, for some $c \in \mathbb{C}$ (in particular, $\|F_{z_0}\|^p_{\mathcal{F}^p} = 2/p$); see e.g. [56, Theorem 2.7].

As a consequence we obtain at once the following sharp uncertainty principle for the STFT.

Corollary 5.4 *Let $1 \leq p < \infty$. If for some $\epsilon \in (0,1)$, some function $f \in M^p(\mathbb{R})\setminus\{0\}$ and some $\Omega \subset \mathbb{R}^2$ we have $\int_\Omega |\mathcal{V}f(x,\omega)|^p\, dx d\omega \geq$*

$(1-\epsilon)\|f\|^p_{M^p}$, *then necessarily*

$$|\Omega| \geq \frac{2}{p}\log(1/\epsilon), \tag{5.8}$$

with equality if and only if Ω *is a ball and* f *has the form* (5.7)*, where* φ *is the Gaussian in* (1.1) *and* (x_0, ω_0) *is the center of the ball.*

We point out that, in the case $p = 1$, the following rougher –but valid for any window in $M^1(\mathbb{R})\backslash\{0\}$– lower bound

$$|\Omega| \geq 4(1-\epsilon)^2$$

was obtained in [35, Proposition 2.5.2]. Arguing as in Sect. 4 it would not be difficult to suitably generalise Theorem 5.3 and Corollary 5.4 in arbitrary dimension.

5.3 Changing window

Theorem 4.1 can be suitably reformulated when the Gaussian window φ in (4.2) is dilated or, more generally, replaced by $\mu(\mathcal{A})\varphi$, where $\mu(\mathcal{A})$ is a metaplectic operator associated with a symplectic matrix $\mathcal{A} \in Sp(d,\mathbb{R})$ (recall that, in dimension 1, $Sp(1,\mathbb{R}) = SL(2,\mathbb{R})$ is the special linear group of 2×2 real matrices with determinant 1).

We address to [35, Section 9.4] for a detailed introduction to the metaplectic representation. Roughly speaking one associates, with any matrix $\mathcal{A} \in Sp(d,\mathbb{R})$, a unitary operator $\mu(\mathcal{A})$ on $L^2(\mathbb{R}^d)$ defined up to a phase factor, providing a projective unitary representation of $Sp(d,\mathbb{R})$ on $L^2(\mathbb{R}^d)$. In more concrete terms, we know that $Sp(d,\mathbb{R})$ is generated by matrices of the type (in block-matrix notation)

$$\mathcal{A}_1 = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \qquad \mathcal{A}_2 = \begin{pmatrix} A & 0 \\ 0 & A^{*-1} \end{pmatrix} \qquad \mathcal{A}_3 = \begin{pmatrix} I & 0 \\ C & I \end{pmatrix}$$

where $A \in GL(d,\mathbb{R})$, and C is real and symmetric (I denoting the identity matrix). The corresponding operators are then given by $\mu(\mathcal{A}_1) = \mathcal{F}$ (Fourier transform), $\mu(\mathcal{A}_2)f(x) = |\det A|^{-1/2} f(A^{-1}x)$ and $\mu(\mathcal{A}_3)f(x) = e^{\pi i Cx\cdot x} f(x)$ (up to a phase factor). Now, the relevant property of the STFT is its symplectic covariance (see [34, Lemma 9.4.3]):

$$|\mathcal{V}_{\mu(\mathcal{A})\varphi}(\mu(\mathcal{A})f)(x,\omega)| = |\mathcal{V}_\varphi(f)(\mathcal{A}^{-1}(x,\omega))|.$$

As a consequence, if we define, for $g, f \in L^2(\mathbb{R}^d)\setminus\{0\}$, the quotients

$$\Phi_{\Omega,g}(f) := \frac{\int_\Omega |\mathcal{V}_g f(x,\omega)|^2\, dx d\omega}{\int_{\mathbb{R}^{2d}} |\mathcal{V}_g f(x,\omega)|^2\, dx d\omega},$$

we obtain (since $\det \mathcal{A} = 1$)

$$\Phi_{\Omega,\mu(\mathcal{A})\varphi}(\mu(\mathcal{A})f) = \Phi_{\mathcal{A}^{-1}(\Omega),\varphi}(f).$$

Hence, since $\mathcal{A}$ is measure preserving and $\mu(\mathcal{A})$ is a unitary operator, we deduce at once from Theorem 4.1 that for every measurable subset $\Omega \subset \mathbb{R}^{2d}$ of finite measure and every $f \in L^2(\mathbb{R}^d)\setminus\{0\}$,

$$\frac{\int_\Omega |\mathcal{V}_{\mu(\mathcal{A})\varphi} f(x,\omega)|^2\, dx d\omega}{\|f\|_{L^2}^2} \le \frac{\gamma(d, c_\Omega)}{(d-1)!}$$

with $c_\Omega = \pi(|\Omega|/\boldsymbol{\omega}_{2d})^{1/d}$. Moreover, the equality occurs if and only if $f = \mu(\mathcal{A})M_{\omega_0}T_{x_0}\varphi$ (recall (4.2)) and Ω is equivalent, up to a set of measure zero, to $\mathcal{A}(B)$ for some ball $B \subset \mathbb{R}^{2d}$ centered at (x_0, ω_0), where $T_{x_0} f(x) = f(x - x_0)$ and $M_{\omega_0} f(x) = e^{2\pi i x\cdot\omega_0} f(x)$.

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Appendix

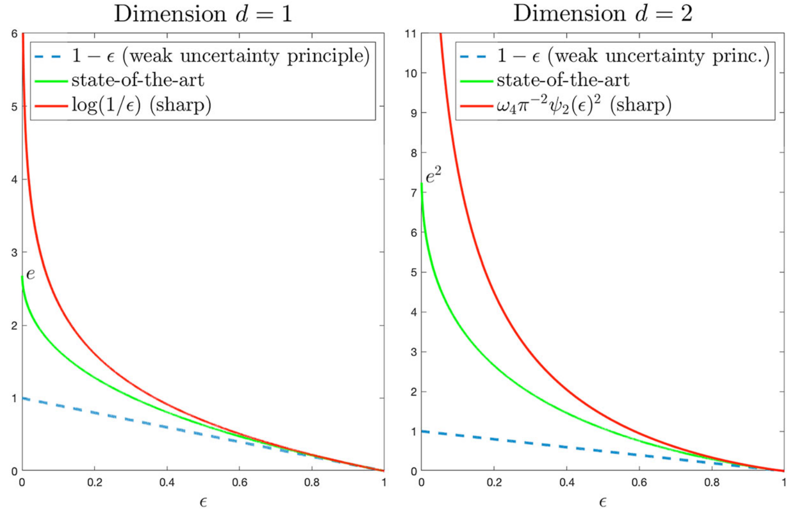

Fig. 1 Left: in dimension 1, assuming that $\Omega \subset \mathbb{R}^2$ captures a fraction $1-\epsilon$ of the energy of some function $f \in L^2(\mathbb{R})$, comparison between the lower bound for $|\Omega|$ in (1.7) (state-of-the-art), the sharp lower bound $\log(1/\epsilon)$ in (1.6) and the so-called *weak uncertainty principle* $|\Omega| \geq 1-\epsilon$ [34, Proposition 3.3.1] (which follows at once from the elementary estimate $\|\mathcal{V}f\|_{L^\infty} \leq \|f\|_{L^2}$). Right: the same comparison in dimension $d = 2$. Here the state-of-the-art is represented by (4.9), whereas the sharp bound $|\Omega| \geq \boldsymbol{\omega}_4\pi^{-2}\psi_2(\epsilon)^2$ is given in (4.8) ($\boldsymbol{\omega}_4 = \pi^2/2$)

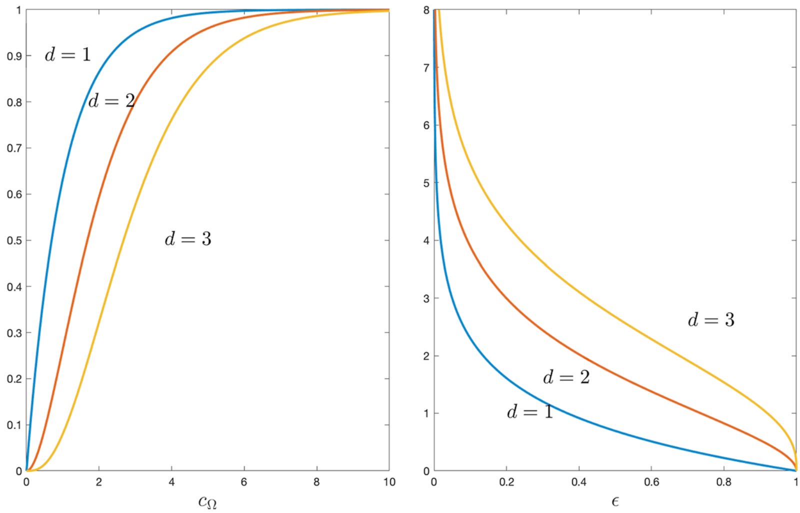

Fig. 2 Left: The upper bound $\gamma(d, c_\Omega)/(d-1)!$ in (4.5), for $d = 1, 2, 3$, as a function of $c_\Omega = \pi(|\Omega|/\boldsymbol{\omega}_{2d})^{1/d}$. Right: The lower bound $\psi_d(\epsilon)$ for c_Ω in (4.8), for $d = 1, 2, 3$. Recall, $\psi_d(\epsilon)$ is the inverse function of $1 - \gamma(d, s)/(d-1)!$, in particular $\psi_1(\epsilon) = \log(1/\epsilon)$

References

1. Abreu, L.D., Dörfler, M.: An inverse problem for localization operators. Inverse Prob. **28**(11), 115001 (2012)
2. Abreu, L.D., Gröchenig, K., Romero, J.L.: On accumulated spectrograms. Trans. Am. Math. Soc. **368**(5), 3629–3649 (2016)
3. Abreu, L.D., Pereira, J.A.M., Romero, J.L.: Sharp rates of convergence for accumulated spectrograms. Inverse Prob. **33**(11), 115008 (2017)
4. Abreu, L.D., Speckbacher, M.: Deterministic guarantees for L^1-reconstruction: a large sieve approach with geometric flexibility. In: IEEE Proceedings SampTA (2019)
5. Abreu, L.D., Speckbacher, M.: Donoho-Logan large sieve principles for modulation and polyanalytic Fock spaces. Bull. Sci. Math. **171**, 103032 (2021)
6. Almgren, F.J., Jr., Lieb, E.H.: Symmetric decreasing rearrangement is sometimes continuous. J. Am. Math. Soc. **2**(4), 683–773 (1989)
7. Baernstein, A.: II. Symmetrization in analysis, Volume 36 of New Mathematical Monographs. Cambridge University Press, Cambridge. With David Drasin and Richard S. Laugesen, with a foreword by Walter Hayman (2019)
8. Berezin, F.A.: Wick and anti-Wick symbols of operators. Mat. Sb. (N.S.) **86**(128), 578–610 (1971)
9. Berezin, F.A., Shubin, M.A.: The Schrödinger Equation, Volume 66 of Mathematics and its Applications (Soviet Series). Kluwer Academic Publishers Group, Dordrecht (1991) [Translated from the 1983 Russian edition by Yu. Rajabov, D. A. Leĭtes and N. A. Sakharova and revised by Shubin. With contributions by G. L. Litvinov and Leĭtes]
10. Boggiatto, P., Carypis, E., Oliaro, A.: Two aspects of the Donoho–Stark uncertainty principle. J. Math. Anal. Appl. **434**(2), 1489–1503 (2016)
11. Bombieri, E.: A note on the large sieve. Acta Arith. **18**, 401–404 (1971)
12. Bonami, A., Demange, B.: A survey on uncertainty principles related to quadratic forms. Collect. Math. 1–36 (2006)
13. Brothers, J.E., Ziemer, W.P.: Minimal rearrangements of Sobolev functions. J. Reine Angew. Math. **384**, 153–179 (1988)
14. Bucur, D., Buttazzo, G.: Variational methods in shape optimization problems. In: Progress in Nonlinear Differential Equations and their Applications, vol. 65. Birkhäuser Boston Inc, Boston (2005)
15. Cianchi, A., Fusco, N.: Functions of bounded variation and rearrangements. Arch. Ration. Mech. Anal. **165**(1), 1–40 (2002)
16. Cordero, E., Rodino, L., Gröchenig, K.: Localization operators and time-frequency analysis. In: Harmonic Wavelet and p-adic Analysis, pp. 83–110. World Sci. Publ, Hackensack (2007)
17. Cowling, M.G., Price, J.F.: Bandwidth versus time concentration: the Heisenberg–Pauli–Weyl inequality. SIAM J. Math. Anal. **15**(1), 151–165 (1984)
18. Daubechies, I.: Time-frequency localization operators: a geometric phase space approach. IEEE Trans. Inform. Theory **34**(4), 605–612 (1988)
19. de Gosson, M.A.: The symplectic camel and the uncertainty principle: the tip of an iceberg? Found. Phys. **39**(2), 194–214 (2009)
20. De Mari, F., Feichtinger, H.G., Nowak, K.: Uniform eigenvalue estimates for time-frequency localization operators. J. Lond. Math. Soc. (2) **65**(3), 720–732 (2002)
21. Demange, B.: Uncertainty principles for the ambiguity function. J. Lond. Math. Soc. (2) **72**(3), 717–730 (2005)
22. Donoho, D.L., Logan, B.F.: Signal recovery and the large sieve. SIAM J. Appl. Math. **52**(2), 577–591 (1992)
23. Donoho, D.L., Stark, P.B.: Uncertainty principles and signal recovery. SIAM J. Appl. Math. **49**(3), 906–931 (1989)

24. Evans, L.C., Gariepy, R.F.: Measure Theory and Fine Properties of Functions. Textbooks in Mathematics, revised CRC Press, Boca Raton (2015)
25. Elliott, H., Lieb (1978) Proof of an entropy conjecture of Wehrl. Commun. Math. Phys. **62**(1), 35–41. https://doi.org/10.1007/BF01940328
26. Fefferman, C.L.: The uncertainty principle. Bull. Am. Math. Soc. (N.S.) **9**(2), 129–206 (1983)
27. Fernández, C., Galbis, A.: Annihilating sets for the short time Fourier transform. Adv. Math. **224**(5), 1904–1926 (2010)
28. Folland, G.B.: Harmonic Analysis in Phase Space. Annals of Mathematics Studies, vol. 122. Princeton University Press, Princeton (1989)
29. Folland, G.B., Sitaram, A.: The uncertainty principle: a mathematical survey. J. Fourier Anal. Appl. **3**(3), 207–238 (1997)
30. Fuchs, W.H.J.: On the magnitude of Fourier transforms. In: Proceedings of the International Congress of Mathematicians, Vol. II, pp. 106–107. North-Holland Publishing C.O., Amsterdam (1954)
31. Fusco, N.: The quantitative isoperimetric inequality and related topics. Bull. Math. Sci. **5**(3), 517–607 (2015)
32. Fusco, N., Maggi, F., Pratelli, A.: The sharp quantitative isoperimetric inequality. Ann. Math. (2) **168**(3), 941–980 (2008)
33. Galbis, A.: Norm estimates for selfadjoint Toeplitz operators on the Fock space. Complex Anal. Oper. Theory **16**(1), 15 (2022)
34. Gröchenig, K.: Foundations of Time-Frequency Analysis. Applied and Numerical Harmonic Analysis, Birkhäuser Boston Inc, Boston, MA (2001)
35. Gröchenig, K.: Uncertainty principles for time-frequency representations. In: Advances in Gabor Analysis, Applied and Numerical Harmonic Analysis, pp. 11–30. Birkhäuser Boston, Boston (2003)
36. Gromov, M.: Pseudo holomorphic curves in symplectic manifolds. Invent. Math. **82**(2), 307–347 (1985)
37. Hofer, H., Zehnder, E.: Symplectic invariants and Hamiltonian dynamics. Modern Birkhäuser Classics. Birkhäuser Verlag, Basel (2011). (Reprint of the 1994 edition)
38. Hörmander, L.: The analysis of linear partial differential operators. I. Classics in Mathematics (2003). Distribution theory and Fourier analysis, Reprint of the second (1990) edition. Springer, Berlin
39. Krantz, S.G., Parks, H.R.: A primer of real analytic functions. Birkhäuser Advanced Texts: Basler Lehrbücher. [Birkhäuser Advanced Texts: Basel Textbooks]. Birkhäuser Boston, Inc., Boston, MA, second edition (2002)
40. Landau, H.J.: An overview of time and frequency limiting. In: Fourier Techniques and Applications (Kensington, 1983), pp. 201–220. Plenum, New York (1985)
41. Landau, H.J., Pollak, H.O.: Prolate spheroidal wave functions, Fourier analysis and uncertainty II. Bell Syst. Tech. J. **40**, 65–84 (1961)
42. Lieb, E.H.: Integral bounds for radar ambiguity functions and Wigner distributions. J. Math. Phys. **31**(3), 594–599 (1990)
43. Lieb, E.H., Loss, M.: Analysis. Graduate Studies in Mathematics, vol. 14, 2nd edn. American Mathematical Society, Providence (2001)
44. Mallat, S.: A Wavelet Tour of Signal Processing. The Sparse Way, 3rd edn. Elsevier, Amsterdam. (With contributions from Gabriel Peyré) (2009)
45. Marceca, F., Romero, J.L.: Spectral deviation of concentration operators for the short-time Fourier transform. arXiv:2104.06150
46. Muscalu, C., Schlag, W.: Classical and Multilinear Harmonic Analysis. Cambridge Studies in Advanced Mathematics, vol. 1, 137th edn. Cambridge University Press, Cambridge (2013)

47. Oldfield, J.P.: Two-term Szegő theorem for generalised anti-Wick operators. J. Spectr. Theory **5**(4), 751–781 (2015)
48. Pfander, G.E., Rauhut, H.: Sparsity in time-frequency representations. J. Fourier Anal. Appl. **16**(2), 233–260 (2010)
49. Pfander, G.E., Rauhut, H., Tropp, J.A.: The restricted isometry property for time-frequency structured random matrices. Probab. Theory Relat. Fields **156**(3–4), 707–737 (2013)
50. Serrin, J.: A symmetry problem in potential theory. Arch. Rational Mech. Anal. **43**, 304–318 (1971)
51. Signahl, M., Toft, J.: Mapping properties for the Bargmann transform on modulation spaces. J. Pseudo-Differ. Oper. Appl. **3**(1), 1–30 (2012)
52. Slepian, D.: Some comments on Fourier analysis, uncertainty and modeling. SIAM Rev. **25**(3), 379–393 (1983)
53. Tao, T.: An uncertainty principle for cyclic groups of prime order. Math. Res. Lett. **12**(1), 121–127 (2005)
54. Tataru, D.: Phase space transforms and microlocal analysis. In: Phase Space Analysis of Partial Differential Equations. Vol. II, Pubbl. Cent. Ric. Mat. Ennio De Giorgi, pp. 505–524. Scuola Norm. Sup., Pisa (2004)
55. Wong, M.W.: Wavelet Transforms and Localization Operators. Operator Theory: Advances and Applications, vol. 136. Birkhäuser Verlag, Basel (2002)
56. Zhu, K.: Analysis on Fock Spaces. Graduate Texts in Mathematics, vol. 263. Springer, New York (2012)