Wave Packet Analysis of Feynman Path Integrals

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Preface

This book provides an accessible and essentially self-contained presentation of some mathematical aspects of Feynman path integrals in non-relativistic quantum mechanics. In spite of the primary role in the advancement of modern theoretical physics and the wide range of applications, path integrals are still a source of challenging problems for mathematicians. From this standpoint, path integrals can be roughly viewed as approximation formulas for an operator (usually the propagator of a Schrödinger-type evolution equation) involving a suitably designed sequence of operators and different levels of convergence.

There are many excellent treatises on the mathematics of path integrals, regarded from different angles and for different approximation schemes [4, 39, 106, 154, 157, 174, 209], but the results contained in this book have never appeared before in a monograph. Indeed, they reflect the topics addressed in the research activity of the authors over the last years. In keeping with the spirit of harmonic analysis, we focus on the issue of the pointwise convergence of the integral kernels of the approximation operators, as well as the convergence in the spaces of bounded operators on L^2 and on L^p-based Sobolev spaces, $1 < p < \infty$, with optimal loss of derivatives. In addition to the case of smooth potentials, we consider low-regular potentials in Kato-Sobolev spaces or in the so-called Sjöstrand class, which contains the class of potentials considered by S. Albeverio et al. [1–4] and K. Itô [146], namely those coinciding with the Fourier transform of a complex (finite) measure.

In short, the analysis is based on the decomposition of functions and operators by means of the so-called Gabor wave packets, using a machinery developed mainly (but not only) by H. Feichtinger and K. Gröchenig and collaborators over the last thirty years in the context of time-frequency analysis and, with a different terminology, by several authors working on partial differential equations - especially A. Boulkhemair, C. Fefferman, J. Sjöstrand and D. Tataru. This framework allows one to lift the original problem from the configuration space to the phase space, the subtle oscillation issues being embedded into this lifting procedure. The outcome of this approach involves classes of integral operators in phase space - with continuous and band-dominated kernel - which turn out to be much more tamable than the original pseudodifferential and (global) Fourier integral operators with regard to boundedness, composition and inversion properties.

These techniques have shown to be surprisingly successful in dealing with path integrals. For instance, they were recently used in [184] to prove the *pointwise convergence of the integral kernels* – in quantum parlance, the transition amplitudes in the position representation – in the Feynman-Trotter formula for low-regular potentials in the Sjöstrand class. This is a longstanding problem that was actually raised by Feynman himself [98, 100], but so far a positive answer was given only for certain classes of *smooth* potentials by D. Fujiwara and collaborators in highly technical works [105, 106]. In Chapter 6 we will discuss the above-mentioned result, which is moreover global in time (except for a certain discrete subset of exceptional times). We will also consider even more general low-regular, possibly non-local, potentials.

Besides the specific topic, which seems worthy of further exploration, this book shall be intended as an attempt to build a bridge between the communities of people working in time-frequency analysis and mathematical/theoretical physics, as well as to provide an exposition of the present novel approach along with its basic toolkit. Having in mind a researcher or a Ph.D. student as a reader, we collected in Part I the necessary background, in the most suitable form for our purposes, following a smooth pedagogical pattern. In particular, proofs of preliminary results are given whenever they are instructive to make the reader confident enough with the underlying techniques. Then Part II covers the analysis of path integrals with the above-mentioned convergence results. The book can also be used for a Ph.D. (or self-study) course, covering for example Part I (except for the sections "Complements") and Chapters 6 and 7. For the benefit of the reader, the Index of Notation is divided into categories, with a short explanation next to most entries, while the bibliographical entries are organized using the alphabetic BibLaTeX style.

We would like to take this opportunity to warmly thank several colleagues and friends for uncountable inspiring discussions during a longstanding collaboration; in particular Elena Cordero, Hans Feichtinger, Maurice de Gosson, Karlheinz Gröchenig and Luigi Rodino. We also wish to express our sincere gratitude to Sergio Albeverio, Maurice de Gosson (again) and Franz Luef for carefully reading part of the material contained in this monograph and to Riccardo Adami, Luigi Ambrosio, Ernesto De Vito, Daisuke Fujiwara, Massimiliano Gubinelli, Naoto Kumano-go, Sonia Mazzucchi, Luigi Ettore Picasso, Nicola Pinamonti, Paolo Tilli, Stefan Waldmann and Kenji Yajima for discussions or

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Outline

We briefly describe the organization of the material in this monograph.

First, in **Chapter 1** the reader can find an itinerary through the topics of the book, that is an essential introduction to Gabor analysis and mathematical path integrals. We try to outline the key points of wave packet analysis and to describe the state of the art in rigorous Feynman path integrals from an operator-theoretic point of view, without any claim of completeness but only for motivational purposes. Indeed, we provide a description of the problems that we have taken into account and an exposition of the main results that will be proved.

The rest of the monographs is organized in two parts.

- **Part I** comprises the background material on time-frequency analysis. In an attempt of providing a self-contained presentation, in **Chapter 2** we fix the notation and recall the general notions of analysis that are needed below, whereas **Chapter 3** and **Chapter 4** are devoted to the basic results of Gabor analysis of functions/distributions and operators respectively. In **Chapter 5** we also provide a reformulation of this material in the spirit of semiclassical analysis.

- In **Part II** we study the problem of the convergence for Feynman path integrals. More in detail:

  - The problem of pointwise convergence at the level of integral kernels for Feynman-Trotter approximate propagators is the focus of **Chapter 6**.
  - In **Chapter 7** we deal with the problem of convergence in the norm operator topology in L^2 for a certain family of rough parametrices and for a class of potentials with low regularity.
  - **Chapter 8** is devoted again to convergence results in the norm operator topology in L^2 for a refined class of parametrices and for a class of potentials with low Sobolev-type regularity.

- Finally, in **Chapter 9** we are confronted with convergence results in the same spirit as above in the L^p setting, $1 < p < \infty$, characterized by an unavoidable loss of derivatives.

Index of Notation

Time-frequency representations