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NORM ESTIMATES FOR τ-PSEUDODIFFERENTIAL OPERATORS IN WIENER AMALGAM AND MODULATION SPACES

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ABSTRACT. We study continuity properties on modulation spaces for τ-pseudo–differential operators $\mathrm{Op}_\tau(a)$ with symbols a in Wiener amalgam spaces. We obtain boundedness results for $\tau \in (0,1)$ whereas, in the end-points $\tau = 0$ and $\tau = 1$, the corresponding operators are in general unbounded. Furthermore, for $\tau \in (0,1)$, we exhibit a function of τ which is an upper bound for the operator norm. The continuity properties of $\mathrm{Op}_\tau(a)$, for any $\tau \in [0,1]$, with symbols a in modulation spaces are well known. Here we find an upper bound for the operator norm which does not depend on the parameter $\tau \in [0,1]$, as expected.

Key ingredients are uniform continuity estimates for τ-Wigner distributions.

1. INTRODUCTION

Pseudodifferential operators are mathematical tools used extensively in the theory of partial differential equations, engineering and quantum mechanics. Since their first appearance in the works by Kohn, Nirenberg [20] and Hörmander [19], they have been widely studied in the framework of classical analysis by plenty of authors, with privileged symbol classes being the so-called Hörmander class [19]. In this context, we also refer to the textbooks [23, 26, 28, 33].

Starting from the end of the 90's and during the last 20 years they have been considered in the context of time-frequency analysis. Many outcomes have been obtained, showing in particular that operators with rough symbols (functions not even differentiable or tempered distributions) may be bounded on $L^2(\mathbb{R}^d)$. The

contributions are so many that we are not able to cite them all. See, for instance, [2, 5, 8, 10, 16, 17, 18, 21, 22, 25, 27, 29, 30, 31, 32].

Assume $\tau \in [0,1]$, $a \in \mathcal{S}'(\mathbb{R}^{2d})$, then the τ-pseudodifferential operator $\mathrm{Op}_\tau(a)$ with symbol a can be defined by

$$(1) \quad \mathrm{Op}_\tau(a)f(x) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{2\pi i(x-y)\xi} a((1-\tau)x + \tau y, \xi) f(y)\, dy d\xi, \quad f \in \mathcal{S}(\mathbb{R}^d).$$

If $\tau = 0$, the corresponding operator $\mathrm{Op}_0(a)$ is called the Kohn-Niremberg operator and can be rewritten as

$$(2) \qquad \mathrm{Op}_0(a)f(x) = \int_{\mathbb{R}^d} a(x,\xi)\hat{f}(\xi)e^{2\pi i x\xi} d\xi, \qquad f \in \mathcal{S}(\mathbb{R}^d).$$

For $\tau = 1/2$, $\mathrm{Op}_{1/2}(a)$ (proposed by Weyl in [34]) is called Weyl operator and takes the form

$$(3) \qquad \mathrm{Op}_{1/2}(a)f(x) = \int_{\mathbb{R}^{2d}} a\left(\frac{x+y}{2}, \xi\right) f(y)e^{2\pi i(x-y)\xi} dy d\xi, \qquad f \in \mathcal{S}(\mathbb{R}^d).$$

In this paper we continue the study of boundedness properties of pseudodifferential operators using tools from time-frequency analysis. The main ingredients are the time-frequency representations related to the definition of τ-pseudodifferential operators. For $\tau \in [0,1]$, the (cross-)τ-Wigner distribution (τ-WD) of signals $f, g \in L^2(\mathbb{R}^d)$ is defined by

$$(4) \qquad W_\tau(f,g)(x,\xi) = \int_{\mathbb{R}^d} e^{-2\pi i t\xi} f(x+\tau t)\overline{g(x-(1-\tau)t)} dt, \quad (x,\xi) \in \mathbb{R}^{2d}.$$

For $f = g$, $W_\tau f := W_\tau(f,f)$ is called the τ-Wigner distribution of f. Note that $W_\tau f$ is a quadratic time-frequency representation which is a generalization of the well known Wigner distribution, recaptured in the case $\tau = 1/2$:

$$W_{1/2}(f,g)(x,\xi) = W(f,g)(x,\xi) = \int_{\mathbb{R}^d} f\left(x+\frac{t}{2}\right)\overline{g\left(x-\frac{t}{2}\right)}e^{-2\pi i t\xi}\, dt.$$

For $\tau = 0$, $W_0(f,g)$ is named (cross-)Rihaczek distribution

$$(5) \qquad W_0(f,g)(x,\xi) = \mathcal{R}(f,g)(x,\xi) = e^{-2\pi i x\cdot\xi} f(x)\overline{\hat{g}(\xi)};$$

and for $\tau = 1$, $W_1(f,g)$ is the (cross-)conjugate Rihaczek distribution

$$(6) \qquad W_1(f,g)(x,\xi) = \mathcal{R}^*(f,g)(x,\xi) = \overline{\mathcal{R}(g,f)(x,\xi)} = e^{2\pi i x\cdot\xi}\overline{g(x)}\hat{f}(\xi).$$

Given a symbol $a \in \mathcal{S}'(\mathbb{R}^{2d})$, the τ-pseudodifferential operator $\mathrm{Op}_\tau(a)$ in (1) can be defined weakly as a duality between the symbol a and the τ-WD $W_\tau(g,f)$ as follows

$$\langle \mathrm{Op}_\tau(a)f, g\rangle = \langle a, W_\tau(g,f)\rangle, \qquad f, g \in \mathcal{S}(\mathbb{R}^d).$$

Inspired by the work of Boulkhemair [5], we continue his investigation considering symbols in the new framework of Wiener amalgam spaces. Such spaces can be

viewed as $L^q(L^p)$-norm of a time-frequency representation: the short-time Fourier transform (STFT) $V_g f$ of a signal $f \in \mathcal{S}'(\mathbb{R}^d)$ with respect to a window function $g \in \mathcal{S}(\mathbb{R}^d)$, defined by

$$(7) \qquad V_g f(z) = \langle f, \pi(z) g \rangle = \mathcal{F}[f T_x g](\xi) = \int_{\mathbb{R}^d} f(t) \, \overline{g(t-x)} \, e^{-2\pi i t \xi} \, dt,$$

for $z = (x, \xi) \in \mathbb{R}^d \times \mathbb{R}^d$. For simplicity, we recall their definition in the un-weighted case, referring to the next section for a more general definition and related properties. A tempered distribution $f \in \mathcal{S}'(\mathbb{R}^d)$ is in the Wiener amalgam space $W(\mathcal{F}L^p, L^q)(\mathbb{R}^d)$, $1 \leq p, q \leq \infty$, if

$$\|f\|_{W(\mathcal{F}L^p, L^q)(\mathbb{R}^d)} := \left(\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |V_g f(x, \xi)|^p \, d\xi \right)^{q/p} dx \right)^{1/q} < \infty.$$

Roughly speaking, a distribution f is in the space $W(\mathcal{F}L^p, L^q)(\mathbb{R}^d)$ if *locally* it behaves like a function in $\mathcal{F}L^p(\mathbb{R}^d)$ and *globally* decays as a function in $L^q(\mathbb{R}^d)$. Such spaces capture the different behaviour of functions/distributions on local and global levels. For instance, it can be shown that the delta distribution δ is in $W(\mathcal{F}L^\infty, L^1)(\mathbb{R}^d)$: its Fourier transform $\mathcal{F}\delta = 1$ belongs to $L^\infty(\mathbb{R}^d)$ and the compact support guarantees whatever decay at infinity.

Modulation spaces are closely related to such Wiener spaces. Indeed the modulation space $M^{p,q}(\mathbb{R}^d)$ can be defined by

$$M^{p,q}(\mathbb{R}^d) = \mathcal{F}^{-1} W(\mathcal{F}L^p, L^q)(\mathbb{R}^d),$$

where $\mathcal{F}^{-1}$ is the inverse Fourier transform.

Sufficient and necessary conditions for boundedness properties of pseudodifferential operators with symbols in modulation spaces and acting on the same spaces have been found in many papers, cf. [6, 7, 31, 32] and the bibliography therein. Here such conditions do not depend on the parameter $\tau \in [0, 1]$, see Section 5 for details in this framework.

In our context we continue to study boundedness properties on modulation spaces but the symbols are in the Wiener ones. Here the continuity properties do depend on the parameter τ.

For $1 \leq r_1, r_2 \leq \infty$, we introduce the function

$$(8) \qquad \alpha_{(r_1, r_2)}(\tau) = \frac{1}{\tau^{d\left(\frac{1}{r_1'} + \frac{1}{r_2}\right)} (1 - \tau)^{d\left(\frac{1}{r_1} + \frac{1}{r_2'}\right)}}, \qquad \tau \in (0, 1).$$

Observe that the function $\alpha_{(r_1, r_2)}(\tau)$ is unbounded on $(0, 1)$. Indeed, for $(r_1, r_2) \notin \{(1, \infty), (\infty, 1)\}$,

$$\lim_{\tau \to 0^+} \alpha_{(r_1, r_2)}(\tau) = \lim_{\tau \to 1^-} \alpha_{(r_1, r_2)}(\tau) = +\infty.$$

For $(r_1, r_2) = (1, \infty)$ we have $\lim_{\tau \to 1^-} \alpha_{(1,\infty)}(\tau) = +\infty$ whereas, for $(r_1, r_2) = (\infty, 1)$, $\lim_{\tau \to 0^+} \alpha_{(\infty,1)}(\tau) = +\infty$. An unweighted version of our main result, cf. Theorem 4.3 below, can be read as follows:

Theorem 1.1. *Suppose that $1 \le p, q, r_1, r_2 \le \infty$ satisfy*

$$q \le p', \quad \max\{r_1, r_2, r_1', r_2'\} \le p.$$

Let a be a symbol in $W(\mathcal{F}L^p, L^q)(\mathbb{R}^{2d})$. For $\tau \in (0,1)$, every τ-pseudodifferential operator $\mathrm{Op}_\tau(a)$ is a bounded operator on $M^{r_1,r_2}(\mathbb{R}^d)$. Moreover, there exists a constant $C > 0$ independent of τ such that

$$(9) \qquad \|\mathrm{Op}_\tau(a)f\|_{M^{r_1,r_2}} \le C\alpha_{(r_1,r_2)}(\tau)\|a\|_{W(\mathcal{F}L^p,L^q)}\|f\|_{M^{r_1,r_2}}, \quad \tau \in (0,1).$$

Hence we have found an upper bound for the operator norm:

$$\|\mathrm{Op}_\tau(a)\|_{\mathcal{B}(M^{r_1,r_2})} \le C\alpha_{(r_1,r_2)}(\tau)\|a\|_{W(\mathcal{F}L^p,L^q)}.$$

The unboundedness of the function $\alpha_{(r_1,r_2)}(\tau)$ in the end-points suggests that the boundedness results above fail in the case of Kohn-Nirenberg operators $\mathrm{Op}_0(a)$ and of operators *with right symbol* $\mathrm{Op}_1(a)$ (also called anti-Kohn-Nirenberg operators). Indeed, we exhibit precise counterexamples in Proposition 4.4 below.

The paper is organized as follows. Section 2 is focused on the preliminary definitions and properties of τ-Wigner distributions and the involved function spaces. In Section 3, we study the continuity properties of $W_\tau(f, g)$ in the Wiener amalgam spaces, obtaining uniform estimates with respect to the parameter τ. Section 4 is devoted to the proof of the main theorem: Theorem 4.3. We also treat the cases $\tau = 0$ and $\tau = 1$, showing examples of unbounded operators. Section 5 provides some useful remarks on the continuity results of $\mathrm{Op}_\tau(a)$ with symbol in modulation spaces.

Notation. We define the scalar product on $\mathbb{R}^d$ by $xy = x \cdot y$. The Schwartz class is denoted by $\mathcal{S}(\mathbb{R}^d)$ and its dual, the space of tempered distributions, by $\mathcal{S}'(\mathbb{R}^d)$. The brackets $\langle \cdot, \cdot \rangle$ stand for the inner product on $L^2(\mathbb{R}^d)$ or for duality pairing between a tempered distribution in $\mathcal{S}'$ and a function in $\mathcal{S}$ (for convention it is antilinear in the second argument).
We write $f \lesssim g$ to indicate $f(x) \le Cg(x)$ for every x and some constant C, and similarly for $\gtrsim$. The notation $f \asymp g$ stands for $f \lesssim g$ and $f \gtrsim g$. We use a normalized Fourier transform

$$\mathcal{F}f(\xi) = \int_{\mathbb{R}^d} e^{-2\pi i x \xi} f(x) \, dx.$$

The translation operator T_x of a function f on $\mathbb{R}^d$ is defined as $T_x f(t) = f(t-x)$ and the modulation operator $M_\xi f(t) = e^{2\pi i \xi t} f(t)$. For $z = (x, \xi)$, we denote the so-called time-frequency shift acting on a function or distribution as $\pi(z)f(t) = M_\xi T_x f(t)$.

The reflection operator is defined as $\mathcal{I}f(x) = f(-x)$. For $1 \le p \le \infty$, the conjugate exponent p' of p is the one that satisfies $1/p + 1/p' = 1$.

2. Time-frequency Representations and Function Spaces

Denote by J the canonical symplectic matrix in $\mathbb{R}^{2d}$:

$$(10) \qquad J = \begin{pmatrix} 0_{d \times d} & I_{d \times d} \\ -I_{d \times d} & 0_{d \times d} \end{pmatrix} \in Sp(2d, \mathbb{R}),$$

where the symplectic group $Sp(2d, \mathbb{R})$ is defined by

$$Sp(d, \mathbb{R}) = \left\{ M \in GL(2d, \mathbb{R}) : \ M^{\top} J M = J \right\}.$$

In the sequel we shall heavily use the following symplectic matrix

$$(11) \qquad \mathcal{A}_{\tau} = \begin{pmatrix} 0_{d \times d} & (\frac{1-\tau}{\tau})^{1/2} I_{d \times d} \\ -(\frac{\tau}{1-\tau})^{1/2} I_{d \times d} & 0_{d \times d} \end{pmatrix}, \quad \tau \in (0, 1).$$

The main properties of $\mathcal{A}_{\tau}$ are detailed below. Their proof is attained by easy computations.

Lemma 2.1. *For any $\tau \in (0, 1)$, the matrix $\mathcal{A}_{\tau}$ in (11) enjoys the following properties:*

(i) $\mathcal{A}_{\tau} \in Sp(d, \mathbb{R})$; *in particular,* $\mathcal{A}_{1/2} = J$.
(ii) $\mathcal{A}_{\tau}^{\top} = -\mathcal{A}_{1-\tau}$, $\mathcal{A}_{\tau}^{-1} = -\mathcal{A}_{\tau}$.
(iii) $\mathcal{A}_{1-\tau} \mathcal{A}_{\tau} = \mathcal{A}_{\tau}^{\top} \mathcal{A}_{\tau}^{-1} = I_{2d \times 2d} - \mathcal{B}_{\tau}$, *where*

$$(12) \qquad \mathcal{B}_{\tau} = \begin{pmatrix} \frac{1}{1-\tau} I_{d \times d} & 0_{d \times d} \\ 0_{d \times d} & \frac{1}{\tau} I_{d \times d} \end{pmatrix}.$$

(iv) $\sqrt{\tau(1-\tau)} (\mathcal{A}_{\tau} + \mathcal{A}_{1-\tau}) = \sqrt{\tau(1-\tau)} \mathcal{B}_{\tau} \mathcal{A}_{\tau} = J$.

2.1. τ-Wigner Distributions and their Short-Time Fourier Transforms.
We list now some useful features enjoyed by the τ-WD which we will use later (cf. [3, 9]).

Proposition 2.2. *For $\tau \in [0, 1]$, $f, g, f_i, g_i \in L^2(\mathbb{R}^d)$, $i = 1, 2$, we have*
(i) $W_{1-\tau}(f, g) = \overline{W_{\tau}(g, f)}$.
(ii) $W_{\tau} f(x, \xi) = W_{1-\tau} \hat{f}(\xi, -x)$. *Equivalently*

$$W_{\tau} \hat{f}(z) = W_{1-\tau} f(Jz),$$

where J is the canonical symplectic matrix in (10).
(iii) Moyal's Formula for τ-WD:

$$(13) \qquad \langle W_{\tau}(f_1, g_1), W_{\tau}(f_2, g_2) \rangle = \langle f_1, f_2 \rangle \overline{\langle g_1, g_2 \rangle}.$$

(iv) Covariance property for the τ-WD:

$$(14) \quad W_{\tau}(\pi(w)f, \pi(w)g)(z) = T_w W_{\tau}(f, g)(z) = W_{\tau}(f, g)(z - w), \quad w, z \in \mathbb{R}^{2d}.$$

To study continuity properties of the τ-WD on modulation and Wiener spaces, we need to compute its Short-time Fourier transform (STFT). Recall that the STFT of a signal $f \in \mathcal{S}'(\mathbb{R}^d)$ with respect a fixed window function $g \in \mathcal{S}(\mathbb{R}^d)$ is defined in (7). Important properties of STFT we shall use are as follows.

Proposition 2.1. *For $f, f_i, g, g_i \in L^2(\mathbb{R}^d)$, $i = 1, 2$, we have:*
(i) Orthogonality relations for the STFT:

$$(15) \qquad \langle V_{g_1} f_1, V_{g_2} f_2 \rangle_{L^2(\mathbb{R}^{2d})} = \langle f_1, f_2 \rangle_{L^2(\mathbb{R}^d)} \overline{\langle g_1, g_2 \rangle}_{L^2(\mathbb{R}^d)}.$$

(ii) STFT of time-frequency shifts:

$$(16) \qquad V_{M_\omega T_u g}(M_\omega T_u f)(x, \xi) = e^{2\pi i(\omega x - \xi u)} V_g f(x, \xi), \qquad u, x, \xi, \omega \in \mathbb{R}^d.$$

(iii) For $g_0, g, \gamma \in \mathcal{S}(\mathbb{R}^d)$ such that $\langle \gamma, g \rangle \neq 0$, $f \in \mathcal{S}'(\mathbb{R}^d)$,

$$(17) \qquad |V_{g_0} f(x, \xi)| \leq \frac{1}{|\langle \gamma, g \rangle|}(|V_g f| * |V_{g_0} \gamma|)(x, \xi), \qquad (x, \xi) \in \mathbb{R}^{2d}.$$

(iv) Fundamental identity of time-frequency analysis:

$$(18) \qquad V_g f(x, \xi) = e^{-2\pi i x \xi} V_{\hat{g}} \hat{f}(\xi, -x), \quad (x, \xi) \in \mathbb{R}^{2d}.$$

For $\tau \in (0, 1)$, the τ-Wigner distribution can be rephrased as a STFT, the key ingredient is the operator A_τ below.

Definition 2.2. *For $\tau \in (0, 1)$, we define the operator A_τ by*

$$(19) \qquad A_\tau : f(t) \longmapsto \mathcal{I}f\left(\frac{1 - \tau}{\tau} t\right).$$

Then (cf. [3, Lemma 6.2]):

Lemma 2.3. *For $\tau \in (0, 1)$, $f, g \in L^2(\mathbb{R}^d)$, we have*

$$(20) \qquad W_\tau(f, g)(x, \xi) = \frac{1}{\tau^d} e^{2\pi i \frac{1}{\tau} x \xi} V_{A_\tau g} f\left(\frac{1}{1-\tau} x, \frac{1}{\tau} \xi\right), \quad (x, \xi) \in \mathbb{R}^{2d}.$$

The proof of the following lemma is a matter of computation.

Lemma 2.4. *For $\tau \in (0, 1)$, $z = (z_1, z_2) \in \mathbb{R}^{2d}$, the operators A_τ and $\pi(z)$ commute as follows*

$$(21) \qquad \pi(z) A_\tau = A_\tau \pi\left(-\frac{1-\tau}{\tau} z_1, -\frac{\tau}{1-\tau} z_2\right),$$

$$(22) \qquad A_\tau \pi(z) = \pi\left(-\frac{\tau}{1-\tau} z_1, -\frac{1-\tau}{\tau} z_2\right) A_\tau.$$

In the next lemmas we calculate the STFT of $W_\tau(g, f)$, generalizing [14, Lemma 4.3.1].

Lemma 2.5. *Consider $\tau \in (0,1)$. Let $\varphi_1, \varphi_2 \in \mathcal{S}(\mathbb{R}^d)$, $f, g \in \mathcal{S}(\mathbb{R}^d)$ and set $\Phi_\tau = W_\tau(\varphi_1, \varphi_2)$. Then,*
(23)
$$V_{\Phi_\tau} W_\tau(g,f)(z,\zeta) = e^{-2\pi i z_2 \zeta_2} V_{\varphi_1} g(z_1 - \tau\zeta_2, z_2 + (1-\tau)\zeta_1) \overline{V_{\varphi_2} f(z_1 + (1-\tau)\zeta_2, z_2 - \tau\zeta_1)}$$

where $z = (z_1, z_2), \zeta = (\zeta_1, \zeta_2) \in \mathbb{R}^{2d}$.

Proof. Using the covariance property (14) and the representation of the τ-Wigner distribution as a STFT in (20),

$$
\begin{aligned}
V_{\Phi_\tau} W_\tau(g,f)(z,\zeta) &= \langle W_\tau(g,f), M_\zeta T_z W_\tau(\varphi_1, \varphi_2) \rangle \\
&= \langle W_\tau(g,f), M_\zeta W_\tau(\pi(z)\varphi_1, \pi(z)\varphi_2) \rangle \\
&= \frac{1}{\tau^{2d}} \int_{\mathbb{R}^{2d}} V_{A_\tau f} g\left(\frac{1}{1-\tau}x, \frac{1}{\tau}\xi\right) e^{-2\pi i (x,\xi) \cdot (\zeta_1, \zeta_2)} \overline{V_{A_\tau \pi(z)\varphi_2} \pi(z)\varphi_1 \left(\frac{1}{1-\tau}x, \frac{1}{\tau}\xi\right)} dxd\xi \\
&= \frac{(1-\tau)^d}{\tau^d} \int_{\mathbb{R}^{2d}} V_{A_\tau f} g(x, \xi) e^{-2\pi i (\zeta_1, \zeta_2) \cdot ((1-\tau)x, \tau\xi)} \overline{V_{A_\tau \pi(z)\varphi_2} \pi(z)\varphi_1 (x, \xi)} dxd\xi.
\end{aligned}
$$

To shorten notation, we write

$$c_\tau = \frac{(1-\tau)^d}{\tau^d}.$$

Using formula (16), the orthogonality relations (15) and the commutation relations between π and A_τ in Lemma 2.4, we compute

$$
\begin{aligned}
& V_{\Phi_\tau} W_\tau(g,f)(z,\zeta) \\
&= c_\tau \int_{\mathbb{R}^{2d}} V_{\pi(\tau\zeta_2, -(1-\tau)\zeta_1)A_\tau f} \pi(\tau\zeta_2, -(1-\tau)\zeta_1) g \overline{V_{A_\tau \pi(z)\varphi_2} \pi(z)\varphi_1}(x,\xi) dxd\xi \\
&= c_\tau \langle \pi(\tau\zeta_2, -(1-\tau)\zeta_1) g, \pi(z_1, z_2)\varphi_1 \rangle \overline{\langle \pi(\tau\zeta_2, -(1-\tau)\zeta_1) A_\tau f, A_\tau \pi(z_1, z_2)\varphi_2 \rangle} \\
&= c_\tau e^{-2\pi i \tau z_2 \zeta_2} \langle g, \pi(z_1 - \tau\zeta_2, z_2 + (1-\tau)\zeta_1)\varphi_1 \rangle \\
&\qquad \times \overline{\langle A_\tau f, \pi(-\tau\zeta_2, (1-\tau)\zeta_1) A_\tau \pi(z_1, z_2)\varphi_2 \rangle} \\
&= c_\tau e^{-2\pi i \tau z_2 \zeta_2} \langle g, \pi(z_1 - \tau\zeta_2, z_2 + (1-\tau)\zeta_1)\varphi_1 \rangle \\
&\qquad \times \overline{\langle A_\tau f, A_\tau \pi((1-\tau)\zeta_2, -\tau\zeta_1) \pi(z_1, z_2)\varphi_2 \rangle} \\
&= e^{-2\pi i \tau z_2 \zeta_2} e^{-2\pi i (1-\tau) z_2 \zeta_2} \langle g, \pi(z_1 - \tau\zeta_2, z_2 + (1-\tau)\zeta_1)\varphi_1 \rangle \\
&\qquad \times \overline{\langle f, \pi(z_1 + (1-\tau)\zeta_2, z_2 - \tau\zeta_1)\varphi_2 \rangle} \\
&= e^{-2\pi i z_2 \zeta_2} V_{\varphi_1} g(z_1 - \tau\zeta_2, z_2 + (1-\tau)\zeta_1) \overline{V_{\varphi_2} f(z_1 + (1-\tau)\zeta_2, z_2 - \tau\zeta_1)}.
\end{aligned}
$$

The claim is proved. $\qquad\square$

Formula (23) can be equivalently written as

(24)
$$V_{\Phi_\tau} W_\tau(g, f)(z, \zeta) = e^{-2\pi i z_2 \zeta_2} V_{\varphi_1} g(z + \sqrt{\tau(1-\tau)} \mathcal{A}_\tau^T \zeta) \overline{V_{\varphi_2} f(z + \sqrt{\tau(1-\tau)} \mathcal{A}_\tau \zeta)},$$

where $\mathcal{A}_\tau$ is a symplectic matrix defined in (11).

The previous lemma does not cover the case $\tau = 0$ and $\tau = 1$, which are treated below.

Lemma 2.6 (*STFT of the Rihaczek distribution*)**.** *Let* $\varphi_1, \varphi_2 \in \mathcal{S}(\mathbb{R}^d)$, $f, g \in \mathcal{S}(\mathbb{R}^d)$ *and set* $\Phi_0 = W_0(\varphi_1, \varphi_2)$. *Then,*

(25)
$$V_{\Phi_0} W_0(g, f)(z, \zeta) = e^{-2\pi i z_2 \zeta_2} V_{\varphi_1} g(z_1, z_2 + \zeta_1) \overline{V_{\varphi_2} f(z_1 + \zeta_2, z_2)},$$

where $z = (z_1, z_2), \zeta = (\zeta_1, \zeta_2) \in \mathbb{R}^{2d}$.

Proof. We use the definition in (5) and formula (16) in the following computations:

$$V_{\Phi_0} W_0(g, f)(z, \zeta) = \langle W_0(g, f), M_\zeta T_z W_0(\varphi_1, \varphi_2) \rangle$$
$$= \int_{\mathbb{R}^{2d}} e^{-2\pi i x \xi} g(x) \overline{\hat{f}(\xi)} e^{-2\pi i (x \zeta_1 + \xi \zeta_2)} e^{2\pi i (x - z_1)(\xi - z_2)} \overline{\varphi_1(x - z_1) \widehat{\varphi_2}(\xi - z_2)} \, dx d\xi$$
$$= e^{2\pi i z_1 z_2} \int_{\mathbb{R}^d} e^{-2\pi i x(z_2 + \zeta_1)} g(x) \overline{\varphi_1(x - z_1)} \, dx \int_{\mathbb{R}^d} \overline{\hat{f}(\xi)} e^{-2\pi i \xi(z_1 + \zeta_2)} \overline{\widehat{\varphi_2}(\xi - z_2)} \, d\xi$$
$$= e^{2\pi i z_1 z_2} V_{\varphi_1} g(z_1, z_2 + \zeta_1) \overline{V_{\widehat{\varphi_2}} \hat{f}(z_2, -(z_1 + \zeta_2))}$$
$$= e^{2\pi i z_1 z_2} V_{\varphi_1} g(z_1, z_2 + \zeta_1) \overline{V_{\varphi_2} f(z_1 + \zeta_2, z_2)} e^{-2\pi i z_2(z_1 + \zeta_2)}$$
$$= e^{-2\pi i z_2 \zeta_2} V_{\varphi_1} g(z_1, z_2 + \zeta_1) \overline{V_{\varphi_2} f(z_1 + \zeta_2, z_2)},$$

as desired. □

Corollary 2.7 (*STFT of the conjugate-Rihaczek distribution*)**.** *Let* $\varphi_1, \varphi_2 \in \mathcal{S}(\mathbb{R}^d)$, $f, g \in \mathcal{S}(\mathbb{R}^d)$ *and set* $\Phi_1 = W_1(\varphi_1, \varphi_2)$. *Then,*

(26)
$$V_{\Phi_1} W_1(g, f)(z, \zeta) = e^{-2\pi i z_2 \zeta_2} V_{\varphi_1} g(z_1 - \zeta_2, z_2) \overline{V_{\varphi_2} f(z_1, z_2 - \zeta_1)},$$

where $z = (z_1, z_2), \zeta = (\zeta_1, \zeta_2) \in \mathbb{R}^{2d}$.

Proof. Using the connection between the Rihaczek and the conjugate-Rihaczek distribution in (6) and the result of Lemma 2.6 we can write

$$
\begin{aligned}
V_{\Phi_1} W_1(g,f)(z,\zeta) &= \langle W_1(g,f), M_\zeta T_z W_1(\varphi_1,\varphi_2)\rangle \\
&= \overline{\langle \overline{W_0(f,g)}, M_\zeta T_z \overline{W_0(\varphi_2,\varphi_1)}\rangle} \\
&= \overline{\langle W_0(f,g), M_{-\zeta} T_z W_0(\varphi_2,\varphi_1)\rangle} \\
&= \overline{V_{W_0(\varphi_2,\varphi_1)} W_0(f,g)(z,-\zeta)} \\
&= \overline{e^{2\pi i z_2 \zeta_2} V_{\varphi_2} f(z_1, z_2 - \zeta_1) \overline{V_{\varphi_1} g(z_1 - \zeta_2, z_2)}} \\
&= e^{-2\pi i z_2 \zeta_2} V_{\varphi_1} g(z_1 - \zeta_2, z_2) \overline{V_{\varphi_2} f(z_1, z_2 - \zeta_1)}.
\end{aligned}
$$

The proof is completed. $\qquad\square$

Remark 2.3. *(i) Heuristically, formulae (25) and (26) can be inferred by putting $\tau = 0$ and $\tau = 1$ respectively in the expression (23).*
(ii) The STFT of a multilinear version of the Rihaczek distribution was computed in [2, Lemma 3.3], cf. formula (3.3). However, there is a flaw in the phase factor of that formula. Indeed, the exponential $e^{2\pi i u_0 \cdot (u_1 + \cdots + u_m)}$ should be replaced by $e^{2\pi i \sum_{i=1}^m u_i \cdot v_i}$, as the linear case $m = 1$ in (25) shows.

2.2. Generalized Gaussian Functions.

In order to compute the norm of W_τ in Wiener amalgam spaces, generalized Gaussian functions will play a crucial role. Given $a, b, c > 0$, the generalized Gaussian function is defined as

$$
(27) \qquad f_{a,b,c}(x,\xi) = e^{-\pi a x^2} e^{-\pi b \xi^2} e^{2\pi i c x \xi}, \quad (x,\xi) \in \mathbb{R}^{2d}.
$$

In the sequel, we will employ the STFT of a generalized Gaussian function, computed in [6, Proposition 2.2]:

Proposition 2.4. *For $\Phi(x,\xi) = e^{-\pi(x^2 + \xi^2)}$, $z = (z_1, z_2)$, $\zeta = (\zeta_1, \zeta_2) \in \mathbb{R}^{2d}$, we obtain*

$$
V_\Phi f_{a,b,c}(z,\zeta) = C(a,b,c) e^{-\pi \frac{[a(b+1)+c^2]z_1^2 + [(a+1)b+c^2]z_2^2 + (b+1)\zeta_1^2 + (a+1)\zeta_2^2 - 2c(z_1\zeta_2 + z_2\zeta_1)}{(a+1)(b+1)+c^2}}
$$

$$
(28) \qquad \times\ e^{-\frac{2\pi i}{a+1}\left[z_1\zeta_1 + (cz_1 - (a+1)\zeta_2)\frac{c\zeta_1 + (a+1)z_2}{(a+1)(b+1)+c^2}\right]},
$$

with $C(a,b,c) = [(a+1)(b+1) + c^2]^{-d/2}$.

The τ-Wigner distribution of the Gaussian function $\varphi(t) = e^{-\pi t^2}$ is in turn a generalized Gaussian function, as showed in the next lemma.

Lemma 2.8. *Consider $\varphi_1(t) = \varphi_2(t) = \varphi(t) = e^{-\pi t^2}$, $t \in \mathbb{R}^d$, and $\tau \in [0,1]$. Then*

$$
W_\tau \varphi(x,\xi) = \frac{1}{(2\tau^2 - 2\tau + 1)^{d/2}} e^{-\pi \frac{1}{2\tau^2 - 2\tau + 1} x^2} e^{-\pi \frac{1}{2\tau^2 - 2\tau + 1} \xi^2} e^{2\pi i \frac{2\tau - 1}{2\tau^2 - 2\tau + 1} x\xi},
$$

for all $(x, \xi) \in \mathbb{R}^{2d}$.

Proof. Using the definition of the τ-WD in (4),

$$
\begin{aligned}
W_\tau \varphi(x, \xi) &= \int_{\mathbb{R}^d} e^{-2\pi i \xi t} e^{-\pi (x + \tau t)^2} e^{-\pi (x - (1-\tau)t)^2} dt \\
&= \int_{\mathbb{R}^d} e^{-2\pi i \xi t} e^{-2\pi x^2} e^{-\pi[(2\tau^2 - 2\tau + 1)t^2 + 2(2\tau - 1)xt]} dt \\
&= e^{-2\pi x^2 + \pi (\frac{2\tau - 1}{\sqrt{2\tau^2 - 2\tau + 1}})^2 x^2} \int_{\mathbb{R}^d} e^{-2\pi i \xi t} e^{-\pi (\sqrt{2\tau^2 - 2\tau + 1} t + \frac{2\tau - 1}{\sqrt{2\tau^2 - 2\tau + 1}} x)^2} dt.
\end{aligned}
$$

We perform the following change of variables

$$
\sqrt{2\tau^2 - 2\tau + 1}\, t + \frac{2\tau - 1}{\sqrt{2\tau^2 - 2\tau + 1}} x = y,
$$

so that, naming $c(\tau) = 2\tau^2 - 2\tau + 1 > 0$,

$$
\begin{aligned}
W_\tau \varphi(x, \xi) &= \frac{1}{c(\tau)^{\frac{d}{2}}} e^{-\pi \left(2 - \frac{(2\tau - 1)^2}{c(\tau)}\right) x^2} \int_{\mathbb{R}^d} e^{-2\pi i \xi \frac{y}{\sqrt{c(\tau)}}} e^{2\pi i \frac{2\tau - 1}{c(\tau)} \xi x} e^{-\pi y^2} dy \\
&= \frac{1}{c(\tau)^{\frac{d}{2}}} e^{-\pi \frac{1}{c(\tau)} x^2} e^{2\pi i \frac{2\tau - 1}{c(\tau)} \xi x} e^{-\pi \frac{1}{c(\tau)} \xi^2},
\end{aligned}
$$

as desired. $\square$

2.3. **Weights and Function Spaces.** In time-frequency analysis, weight functions play an important role, since they describe the growth and the decay of a signal f on the time-frequency plane $\mathbb{R}^{2d}$. For a complete survey on weights, we refer to [15]. A weight function is a positive, locally integrable function on $\mathbb{R}^{2d}$. In the sequel, we will need the following types of weight functions.

Definition 2.5. *Let v and m be positive functions on $\mathbb{R}^{2d}$.*

(i) *A weight v is called submultiplicative if*

$$
v(z_1 + z_2) \leq v(z_1) v(z_2), \qquad \forall z_1, z_2 \in \mathbb{R}^{2d}.
$$

(ii) *Let v be a submultiplicative weight, a positive function m on $\mathbb{R}^{2d}$ is called a v-moderate weight, if there exists a constant $C > 0$, such that*

$$
m(z_1 + z_2) \leq C v(z_1) m(z_2), \qquad \forall z_1, z_2 \in \mathbb{R}^{2d}.
$$

Let $\mathcal{M}_v(\mathbb{R}^{2d})$ be the space of all v-moderate weights. An important feature of submultiplicative weights is that they have at most an exponential growth, cf. [15, Lemma 4.2]):

Lemma 2.9. *If v is submultiplicative and even weight, then there exist constants $C, a > 0$ such that*

$$v(x) \leq C e^{a|x|}, \qquad \forall x \in \mathbb{R}^{2d}.$$

From now on, we assume that v is a continuous, positive, even, submultiplicative weight, i.e., $v(0) = 1$, $v(z) = v(-z)$ and $v(z_1 + z_2) \leq v(z_1)v(z_2)$, for all $z_1, z_2 \in \mathbb{R}^{2d}$. In what follows, we will use weight $1/v$, which is a v-moderate weight:

$$v(x) = v(x + y - y) \leq v(x + y)v(y) \quad \Rightarrow \quad \frac{1}{v(x + y)} \leq v(y)\frac{1}{v(x)}.$$

Weight functions occur in the definition of general modulation spaces and Wiener amalgam spaces, where they offer a good device to measure a joint time-frequency concentration of a function or distribution. The definition of these function spaces relies on imposing a suitable norm on the short-time Fourier transform, defined in (7). For their basic properties we refer to [11, 12, 13] and the textbooks [9, 14]. Given a non-zero window $g \in \mathcal{S}(\mathbb{R}^d)$, a v-moderate weight function m on $\mathbb{R}^{2d}$, $1 \leq p, q \leq \infty$, the *modulation space* $M_m^{p,q}(\mathbb{R}^d)$ consists of all tempered distributions $f \in \mathcal{S}'(\mathbb{R}^d)$ such that $V_g f \in L_m^{p,q}(\mathbb{R}^{2d})$ (weighted mixed-norm spaces). The norm on $M_m^{p,q}(\mathbb{R}^d)$ is defined by

$$\|f\|_{M_m^{p,q}} = \|V_g f\|_{L_m^{p,q}} = \left(\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |V_g f(x, \xi)|^p m(x, \xi)^p \, dx \right)^{q/p} d\xi \right)^{1/q}$$

(obvious modifications for $p = \infty$ or $q = \infty$). If $p = q$, we write $M_m^p(\mathbb{R}^d)$ instead of $M_m^{p,p}(\mathbb{R}^d)$, and if $m(z) \equiv 1$ on $\mathbb{R}^{2d}$, then we write $M^{p,q}(\mathbb{R}^d)$ and $M^p(\mathbb{R}^d)$ for $M_m^{p,q}(\mathbb{R}^d)$ and $M_m^{p,p}(\mathbb{R}^d)$.

The space $M_m^{p,q}(\mathbb{R}^d)$ is a Banach space whose definition is independent of the choice of the window g, in the sense that different non-zero window functions yield equivalent norms. The modulation space $M^{\infty,1}(\mathbb{R}^{2d})$ is also called the Sjöstrand's class [24]. We recall the inclusion properties of modulation spaces. Suppose m_1, m_2 weight functions with $m_2 \lesssim m_1$. Then, for $1 \leq p_1, p_2, q_1, q_2 \leq \infty$, with $p_1 \leq p_2$, $q_1 \leq q_2$,

$$(29) \qquad \mathcal{S}(\mathbb{R}^d) \subseteq M_{m_1}^{p_1,q_1}(\mathbb{R}^d) \subseteq M_{m_2}^{p_2,q_2}(\mathbb{R}^d) \subseteq \mathcal{S}'(\mathbb{R}^d).$$

Note that for any $p, q \in [1, \infty]$ and any $m \in \mathcal{M}_v(\mathbb{R}^{2d})$, the inner product $\langle \cdot, \cdot \rangle$ on $\mathcal{S}(\mathbb{R}^d) \times \mathcal{S}(\mathbb{R}^d)$ extends to a continuous sesquilinear map $M_m^{p,q}(\mathbb{R}^d) \times M_{1/m}^{p',q'}(\mathbb{R}^d) \to \mathbb{C}$. Given even weigh functions u, w on $\mathbb{R}^d$, the Wiener amalgam space $W(\mathcal{F}L_u^p, L_w^q)(\mathbb{R}^d)$ consist of all distributions $f \in \mathcal{S}'(\mathbb{R}^d)$ such that

$$\|f\|_{W(\mathcal{F}L_u^p, L_w^q)(\mathbb{R}^d)} := \left(\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |V_g f(x, \xi)|^p u^p(\xi) \, d\xi \right)^{q/p} w^q(x) dx \right)^{1/q} < \infty$$

where if $p = \infty$ or $q = \infty$, then we use the supremum norm.

The Wiener amalgam spaces $W(\mathcal{F}L_u^p, L_w^q)(\mathbb{R}^d)$ are the image of modulation spaces $M_m^{p,q}(\mathbb{R}^d)$ under the Fourier transform

$$(30) \qquad \mathcal{F}(M_{u \otimes w}^{p,q})(\mathbb{R}^d) = W(\mathcal{F}L_u^p, L_w^q)(\mathbb{R}^d).$$

Indeed, using Parseval identity in (7) and the fundamental identity (18), we can write $|V_g f(x, \xi)| = |V_{\hat{g}}\hat{f}(\xi, -x)| = |\mathcal{F}(\hat{f}\, T_\xi \overline{\hat{g}})(-x)|$ and (recall $u(x) = u(-x)$)

$$\|f\|_{M_{u \otimes w}^{p,q}} = \left(\int_{\mathbb{R}^d} \|\hat{f}\, T_\xi \overline{\hat{g}}\|_{\mathcal{F}L_u^p}^q w^q(\xi)\, d\xi \right)^{1/q} = \|\hat{f}\|_{W(\mathcal{F}L_u^p, L_w^q)}.$$

Hence Wiener amalgam spaces are Banach spaces and their definition is independent of the choice of g.

Modulation and Wiener amalgam space norms of signals are weighted mixed-norm spaces of their short-time Fourier transforms. Hence their properties are based on those of the spaces $L_m^{p,q}$. Let us recall the convolution product of mixed-norm spaces [1]:

Lemma 2.10. *For $1 \leq p_i, q_i, r, s \leq \infty$, $i = 1, 2$, $m \in \mathcal{M}_v(\mathbb{R}^{2d})$, $F \in L_v^{p_1, q_1}(\mathbb{R}^{2d})$, $G \in L_m^{p_2, q_2}(\mathbb{R}^{2d})$, we have $F * G \in L_m^{r,s}(\mathbb{R}^{2d})$, with $1/p_1 + 1/p_2 = 1 + 1/r$, $1/q_1 + 1/q_2 = 1 + 1/s$ and*

$$(31) \qquad \|F * G\|_{L_m^{r,s}} \leq \|F\|_{L_v^{p_1,q_1}} \|G\|_{L_m^{p_2,q_2}}.$$

We say that a measurable function f on $\mathbb{R}^{4d}$ is in the space $L_z^\infty(L_{\zeta,m}^1)(\mathbb{R}^{4d})$, with m weight function on $\mathbb{R}^{2d}$, if

$$(32) \qquad \|f\|_{L_z^\infty(L_{\zeta,m}^1)} = \sup_{z \in \mathbb{R}^{2d}} \int_{\mathbb{R}^{2d}} |f(z, \zeta)| m(\zeta) d\zeta < \infty.$$

When working on the STFT of τ-WD, we will use the following Young-type inequality:

Lemma 2.11. *If $m \in \mathcal{M}_v(\mathbb{R}^{2d})$, $f \in L_{1 \otimes v}^1(\mathbb{R}^{4d})$ and $g \in L_z^\infty(L_{\zeta,m}^1)(\mathbb{R}^{4d})$, then $f * g \in L_z^\infty(L_{\zeta,m}^1)(\mathbb{R}^{4d})$, with*

$$\|f * g\|_{L_z^\infty(L_{\zeta,m}^1)} \leq \|f\|_{L_{1 \otimes v}^1} \|g\|_{L_z^\infty(L_{\zeta,m}^1)}.$$

Proof. Using the definition of $L_z^\infty(L_{\zeta,m}^1)$-norm in (32),

$$I := \|f * g\|_{L_z^\infty(L_{\zeta,m}^1)} = \sup_{z \in \mathbb{R}^{2d}} \int_{\mathbb{R}^{2d}} |f * g|(z,\zeta)m(\zeta)d\zeta$$

$$= \sup_{z \in \mathbb{R}^{2d}} \int_{\mathbb{R}^{2d}} \left| \int_{\mathbb{R}^{4d}} f(y,\eta)g(z-y,\zeta-\eta)dyd\eta \right| m(\zeta)d\zeta$$

$$\leq \sup_{z \in \mathbb{R}^{2d}} \int_{\mathbb{R}^{2d}} \int_{\mathbb{R}^{2d}} \left(\int_{\mathbb{R}^{2d}} |f|(y,\eta)|g|(z-y,\zeta-\eta)d\eta \right) m(\zeta)dyd\zeta$$

$$= \sup_{z \in \mathbb{R}^{2d}} \int_{\mathbb{R}^{2d}} \int_{\mathbb{R}^{2d}} (|f|(y,\cdot) * |g|(z-y,\cdot))(\zeta)m(\zeta)dyd\zeta.$$

By Young's inequality (31),

$$I = \sup_{z \in \mathbb{R}^{2d}} \int_{\mathbb{R}^{2d}} \||f|(y,\cdot)\|_{L_v^1} \||g|(z-y,\cdot)\|_{L_m^1} dy$$

$$\leq \int_{\mathbb{R}^{2d}} \||f|(y,\cdot)\|_{L_v^1} \sup_{z \in \mathbb{R}^d} \||g|(z-y,\cdot)\|_{L_m^1} dy$$

$$= \|g\|_{L_z^\infty(L_{\zeta,m}^1)} \|f\|_{L_{1 \otimes v}^1},$$

as claimed. $\qquad\square$

A particular case of Lemma 2.10 gives:

Lemma 2.12. *Suppose $m \in \mathcal{M}_v(\mathbb{R}^{2d})$, $f \in L_{1 \otimes v}^1(\mathbb{R}^{4d})$ and $g \in L_{1 \otimes m}^2(\mathbb{R}^{4d})$. Then $f * g \in L_{1 \otimes m}^2(\mathbb{R}^{4d})$, with*

$$\|f * g\|_{L_{1 \otimes m}^2} \leq \|f\|_{L_{1 \otimes v}^1} \|g\|_{L_{1 \otimes m}^2}.$$

3. Boundedness properties of τ-Wigner Distributions

This section is devoted to investigate the continuity properties of τ-Wigner distributions in the realm of Wiener and modulation spaces. For a submultiplicative weight v, we set

$$(33) \qquad\qquad v_J(z) = v(Jz),$$

where J denotes the canonical symplectic matrix (10).

Lemma 3.1. *Assume that $m \in \mathcal{M}_v(\mathbb{R}^{2d})$, $1 \leq p_1, p_2 \leq \infty$, $f \in M_m^{p_1,p_2}(\mathbb{R}^d)$, $g \in M_{1/m}^{p_1',p_2'}(\mathbb{R}^d)$. Then for every $\tau \in (0,1)$, the τ-Wigner distribution $W_\tau(g,f)$ is in $W(\mathcal{F}L_{1/v_J}^1, L^\infty)(\mathbb{R}^{2d})$, with*

$$(34) \qquad \|W_\tau(g,f)\|_{W(\mathcal{F}L_{1/v_J}^1,L^\infty)} \leq C\alpha_{(p_1,p_2)}(\tau)\|f\|_{M_m^{p_1,p_2}}\|g\|_{M_{1/m}^{p_1',p_2'}},$$

where the function $\alpha_{(p_1,p_2)}(\tau)$ is defined in (8) and $C > 0$ is independent of τ.

Proof. We compute the STFT of $W_\tau(g, f)$ with respect to the window function $\Phi_\tau \in \mathcal{S}(\mathbb{R}^{2d})$ defined in Lemma 2.5. Using that lemma and the properties of the matrix $\mathcal{A}_\tau$ in Lemma 2.1, by performing the change of variables $\sqrt{\tau(1-\tau)}\mathcal{A}_\tau\zeta = \eta$, we deduce

$$\int_{\mathbb{R}^{2d}} |V_{\Phi_\tau} W_\tau(g, f)|(z, \zeta)\frac{1}{v(J\zeta)}d\zeta$$

$$= \int_{\mathbb{R}^{2d}} |V_{\varphi_1}g(z + \sqrt{\tau(1-\tau)}\mathcal{A}_\tau^T\zeta)||V_{\varphi_2}f(z + \sqrt{\tau(1-\tau)}\mathcal{A}_\tau\zeta)|\frac{1}{v(\sqrt{\tau(1-\tau)}\mathcal{B}_\tau\mathcal{A}_\tau\zeta)}d\zeta$$

$$= \frac{1}{[\tau(1-\tau)]^d} \int_{\mathbb{R}^{2d}} |V_{\varphi_1}g(z + \mathcal{A}_{1-\tau}\mathcal{A}_\tau\eta)||V_{\varphi_2}f(z + \eta)|\frac{1}{v(\mathcal{B}_\tau\eta)}d\eta.$$

Since m is a v-moderate weight, we can find a positive constant C, independent of τ, such that

$$(35) \qquad\qquad \frac{1}{v(\mathcal{B}_\tau\eta)} \leq C\frac{m(z + \eta)}{m(z + \mathcal{A}_{1-\tau}\mathcal{A}_\tau\eta)},$$

so that

$$\int_{\mathbb{R}^{2d}} |V_{\Phi_\tau} W_\tau(g, f)|(z, \zeta)\frac{1}{v(J\zeta)}d\zeta$$

$$\leq C\frac{1}{[\tau(1-\tau)]^d} \int_{\mathbb{R}^{2d}} |V_{\varphi_1}g(z + \mathcal{A}_{1-\tau}\mathcal{A}_\tau\eta)||V_{\varphi_2}f(z + \eta)|\frac{m(z + \eta)}{m(z + \mathcal{A}_{1-\tau}\mathcal{A}_\tau\eta)}d\eta.$$

Consequently,

$$\|W_\tau(g, f)\|_{W(\mathcal{F}L^1_{1/v_J}, L^\infty)}$$

$$\asymp \sup_{z\in\mathbb{R}^{2d}} \int_{\mathbb{R}^{2d}} |V_{\varphi_1}g(z + \sqrt{\tau(1-\tau)}\mathcal{A}_\tau^T\zeta)||V_{\varphi_2}f(z + \sqrt{\tau(1-\tau)}\mathcal{A}_\tau\zeta)|\frac{1}{v(\sqrt{\tau(1-\tau)}\mathcal{B}_\tau\mathcal{A}_\tau\zeta)}d\zeta$$

$$\leq C\frac{1}{[\tau(1-\tau)]^d} \sup_{z\in\mathbb{R}^{2d}} \int_{\mathbb{R}^{2d}} |V_{\varphi_1}g(z + \mathcal{A}_{1-\tau}\mathcal{A}_\tau\eta)||V_{\varphi_2}f(z + \eta)|\frac{m(z + \eta)}{m(z + \mathcal{A}_{1-\tau}\mathcal{A}_\tau\eta)}d\eta$$

$$\leq C\frac{1}{[\tau(1-\tau)]^d}\|V_{\varphi_1}fm\|_{L^{p_1, p_2}}\|V_{\varphi_2}g\frac{1}{m}(z + \mathcal{A}_{1-\tau}\mathcal{A}_\tau\cdot)\|_{L^{p_1', p_2'}}$$

$$\lesssim \frac{1}{[\tau(1-\tau)]^d}\left(\frac{1-\tau}{\tau}\right)^{d\left(\frac{1}{p_2} - \frac{1}{p_1}\right)}\|f\|_{M_m^{p_1, p_2}}\|g\|_{M_{1/m}^{p_1', p_2'}}.$$

The claim is proved. □

The previous estimate is not uniform with respect to τ, in the sense that the $W(\mathcal{F}L^1_{1/v_J}, L^\infty)$-norm of the τ-WD has been calculated by using a window function Φ_τ depending on τ. The next goal is to find an upper bound of this norm independent of τ. We will need the following result.

Lemma 3.2. *Consider* $\Phi(x, \xi) = e^{-\pi(x^2 + \xi^2)}$, $(x, \xi) \in \mathbb{R}^{2d}$, *and* $\Phi_\tau = W_\tau(\varphi, \varphi)$, *where* $\varphi(t) = e^{-\pi t^2}$, $t \in \mathbb{R}^d$. *Then, for* v_J *in* (33), *there exists a constant* $C > 0$ *such that*

$$(36) \qquad \|V_\Phi \Phi_\tau\|_{L^1_{1 \otimes v_J}} \leq C, \quad \forall \tau \in [0, 1].$$

Consequently,

$$(37) \qquad \|\Phi_\tau\|_{M^1_{1 \otimes v_J}} \leq C, \quad \forall \tau \in [0, 1].$$

Proof. Using Lemma 2.8 and formula (28), with $z = (z_1, z_2), \zeta = (\zeta_1, \zeta_2) \in \mathbb{R}^{2d}$, we compute

$$|V_\Phi \Phi_\tau|(z, \zeta) = \frac{1}{(2\tau^2 - 2\tau + 1)^{d/2}} \frac{(2\tau^2 - 2\tau + 1)^{d/2}}{(2\tau^2 - 2\tau + 5)^{d/2}}$$

$$\times e^{-\pi \frac{\frac{3}{2\tau^2 - 2\tau + 1}(z_1^2 + z_2^2) + \frac{2\tau^2 - 2\tau + 2}{2\tau^2 - 2\tau + 1}(\zeta_1^2 + \zeta_2^2) - 2\frac{2\tau - 1}{2\tau^2 - 2\tau + 1}(z_1 \zeta_2 + z_2 \zeta_1)}{\frac{2\tau^2 - 2\tau + 5}{2\tau^2 - 2\tau + 1}}}$$

$$= \frac{1}{(2\tau^2 - 2\tau + 5)^{d/2}} e^{-\pi \frac{3(z_1^2 + z_2^2) + (2\tau^2 - 2\tau + 2)(\zeta_1^2 + \zeta_2^2) + (2 - 4\tau)(z_1 \zeta_2 + z_2 \zeta_1)}{2\tau^2 - 2\tau + 5}}.$$

Observing that

$$\frac{1}{(2\tau^2 - 2\tau + 5)^{d/2}} \leq \max_{\tau \in (0,1)} \frac{1}{(2\tau^2 - 2\tau + 5)^{d/2}} = \left(\frac{2}{9}\right)^{d/2},$$

by Lemma 2.9, we have,

$$\|V_\Phi \Phi_\tau\|_{L^1_{1 \otimes v_J}} \leq \left(\frac{2}{9}\right)^{d/2}$$

$$\times \int_{\mathbb{R}^{2d}} \int_{\mathbb{R}^{2d}} e^{-\pi \frac{3(z_1^2 + z_2^2) + (2\tau^2 - 2\tau + 2)(\zeta_1^2 + \zeta_2^2) + (2 - 4\tau)(z_1 \zeta_2 + z_2 \zeta_1)}{2\tau^2 - 2\tau + 5}} v_J(\zeta) d\zeta_1 d\zeta_2 dz_1 dz_2$$

$$\leq C \int_{\mathbb{R}^{2d}} \int_{\mathbb{R}^{2d}} e^{-\pi \frac{3(z_1^2 + z_2^2) + (2\tau^2 - 2\tau + 2)(\zeta_1^2 + \zeta_2^2) + (2 - 4\tau)(z_1 \zeta_2 + z_2 \zeta_1)}{2\tau^2 - 2\tau + 5}} e^{a|J\zeta|} d\zeta_1 d\zeta_2 dz_1 dz_2$$

$$= C \int_{\mathbb{R}^{2d}} e^{-\pi \frac{3(z_1^2 + z_2^2)}{2\tau^2 - 2\tau + 5}} I_1 dz_1 dz_2,$$

where

$$I_1 := \int_{\mathbb{R}^{2d}} e^{-\pi \frac{(2\tau^2 - 2\tau + 2)(\zeta_1^2 + \zeta_2^2) + (2 - 4\tau)(z_1 \zeta_2 + z_2 \zeta_1)}{2\tau^2 - 2\tau + 5}} e^{a|J\zeta|} d\zeta_1 d\zeta_2.$$

The integral I_1 can be computed as follows

$$I_1 = \int_{\mathbb{R}^{2d}} e^{-\pi \frac{(2\tau^2 - 2\tau + 2)(\zeta_1^2 + \zeta_2^2) + (2 - 4\tau)(z_1\zeta_2 + z_2\zeta_1)}{2\tau^2 - 2\tau + 5}} e^{a|J\zeta|} d\zeta_1 d\zeta_2$$

$$\leq \int_{\mathbb{R}^{2d}} e^{-\pi \frac{(2\tau^2 - 2\tau + 2)(\zeta_1^2 + \zeta_2^2) + (2 - 4\tau)(z_1\zeta_2 + z_2\zeta_1)}{2\tau^2 - 2\tau + 5}} e^{a(|\zeta_1| + |\zeta_2|)} d\zeta_1 d\zeta_2$$

$$= \left(\int_{\mathbb{R}^d} e^{-\pi \frac{(2\tau^2 - 2\tau + 2)\zeta_1^2 + (2 - 4\tau)z_2\zeta_1}{2\tau^2 - 2\tau + 5}} e^{a|\zeta_1|} d\zeta_1 \right) \left(\int_{\mathbb{R}^d} e^{-\pi \frac{(2\tau^2 - 2\tau + 2)\zeta_2^2 + (2 - 4\tau)z_1\zeta_2}{2\tau^2 - 2\tau + 5}} e^{a|\zeta_2|} d\zeta_2 \right).$$

We calculate the integral with respect to the variable ζ_1 (the other integral is analogous):

$$\int_{\mathbb{R}^d} e^{-\pi \frac{(2\tau^2 - 2\tau + 2)\zeta_1^2 + (2 - 4\tau)z_2\zeta_1}{2\tau^2 - 2\tau + 5}} e^{a|\zeta_1|} d\zeta_1 = \int_{\mathbb{R}^d} e^{-\pi \frac{(2\tau^2 - 2\tau + 2)\zeta_1^2 + (2 - 4\tau)z_2\zeta_1 + \frac{(1 - 2\tau)^2 z_2^2}{2\tau^2 - 2\tau + 2} - \frac{(1 - 2\tau)^2 z_2^2}{2\tau^2 - 2\tau + 2}}{2\tau^2 - 2\tau + 5}} e^{a|\zeta_1|} d\zeta_1$$

$$= e^{\pi \frac{(1 - 2\tau)^2 z_2^2}{(2\tau^2 - 2\tau + 2)(2\tau^2 - 2\tau + 5)}}$$

$$\times \int_{\mathbb{R}^d} e^{-\pi \frac{(\sqrt{2\tau^2 - 2\tau + 2}\zeta_1 + \frac{1 - 2\tau}{\sqrt{2\tau^2 - 2\tau + 2}} z_2)^2}{2\tau^2 - 2\tau + 5}} e^{a|\zeta_1|} d\zeta_1$$

$$= e^{\pi \frac{(1 - 2\tau)^2 z_2^2}{(2\tau^2 - 2\tau + 2)(2\tau^2 - 2\tau + 5)}} \underbrace{\int_{\mathbb{R}^d} e^{-\pi \frac{((2\tau^2 - 2\tau + 2)\zeta_1 + (1 - 2\tau)z_2)^2}{(2\tau^2 - 2\tau + 5)(2\tau^2 - 2\tau + 2)}} e^{a|\zeta_1|} d\zeta_1}_{:= I_3}.$$

In I_3 we perform the following change of variables

$$(2\tau^2 - 2\tau + 2)\zeta_1 + (1 - 2\tau)z_2 = \eta_1,$$

so that,

$$I_3 = \frac{1}{(2\tau^2 - 2\tau + 2)^d} \int_{\mathbb{R}^d} e^{-\pi \frac{\eta_1^2}{(2\tau^2 - 2\tau + 5)(2\tau^2 - 2\tau + 2)}} e^{\frac{a}{2\tau^2 - 2\tau + 2}|\eta_1 - (1 - 2\tau)z_2|} d\eta_1$$

$$\leq C_1^d e^{\frac{a|1 - 2\tau|}{2\tau^2 - 2\tau + 2}|z_2|} \int_{\mathbb{R}^d} e^{-\pi C_2 \eta_1^2} e^{aC_1|\eta_1|} d\eta_1,$$

where

$$C_1 = \max_{\tau \in [0,1]} \frac{1}{(2\tau^2 - 2\tau + 2)} = \frac{2}{3}, \quad C_2 = \min_{\tau \in [0,1]} \frac{1}{(2\tau^2 - 2\tau + 5)(2\tau^2 - 2\tau + 2)} = \frac{1}{10}.$$

Using $\lim_{|\eta_1|\to\infty} e^{-\pi\frac{C_1}{2}\eta_1^2}e^{aC_2|\eta_1|} = 0$, for every $\epsilon > 0$ there exists $R > 0$ such that $e^{-\pi\frac{C_1}{2}\eta_1^2}e^{aC_2|\eta_1|} \leq \epsilon$, for all $|\eta_1|$ with $|\eta_1| > R$. Hence

$$I_3 \leq C_1^d e^{\frac{a|1-2\tau|}{2\tau^2-2\tau+2}|z_2|} \int_{\mathbb{R}^d} e^{-\pi C_2\eta_1^2}e^{aC_1|\eta_1|}d\eta_1$$

$$= C_1^d e^{\frac{a|1-2\tau|}{2\tau^2-2\tau+2}|z_2|} \left(\int_{\{\eta_1\in\mathbb{R}^d:|\eta_1|\leq R\}} e^{-\pi C_2\eta_1^2}e^{aC_1|\eta_1|}d\eta_1 + \int_{\{\eta_1\in\mathbb{R}^d:|\eta_1|>R\}} e^{-\pi C_2\eta_1^2}e^{aC_1|\eta_1|}d\eta_1 \right)$$

$$\leq C_1^d e^{\frac{a|1-2\tau|}{2\tau^2-2\tau+2}|z_2|} \left(e^{aC_1 R} \int_{\mathbb{R}^d} e^{-\pi C_2\eta_1^2}d\eta_1 + \epsilon \int_{\mathbb{R}^d} e^{-\pi\frac{C_2}{2}\eta_1^2}d\eta_1 \right) = \tilde{C}e^{\frac{a|1-2\tau|}{2\tau^2-2\tau+2}|z_2|} < \infty,$$

where $\tilde{C}$ is a constant independent of τ. In conclusion, the integral I_1 can be majorized as

$$I_1 \leq 2\tilde{C}e^{\pi\frac{(1-2\tau)^2 z_2^2}{(2\tau^2-2\tau+2)(2\tau^2-2\tau+5)}+\frac{a|1-2\tau|}{2\tau^2-2\tau+2}|z_2|}e^{\pi\frac{(1-2\tau)^2 z_1^2}{(2\tau^2-2\tau+2)(2\tau^2-2\tau+5)}+\frac{a|1-2\tau|}{2\tau^2-2\tau+2}|z_1|}.$$

Thus, there exists a constant $M_1 > 0$ independent of τ such that

$$\|V_\Phi\Phi_\tau\|_{L^1_{1\otimes v_J}} \leq M_1 \int_{\mathbb{R}^d} e^{-\pi\frac{3z_1^2}{2\tau^2-2\tau+5}}e^{\pi\frac{(1-2\tau)^2 z_1^2}{(2\tau^2-2\tau+2)(2\tau^2-2\tau+5)}+\frac{a|1-2\tau|}{2\tau^2-2\tau+2}|z_1|}dz_1$$

$$\times \int_{\mathbb{R}^d} e^{-\pi\frac{3z_2^2}{2\tau^2-2\tau+5}}e^{\pi\frac{(1-2\tau)^2 z_2^2}{(2\tau^2-2\tau+2)(2\tau^2-2\tau+5)}+\frac{a|1-2\tau|}{2\tau^2-2\tau+2}|z_2|}dz_2$$

$$= 2M_1 \int_{\mathbb{R}^d} e^{-\pi\frac{3z_1^2}{2\tau^2-2\tau+5}}e^{\pi\frac{(1-2\tau)^2 z_1^2}{(2\tau^2-2\tau+2)(2\tau^2-2\tau+5)}+\frac{a|1-2\tau|}{2\tau^2-2\tau+2}|z_1|}dz_1.$$

The integral with respect the variable z_1 is computed analogously to the one for ζ_1 above. The estimate (37) follows by

$$\|\Phi_\tau\|_{M^1_{1\otimes v_J}} \asymp \|V_\Phi\Phi_\tau\|_{L^1_{1\otimes v_J}} \leq C,$$

as desired. □

Proposition 3.1. *Under the assumptions of Lemma 3.1, there exists a constant $C > 0$ independent of τ such that*

$$(38) \qquad \|W_\tau(g,f)\|_{W(\mathcal{F}L^1_{1/v_J},L^\infty)} \leq C\alpha_{(p_1,p_2)}(\tau)\|f\|_{M^{p_1,p_2}_m}\|g\|_{M^{p'_1,p'_2}_{1/m}}, \qquad \tau \in (0,1).$$

Proof. Changing window in the computation of the STFT as in (17), using Lemmas 2.11, 3.2 and Moyal's formula (13), we have

$$\|V_\Phi W_\tau(g,f)\|_{L_z^\infty(L_{\zeta,1/v_J}^1)} \leq \frac{1}{|\langle \Phi_\tau, \Phi_\tau \rangle|} \||V_{\Phi_\tau} W_\tau(g,f)| * |V_\Phi \Phi_\tau|\|_{L_z^\infty(L_{\zeta,1/v_J}^1)}$$

$$\leq \frac{1}{\|\varphi\|^2 \|\varphi\|^2} \|V_{\Phi_\tau} W_\tau(g,f)\|_{L_z^\infty(L_{\zeta,1/v_J}^1)} \|V_\Phi \Phi_\tau\|_{L_{1\otimes v_J}^1}$$

$$\leq C\alpha_{(p_1,p_2)}(\tau)\|f\|_{M_m^{p_1,p_2}} \|g\|_{M_{1/m}^{p_1',p_2'}}.$$

This completes the proof.   □

Repeating the pattern of Lemma 3.1 and Proposition 3.1 in the Wiener amalgam space $W(\mathcal{F}L_{1/v_J}^2, L^2)(\mathbb{R}^{2d})$, we can state the following.

Proposition 3.2. *Let* $m \in \mathcal{M}_v(\mathbb{R}^{2d})$, $f \in M_m^2(\mathbb{R}^d)$ *and* $g \in M_{1/m}^2(\mathbb{R}^d)$. *For* $\tau \in (0,1)$, *the* τ-*WD* $W_\tau(g,f)$ *is in* $W(\mathcal{F}L_{1/v_J}^2, L^2)(\mathbb{R}^{2d})$, *with the uniform estimate*

$$(39) \qquad \|W_\tau(g,f)\|_{W(\mathcal{F}L_{1/v_J}^2, L^2)} \leq C\|f\|_{M_m^2} \|g\|_{M_{1/m}^2},$$

where the positive constant C *is independent of* τ.

Proof. First Step. We use Lemma 2.5, Young's Inequality $L^1 * L^1 \subset L^1$ and the change of variables $\mathcal{B}_\tau \eta \to \eta$, to compute

$$\|W_\tau(g,f)\|_{W(\mathcal{F}L_{1/v_J}^2, L^2)}$$

$$\asymp \left(\int_{\mathbb{R}^{2d}} \int_{\mathbb{R}^{2d}} |V_{\varphi_1} g(z + \sqrt{\tau(1-\tau)}\mathcal{A}_\tau^T \zeta)|^2 |V_{\varphi_2} f(z + \sqrt{\tau(1-\tau)}\mathcal{A}_\tau \zeta)|^2 \frac{1}{v^2(J\zeta)} d\zeta dz \right)^{\frac{1}{2}}$$

$$\leq C\frac{1}{[\tau(1-\tau)]^d} \left(\int_{\mathbb{R}^{2d}} \int_{\mathbb{R}^{2d}} |V_{\varphi_1} g(z + \eta - \mathcal{B}_\tau \eta)|^2 |V_{\varphi_2} f(z+\eta)|^2 \frac{m^2(z+\eta)}{m^2(z+\eta-\mathcal{B}_\tau \eta)} d\eta dz \right)^{\frac{1}{2}}$$

$$= C\frac{1}{[\tau(1-\tau)]^d} \left(\int_{\mathbb{R}^{2d}} (|V_{\varphi_2} f|^2 m^2) * (|V_{\varphi_1} g|^2 \frac{1}{m^2})(\mathcal{B}_\tau \eta)\, d\eta \right)^{\frac{1}{2}}$$

$$\lesssim \||V_\varphi f|^2 m^2\|_1 \||V_\varphi g|^2 \frac{1}{m^2}\|_1$$

$$\lesssim \|f\|_{M_m^2} \|g\|_{M_{1/m}^2}.$$

Second Step. Consider now $\Phi \in \mathcal{S}(\mathbb{R}^{2d})$. Then the same pattern as in the proof of Proposition 3.1, with Lemma 2.11 replaced by Lemma 2.12, gives the uniform estimate (39).   □

The previous issue can be rephrased in terms of modulation spaces as follows (cf. (30)).

Corollary 3.3. *For $\tau \in (0,1)$, $m \in \mathcal{M}_v(\mathbb{R}^{2d})$, $f \in M_m^2(\mathbb{R}^d)$, $g \in M_{1/m}^2(\mathbb{R}^d)$, the τ-WD belongs to $M_{1/v_J \otimes 1}^2(\mathbb{R}^{2d})$ with*

$$\|W_\tau(g,f)\|_{M_{1/v_J \otimes 1}^2} \leq C\|f\|_{M_m^2}\|g\|_{M_{1/m}^2},$$

with $C > 0$ independent of τ.

4. MAIN RESULT

This section is devoted to the proof of Theorem 4.3. We will start with two preliminary results about τ-pseudodifferential operators acting on modulation spaces and having symbols in $W(\mathcal{F}L_{1/v_J}^1, L^\infty)(\mathbb{R}^{2d})$ and $W(\mathcal{F}L_{1/v_J}^2, L^2)(\mathbb{R}^{2d})$, respectively. Then, by means of complex interpolation between Wiener amalgam spaces, we shall reach our goal.

Proposition 4.1. *Suppose that $m \in \mathcal{M}_v(\mathbb{R}^{2d})$ and consider a symbol function $a \in W(\mathcal{F}L_{v_J}^\infty, L^1)(\mathbb{R}^{2d})$. Then for every $\tau \in (0,1)$, the τ-pseudodifferential operator $\mathrm{Op}_\tau(a)$ is bounded on $M_m^{p_1,p_2}(\mathbb{R}^d)$, for every $1 \leq p_1, p_2 \leq \infty$, with*

$$(40) \qquad \|\mathrm{Op}_\tau(a)f\|_{M_m^{p_1,p_2}} \leq C\alpha_{(p_1,p_2)}(\tau)\|a\|_{W(\mathcal{F}L_{v_J}^\infty, L^1)}\|f\|_{M_m^{p_1,p_2}}$$

($C > 0$ does not depend on τ).

Proof. For every $f \in M_m^{p_1,p_2}(\mathbb{R}^d)$ and $g \in M_{1/m}^{p_1',p_2'}(\mathbb{R}^d)$, we can write

$$|\langle \mathrm{Op}_\tau(a)f, g\rangle| = |\langle a, W_\tau(g,f)\rangle| \leq \|V_\Phi a\|_{L_{\bar{z}}^1(L_{v_J,\zeta}^\infty)}\|V_\Phi W_\tau(g,f)\|_{L_{\bar{z}}^\infty(L_{1/v_J,\zeta}^1)}.$$

Observing that

$$\|W_\tau(g,f)\|_{W(\mathcal{F}L_{1/v_J}^1, L^\infty)} \asymp \|V_\Phi W_\tau(g,f)\|_{L_{\bar{z}}^\infty(L_{1/v_J,\zeta}^1)}$$

and using Proposition 3.1, we conclude the proof. $\qquad\square$

Proposition 4.2. *Let $m \in \mathcal{M}_v(\mathbb{R}^{2d})$, $a \in W(\mathcal{F}L_{v_J}^2, L^2)(\mathbb{R}^{2d})$ and $\tau \in (0,1)$. Then the operator $\mathrm{Op}_\tau(a)$ is bounded on M_m^2 with*

$$(41) \qquad \|\mathrm{Op}_\tau(a)f\|_{M_m^2} \leq C\|a\|_{W(\mathcal{F}L_{v_J}^2, L^2)}\|f\|_{M_m^2},$$

where the constant $C > 0$ is independent of τ.

Proof. The proof is similar to the one of Proposition 4.1, where Proposition 3.1 is replaced by 3.2. $\qquad\square$

Propositions 4.1 and 4.2 are the main ingredients in the proof of Theorem 4.3, which generalizes [10, Theorem 3.1] in the case of τ-pseudodifferential operators.

Theorem 4.3. *Suppose that $1 \leq p, q, r_1, r_2 \leq \infty$ satisfy*

$$(42) \qquad\qquad q \leq p'$$

and

$$(43) \qquad\qquad \max\{r_1, r_2, r_1', r_2'\} \leq p.$$

Let $m \in \mathcal{M}_v(\mathbb{R}^{2d})$ and $a \in W(\mathcal{F}L^p_{v_J}, L^q)(\mathbb{R}^{2d})$. For $\tau \in (0,1)$, every τ-pseudodifferential operator $\mathrm{Op}_\tau(a)$ is a bounded operator on $M^{r_1, r_2}_m(\mathbb{R}^d)$. Moreover, there exists a constant $C > 0$ independent of τ such that

$$(44) \qquad \|\mathrm{Op}_\tau(a)f\|_{M^{r_1,r_2}_m} \leq C\alpha_{(r_1,r_2)}(\tau)\|a\|_{W(\mathcal{F}L^p_{v_J}, L^q)}\|f\|_{M^{r_1,r_2}_m}, \quad \tau \in (0,1).$$

Proof. The key tool is the complex interpolation between Wiener amalgam and modulation spaces. We regard Op_τ as the bilinear map $(a, f) \mapsto \mathrm{Op}_\tau(a)f$. Proposition 4.1 and Proposition 4.2 give the continuity of the τ-pseudodifferential operator Op_τ on the following function spaces

$$W(\mathcal{F}L^\infty_{v_J}, L^1)(\mathbb{R}^{2d}) \times M^{p_1,p_2}_m(\mathbb{R}^d) \to M^{p_1,p_2}_m(\mathbb{R}^d),$$
$$W(\mathcal{F}L^2_{v_J}, L^2)(\mathbb{R}^{2d}) \times M^2_m(\mathbb{R}^d) \to M^2_m(\mathbb{R}^d),$$

for $1 \leq p_1, p_2 \leq \infty$. Using the complex interpolation between Wiener amalgam and modulation spaces [12], for $\theta \in [0,1]$, we have

$$[W(\mathcal{F}L^\infty_{v_J}, L^1), W(\mathcal{F}L^2_{v_J}, L^2)]_\theta = W(\mathcal{F}L^p_{v_J}, L^{p'}),$$

with $2 \leq p \leq \infty$, and $[M^{p_1,p_2}_m, M^2_m]_\theta = M^{r_1,r_2}_m$, with

$$(45) \qquad \frac{1}{r_1} = \frac{1-\theta}{p_1} + \frac{\theta}{2} = \frac{1-\theta}{p_1} + \frac{1}{p}$$

and

$$(46) \qquad \frac{1}{r_2} = \frac{1-\theta}{p_2} + \frac{\theta}{2} = \frac{1-\theta}{p_2} + \frac{1}{p}$$

so that $r_1, r_2 \leq p$. Similarly, we obtain $r_1', r_2' \leq p$, and thus the relation (43). Due to inclusion relations for Wiener amalgam spaces, we relax the assumptions on symbols, so that the symbol a may belong to $W(\mathcal{F}L^p_{v_J}, L^q)(\mathbb{R}^{2d})$, with $q \leq p'$,

which gives (42). Finally, the norm is provided by

$$\| \operatorname{Op}_\tau \|_{\mathcal{B}(W(\mathcal{F}L^p_{v_J}, L^q) \times M^{r_1, r_2}_m, M^{r_1, r_2}_m)} \leq \| \operatorname{Op}_\tau \|^{1-\theta}_{\mathcal{B}(W(\mathcal{F}L^\infty_{v_J}, L^1) \times M^{p_1, p_2}_m, M^{p_1, p_2}_m)}$$

$$\times \| \operatorname{Op}_\tau \|^{\theta}_{\mathcal{B}(W(\mathcal{F}L^2_{v_J}, L^2) \times M^2_m, M^2_m)}$$

$$\leq C \frac{1}{\tau^{d(1-\theta)\left(1 - \frac{1}{p_1} + \frac{1}{p_2}\right)} (1 - \tau)^{d(1-\theta)\left(1 + \frac{1}{p_1} - \frac{1}{p_2}\right)}}$$

$$\leq C \frac{1}{\tau^{d\left(1 - \frac{1}{p_1} + \frac{1}{p_2}\right)} (1 - \tau)^{d\left(1 + \frac{1}{p_1} - \frac{1}{p_2}\right)}},$$

since $1 - \theta \leq 1$. This concludes the proof. $\qquad\square$

We finally consider the end-points $\tau = 0$ and $\tau = 1$, for which the boundedness results stated above do not hold in general. We remark that the modulation space $M^2(\mathbb{R}^d)$ is simply the Lebesgue space $L^2(\mathbb{R}^d)$. The following example generalizes a 1-dimensional example exhibited by Boulkhemair in [5].

Proposition 4.4. *There exists a symbol $a \in W(\mathcal{F}L^\infty, L^1)(\mathbb{R}^{2d})$ such that the corresponding Kohn-Nirenberg $\operatorname{Op}_0(a)$ and anti-Kohn-Nirenberg $\operatorname{Op}_1(a)$ operators are not bounded on $L^2(\mathbb{R}^d)$.*

Proof. Consider the symbol function

$$(47) \qquad a(x_1, \ldots, x_d, \xi_1, \ldots, \xi_d) = x_1^{-1/2} \ldots x_d^{-1/2} \chi_{(0,1]}(x_1) \ldots \chi_{(0,1]}(x_d) e^{-\pi \xi^2},$$

with $\xi^2 = \xi_1^2 + \cdots + \xi_d^2$. An easy computation shows that $a \in L^1(\mathbb{R}^{2d}) = W(L^1, L^1)(\mathbb{R}^{2d}) \subset W(\mathcal{F}L^\infty, L^1)(\mathbb{R}^{2d})$. Let us show that the Kohn-Niremberg $\operatorname{Op}_0(a)$ is unbounded on $L^2(\mathbb{R}^d)$. Consider the Gaussian function $f(t) = e^{\pi t^2} \in L^2(\mathbb{R}^d)$, then $\operatorname{Op}_0(a)f \notin L^2(\mathbb{R}^d)$. Indeed, by a tensor product argument, we reduce to compute the following one-dimensional integral:

$$\int_{\mathbb{R}} e^{2\pi i x \xi} x^{-1/2} \chi_{(0,1]}(x) e^{-\pi \xi^2} e^{-\pi \xi^2} d\xi = \frac{1}{\sqrt{2}} x^{-1/2} \chi_{(0,1]}(x) e^{-\pi \frac{x^2}{2}},$$

whose result is a function that does not belong to $L^2(\mathbb{R})$.

To prove that the anti-Kohn-Nirenberg operator $\operatorname{Op}_1(a)$, where a is defined in (47), is unbounded on $L^2(\mathbb{R}^d)$, it is sufficient to observe that its adjoint operator is the Kohn-Niremberg one: $(\operatorname{Op}_1(a))^* = \operatorname{Op}_0(a)$, as detailed below:

$$\langle \operatorname{Op}_1(a)f, g \rangle = \langle a, \mathcal{R}^*(g, f) \rangle = \langle a, \overline{\mathcal{R}(f, g)} \rangle = \langle \mathcal{R}(f, g), \overline{a} \rangle = \langle f, \operatorname{Op}_0(a)g \rangle.$$

This proves our claim. $\qquad\square$

5. Remarks on boundedness results for symbols in modulation spaces

We address this section to study the boundedness results for τ-pseudodifferential operators with symbols in weighted modulation spaces. Recall (cf. [19] and [31, Remark 1.5]) that for every choice $\tau_1, \tau_2 \in [0,1]$, $a_1, a_2 \in \mathcal{S}'(\mathbb{R}^{2d})$,

$$(48) \qquad \mathrm{Op}_{\tau_1}(a_1) = \mathrm{Op}_{\tau_2}(a_2) \Leftrightarrow \widehat{a_2}(\xi_1, \xi_2) = e^{-2\pi i(\tau_2 - \tau_1)\xi_1\xi_2}\widehat{a_1}(\xi_1, \xi_2).$$

For $t > 0$ define $H_t(x, \xi) = e^{2\pi itx\xi}$ and observe that

$$(49) \qquad \mathcal{F}H_t(\zeta_1, \zeta_2) = \frac{1}{t^d}e^{-2\pi i\frac{1}{t}\zeta_1\zeta_2}.$$

So, for $\tau_1 \neq \tau_2$, by (49),

$$(50) \qquad a_2(x, \xi) = \frac{1}{|\tau_1 - \tau_2|^d}e^{2\pi i(\tau_2 - \tau_1)\Psi} * a_1(x, \xi),$$

where $\Psi(x, \xi) = x\xi$. Toft in [31, Proposition 1.2 (5)] proved that the mapping $a \mapsto T_\Phi a = e^{2\pi i\Phi} * a$ is a homeomorphism on $M^{p,q}(\mathbb{R}^{2d})$, $1 \leq p, q \leq \infty$. This implies that results for Weyl operators with symbols in modulation spaces are still true for any τ-operator. The main goal of this section is to show uniform estimates for τ-operators with symbol in weighted modulation spaces. Following the pattern of the previous section, we first compute the norm of the τ-WD in weighted modulation spaces.

The next Proposition extends the sufficient conditions of [6, Theorem 1.1] in the case of τ-Wigner distributions.

Proposition 5.1. *Assume that $p_1, p_2, q_1, q_2, p, q \in [0,1]$ satisfy*

$$(51) \qquad\qquad p_i, q_i \leq q, \qquad i = 1, 2,$$

and

$$(52) \qquad \frac{1}{p_1} + \frac{1}{p_2} \geq \frac{1}{p} + \frac{1}{q}, \qquad \frac{1}{q_1} + \frac{1}{q_2} \geq \frac{1}{p} + \frac{1}{q}.$$

Consider $m \in \mathcal{M}_v$, $f \in M_m^{p_1,q_1}(\mathbb{R}^d)$ and $g \in M_{1/m}^{p_2,q_2}(\mathbb{R}^d)$. Then, for any $\tau \in [0,1]$, $W_\tau(g, f) \in M_{1\otimes 1/v_J}^{p,q}(\mathbb{R}^{2d})$. Furthermore, there exists a constant $C > 0$, independent of $\tau \in [0,1]$, such that

$$\|W_\tau(g, f)\|_{M_{1\otimes 1/v_J}^{p,q}} \leq C\|f\|_{M_m^{p_1,q_1}}\|g\|_{M_{1/m}^{p_2,q_2}}, \quad \forall\tau \in [0,1].$$

Proof. We separate the proof in three cases: $\tau \in (0,1)$, $\tau = 0$ and $\tau = 1$.
Case $\tau \in (0,1)$. *Assume $p \leq q < \infty$. Making the change of variables $z +$

$\sqrt{\tau(1-\tau)}\mathcal{A}_\tau\zeta = y$ and using item (iv) of Lemma 2.1, the integral with respect the variable z becomes

$\|W_\tau(g,f)\|_{M^{p,q}_{1\otimes 1/v_J}}$

$$= \left(\int_{\mathbb{R}^{2d}}\left(\int_{\mathbb{R}^{2d}}|V_{\varphi_1}g(z+\sqrt{\tau(1-\tau)}\mathcal{A}^T_\tau\zeta)|^p|V_{\varphi_2}f(z+\sqrt{\tau(1-\tau)}\mathcal{A}_\tau\zeta)|^p dz\right)^{\frac{q}{p}}\frac{1}{v^q(J\zeta)}d\zeta\right)^{\frac{1}{q}}$$

$$= \left(\int_{\mathbb{R}^{2d}}\left(\int_{\mathbb{R}^{2d}}|V_{\varphi_1}g(y-J\zeta)|^p|V_{\varphi_2}f(y)|^p\frac{1}{v^p(J\zeta)}dy\right)^{\frac{q}{p}}d\zeta\right)^{\frac{1}{q}}$$

$$\leq C\left(\int_{\mathbb{R}^{2d}}\left(\int_{\mathbb{R}^{2d}}|V_{\varphi_1}g(y-J\zeta)|^p|V_{\varphi_2}f(y)|^p\frac{m^p(y)}{m^p(y-J\zeta)}dy\right)^{\frac{q}{p}}d\zeta\right)^{\frac{1}{q}}$$

$$= C\left(\int_{\mathbb{R}^{2d}}\left((|\mathcal{I}V_{\varphi_1}g|^p\frac{1}{m^p})*(|V_{\varphi_2}f|^pm^p)(J\zeta)\right)^{\frac{q}{p}}d\zeta\right)^{\frac{1}{q}}$$

$$= C\||(\mathcal{I}V_{\varphi_1}g|^p\frac{1}{m^p})*(|V_{\varphi_2}f|^pm^p)\|^{1/p}_{L^{q/p}},$$

where $\mathcal{I}$ is the reflection operator. The rest goes exactly as in the proof of Theorem 3.1 in [6], obtaining the estimate

$$\|W_\tau(g,f)\|^p_{M^{p,q}_{1\otimes 1/v_J}} \lesssim \|f\|_{M^{p_1,q_1}_m}\|f\|_{M^{p_2,q_2}_{1/m}}.$$

Using Lemma 3.2, there exists a positive constant C independent of $\tau \in (0,1)$ such that

$$\|W_\tau(g,f)\|_{M^{p,q}_{1\otimes 1/v_J}} \leq \frac{1}{|\langle\Phi_\tau,\Phi_\tau\rangle|}\|V_{\Phi_\tau}W_\tau(g,f)*V_\Phi\Phi_\tau\|_{L^{p,q}_{1\otimes 1/v_J}} \leq C\|f\|_{M^{p_1,q_1}_m}\|g\|_{M^{p_2,q_2}_{1/m}},$$

concluding the proof for $\tau \in (0,1)$, $p \leq q < \infty$.
Assume $p = q = \infty$. We have

$$\|W_\tau(g,f)\|_{M^\infty_{1\otimes 1/v_J}} = \sup_{z,\zeta\in\mathbb{R}^{2d}} |V_{\varphi_1}g(z+\sqrt{\tau(1-\tau)}\mathcal{A}_\tau^T\zeta)||V_{\varphi_2}f(z+\sqrt{\tau(1-\tau)}\mathcal{A}_\tau\zeta)|\frac{1}{v(J\zeta)}$$

$$= \sup_{\zeta\in\mathbb{R}^{2d}}\sup_{z\in\mathbb{R}^{2d}} |V_{\varphi_1}g(z-J\zeta)||V_{\varphi_2}f(z)|\frac{1}{v(J\zeta)},$$

$$\leq C\sup_{\zeta\in\mathbb{R}^{2d}}\sup_{z\in\mathbb{R}^{2d}} |V_{\varphi_1}g(z-J\zeta)||V_{\varphi_2}f(z)|\frac{m(z)}{m(z-J\zeta)},$$

$$= C\sup_{z\in\mathbb{R}^{2d}} |V_{\varphi_2}f(z)|m(z)\left(\sup_{\zeta\in\mathbb{R}^{2d}} |V_{\varphi_1}g(z-J\zeta)|\frac{1}{m(z-J\zeta)}\right),$$

$$= C\||V_{\varphi_2}f|m\|_{L^\infty}\||V_{\varphi_2}g|\frac{1}{m}\|_{L^\infty}$$

$$= \|f\|_{M^\infty_{1/v}}\|g\|_{M^\infty_v}$$

$$\leq \|f\|_{M^{p_1,q_1}_{1/v}}\|g\|_{M^{p_2,q_2}_v},$$

for every $1\leq p_i, q_i \leq \infty$. The conclusion follows again by Lemma 3.2 and Young's inequality.

Assume $p > q$. Using the inclusion relations for modulation spaces, we majorize

$$\|W_\tau(g,f)\|_{M^{p,q}_{1\otimes 1/v_J}} \leq \|W_\tau(g,f)\|_{M^{q,q}_{1\otimes 1/v_J}} \leq C\|f\|_{M^{p_1,q_1}_m}\|g\|_{M^{p_2,q_2}_{1/m}},$$

for every $1\leq p_i, q_i \leq q$, $i = 1,2$.

Case $\tau = 0$. In this case, we obtain at once a uniform estimate. Indeed, using Lemma 2.6,

$$\|W_0(g,f)\|_{M^{p,q}_{1\otimes 1/v_J}}$$

$$= \left(\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d}|V_{\varphi_1}g(z_1,z_2+\zeta_1)|^p|V_{\varphi_2}f(z_1+\zeta_2,z_2)|^p dz_1 dz_2\right)^{q/p}\frac{1}{v_J^q(\zeta)}d\zeta_1 d\zeta_2\right)^{1/q}$$

$$= \left(\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d}|V_{\varphi_1}g(z_1-\zeta_2,z_2+\zeta_1)|^p|V_{\varphi_2}f(z_1,z_2)|^p\frac{1}{v^p(J\zeta)}dz_1 dz_2\right)^{q/p}d\zeta_1 d\zeta_2\right)^{1/q}$$

$$\leq C\left(\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d}|V_{\varphi_1}g(z-J\zeta)|^p|V_{\varphi_2}f(z)|^p\frac{m^p(z)}{m^p(x-J\zeta)}dz\right)^{q/p}d\zeta\right)^{1/q}$$

$$= C\|(|V_{\varphi_1}g|^p(1/m^p))*(|V_{\varphi_2}f|^p m^p)\|_{L^{q/p}}.$$

Then we proceed as in Case $\tau\in(0,1)$.

Case $\tau = 1$. The proof is analogous to the one of Case $\tau = 0$. We are done. □

The boundedness results for τ-WDs transfer to τ-pseudodifferential operators as follows.

Theorem 5.1. *Let $1 \leq p_1, p_2, q_1, q_2, p \leq \infty$ be indices such that*

(53)
$$p_1, p'_2, q_1, q'_2 \leq q',$$

and

(54)
$$\frac{1}{p_1} + \frac{1}{p'_2} \geq \frac{1}{p'} + \frac{1}{q'}, \quad \frac{1}{q_1} + \frac{1}{q'_2} \geq \frac{1}{p'} + \frac{1}{q'}.$$

Let $m \in \mathcal{M}_v(\mathbb{R}^{2d})$. For every $\tau \in [0,1]$, the τ-pseudodifferential operator $\mathrm{Op}_\tau(a)$, with symbol $a \in M^{p,q}_{1 \otimes v_J}(\mathbb{R}^{2d})$, is a bounded operator from $M^{p_1,q_1}_m(\mathbb{R}^d)$ to $M^{p_2,q_2}_m(\mathbb{R}^d)$, with

$$\| \mathrm{Op}_\tau(a) f \|_{M^{p_2,q_2}_m} \leq C \|a\|_{M^{p,q}_{1 \otimes v_J}} \|f\|_{M^{p_1,q_1}_m},$$

and $C > 0$ is independent of τ.

Proof. If $f \in M^{p_1,q_1}_m(\mathbb{R}^d)$ and $g \in M^{p'_2,q'_2}_{1/m}(\mathbb{R}^d)$, then $W_\tau(g,f) \in M^{p',q'}_{1 \otimes \frac{1}{v_J}}(\mathbb{R}^{2d})$, by Proposition 5.1, provided that (53) and (54) hold. Thereby there exists a positive constant C such that for any $\tau \in [0,1]$,

$$|\langle \mathrm{Op}_\tau(a)f, g \rangle| = |\langle a, W_\tau(g,f) \rangle|$$
$$\leq C \|a\|_{M^{p,q}_{1 \otimes v_J}} \|f\|_{M^{p_1,q_1}_m} \|g\|_{M^{p'_2,q'_2}_{1/m}},$$

as desired. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

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