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CONFORMAL GEOMETRY OF ISOTROPIC CURVES IN THE COMPLEX QUADRIC

EMILIO MUSSO AND LORENZO NICOLODI

Abstract. Let $\mathbb{Q}_3$ be the complex 3-quadric endowed with its standard complex conformal structure. We study the complex conformal geometry of isotropic curves in $\mathbb{Q}_3$. By an isotropic curve we mean a nonconstant holomorphic map from a Riemann surface into $\mathbb{Q}_3$, null with respect to the conformal structure of $\mathbb{Q}_3$. The relations between isotropic curves and a number of relevant classes of surfaces in Riemannian and Lorentzian spaceforms are discussed.

1. Introduction

Let $\mathbb{Q}_3$ be the nonsingular complex hyperquadric of $\mathbb{CP}^4$, regarded as the Grassmannian of Lagrangian 2-planes of $\mathbb{C}^4$ and equipped with its canonical $\mathrm{Sp}(2,\mathbb{C})$-invariant complex conformal structure $\mathbf{g}$ (cf. [23, 26, 30]). An *isotropic curve* is a nonconstant holomorphic map $f : S \to \mathbb{Q}_3$ from a Riemann surface S into $\mathbb{Q}_3$, such that $f^*(\mathbf{g}) = 0$. (The complex null geodesics, i.e., the projective lines in $\mathbb{CP}^4$ contained in $\mathbb{Q}_3$, are excluded from our consideration.) The purpose of this paper is to provide a systematic study of the complex conformal geometry of isotropic curves. This topic was inspired by Chern's article [13] and Bryant's papers [6, 8, 9].

The main motivation comes from differential geometry, and more specifically from surface geometry. In fact, the point of view of isotropic curves provides a unifying framework for a number of different classes of immersed surfaces according to the following scheme: an isotropic curve is naturally associated to any immersion of the class; and conversely an immersion can be recovered from the isotropic curve. Examples that fit into this scheme include minimal surfaces in Euclidean space $\mathbb{R}^3$, maximal surfaces in Minkowski space $\mathbb{R}^{1,2}$, surfaces of constant mean curvature one (CMC 1) and flat fronts in hyperbolic space $\mathcal{H}^3$, spacelike CMC 1 surfaces and flat fronts in de Sitter space $\mathcal{H}^{1,2}$, and superminimal immersions in the four-sphere S^4. For minimal surfaces, this amounts essentially to the classical description of a minimal surface $f : S \to \mathbb{R}^3$ as the real part of a holomorphic null curve $F : S \to \mathbb{C}^3$ (cf. [15, 18, 43]). For the case of maximal surfaces in $\mathbb{R}^{1,2}$, see [27, 31, 36]; for the case of CMC 1 surfaces in hyperbolic space, besides Bryant's seminal paper [7], see [4, 37, 39, 38, 41]; for flat fronts in hyperbolic 3-space, see [16, 17, 28, 29, 34, 38]; for spacelike CMC 1 surfaces and flat fronts in de Sitter space, see [31, 36]. Finally,

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for the case of superminimal surfaces in S^4, see [6, 13]. Other references will be given throughout the paper.

Description of results. Section 2 collects some background material about the holomorphic conformal structure of $\mathbb{Q}_3$ and the contact structure of $\mathbb{CP}^3$. In particular, we briefly discuss the projections of suitable Zariski open sets of $\mathbb{Q}_3$ onto $\mathbb{R}^3$, $\mathbb{R}^{1,2}$, $\mathcal{H}^3$ and $\mathcal{H}^{1,2}$, the projections of suitable Zariski open sets of $\mathbb{CP}^3$ onto $\mathcal{H}^3$ and $\mathcal{H}^{1,2}$, and reformulate the twistor projections of $\mathbb{CP}^3$ onto S^4.

Section 3 is concerned with complex projective structures on a Riemann surface (cf. [12, 25, 32, 35]). We provide the needed background on projective structures and show that naturally associated to a projective structure on a Riemann surface there is an operator $\mathfrak{d}$ from quartic to quadratic meromorphic differentials.

Section 4 studies the complex conformal geometry of isotropic curves by the method of moving frames and discusses the relation of isotropic curves with surface geometry. First, we briefly recall the construction of the branched conformal (spacelike) immersions and fronts in $\mathbb{R}^3$, $\mathbb{R}^{2,1}$, $\mathcal{H}^3$, $\mathcal{H}^{2,1}$, and S^4 associated to an isotropic curve f, the *surfaces tamed* by f. The main properties of these surfaces are collected in Theorem A. Next, generalizing the classical Goursat transformation for minimal surfaces (cf. [18, 24]), we define the conformal Goursat transformation for surfaces tamed by isotropic curves. The close relationship between isotropic curves and projective structures on a Riemann surface is established in Theorem B, which shows that to any isotropic curve $f : S \to \mathbb{Q}_3$ one can associate a meromorphic projective structure and a meromorphic quartic differential δ on S. We then characterize the conformal cycles, i.e., the isotropic curves all of whose points are zeros of the quartic differential (heptactic points), and describe their corresponding tamed surfaces. Next, we introduce the notion of osculating cycle for an isotropic curve f and geometrically characterize heptactic points in terms of the order of contact of the isotropic curve with the osculating cycle. In Theorem C, we solve the equivalence problem for generic isotropic curves in terms of the meromorphic differentials δ and $\mathfrak{d}(\delta)$. The meromorphic function $\mathfrak{d}^2(\delta)^2/\delta$ is called the bending of the isotropic curve. In Theorem D, viewing $\mathbb{Q}_3$ as a homogeneous space of $\mathrm{Sp}(2,\mathbb{C})$, we address the question of deformation and rigidity for isotropic curves according to the general theory of deformation of submanifolds in homogeneous spaces as formulated by Cartan [10] and further developed by Griffiths and Jensen [19, 22, 23]. We prove that a generic isotropic curve is deformable of order four and rigid to order five.

Section 5 is devoted to the study of isotropic curves with constant bending. Referring to the classical notion of W-curve in projective 3-space (cf. [1, 13]), we introduce the notion of isotropic W-curves and describe their tamed surfaces. In the main theorem of the section, Theorem E, we prove that a compact isotropic curve with constant bending is a branched reparametrization of an isotropic W-curve. To better illustrate the connections between the theory of isotropic curves and surface geometry, a number of examples is considered throughout the paper. For figures and for some symbolic computations, we used the software *Mathematica*.

2. Preliminaries

2.1. **The complex symplectic group.** Consider $\mathbb{C}^4$ with the symplectic form $\omega = dz^1 \wedge dz^3 + dz^2 \wedge dz^4$. Let $\mathrm{Sp}(2,\mathbb{C})$ be the symplectic group of ω and $\mathfrak{sp}(2,\mathbb{C})$ its Lie algebra. Denote by $(\mathbf{e}_1, \dots, \mathbf{e}_4)$ the natural basis of $\mathbb{C}^4$ and by $A_j : \mathrm{Sp}(2,\mathbb{C}) \to \mathbb{C}^4$

the holomorphic map taking $A \in \mathrm{Sp}(2,\mathbb{C})$ to $A\mathbf{e}_j$, $j = 1,\dots,4$. The Maurer–Cartan form φ of $\mathrm{Sp}(2,\mathbb{C})$ is the $\mathfrak{sp}(2,\mathbb{C})$-valued holomorphic 1-form such that $dA_i = \varphi_i^j A_j$, $i = 1,\dots,4$.[1] Let $\mathfrak{C}^5$ be the complex vector space spanned by the skew-symmetric matrices

$$\mathrm{L}_1 = \mathbf{e}_2^1 - \mathbf{e}_1^2,\ \mathrm{L}_2 = \mathbf{e}_4^1 - \mathbf{e}_1^4,\ \mathrm{L}_3 = \tfrac{1}{\sqrt{2}}(\mathbf{e}_3^1 - \mathbf{e}_1^3 - \mathbf{e}_4^2 + \mathbf{e}_2^4),\ \mathrm{L}_4 = \mathbf{e}_3^2 - \mathbf{e}_2^3,\ \mathrm{L}_5 = \mathbf{e}_4^3 - \mathbf{e}_3^4,$$

where $\mathbf{e}_i^j$ denotes the elementary 4×4 matrix with 1 in the (i,j) place and 0 elsewhere, $i,j = 1,\dots,4$. On $\mathfrak{C}^5$, we consider the nondegenerate bilinear form $g_{\mathfrak{C}}(X,Y) = \frac{1}{2}\mathrm{tr}(JXJY)$, where $J = \mathbf{e}_1^3 + \mathbf{e}_2^4 - \mathbf{e}_3^1 - \mathbf{e}_4^2$. Then, $(\mathrm{L}_1,\dots,\mathrm{L}_5)$ is a basis of $\mathfrak{C}^5$ such that

$$(g_{\mathfrak{C}}(\mathrm{L}_a,\mathrm{L}_b))_{1\le a,b\le 5} = \mathbf{b}_1^5 + \mathbf{b}_2^4 + \mathbf{b}_3^3 + \mathbf{b}_4^2 + \mathbf{b}_5^1,$$

where $\mathbf{b}_a^b$ denote the elementary 5×5 matrices. For every $A \in \mathrm{Sp}(2,\mathbb{C})$, the linear map $\mathtt{L}_A : \mathfrak{C}^5 \ni X \mapsto AX{}^tA \in \mathfrak{C}^5$ is an orthogonal transformation and $\mathtt{L} : \mathrm{Sp}(2,\mathbb{C}) \ni A \mapsto \mathtt{L}_A \in \mathrm{O}(\mathfrak{C}^5, g_{\mathfrak{C}})$ is a spin covering homomorphism. Let $\mathfrak{R}^5$ be the 5-dimensional real subspace of $\mathfrak{C}^5$ spanned by

$$\mathrm{E}_1 = \tfrac{1}{\sqrt{2}}(\mathrm{L}_1 + \mathrm{L}_5),\ \mathrm{E}_2 = \tfrac{i}{\sqrt{2}}(\mathrm{L}_1 - \mathrm{L}_5),\ \mathrm{E}_3 = \mathrm{L}_3,\ \mathrm{E}_4 = \tfrac{1}{\sqrt{2}}(\mathrm{L}_2 + \mathrm{L}_4),\ \mathrm{E}_5 = \tfrac{i}{\sqrt{2}}(\mathrm{L}_2 - \mathrm{L}_4).$$

Restricting $g_{\mathfrak{C}}$ to $\mathfrak{R}^5$ we get a positive definite scalar product, denoted by $g_{\mathfrak{R}}$. By construction, $(\mathrm{E}_1,\dots,\mathrm{E}_5)$ is an orthogonal basis. The subspace $\mathfrak{R}^5$ is $\mathrm{Sp}(2)$-invariant and $\mathtt{L} : \mathrm{Sp}(2) \ni A \mapsto \mathtt{L}_A \in \mathrm{O}(\mathfrak{R}^5, g_{\mathfrak{R}})$ is a spin homomorphism. Consider $\mathrm{S}^3(\mathbb{C}^2)$, the third symmetric power of $\mathbb{C}^2$, and let $\mathtt{F} : \mathrm{S}^3(\mathbb{C}^2) \to \mathbb{C}^4$ be the isomorphism defined by $\mathtt{F}(\mathbf{a}_{111}) = \mathbf{e}_1$, $\mathtt{F}(\mathbf{a}_{112}) = -\sqrt[3]{2/9}\mathbf{e}_2$, $\mathtt{F}(\mathbf{a}_{222}) = \sqrt[3]{1/6}\mathbf{e}_3$, $\mathtt{F}(\mathbf{a}_{122}) = \mathbf{e}_4$, where $(\mathbf{a}_1,\mathbf{a}_2)$ is the natural basis of $\mathbb{C}^2$ and $\mathbf{a}_{ijk}$ is the symmetric tensor product $\mathbf{a}_j\mathbf{a}_j\mathbf{a}_k$. Let $\widehat{\mathtt{S}}$ be the representation of $\mathrm{SL}(2,\mathbb{C})$ on $\mathrm{S}^3(\mathbb{C}^2)$ induced by the standard representation of $\mathrm{SL}(2,\mathbb{C})$ on $\mathbb{C}^2$. Then $\mathtt{S} : \mathrm{SL}(2,\mathbb{C}) \ni X \mapsto \mathtt{F}\circ\widehat{\mathtt{S}}(X)\circ\mathtt{F}^{-1} \in \mathrm{Sp}(2,\mathbb{C})$ is a Lie group monomorphism. The subgroup $\mathtt{S}(\mathrm{SL}(2,\mathbb{C}))$ will be denoted by H.

2.2. **The conformal structure of $\mathbb{Q}_3$.** Let $\mathbb{Q}_3$ be the compact complex 3-fold of Lagrangian 2-planes of $\mathbb{C}^4$. The map $\pi : \mathrm{Sp}(2,\mathbb{C}) \ni A \mapsto [A_1 \wedge A_2] \in \mathbb{Q}_3$ is a holomorphic principal bundle with structure group

$$\mathrm{G}_0 = \{A \in \mathrm{Sp}(2,\mathbb{C}) \mid [A_1 \wedge A_2] = [\mathbf{e}_1 \wedge \mathbf{e}_2]\}.$$

Let $\mathcal{T} = \{(P,\mathbf{v}) \in \mathbb{Q}_3 \times \mathbb{C}^4 \mid \mathbf{v} \in P\}$ be the tautological bundle of $\mathbb{Q}_3$ and $\widetilde{\mathcal{T}}$ the quotient bundle $(\mathbb{Q}_3 \times \mathbb{C}^4)/\mathcal{T}$. If $A : U \subset \mathbb{Q}_3 \to \mathrm{Sp}(2,\mathbb{C})$ is a holomorphic cross section of π, then $(\varphi_1^3, \varphi_1^4, \varphi_2^4)$ is a holomorphic coframe and $A_1 \wedge A_2$ is a trivialization of $\bigwedge^2(\mathcal{T})$.[2] For every $\mathbf{v} \in \mathbb{C}^4$ and $P \in U$, we denote by $|[\mathbf{v}]|_P$ the equivalence class of $\mathbf{v}$ in $\widetilde{\mathcal{T}}\,|_P$. Then $U \ni P \mapsto |[A_3|_P]|_P \wedge |[A_4|_P]|_P$ is a local trivialization of $\bigwedge^2(\widetilde{\mathcal{T}})$. Let $(A_1 \wedge A_2)^*$ be the dual of $A_1 \wedge A_2$. Then,

$$\mathbf{g}_{|U} = (\varphi_1^3\varphi_2^4 - (\varphi_1^4)^2) \otimes (A^1 \wedge A^2)^* \otimes (|[A_3]| \wedge |[A_4]|))$$

is independent of the choice of A and defines a nondegenerate holomorphic section $\mathbf{g}$ of $\mathrm{S}^{(2,0)}(\mathbb{Q}_3) \otimes \bigwedge^2(\mathcal{T})^* \otimes \bigwedge^2(\widetilde{\mathcal{T}})$. Then $\mathbf{g}$ determines a holomorphic $\mathrm{Sp}(2,\mathbb{C})$-invariant conformal structure on $\mathbb{Q}^3$ (cf. [26, 30]).

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[1] Summation over repeated indices is assumed.

[2] By slightly abusing notation, we will simply write φ_j^i instead of $A^*(\varphi_j^i)$.

2.2.1. *The Plücker map.* Let $\mathbf{x}\wedge\mathbf{y}\in\bigwedge^2(\mathbb{C}^4)$ be a Lagrangian bivector. Then the 4×4 skew-symmetric matrix $\mathtt{l}(\mathbf{x}\wedge\mathbf{y})=\mathbf{x}\,{}^t\mathbf{y}-\mathbf{y}\,{}^t\mathbf{x}$ belongs to the nullcone of $\mathfrak{C}^5$. The *Plücker map* $\lambda:\mathbb{Q}_3\ni[\mathbf{x}\wedge\mathbf{y}]\mapsto[\mathtt{l}(\mathbf{x}\wedge\mathbf{y})]\in\mathbb{P}\mathfrak{C}^5$ is a conformal biholomorphism of $\mathbb{Q}_3$ onto the null quadric $\mathcal{Q}_3$ of $\mathbb{P}\mathfrak{C}^5$. The components of $\mathtt{l}$ with respect to $(\mathrm{L}_1,\dots,\mathrm{L}_5)$ will be denoted by $\mathtt{l}^a$, $a=1,\dots,5$.

2.2.2. *Affine and unimodular conformal charts.* On $\mathbb{C}^3$ with coordinates (w^1,w^2,w^3), consider the quadratic form $g_{\mathbb{C}^3}=dw^1dw^3-(dw^2)^2$. Let $\mathcal{U}\subset\mathbb{Q}_3$ be the complement of the hyperplane section $\mathcal{A}=\{[\mathbf{x}\wedge\mathbf{y}]\in\mathbb{Q}_3\mid \mathtt{l}^1(\mathbf{x}\wedge\mathbf{y})=0\}$. Then

$$\jmath:\mathcal{U}\ni[\mathbf{x}\wedge\mathbf{y}]\mapsto\frac{1}{\mathtt{l}^1(\mathbf{x}\wedge\mathbf{y})}\left(-\mathtt{l}^4(\mathbf{x}\wedge\mathbf{y}),\mathtt{l}^3(\mathbf{x}\wedge\mathbf{y}),\mathtt{l}^2(\mathbf{x}\wedge\mathbf{y})\right)\in\mathbb{C}^3$$

is a conformal biholomorphism, the *affine conformal chart* of $\mathbb{Q}_3$. On $\mathrm{SL}(2,\mathbb{C})$, consider the bi-invariant quadratic form $\widehat{g}=dx^1_1dx^2_2-dx^2_1dx^1_2$. Let $\widetilde{\mathcal{U}}\subset\mathbb{Q}_3$ be the complement of the hyperplane section $\mathcal{B}=\{[\mathbf{x}\wedge\mathbf{y}]\in\mathbb{Q}_3\mid \mathtt{l}^3(\mathbf{x}\wedge\mathbf{y})=0\}$. The map $\tilde{\jmath}:\widetilde{\mathcal{U}}\to\mathrm{SL}(2,\mathbb{C})$ defined by

$$\tilde{\jmath}([\mathbf{x}\wedge\mathbf{y}])=\frac{\sqrt{2}}{\mathtt{l}^3(\mathbf{x}\wedge\mathbf{y})}\left(\mathtt{l}^1(\mathbf{x}\wedge\mathbf{y})\mathbf{a}^1_1-\mathtt{l}^2(\mathbf{x}\wedge\mathbf{y})\mathbf{a}^2_1-\mathtt{l}^4(\mathbf{x}\wedge\mathbf{y})\mathbf{a}^1_2-\mathtt{l}^5(\mathbf{x}\wedge\mathbf{y})\mathbf{a}^2_2\right),$$

where $\mathbf{a}^s_r$, $r,s=1,2$, denote the elementary 2×2 matrices, is a conformal biholomorphism, the *unimodular conformal chart* of $\mathbb{Q}_3$.

2.3. **The contact structure of $\mathbb{CP}^3$.** On $\mathbb{CP}^3$ we consider the $\mathrm{Sp}(2,\mathbb{C})$-invariant contact structure induced by $\zeta=-z^3dz^1-z^1dz^3+z^4dz^2-z^2dz^4$.

2.3.1. *The unimodular contact chart.* On $\mathrm{PSL}(2,\mathbb{C})$, consider the left-invariant contact form $\widetilde{\zeta}=x^2_2dx^1_1-x^1_1dx^2_2+x^2_1dx^1_2-x^1_2dx^2_1$. Let $\mathcal{V}\subset\mathbb{CP}^3$ be the complement of the 2-dimensional quadric $\mathcal{Q}_2=\{[\xi]\in\mathbb{CP}^3\mid \xi^1\xi^3-\xi^2\xi^4=0\}$. The map

$$\jmath^\sharp:\mathcal{V}\ni[\xi]\mapsto\left[\left(\xi^2\mathbf{a}^1_1+\xi^1\mathbf{a}^2_1+\xi^3\mathbf{a}^1_2+\xi^4\mathbf{a}^2_2\right)\right]\in\mathrm{PSL}(2,\mathbb{C})$$

is a contact biholomorphism, the *unimodular contact chart* of $\mathbb{CP}^3$.

2.4. **Fibrations over Riemannian and Lorentzian space forms.**

2.4.1. *Hyperbolic and de Sitter projections.* Let $\mathfrak{h}(2,\mathbb{C})$ be the space of 2×2 hermitian matrices with scalar product $2(\alpha,\beta)=\det(\alpha)+\det(\beta)-\det(\alpha+\beta)$. The hyperbolic and de Sitter spaces can be realized as $\mathcal{H}^3=\{\alpha\in\mathfrak{h}(2,\mathbb{C})\mid\det(\alpha)=1,\,\mathrm{tr}(\alpha)>0\}$ and $\mathcal{H}^{1,2}=\{\alpha\in\mathfrak{h}(2,\mathbb{C})\mid\det(\alpha)=-1\}$, equipped with the Riemannian and Lorentzian structures inherited from $(\cdot,\cdot)$. The projections $\pi_{\mathcal{H}^3}:\mathrm{PSL}(2,\mathbb{C})\ni[B]\mapsto B\,{}^t\bar{B}\in\mathcal{H}^3$ and $\pi_{\mathcal{H}^{1,2}}:\mathrm{PSL}(2,\mathbb{C})\ni[B]\mapsto B(\mathbf{a}^1_1-\mathbf{a}^2_2)\,{}^t\bar{B}\in\mathcal{H}^{1,2}$ are, respectively, the bundle of oriented orthonormal frames of $\mathcal{H}^3$ and that of oriented and future-directed orthonormal frames of $\mathcal{H}^{1,2}$. In turn,

$$\begin{cases}\tilde{\ell}_{\mathcal{H}^3}=\pi_{\mathcal{H}^3}\circ[\tilde{\jmath}]:\widetilde{\mathcal{U}}\to\mathcal{H}^3,\\ \tilde{\ell}_{\mathcal{H}^{1,2}}=\pi_{\mathcal{H}^{2,1}}\circ[\tilde{\jmath}]:\widetilde{\mathcal{U}}\to\mathcal{H}^{1,2},\\ \ell^\sharp_{\mathcal{H}^3}=\pi_{\mathcal{H}^3}\circ\jmath^\sharp:\mathcal{V}\to\mathcal{H}^3,\\ \ell^\sharp_{\mathcal{H}^{1,2}}=\pi_{\mathcal{H}^{2,1}}\circ\jmath^\sharp:\mathcal{V}\to\mathcal{H}^{1,2}\end{cases}$$

give on $\widetilde{\mathcal{U}}\subset\mathbb{Q}_3$ and $\mathcal{V}\subset\mathbb{CP}^3$ structures of principal bundles either on $\mathcal{H}^3$ or on $\mathcal{H}^{1,2}$. The structure groups are $\mathrm{SU}(2)$ or $\mathrm{SU}(1,1)$, respectively.

2.4.2. *Euclidean and Minkowski projections.* Let $\mathbb{R}^3$ and $\mathbb{R}^{1,2}$ denote Euclidean and Minkowski 3-space, respectively. The maps

$$\begin{cases} \pi_{\mathbb{R}^3} : \mathbb{C}^3 \ni (z_1, z_2, z_3) \mapsto \left(\frac{\mathrm{Re}(z_1+z_3)}{2}, \frac{\mathrm{Im}(z_1-z_3)}{2}, \mathrm{Re}(z_2)\right) \in \mathbb{R}^3, \\ \pi_{\mathbb{R}^{1,2}} : \mathbb{C}^3 \ni (z_1, z_2, z_3) \mapsto \left(\mathrm{Re}(z_2), \frac{\mathrm{Im}(z_1+z_3)}{2}, \frac{\mathrm{Im}(z_1-z_3)}{2}\right) \in \mathbb{R}^{1,2} \end{cases}$$

give on $\mathbb{C}^3$ two structures of real vector bundle. Consequently, $\ell_{\mathbb{R}^3} = \pi_{\mathbb{R}^3} \circ \jmath$ and $\ell_{\mathbb{R}^{1,2}} = \pi_{\mathbb{R}^{1,2}} \circ \jmath$ make $\mathcal{U}$ a real vector bundle over $\mathbb{R}^3$ or over $\mathbb{R}^{1,2}$.

2.4.3. *The twistor fibration.* A 3-dimensional linear subspace $\mathfrak{p} \subset \mathfrak{C}^5$ is said to be *parabolic* if $\dim \mathrm{Ker}(g_{\mathfrak{C}}|_{\mathfrak{p}\times\mathfrak{p}}) = 2$. The totality of all parabolic subspaces, denoted by $\mathfrak{P}^3$, is a compact complex 3-fold acted upon transitively both by $\mathrm{Sp}(2,\mathbb{C})$ and $\mathrm{Sp}(2)$. If $[\xi] \in \mathbb{CP}^3$, then $\mathfrak{p}_\xi = \{X \in \mathfrak{C}^5 \mid XJ(\xi\,{}^t\xi) - (\xi\,{}^t\xi)JX = 0\}$ belongs to $\mathfrak{P}^3$. The map $\mathfrak{p} : \mathbb{CP}^3 \ni \xi \mapsto \mathfrak{p}_\xi \in \mathfrak{P}^3$ is an equivariant biholomorphism. Let S^4 be the unit sphere of $\mathfrak{R}^5$. Consider the $\mathrm{SU}(2)$-bundle $\pi_{\mathfrak{P}} : \mathrm{Sp}(2) \ni A \mapsto [\mathtt{L}(A)_1 \wedge \mathtt{L}(A)_2 \wedge \mathtt{L}(A)_3] \in \mathfrak{P}^3$. By construction, the map $\mathrm{Sp}(2) \ni A \mapsto \mathtt{L}(A)_3 \in S^4$ is constant along the fibers of $\pi_{\mathfrak{P}}$, and hence it descends to a map $\mathfrak{w} : \mathfrak{P}^3 \to S^4$ which, upon the identification of $\mathbb{CP}^3$ with $\mathfrak{P}^3$, amounts to the *twistor fibration* (cf. [6]).

3. Projective structures

We will briefly recall the notion of a complex projective structure on a Riemann surface. We will adapt to this specific case the general definition given by S. Kobayashi in [25], which goes back to E. Cartan [12].

Definition 1. Let S be a Riemann surface and let $\mathrm{SL}(2,\mathbb{C})_1$ be the group of upper triangular 2×2 unimodular matrices. A *complex projective structure* on S is a holomorphic principal $\mathrm{SL}(2,\mathbb{C})_1$-bundle $P \to S$, equipped with a holomorphic Cartan connection η.

Lemma 1. *Let (P,η) be a projective structure. About any point $p_0 \in S$, there exist a coordinate chart (U,z) and a cross section $\mathfrak{p} : U \to P$, such that $\mathfrak{p}^*(\eta) = \mathbf{a}_2^1 dz$.*

Proof. Let $\mathfrak{q} : V \to P$ be a section of P, defined on a neighborhood of p_0. Possibly shrinking V, there is a coordinate $w : V \to \mathbb{C}$, such that $\mathfrak{q}^*(\eta) = (x_1^1(\mathbf{a}_1^1 - \mathbf{a}_2^2) + \mathbf{a}_2^1 + x_2^1\mathbf{a}_1^2)dw$. By the existence and uniqueness theorem for holomorphic ODE (see for instance [21, p. 46]), there exists a holomorphic function $g : U \to \mathbb{C}$, defined on a smaller neighborhood $U \subset V$ of p_0, such that $g'' - g^2 + (x_1^1)' + (x_1^1)^2 + x_1^1 = 0$, where h' denotes the derivative of a function h with respect to dw. Let $b = e^{-g}(x_1^1 + g')$ and define $z : U \to \mathbb{C}$ by $dz = e^{2g}dw$. Possibly shrinking U, z is a coordinate. Put $\mathfrak{p} = \mathfrak{q} \cdot (e^g\mathbf{a}_1^1 + e^{-g}\mathbf{a}_2^2 + b\mathbf{a}_1^2)$. Then, (U,z) and $\mathfrak{p}$ satisfy the required condition. □

Definition 2. A chart (U,z) is said to be *adapted* to (P,η) if there exists a cross section $\mathfrak{p} : U \to P$, such that $\mathfrak{p}^*(\eta) = \mathbf{a}_2^1 dz$. We call $\mathfrak{p}$ a *flat section* of P. By Lemma 1, S can be covered by an atlas $\mathcal{P}$ of adapted charts. An atlas on S is said to be *projective* if its transition functions are restrictions of Möbius transformations.

Lemma 2. *The atlas of the charts adapted to (P,η) is projective.*

Proof. It suffices to prove that the transition function between two adapted local coordinates z and w, defined on the same simply connected open neighborhood U, is the restriction of a Möbius transformation. Let $\mathfrak{p}, \mathfrak{q} : U \to P$ be the cross sections

such that $\mathfrak{p}^*(\eta) = \mathbf{a}_2^1 dz$ and $\mathfrak{q}^*(\eta) = \mathbf{a}_2^1 dw$. Consider $x = e^g \mathbf{a}_1^1 + e^{-g}\mathbf{a}_2^2 + b\mathbf{a}_1^2 : U \to \mathrm{SL}(2,\mathbb{C})_1$, such that $\mathfrak{q} = \mathfrak{p} \cdot x$. Then,

$$\mathbf{a}_2^1 dw = e^{2g} dz\, \mathbf{a}_2^1 + (dg - e^g b dz)\mathbf{a}_1^1 - (dg - e^g b dz)\mathbf{a}_2^2 + ((e^{-g}(db + bdg) - b^2 dz)\mathbf{a}_1^2.$$

This implies $bdz = e^{-g}dg$ and $d(dg/dz)dz - (dg)^2 = 0$. From the second equation, it follows that $g = c_2 - \log(z + c_1)$, where c_1, c_2 are two constants of integration. Then, $dw = e^{2g}dz$ yields $w = (z + c_1)^{-1}(c_3 z + (c_1 c_3 - e^{2c_2}))$, for some $c_3 \in \mathbb{C}$. This concludes the proof. □

Remark 1. In the literature [32, 35], a projective structure on a Riemann surface S is often defined in terms of a projective atlas. It is not difficult to prove that every projective atlas is originated by a projective structure in the sense of Definition 1.

Let (P, η) be a projective structure on S and let $\mathcal{P}$ be its the projective atlas. Let $\mathcal{M}^q$ be the sheaf of meromorphic differentials of order q. For each $(U, z) \in \mathcal{P}$, we define[3] $\mathfrak{d}_{(U,z)} : \mathcal{M}^4|_U \to \mathcal{M}^2|_U$ by

$$\mathfrak{d}_{(U,z)}(\mathrm{Z}dz^4) = \big(\frac{1}{2}\frac{\mathrm{Z}''}{\mathrm{Z}} - \frac{9}{16}\frac{\mathrm{Z}'^2}{\mathrm{Z}^2}\big)dz^2 \tag{1}$$

Lemma 3. *The second order operator $\mathfrak{d}_{(U,z)}$ is independent of the choice of the projective chart.*

Proof. Let $\tilde{z}$ and z be two coordinates on the neighborhood U, with transition function $h = \tilde{z} \circ z^{-1}$, and let $\mathfrak{d}_{(U,z)}$, $\mathfrak{d}_{(U,\tilde{z})}$ be defined as in (1). It is a computational matter to check that

$$\mathfrak{d}_{(U,z)} = \mathfrak{d}_{(U,\tilde{z})} + 2\,\mathcal{S}_z(h)dz^2, \tag{2}$$

where $\mathcal{S}_z(h) = (\frac{h''}{h'})' - \frac{1}{2}(\frac{h''}{h'})^2$ is the Schwarzian derivative of h with respect to z. If both charts are projective, then h is a Möbius transformation, which implies $\mathcal{S}_z(h) = 0$, and hence the result. □

Definition 3. Let (P, η) be a projective structure on S and let $\mathcal{P}$ be its projective atlas. Then, there exists a second order differential operator $\mathfrak{d} : \mathcal{M}^4 \to \mathcal{M}^2$, such that $\mathfrak{d}|_U = \mathfrak{d}_{(U,z)}$, for every $(U, z) \in \mathcal{P}$.

To apply projective structures in the study of isotropic curves we need a slightly more general notion.

Definition 4. Let $\mathrm{D} \subset S$ be a discrete set. A *meromorphic projective structure* on S is a projective structure (P, η) on $S \setminus D$, satisfying the following condition: for every $p_0 \in D$, there exist an open neighborhood U, with $U \cap D = \{p_0\}$, and a section $\mathfrak{p} : U \setminus \{p_0\} \to P$, such that $\mathfrak{p}^*(\eta)$ is meromorphic on U. A point $p_0 \in D$ is a *removable singularity* if P and η can be extended across p_0. If the points of D are not removable singularities, we say that D is the *singular locus* of (P, η). We can use (2) to define the operator $\mathfrak{d} : \mathcal{M}^4 \to \mathcal{M}^2$ also for meromorphic projective structures.

[3]We implicitly assume that $\mathfrak{d}_{(U,z)}(0) = 0$.

4. Isotropic curves in $\mathbb{Q}_3$

4.1. **Isotropic curves and Legendre associates.** A nonconstant holomorphic map $f : S \to \mathbb{Q}_3$ from a connected Riemann surface S into the complex quadric $\mathbb{Q}_3$ is said to be an *isotropic curve* if $f^*(\mathbf{g}) = 0$. Given $[\xi] \in \mathbb{CP}^3$, the pencil $\{P \in \mathbb{Q}_3 \mid [\xi] \subset P\}$ is a complex null geodesic [30], referred to as a *null ray*. We will only consider isotropic curves which are not null rays.

Let f be an isotropic curve and p_0 a point of S. Let (U, z) be a complex chart centered at p_0 and $\mathbf{u}_1$, $\mathbf{u}_2 : U \to \mathbb{C}^4$ be two maps, such that $f|_U = [\mathbf{u}_1 \wedge \mathbf{u}_2]$. Let $\mathfrak{s}(2, \mathbb{C})$ denote the space of symmetric 2×2 matrices and consider the nonconstant map $\dot{\mathbf{m}} = (\dot{m}_{ij}) : U \to \mathfrak{s}(2, \mathbb{C})$, defined by $\dot{m}_{ij} = \omega(\mathbf{u}_i, \mathbf{u}_j')$. Notice that f is isotropic if and only if $\det(\dot{\mathbf{m}}) = 0$. Possibly shrinking U, there exist a nowhere zero map $\dot{\mathtt{m}} : U \to \mathfrak{s}(2, \mathbb{C})$ and a nonnegative integer k_1, such that $\dot{\mathbf{m}} = z^{k_1}\dot{\mathtt{m}}$. The integer k_1 is the ramification index of f at p_0. Possibly switching $\mathbf{u}_1$ and $\mathbf{u}_2$ and shrinking U, we may assume that $\dot{\mathtt{m}}_{22}$ is nowhere zero. Then, $[-\dot{\mathtt{m}}_{22}\mathbf{u}_1 + \dot{\mathtt{m}}_{12}\mathbf{u}_2] : U \to \mathbb{CP}^3$ does not depend on the choice of $\mathbf{u}_1$ and $\mathbf{u}_2$. Thus there exists a holomorphic map $f^\sharp : S \to \mathbb{CP}^3$, such that $f^\sharp|_U = [-\dot{\mathtt{m}}_{22}\mathbf{u}_1 + \dot{\mathtt{m}}_{12}\mathbf{u}_2]$. By construction, $f^\sharp$ is a Legendre map and $f^\sharp(S)$ is not contained in any contact line of $\mathbb{CP}^3$. We call $f^\sharp$ the *Legendre associate* of f. The divisors of the critical points of f and $f^\sharp$ will be denoted by Δ_f and $\Delta_{f^\sharp}$, respectively.

Remark 2. Conversely, if $f^\sharp : S \to \mathbb{CP}^3$ is a Legendrian curve not contained in a contact line, then $f = [f^\sharp \wedge f^{\sharp\,\prime}] : S \to \mathbb{Q}_3$ is an isotropic curve which is not contained in any isotropic ray. This and the result of Bryant [6, Theorem G], asserting that any compact connected Riemann surface can be holomorphically embedded in $\mathbb{CP}^3$ as a Legendrian curve, imply that for every compact Riemann surfaces S there exists a generically one-to-one isotropic curve $f : S \to \mathbb{Q}_3$.

4.2. **Isotropic curves and classical surface theory.** For a given isotropic curve $f : S \to \mathbb{Q}_3$, consider the discrete subsets $\mathrm{E}_f = f(S) \cap \mathcal{A}$, $\widetilde{\mathrm{E}}_f = f(S) \cap \mathcal{B}$, and $\mathrm{E}_f^\sharp = f^\sharp(S) \cap \mathcal{Q}$. If S is compact, these are finite subsets. Let $S_f = S \setminus \mathrm{E}_f$, $\widetilde{S}_f = S \setminus \widetilde{\mathrm{E}}_f$, and $S_f^\sharp = S \setminus \mathrm{E}_f^\sharp$. The following theorem is a compendium of results ranging from the 19th century through the first decades of this century [43, 7, 6, 13, 27, 29, 31, 17].

Theorem A. *Let $f : S \to \mathbb{Q}_3$ be an isotropic curve. Then,*

1. *$\phi_{\mathbb{R}^3} := \ell_{\mathbb{R}^3} \circ f : S_f \to \mathbb{R}^3$ is a conformal, branched minimal immersion. The points of E_f are the ends of $\phi_{\mathbb{R}^3}$.*
2. *$\phi_{\mathbb{R}^{1,2}} := \ell_{\mathbb{R}^{1,2}} \circ f : S_f \to \mathbb{R}^{2,1}$ is a conformal, branched maximal immersion. The points of E_f are the ends of $\phi_{\mathbb{R}^3}$.*
3. *$\widetilde{\phi}_{\mathcal{H}^3} := \tilde{\ell}_{\mathcal{H}^3} \circ f : \widetilde{S}_f \to \mathcal{H}^3$ is a conformal, branched CMC 1 immersion. The points of $\widetilde{\mathrm{E}}_f$ are the ends of $\widetilde{\phi}_{\mathcal{H}^3}$.*
4. *$\widetilde{\phi}_{\mathcal{H}^{1,2}} := \tilde{\ell}_{\mathcal{H}^{1,2}} \circ f : \widetilde{S}_f \to \mathcal{H}^{1,2}$ is a conformal, branched spacelike CMC 1 immersion. The points of $\widetilde{\mathrm{E}}_f$ are the ends of $\widetilde{\phi}_{\mathcal{H}^{1,2}}$.*
5. *$\phi^\sharp_{\mathcal{H}^3} := \ell^\sharp_{\mathcal{H}^3} \circ f^\sharp : S_f^\sharp \to \mathcal{H}^3$ is a flat front. The points of $\mathrm{E}_f^\sharp$ are the ends of $\phi^\sharp_{\mathcal{H}^3}$.*
6. *$\phi^\sharp_{\mathcal{H}^{1,2}} := \ell^\sharp_{\mathcal{H}^{1,2}} \circ f^\sharp : S_f^\sharp \to \mathcal{H}^{1,2}$ is a spacelike flat front. The points of $\mathrm{E}_f^\sharp$ are the ends of $\phi^\sharp_{\mathcal{H}^{1,2}}$.*

(7) $\phi^\sharp_{S^4} := \mathfrak{w} \circ f^\sharp : S \to S^4$ *is a branched superminimal immersion.*

The listed maps are said to be the branched (or frontal) immersions tamed *by* f.

Remark 3. According to [6, Theorem F], a Legendre curve from a Riemann surface S to $\mathbb{CP}^3$ not contained in a contact line can be written in the form

$$f^\sharp = [(2g', 2gg', 2hg' - gh', h')], \tag{3}$$

where g and h are meromorphic functions with g nonconstant. In view of Theorem A, one can derive explicit representation formulae for the branched (frontal) immersions tamed by f in terms of the meromorphic functions g and h (see for instance [39, 29, 16, 17, 34]). As an example, in [29], the Bryant representation formula for Legendrian curves in $\mathbb{CP}^3$ was implicitly used to find flat fronts in $\mathcal{H}^3$ with n smooth ends (see Figure 1). The generating meromorphic functions are $g_n(z) = z^{-1}(z^n - 1)^{(2-n)/n}$ and $f_n(z) = z^{-2}(1 + z^n)(z^n - 1)^{(2-n)/n}$, $n \in \mathbb{N}$. They are defined on a covering of $\mathbb{C}$ punctured at the nth roots of the unity.

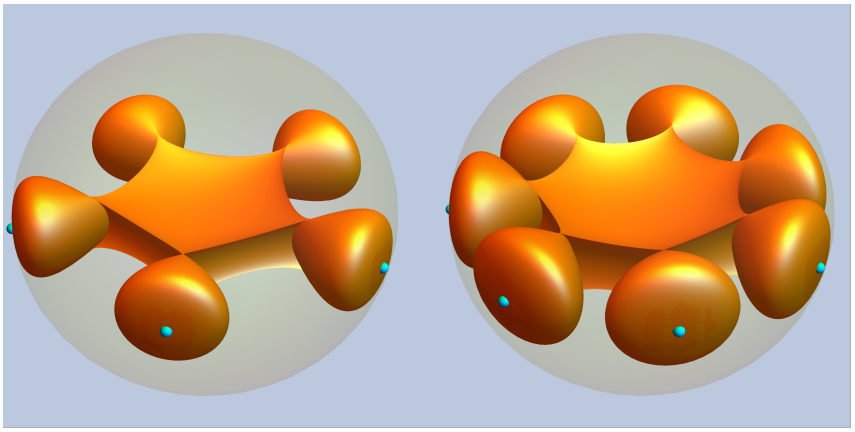

FIGURE 1. Flat fronts in hyperbolic 3-space with 5 and 7 ends, with rotational symmetries of order 5 and 7, originated by the meromorphic functions (g_5, h_5) and (g_7, h_7), respectively.

Definition 5. Let ϕ and $\hat\phi$ be tamed by f and $\hat f$, respectively. We say that ϕ is a *conformal Goursat transform* of $\hat\phi$ if there exists $A \in \mathrm{Sp}(2, \mathbb{C})$, such that $f(S) = A \cdot \hat f(\hat S)$.

Remark 4. The (*classical*) *Goursat transformation* was originally introduced for minimal surfaces in Euclidean space by Goursat [18, 24]. The classical definition can be rephrased as follows.

• Let ϕ and $\hat\phi$ be two branched minimal immersions in $\mathbb{R}^3$ tamed by f and $\hat f$, respectively; ϕ and $\hat\phi$ are (classical) Goursat transforms of each other if there exists an element A of the subgroup $\{X \in \mathrm{Sp}(2, \mathbb{C}) \mid X(\mathbf{e}_1 \wedge \mathbf{e}_2) = \mathbf{e}_1 \wedge \mathbf{e}_2,\ X(\mathbf{e}_3 \wedge \mathbf{e}_4) = \mathbf{e}_3 \wedge \mathbf{e}_4\} \cong \mathrm{SL}(2, \mathbb{C})$, such that $\hat f = A \cdot f$.

This means that the Gauss maps of ϕ and $\hat\phi$ differ by a Möbius transformation of S^2. Taking this point of view, one can define a Goursat transformation for isothermic

surfaces in $\mathbb{R}^3$ [20] which generalizes the classical one but, excluding the case of minimal surfaces, seems not directly related to our definition.

A Goursat transformation for CMC 1 surfaces in $\mathcal{H}^3$ has been considered in [14, 37, 41, 4]. It can be described as follows.

• Let ϕ and $\hat{\phi}$ be two CMC 1 immersed surfaces of $\mathcal{H}^3$ tamed by f and $\hat{f}$; ϕ and $\hat{\phi}$ are (*hyperbolic*) *Goursat transforms* of each other if there exists an element A of the subgroup $\{X \in \mathrm{Sp}(2,\mathbb{C}) \mid X\mathbf{e}_1 = \mathbf{e}_1,\, X\mathbf{e}_3 = \mathbf{e}_3,\, X(\mathbf{e}_2 \wedge \mathbf{e}_4) = \mathbf{e}_2 \wedge \mathbf{e}_4\} \cong \mathrm{SL}(2,\mathbb{C})$, such that $\hat{f} = A \cdot f$.

Therefore, our definition is a natural generalization of both the classical and hyperbolic Goursat transformations.

4.3. **The projective structure of an isotropic curve.** The zeroth order frame bundle along an isotropic curve $f : S \to \mathbb{Q}_3$ is the principal G_0-bundle $\mathcal{F}_0 \to S$,

$$\mathcal{F}_0 = \{(p, A) \in S \times \mathrm{Sp}(2,\mathbb{C}) \mid f(p) = \pi(A) = [A_1 \wedge A_2]\}\,. \tag{4}$$

The holomorphic sections of $\mathcal{F}_0$ are called *symplectic frames* along f. Let H_1 be the 2-dimensional Lie subgroup $\{X \in \mathrm{H} \mid (X\mathbf{e}_1) \wedge \mathbf{e}_1 = 0\}$, isomorphic to $\mathrm{SL}(2,\mathbb{C})_1$.

Theorem B. *Let $f : S \to \mathbb{Q}_3$ be an isotropic curve and $S_\star = S \setminus |\Delta_f| \cup |\Delta_{f^\sharp}|$. There exists a unique reduced bundle $\pi : \mathcal{F} \to S_\star$ of $\mathcal{F}_0|_{S_\star}$ with structure group H_1 such that:*

• *$\mathcal{F} \ni (p, A) \mapsto A \in \mathrm{Sp}(2,\mathbb{C})$ is an integral manifold of the Pfaffian differential system given by*

$$\varphi^3_1 = \varphi^4_1 = \varphi^2_1 - \varphi^4_2 = \varphi^1_1 - 3\varphi^2_2 = 4\varphi^1_2 - 3\varphi^2_4 = \varphi^1_4 = 0, \quad \varphi^4_2 \neq 0. \tag{5}$$

• *There exists a meromorphic quartic differential δ on S such that $\pi^*(\delta) = \varphi^1_3\,(\varphi^4_2)^3$.*

• *$\mathcal{F}$ with the $\mathfrak{sl}(2,\mathbb{C})$-valued 1-form $\eta = \varphi^2_2(\mathbf{a}^1_1 - \mathbf{a}^2_2) - 6^{-1/3}\varphi^4_2\mathbf{a}^1_2 - (3/32)^{1/3}\varphi^2_4\mathbf{a}^2_1$ is a meromorphic projective structure with singular locus $D \subseteq |\Delta_f| \cup |\Delta_{f^\sharp}|$.*

Proof. For the proof of Theorem B, we need the following two lemmas.

Lemma 4. *Let f be an isotropic curve defined on an open disk $\mathrm{D}_\epsilon \subset \mathbb{C}$ centered at the origin. Let k_1 and k_2 be the ramification indices of f and $f^\sharp$ at 0. Possibly shrinking D_ϵ there exists a meromorphic lift $A : \dot{\mathrm{D}}_\epsilon \to \mathrm{Sp}(2,\mathbb{C})$ of f, holomorphic on $\dot{\mathrm{D}}_\epsilon$ and satisfying (5). If $k_1 = k_2 = 0$, A is holomorphic.*

Proof of Lemma 4. Let $(\mathbf{b}_1, \dots, \mathbf{b}_4) \in \mathrm{Sp}(2,\mathbb{C})$, so that $f(0) = [\mathbf{b}_1 \wedge \mathbf{b}_2]$. Then, $f = [\mathbf{u}_1 \wedge \mathbf{u}_2]$, where $\mathbf{u}_1 = \mathbf{b}_1 + m_{11}\mathbf{b}_3 + m_{12}\mathbf{b}_4$ and $\mathbf{u}_2 = \mathbf{b}_2 + m_{12}\mathbf{b}_3 + m_{22}\mathbf{b}_4$. Let $m = m_{11}\mathbf{a}^1_1 + m_{12}(\mathbf{a}^1_2 + \mathbf{a}^2_1) + m_{22}\mathbf{a}^2_2$ and k_1 be the ramification index of f at 0. Then, $dm = z^{k_1}\dot{\mathfrak{m}}dz$, where $\dot{\mathfrak{m}} : \mathrm{D}_\epsilon \to \mathfrak{s}(2,\mathbb{C})$ satisfies $\dot{\mathfrak{m}}(0) \neq 0$ and $\det(\dot{\mathfrak{m}}) = 0$. Without loss of generality, we assume that $\dot{\mathfrak{m}}_{22}(z) \neq 0$, $\forall z$. Let $X_0 \in \mathrm{Sp}(2,\mathbb{C})$ be given by $X_0 = (-\dot{\mathfrak{m}}_{22}\mathbf{e}^1_1 - \dot{\mathfrak{m}}^{-1}_{22}\mathbf{e}^3_3 + \dot{\mathfrak{m}}_{12}\mathbf{e}^1_2 + \dot{\mathfrak{m}}^{-1}_{22}\dot{\mathfrak{m}}_{12}\mathbf{e}^4_3 + \mathbf{e}^2_2 + \mathbf{e}^4_4)_{|0}$ and let $B : \mathrm{D}_\epsilon \to \mathrm{Sp}(2,\mathbb{C})$ be the frame $B = (\mathbf{u}_1, \mathbf{u}_2, \mathbf{b}_3, \mathbf{b}_4)\,X_0$. Then, $B^{-1}dB = z^{k_1}r(a^2\mathbf{e}^1_3 + a(\mathbf{e}^1_4 + \mathbf{e}^2_3) + \mathbf{e}^2_4)dz$, where $r = \dot{\mathfrak{m}}_{22}$ and $a = r^{-1}(\dot{\mathfrak{m}}_{12}(0)\dot{\mathfrak{m}}_{22} - \dot{\mathfrak{m}}_{22}(0)\dot{\mathfrak{m}}_{12})$. Note that a has a zero of order $1 + k_2$ at 0. We assume $(1 \pm ia)(z) \neq 0$, $\forall z$. Let $x_j : \mathrm{D}_\epsilon \to \mathbb{C}$, $j = 0, \dots, 4$, be defined by

$$x_0 = -i\log(\frac{1+ia}{\sqrt{1+a}}), \quad x_1 = -\frac{z^{k_1}r(1+a^2)^2}{a'},$$

$$x_j = \frac{1}{2(r^2(1+a^2)z^{k_1})^{j-1}}\sum_{h=0}^{j-1}\frac{x_{jh}}{z^h}, \quad j = 2, 3, 4,$$

where

$$\begin{cases} x_{20} = \frac{1}{2}(\frac{d}{dz}(\log(\frac{r}{a'})) + \frac{4aa'}{1+a^2}), \\ x_{21} = \frac{k_1}{2}, \\ x_{30} = -\frac{r^2}{5}(\frac{a'''}{a'} - \frac{r''}{r}) + \frac{7r^2}{20}(\frac{d}{dz}(\log(a'))^2 - \frac{r^2}{4}(\frac{d}{dz}(\log(r))^2 - \frac{1}{10}\frac{rr'a''}{a'}, \\ x_{31} = -\frac{k_1 r^2}{10}\frac{d}{dz}(\log(ra')), \\ x_{32} = -\frac{k_1 r^2(4+k_1)}{20} \end{cases}$$

and

$$\begin{cases} x_{40} = \dfrac{1}{20}(4r^3\dfrac{a^{\mathrm{IV}}}{a'} - 2\dfrac{a'''}{a'}(11r^2\dfrac{d}{dz}(\log(a')) + r^2r') + 3(2r^2r'' - 3rr'^2)\dfrac{d}{dz}(\log(a') \\ \qquad\qquad + 3r^2r'(\dfrac{d}{dz}(\log(a'))^2 + 21r^3(\dfrac{d}{dz}(\log(a'))^3 - 4r^2r''' + 3r'(6rr'' - 5(r')^2)), \\ x_{41} = -\frac{k_1 r}{20a'^2}(2r^2a'a''' + 3a'(2rr' - r^2)a'' + 3a'^2(3r'^2 - 2rr''))), \\ x_{42} = -\frac{3}{20}k_1(2+k_1)\frac{d}{dz}(\log(ra'))r^3, \\ x_{43} = -\frac{1}{20}k_1(8 + 6k_1 + k_1^2)r^3. \end{cases}$$

Consider the maps $X, Y : \mathrm{D}_\epsilon \to \mathrm{Sp}(2,\mathbb{C})$, defined by

$$\begin{cases} X = \cos(x_0)I_4 - \sin(x_0)(\mathbf{e}_2^1 - \mathbf{e}_1^2 + \mathbf{e}_4^3 - \mathbf{e}_3^4), \\ Y = \mathbf{e}_2^2 + \mathbf{e}_4^4 + x_1^{-1}\mathbf{e}_3^3 - x_2\mathbf{e}_4^3 + x_3\mathbf{e}_2^3 + x_1(\mathbf{e}_1^1 + x_2\mathbf{e}_1^2 + (x_4 + x_2x_3)\mathbf{e}_1^3 + x_3\mathbf{e}_1^4). \end{cases}$$

The lift BXY satisfies the required properties. □

Lemma 5. *Let A and $\widetilde{A}$ be two lifts of f satisfying* (5)*. Then $A^{-1}\widetilde{A}$ is H_1-valued. Conversely, if A satisfies* (5) *and if X is H_1-valued, then $\widetilde{A} = AX$ satisfies* (5)*.*

Proof of Lemma 5. Let $X = A^{-1}\widetilde{A}$, α and $\widetilde{\alpha}$ be the pull-backs by A and $\widetilde{A}$ of the Maurer–Cartan form. Then,

$$\widetilde{\alpha} = X^{-1}(\alpha X + dX). \tag{6}$$

From $\alpha_1^3 = \alpha_1^4 = 0$ and (6), we get $\widetilde{\alpha}_1^3 = (X_1^2)^2\alpha_2^4$ and $\widetilde{\alpha}_1^4 = X_1^2X_2^2\alpha_2^4$. Since $\widetilde{\alpha}_1^3 = \widetilde{\alpha}_1^4 = 0$ and $\alpha_2^4 \neq 0$, we have $X_1^2 = 0$. From $\alpha_1^3 = \alpha_1^4 = \alpha_1^2 - \alpha_2^4 = 0$, $X_1^2 = 0$ and (6), we have $\widetilde{\alpha}_1^2 - \widetilde{\alpha}_2^4 = (X_2^2)^{-1}(X_1^1 - (X_2^2)^3)\alpha_2^4$. Since $\widetilde{\alpha}_1^2 - \widetilde{\alpha}_2^4 = 0$, then $X_1^1 = (X_2^2)^3$. From $\alpha_1^3 = \alpha_1^4 = \alpha_1^2 - \alpha_2^4 = \alpha_1^1 - 3\alpha_2^2 = 0$, $X_1^2 = X_1^1 - (X_2^2)^3 = 0$ and (6), we obtain $\widetilde{\alpha}_1^1 - 3\widetilde{\alpha}_2^2 = (3X_4^2X_2^2 - 4(X_2^2)^{-1}X_2^1)\alpha_2^4$. Hence $X_2^1 = \frac{3}{4}(X_2^2)^2X_4^2$. From $\alpha_1^3 = \alpha_1^4 = \alpha_1^2 - \alpha_2^4 = \alpha_1^1 - 3\alpha_2^2 = 4\alpha_2^1 - 3\alpha_4^2 = 0$, $X_1^2 = X_1^1 - (X_2^2)^3 = 4X_2^1 - 3(X_2^2)^2X_4^2 = 0$ and (6), we have $4\widetilde{\alpha}_2^1 - 3\widetilde{\alpha}_4^2 = -5(X_3^2X_2^2 + X_4^1(X_2^2)^{-1})\alpha_2^4$. Thus, $X_4^1 + X_3^2(X_2^2)^2 = 0$. From $\alpha_1^3 = \alpha_1^4 = \alpha_1^2 - \alpha_2^4 = \alpha_1^1 - 3\alpha_2^2 = \alpha_4^1 = 0$, $X_1^2 = X_1^1 - (X_2^2)^3 = X_4^1 + X_3^2(X_2^2)^2 = 0$ and (6), we obtain $\widetilde{\alpha}_4^1 = ((X_2^2)^{-1}X_3^1 - \frac{1}{4}X_4^2X_3^2)\alpha_2^4$. Hence,

$$\begin{aligned} X_1^2 = X_1^3 = X_1^4 = X_2^3 = X_2^4 = X_1^1 - (X_2^2)^3 = X_4^1 + X_3^2(X_2^2)^2 \\ = 4X_3^1 - X_2^2X_4^2X_3^2 = 4X_2^1 - 3(X_2^2)^2X_4^2 = 0. \end{aligned}$$

This proves that A is H_1-valued. Retracing the calculations, one sees that if A satisfies (5) and if X is H_1-valued, also AX satisfies (5). □

Lemmas 4 and 5 imply that, for every $p_0 \in S_\star$, there exist an open neighborhood U and a cross section $U \to \mathcal{F}_0$ satisfying (5) and that the transition function of two such sections is H_1-valued. This proves the existence and uniqueness of the reduced bundle $\mathcal{F}$. In addition, if $p_0 \in |\Delta_f| \cup |\Delta_{f^\sharp}|$, then there exist an open neighborhood

U, such that $U \cap |\Delta_f| \cup |\Delta_{f^\sharp}| = \{p_0\}$, and a cross section $A : U \setminus \{p_0\} \to \mathcal{F}$. The point p_0 is either a removable singularity or a pole. We call A a *meromorphic section* of $\mathcal{F}$ at p_0. The pull-back of the Maurer–Cartan form by a meromorphic section is holomorphic on $U \setminus \{p_0\}$ and meromorphic on U.

Let A, $\hat{A}$ be two sections of $\mathcal{F}$ such that $\hat{A} = AX$. Put $X = \mathtt{S}(x)$, where $x : U \cap \hat{U} \to \mathrm{SL}(2,\mathbb{C})_1$ and denote by α and $\hat{\alpha}$ the pull-backs of the Maurer–Cartan form. Then, $\hat{\alpha}^4_2 = (x^1_1)^2\alpha^4_2$ and $\hat{\alpha}^1_3 = (x^1_1)^{-6}\alpha^1_3$. This implies that $\varphi^1_3(\varphi^4_2)^3$ is projectable, i.e., there exists a holomorphic quartic differential δ on $S_\star$, such that $\pi^*(\delta) = \varphi^1_3(\varphi^4_2)^3$. If $p_0 \in |\Delta_f| \cup |\Delta_{f^\sharp}|$ and $A : U \to \mathcal{F}$ is a meromorphic section defined on an open neighborhood of p_0, then $\delta|_U = A^*(\varphi^1_3(\varphi^4_2)^3)$. This implies that δ is meromorphic on U.

Let A, $\hat{A}$ be as above. From $\hat{\alpha} = \mathtt{S}(x)^{-1}\alpha\mathtt{S}(x) + \mathtt{S}(x)^{-1}d\mathtt{S}(x)$, we have $\hat{A}^*(\eta) = x^{-1}A^*(\eta)x + x^{-1}dx$. Taking into account that $A^*(\eta^2_1)$ is nowhere zero, it follows that η is a Cartan connection. □

Definition 6. We call $(\mathcal{F}, \eta)$ the projective structure of f. A point $p_0 \in S$ is said to be *heptactic* if $\delta|_{p_0} = 0$. An isotropic curve with $\delta = 0$ is called a *conformal cycle*.[4] If $\delta \neq 0$, f is said to be of *general type*. The *conformal bending* of an isotropic curve of general type is the meromorphic function $\kappa := \mathfrak{d}(\delta)^2/\delta$.

Remark 5. In his analysis of isotropic curves in $\mathbb{C}^3$, E. Cartan [11, 24] defined, for a generic isotropic curve $f : S \to \mathbb{C}^3$ (*minimal curves* in the classical terminology), a nowhere zero holomorphic 1-form ω on S, the *element of pseudoarc*, and a holomorphic function $\mathtt{k}$, the *curvature*. One can write δ and $\mathfrak{d}(\delta)$ in terms of ω and $\mathtt{k}$. As a result, we obtain $\delta = -\frac{1}{25}(5\mathtt{k}'' - 4\mathtt{k}^2)\omega^4$, where the derivatives are computed with respect to ω. If $z : S \to \mathbb{C}$ is a uniformizing parameter for ω, then f is a conformal cycle if and only if either $\mathtt{k} = 0$, or else $\mathtt{k} = \sqrt[3]{15/2}\wp_{(0,g_3)}(\sqrt[3]{2/15}z)$, where $\wp_{g_2,g_3}$ is the Weierstrass elliptic $\wp$-function with invariants g_2 and g_3. If $\mathtt{k} = 0$, we get a twisted cubic. Assuming $5\mathtt{k}'' - 4\mathtt{k}^2 \neq 0$, we have $\mathfrak{d}(\delta) = \mathtt{h}\omega^2$, where

$$\mathtt{h} = \frac{5\mathtt{k}^{(4)}}{2(5\mathtt{k}''-4\mathtt{k}^2)} - \frac{1125\mathtt{k}^{(3)}(\mathtt{k}^{(3)}-\frac{16}{5}\mathtt{k}')+800\mathtt{k}^{(2)}(\mathtt{k}\mathtt{k}^{(2)}+2\mathtt{k}'^2)+512\mathtt{k}^2(\frac{25}{8}\mathtt{k}'^2-\mathtt{k}^3)}{80(5\mathtt{k}''-4\mathtt{k}^2)^2}.$$

4.4. **Conformal cycles.** The group $\mathrm{SL}(2,\mathbb{C})$ acts on $\mathbb{CP}^3$ and $\mathbb{Q}_3$ via the representation $\mathtt{S}$ (cf. Section 2.1). Apart from fixed points, the orbits of this action are 1-dimensional and conformally congruent to each other. Let $\widehat{\mathcal{C}} \subset \mathbb{CP}^3$ and $\mathcal{C} \subset \mathbb{Q}_3$ denote, respectively, the orbits through $[\mathbf{e}_1]$ and $[\mathbf{e}_1 \wedge \mathbf{e}_2]$. The equivalence problem for conformal cycles is solved by the following.

Proposition 6. *The orbit $\mathcal{C}$ is a rational conformal cycle and $\widehat{\mathcal{C}}$ is its Legendre associate. In addition, any other cycle is conformally congruent to $\mathcal{C}$.*

Proof. The stabilizer of the action of $\mathrm{SL}(2,\mathbb{C})$ at $[\mathbf{e}_1 \wedge \mathbf{e}_2]$ is $\mathrm{SL}(2,\mathbb{C})_1$. Hence $\mathcal{C}$ is biholomorphic to $\mathbb{CP}^1$. The 3-dimensional subgroup $\mathrm{H} = \mathtt{S}(\mathrm{SL}(2,\mathbb{C}))$ is the maximal integral submanifold through I_4 of the left-invariant completely integrable holomorphic Pfaffian differential system on $\mathrm{Sp}(2,\mathbb{C})$ given by

$$\varphi^3_1 = \varphi^4_1 = \varphi^2_1 - \varphi^4_2 = \varphi^1_1 - 3\varphi^2_2 = 4\varphi^1_2 - 3\varphi^2_4 = \varphi^1_4 = \varphi^1_3 = 0. \tag{7}$$

Then, $\{(P, A) \in \mathcal{C} \times \mathrm{H} \mid [A_1 \wedge A_2] = P\}$ is a reduction of the zeroth order frame bundle of $\mathcal{C}$ with structure group H_1. From (7), it follows that this bundle is

[4] We implicitly assume $f(S)$ is not properly contained in any other cycle.

the projective bundle of $\mathcal{C}$. Since $\varphi_3^1 = 0$, the quartic differential of $\mathcal{C}$ vanishes identically. This proves that $\mathcal{C}$ is a a conformal cycle. The punctured curve $\mathcal{C}_* = \mathcal{C}\setminus\{[\mathbf{e}_3\wedge\mathbf{e}_4]\}$ is parametrized by $f(z) = [\mathtt{v}(z)\wedge\mathtt{w}(z)]$, where $\mathtt{v}(z) = \mathbf{e}_1 + \frac{z^3}{3}\mathbf{e}_3 - \frac{z^2}{2}\mathbf{e}_4$ and $\mathtt{w}(z) = \mathbf{e}_2 - \frac{z^2}{2}\mathbf{e}_3 + z\mathbf{e}_4$. The orbit $\widehat{\mathcal{C}}$ is the twisted cubic $z \mapsto [\mathtt{u}(z)]$, $\mathtt{u}(z) = \mathbf{e}_1 + z\mathbf{e}_2 - \frac{z^3}{6}\mathbf{e}_3 + \frac{z^2}{2}\mathbf{e}_4$. Since $\mathtt{w} = \mathtt{u}'$ and $\mathtt{v} = \mathtt{u} - z\mathtt{w}$, it follows that $\widehat{\mathcal{C}}$ is the Legendre associate of $\mathcal{C}$.

Next, let $f : S \to \mathbb{Q}_3$ be any other conformal cycle and $\mathcal{F}_f$ be its projective bundle. Since f is a cycle, the map $\Psi : (p, A) \in \mathcal{F}_f \to A \in \mathrm{Sp}(2,\mathbb{C})$ is an integral manifold of (7). Then, there is $B \in \mathrm{Sp}(2,\mathbb{C})$ such that $\Psi(\mathcal{F}_f) \subset B\cdot\mathrm{H}$. This implies $f(S_\star) \subset B\cdot\mathcal{C}$. By continuity, $f(S) \subset B\cdot\mathcal{C}$ and, by maximality, $f(S) = B\cdot\mathcal{C}$. □

Remark 6. According to [8, 9], it follows from Proposition 6 that conformal cycles exhaust the class of isotropic embeddings of $\mathbb{CP}^1$ into $\mathbb{Q}_3$ of degree 4.

Definition 7. The surfaces associated to a conformal cycle can be viewed as the counterparts of the *Cyclides of Dupin* in the classical *Lie sphere geometry* (see for instance [24]). For this reason they are called *pseudo-Cyclides*. By construction, all pseudo-Cyclides are Goursat transforms of the ones tamed by the *standard cycle*

$$f(z) = [(\mathtt{v}(z)] = \Big[(\mathbf{e}_1 + \frac{z^3}{3}\mathbf{e}_3 - \frac{z^2}{2}\mathbf{e}_4)\wedge(\mathbf{e}_2 - \frac{z^2}{2}\mathbf{e}_3 + z\mathbf{e}_4)\Big].$$

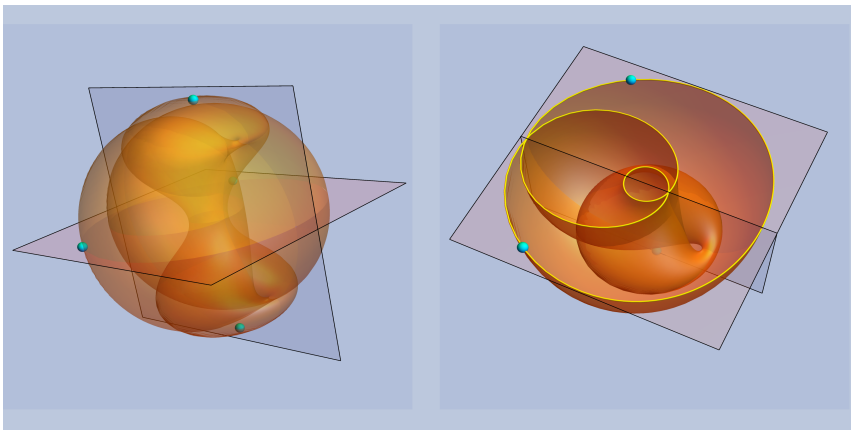

FIGURE 2. A CMC 1 pseudo-Cyclide of $\mathcal{H}^3$ with four embedded ends.

Example 7 (Standard models for pseudo-Cyclides)**.** Up to a homothetic factor, the minimal and maximal surfaces tamed by the standard cycle are the *Enneper surface* and the *conjugate of the maximal space-like Enneper surface of the second kind* [27]. The CMC 1 surface in hyperbolic 3-space tamed by f is a Goursat transform of the rotationally-invariant *Catenoid cousin* with parameter $\mu = 1$ (cf. [7, 24]), while the CMC 1 surface in de Sitter space tamed by f is the *spacelike Catenoid cousin* considered in [31]. The flat front tamed by f is a Goursat transform of the rotationally invariant front with parameter $\mu = -2$ considered in [29]. Finally, the superminimal surface in S^4 tamed by f is a Goursat transform of the Veronese embedding [13]. Figure 2 depicts a CMC 1 pseudo-Cyclide in hyperbolic 3-space. It is not embedded and possesses four smooth ends.

4.5. **Geometric meaning of heptactic points.** The 7-dimensional complex homogeneous space $\mathfrak{C} = \mathrm{Sp}(2,\mathbb{C})/\mathrm{H}$ is the *manifold of conformal cycles* of $\mathbb{Q}_3$. Let f be an isotropic curve, $p \in S_\star$ a generic point, and $A \in \mathcal{F}|_p$ (cf. Theorem B). The cycle $\mathcal{C}_f|_p = A \cdot \mathrm{H} \cdot f(p)$ is independent of the choice of A. We call $\mathcal{C}_f|_p$ the *osculating cycle* of f at p. The holomorphic map $\mathcal{C}_f : \mathrm{S}_\star \ni p \mapsto \mathcal{C}_p \in \mathfrak{C}$ is called the *osculating map* of f. Using the projective structure and the Plücker map, one can prove the following.

Proposition 8. *Let f be an isotropic curve. Then:*

- *f and $\mathcal{C}_f|_p$ have analytic contact of order ≥ 5 at p, for every $p \in \mathrm{S}_\star$.*
- *f and $\mathcal{C}_f|_p$ have analytic contact of order > 5 at $p \in S_\star$ if and only if p is an heptactic point.*
- *The heptactic points of f are the critical points of the osculating map $\mathcal{C}_f$.*

Remark 7. Proposition 8 implies that, if ϕ is tamed by $f : S \to \mathbb{Q}_3$, then the projection of $\mathcal{C}_f|_p$ onto the appropriate Riemannian or Lorentzian spaceform is a pseudo-Cyclide with analytic contact of order at least 5 with ϕ. In addition, ϕ is the envelope of the 2-parameter family of its osculating pseudo-Cyclides. This is reminiscent of a similar property for surfaces in Lie sphere geometry (cf. [3]). Figure 3 reproduces a CMC 1 surface tamed by a generic isotropic curve (in green) and one of its osculating pseudo-Cyclide (the same one depicted in Figure 2).

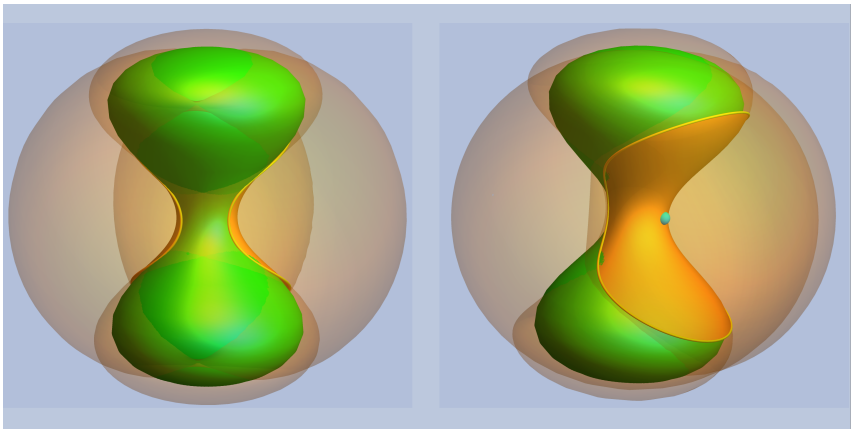

FIGURE 3. A CMC 1 embedded surface tamed by an isotropic curve of general type (green) and one of its osculating pseudo-Cyclides (orange). The pseudo-Cyclide is the same one of Figure 2.

4.6. **The equivalence problem.** Let $f : S \to \mathbb{Q}_3$ and $\hat{f} : \hat{S} \to \mathbb{Q}_3$ be two isotropic curves. We say that $\hat{f}$ is dominated by f if there exist a holomorphic map $h : \hat{S} \to S$ and $X \in \mathrm{Sp}(2,\mathbb{C})$ such that $\hat{f} = X \cdot f \circ h$.

Theorem C. *Let $f : S \to \mathbb{Q}_3$ and $\hat{f} : \hat{S} \to \mathbb{Q}_3$ be two isotropic curves of general type. Then $\hat{f}$ is dominated by f if and only f there exists a holomorphic map $h : \hat{S} \to S$ such that $h^*(\delta) = \hat{\delta}$ and $h^*(\mathfrak{d}\delta) = \mathfrak{d}(\hat{\delta})$.*

Proof. It suffices to show that two isotropic curves $f,\ \hat{f} : S \to \mathbb{Q}_3$ of general type have the same quartic and quadratic differentials if and only if $\hat{f} = X \cdot f$, for some $X \in \mathrm{Sp}(2,\mathbb{C})$. In the proof we consider $\mathcal{F}_f$ and $\mathcal{F}_{\hat{f}}$ as projective bundles, with structure group $\mathrm{SL}(2,\mathbb{C})_1$ acting on the right via the representation $\mathtt{S}$. The proof is organized in a first step, two technical lemmas and a second step.

Step I. Suppose that $\hat{f} = X\, f$. Then, $\mathcal{F}_{\hat{f}} = X\mathcal{F}_f$ and $\mathtt{F} : (p, A) \in \mathcal{F}_f \to (p, X\, A) \in \mathcal{F}_{\hat{f}}$ is a bundle map such that $\mathtt{F}^*(\hat{\varphi}) = \varphi$. Hence, $\mathtt{F}^*(\hat{\eta}) = \eta$ and $\delta_f = \delta_{\hat{f}}$. Since $\mathtt{F}$ preserves the Cartan connections, the projective atlases induced on S by $(\mathcal{F}_f, \eta)$ and $(\mathcal{F}_{\hat{f}}, \hat{\eta})$ do coincide. This implies that $\mathfrak{d}(\delta) = \mathfrak{d}(\hat{\delta})$.

Lemma 9. *Let (P, η) be a singular projective structure on S, let (U, z) be any complex chart of S and let $\mathfrak{p} : U \to P$ be a cross section. Put $\epsilon^i_j dz = \mathfrak{p}^*(\eta^i_j)$. Then*

$$\mathfrak{d}|_U = \mathfrak{d}_{(U,z)} + \mathfrak{r}_{(U,z,\mathfrak{p})} dz^2, \tag{8}$$

where

$$\mathfrak{r}_{(U,z,\mathfrak{p})} = -2\frac{(\epsilon^2_1)''}{\epsilon^2_1} + \frac{(\epsilon^2_1)'}{\epsilon^2_1}\left(3\frac{(\epsilon^2_1)'}{\epsilon^2_1} - 4\epsilon^1_1\right) + 4\left((\epsilon^1_1)' + (\epsilon^1_1)^2 + \epsilon^1_2\epsilon^2_1\right) \tag{9}$$

and $\mathfrak{d}_{(U,z)}$ is defined as in (1)*, although (U, z) is not a projective chart.*

Proof. It is easy to see that $\mathfrak{r}_{(U,z,\mathfrak{p})} dz^2$ does not depend on the choice of the section $\mathfrak{p}$. Let $\tilde{z}$ be another coordinate and h the transition function $\tilde{z} \circ z^{-1}$. A straightforward computation shows that $\mathfrak{r}_{(U,z,\mathfrak{p})} dz^2 = \mathfrak{r}_{(U,\tilde{z},\mathfrak{p})} d\tilde{z}^2 - 2\mathcal{S}_z(h)\, dz^2$. From (2) it follows that the right hand side of (8) does not depend on the choice of the local coordinate and of the cross section. If (U, z) is a projective chart and $\mathfrak{p}$ is a flat section, then $\mathfrak{r}_{(U,z,\mathfrak{p})} = 0$. This implies the result. □

Lemma 10. *Let (P, η) be a singular projective structure on S with singular locus D. Let δ be a meromorphic quartic differential and $S_\sharp = S \setminus D \cup |\Delta_\delta|$. There exists a unique reduced bundle $\widehat{\pi} : \widehat{P} \to S_\sharp$ of $P|_{S_\sharp}$ with structure group $\mathrm{Z}_8 = \{\epsilon \mathbf{a}^1_1 + \epsilon^{-1}\mathbf{a}^2_2 \mid \epsilon^8 = 1\}$, such that $\widehat{\pi}^*(\delta) = 6^{3/4}(\eta^2_1)^4$ and $\eta^1_1 = 0$. In addition, $\widehat{\pi}^*(\mathfrak{d}(\delta)) = 4\eta^2_1\eta^1_2$.*

Proof. Let $\mathtt{D} : P|_{S_\sharp} \to \mathbb{C}$ be the holomorphic function such that $\pi^*(\delta) = \mathtt{D}(\eta^2_1)^4$. By construction $\mathtt{D}(\mathfrak{p} \cdot x) = (x^1_1)^8 \mathtt{D}(\mathfrak{p})$, for every $\mathfrak{p} \in P|_{S_\sharp}$ and every $x \in \mathrm{SL}(2,\mathbb{C})_1$. Thus, $\hat{P} = \{\mathfrak{p} \in P|_{S_\sharp} \mid \mathtt{D}(\mathfrak{p}) = 6^{3/4}\}$ is a reduction of $P|_{S_\sharp}$ with structure group $\mathrm{H} = \{x \in \mathrm{SL}(2,\mathbb{C})_1 \mid (x^1_1)^8 = 1\}$ such that $\hat{\pi}^*(\delta) = 6^{3/4}(\eta^2_1)^4$. The 1-form η^1_1 is tensorial on $\hat{P}$. Thus, $\eta^1_1 = \mathtt{E}\eta^2_1$, where $\mathtt{E}$ is a holomorphic function such that $\mathtt{E}(\mathfrak{p}x) = \epsilon^{-2}(\mathtt{E}(\mathfrak{p}) - \epsilon x^1_2)$. Then, $\widehat{P} = \{\mathfrak{p} \in \hat{P} \mid \mathtt{E}(\mathfrak{p}) = 0\}$ is the unique reduction of $P|_{S_\sharp}$ such that $\widehat{\pi}^*(\delta) = 6^{3/4}(\eta^2_1)^4$, $\eta^1_1 = 0$. Let (U, z) be a complex chart and $\mathfrak{p} : U \to \hat{P}$ be a cross section such that $\delta = 6^{3/4} dz^4$. Then $\epsilon^1_1 = 0$ and $(\epsilon^2_1)^4 = 1$. Using the Lemma 9, we have $\mathfrak{r}_{(U,z,\mathfrak{p})} = 4\epsilon^1_2\epsilon^2_1$. Therefore, $\mathfrak{d}(\delta) = 4\mathfrak{p}^*(\eta^2_1\eta^1_2))$. Taking into account that η^2_1 and η^1_2 are tensorial on $\hat{P}$, we have $\hat{\pi}^*(\mathfrak{d}(\delta)) = 4\eta^2_1\eta^1_2$. □

Step II. Next, assume $\delta = \hat{\delta}$ and $\mathfrak{d}(\delta) = \mathfrak{d}(\hat{\delta})$. Let $D_\star = |\Delta_f| \cup |\Delta_{f^\sharp}| \cup |\Delta_{\hat{f}}| \cup |\Delta_{\hat{f}^\sharp}| \cup |\Delta_\delta|$ and $\mathfrak{p} : \widetilde{S} \to S \setminus D_\star$ be the universal covering. Since $\mathfrak{p}^*(\delta)$ is a nowhere zero holomorphic differential, $\widetilde{S}$ is biholomorphic either to $\mathbb{C}$, or to the unit disk. Then, there exists $z : \widetilde{S} \to \mathbb{C}$ such that $\delta = dz^4$. Consider the Z_8-reductions $\hat{\mathcal{F}}_{f\circ\mathfrak{p}}$ and $\hat{\mathcal{F}}_{\hat{f}\circ\mathfrak{p}}$ of the projective bundles $\mathcal{F}_{f\circ\mathfrak{p}}$ and $\mathcal{F}_{\hat{f}\circ\mathfrak{p}}$ constructed in Proposition 10.

Since $\widetilde{S}$ is simply connected, they are trivial. Pick two global trivialisations A and $\hat{A}$ and denote by α, $\hat{\alpha}$ the pull-backs of the Maurer–Cartan form. Possibly acting on the right with an element of Z_8 we have $\alpha_2^4 = \hat{\alpha}_2^4 = dz$. Since $\alpha_2^4\alpha_4^2 = \hat{\alpha}_2^4\hat{\alpha}_4^2 = \mathfrak{d}(\delta)$, then $\alpha = \hat{\alpha}$. By the Cartan–Darboux congruence Theorem [24], there exists $X \in \mathrm{Sp}(2,\mathbb{C})$ such that $\hat{A} = X\,A$. This implies $\hat{f} \circ \mathtt{p} = Xf \circ \mathtt{p}$. Hence $\hat{f} = X\,f$, as claimed. □

Proposition 11. *Let S be a simply connected Riemann surface. Let δ and γ be two holomorphic differentials of degree four and two, respectively. If $\delta_p \neq 0$, for every $p \in S$, then there exists an isotropic curve $f : S \to \mathbb{Q}_3$ of general type, such that $\delta_f = \delta$ and $\mathfrak{d}(\delta_f) = \gamma$. Moreover, f is unique up to the action by an element of the symplectic group.*

Proof. By the Uniformization Theorem, taking into account that on the Riemann surface S there is a non-null holomorphic differential δ, it follows that S is equivalent to either the complex plane or the unit disk. On S, we can then consider a global holomorphic coordinate $z : S \to \mathbb{C}$ and write $\gamma = \Gamma dz^2$, $\delta = D\,dz^4$ with $D_{|p} \neq 0$, for each $p \in S$. Since S is simply connected, we can choose a fourth root of D, say $\sqrt[4]{D}$. Next, let $E := \Gamma/\sqrt[4]{D}$ and consider the holomorphic $\mathfrak{sp}(2,\mathbb{C})$-valued 1-form $\hat{\varphi} = (\mathbf{e}_2^1 + \mathbf{e}_4^2 + \mathbf{e}_1^3 - \mathbf{e}_3^4)\sqrt[4]{D}dz + (\mathbf{e}_2^4 + \frac{3}{4}(\mathbf{e}_1^2 - \mathbf{e}_4^3))Edx$. Now, $\hat{\varphi}$ satisfies the Maurer–Cartan equation and hence, by the Cartan–Darboux existence theorem, there exists a holomorphic map $A : S \to \mathrm{Sp}(2,\mathbb{C})$ such that $\hat{\varphi} = A^{-1}dA$. The map $f : S \ni p \longmapsto [A_1(p) \wedge A_2(p)] \in \mathbb{Q}_3$ defines an isotropic curve with the required proprieties. The uniqueness assertion follows from Theorem C. □

Theorem C and Proposition 11 can be rephrased in terms of projective structures.

Corollary 12. *Let $f : S \to \mathbb{Q}_3$ and $\hat{f} : \hat{S} \to \mathbb{Q}_3$ be two isotropic curves of general type. Then $\hat{f}$ is dominated by f if and only if there exists a projective map $h : \hat{S} \to S$ such that $h^*(\delta) = \hat{\delta}$.*

Corollary 13. *Let (P,η) be a projective structure on S with singular locus D and let δ be a nonzero meromorphic quartic differential on S. Let $\mathtt{p} : S_\star \to S \setminus D \cup |\Delta_\delta|$ be a universal covering. Then there exists a generic isotropic curve $f : S_\star \to \mathbb{Q}_3$ whose projective structure is equivalent to $(\mathtt{p}^*(P), \mathtt{p}^*(\eta))$ such that $\mathtt{p}^*(\delta) = \delta_f$.*

4.7. **Conformal deformation and rigidity.** We adapt to our specific context, the general concepts of deformation and rigidity [10, 19, 22, 23].

Definition 8. Let f, $\hat{f} : S \to \mathbb{Q}_3$ be two isotropic curves. We say that $\hat{f}$ is a *kth order conformal deformation* of f if there exists a holomorphic map $\mathtt{D} : S \to \mathrm{Sp}(2,\mathbb{C})$, such that $\hat{f}$ and $\mathtt{D}(p) \cdot f$ have analytic contact of order k at p, for every $p \in S$. A deformation is *trivial* if it is congruent to f. We say that f is *deformable of order k* if, for every $p_0 \in S \setminus |\Delta_f| \cup |\Delta_{f^\sharp}| \cup |\Delta_\delta|$, there exist an open neighborhood U of p_0 and an isotropic curve of general type $\hat{f} : U \to \mathbb{Q}_3$, such that $\hat{f}$ is a non-trivial kth order deformation of $f|_U$. Otherwise, f is said to be *rigid to order k*.

Theorem D. *An isotropic curve of general type is deformable of order four and is rigid to order five.*

Proof. We first recall the following.

Fact. Two holomorphic maps ψ, $\hat\psi : S \to \mathbb{CP}^4$ have the same kth order jets at p_0 if and only if for any lifts $\Psi, \hat\Psi : S \to \mathbb{C}^5$ of ψ and $\hat\psi$ and for every complex chart (U, z) with $p_0 \in U$, there exist $\varrho_j \in \big(\Omega^{1,0}(S)|_{p_0}\big)^n$, $n = 0, \dots, k$, such that

$$\delta^j(\hat\Psi)|_{p_0} = \sum_{i=0}^{j} c_i^j \varrho_i \delta^{j-i}(\Psi)|_{p_0}, \quad j = 0, \dots, k, \quad \delta^k(\Psi) = \frac{d\Psi^k}{dz^k}(dz)^k, \tag{10}$$

where $c_i^j \in \mathbb{N}$ are defined by $c_0^j = c_j^j = 1$ and by $c_i^j = c_{i-1}^{j-1} + c_i^{j-1}$, for every $j \geq 2$ and $1 \leq i < j$.

Since the result is local, we may assume the existence of a global section A of the Z_8-bundle $\mathcal{F}_f$. Recall that

$$A^{-1}dA = (\mathbf{e}_2^1 + \mathbf{e}_4^2 + \mathbf{e}_1^3 - \mathbf{e}_3^4)\zeta + \big(\mathbf{e}_2^4 + \frac{3}{4}(\mathbf{e}_1^2 - \mathbf{e}_4^3)\big)\eta \tag{11}$$

where ζ, η are holomorphic 1-forms and $\zeta|_p \neq 0$. Without loss of generality, we may suppose that ζ is nowhere zero. It defines on S a unimodular affine structure consisting of all complex charts (U, z) such that $\zeta = dz$, $\eta = bdz$. Let $\hat f$ be another isotropic curve of general type and $\hat A$ be a cross section of $\mathcal{F}_{\hat f}$. Then

$$\hat A^{-1}d\hat A = \hat a(\mathbf{e}_2^1 + \mathbf{e}_4^2 + \mathbf{e}_1^3 - \mathbf{e}_3^4)dz + \hat b\big(\mathbf{e}_2^4 + \frac{3}{4}(\mathbf{e}_1^2 - \mathbf{e}_4^3)\big)dz, \tag{12}$$

where $\hat a$, $\hat b$ are holomorphic functions, $\hat a \neq 0$. Without loss of generality, we suppose that $\hat a$ is nowhere zero. The quadratic differential

$$\mathfrak{s} = \Big(2\frac{\hat a''}{\hat a} - 3\Big(\frac{d\log\hat a}{dz}\Big)^2\Big)dz^2$$

does not depend on the choice of the unimodular affine chart.

Lemma 14. *$\hat f$ is a fourth order conformal deformation of f if and only if $\gamma_{\hat f} = \gamma_f + \mathfrak{s}$.*

Proof. Let $\lambda : \mathbb{Q}_3 \to \mathcal{Q}_3$ be the Plücker map and $\mathtt{L} : \mathrm{Sp}(2, \mathbb{C}) \to \mathrm{O}(\mathfrak{C}^5, g_{\mathfrak{C}})$ be the spin covering homomorphism. Consider $\psi = \lambda \circ f$, $\hat\psi = \lambda \circ \hat f$ and let $\mathcal{A}, \hat{\mathcal{A}} : S \to \mathrm{O}(\mathfrak{C}^5, g_{\mathfrak{C}})$ be the maps defined by $\mathcal{A} = \mathtt{L} \circ A$, $\hat{\mathcal{A}} = \mathtt{L} \circ \hat A$. Then, $\mathcal{A}_1$ is a lift of ψ and $\hat{\mathcal{A}}_1$ is a lift of $\hat\psi$. In addition, $\mathcal{A}$ and $\hat{\mathcal{A}}$ satisfy $\mathcal{A}^{-1}d\mathcal{A} = \mathtt{N}dz$ and $\hat{\mathcal{A}}^{-1}d\hat{\mathcal{A}} = \hat{\mathtt{N}}dz$, where $\mathtt{N}, \hat{\mathtt{N}} : S \to \mathfrak{o}(\mathfrak{C}^5, g_{\mathfrak{C}})$ are given by

$$\begin{cases} \mathtt{N} = (\mathbf{b}_2^1 + \sqrt{2}(\mathbf{b}_4^3 - \mathbf{b}_3^2) - \mathbf{b}_1^4 - \mathbf{b}_5^4 + \mathbf{b}_2^5) + b(\mathbf{b}_1^2 - \mathbf{b}_4^5 + \frac{3}{3\sqrt{2}}(\mathbf{b}_3^4 - \mathbf{b}_2^3)), \\ \hat{\mathtt{N}} = \hat a(\mathbf{b}_2^1 + \sqrt{2}(\mathbf{b}_4^3 - \mathbf{b}_3^2) - \mathbf{b}_1^4 - \mathbf{b}_5^4 + \mathbf{b}_2^5) + \hat b(\mathbf{b}_1^2 - \mathbf{b}_4^5 + \frac{3}{3\sqrt{2}}(\mathbf{b}_3^4 - \mathbf{b}_2^3)). \end{cases} \tag{13}$$

Let $\mathtt{F}_{(h)}$, $\hat{\mathtt{F}}_{(h)} : S \to \mathbb{C}^5$ be given by the recursive formulae

$$\mathtt{F}_{(0)} = \hat{\mathtt{F}}_{(0)} = \mathbf{b}_1, \quad \mathtt{F}_{(h)} = (\frac{d}{dz} + \mathtt{N})\mathtt{F}_{(h-1)}, \quad \hat{\mathtt{F}}_{(h)} = (\frac{d}{dz} + \hat{\mathtt{N}})\hat{\mathtt{F}}_{(h-1)}, \tag{14}$$

where $(\mathbf{b}_1, \dots, \mathbf{b}_5)$ is the canonical basis of $\mathbb{C}^5$. Define $\mathtt{F}$, $\hat{\mathtt{F}} : S \to \mathrm{GL}(5, \mathbb{C})$ by $\mathtt{F} = (\mathtt{F}_{(0)}, \dots, \mathtt{F}_{(4)})$ and by $\hat{\mathtt{F}} = (\hat{\mathtt{F}}_{(0)}, \dots, \hat{\mathtt{F}}_{(4)})$. Then

$$J^{(4)}(\mathcal{A}_1) = \mathcal{A} \cdot \mathtt{F}, \quad J^{(4)}(\hat{\mathcal{A}}_1) = \hat{\mathcal{A}} \cdot \hat{\mathtt{F}}, \tag{15}$$

where

$$\begin{cases} J^{(4)}(\mathcal{A}_1) = \Big(\mathcal{A}_1, \frac{d\mathcal{A}_1}{dz}, \frac{d^2\mathcal{A}_1}{dz^2}, \frac{d^3\mathcal{A}_1}{dz^3}, \frac{d^4\mathcal{A}_1}{dz^4}\Big), \\ J^{(4)}(\hat{\mathcal{A}}_1) = \Big(\hat{\mathcal{A}}_1, \frac{d\hat{\mathcal{A}}_1}{dz}, \frac{d^2\hat{\mathcal{A}}_1}{dz^2}, \frac{d^3\hat{\mathcal{A}}_1}{dz^3}, \frac{d^4\hat{\mathcal{A}}_1}{dz^4}\Big). \end{cases}$$

From the fact mentioned at the beginning of the proof, $\hat{f}$ is a fourth order deformation of f if and only if there exist a holomorphic map $\mathcal{D} : S \to \mathrm{O}(\mathfrak{C}^5, g_{\mathfrak{C}})$ and holomorphic functions r_j, $j = 0, \dots, 4$, such that

$$J^{(4)}(\hat{\mathcal{A}}_1) = \mathcal{D} \cdot J^{(4)}(\mathcal{A}_1) \cdot \mathrm{R}, \tag{16}$$

where

$$\mathrm{R} = r_0 I_5 + r_1(\mathbf{b}_1^2 + 2\mathbf{b}_2^3 + 3\mathbf{b}_3^4 + 4\mathbf{b}_4^5) + r_2(\mathbf{b}_1^3 + 3\mathbf{b}_2^4 + 6\mathbf{b}_3^5) + r_3(\mathbf{b}_1^4 + 4\mathbf{b}_2^5) + r_4\mathbf{b}_1^5. \tag{17}$$

By construction, we have

$$\mathcal{D} = \hat{\mathcal{A}}\,\hat{\mathtt{F}}\,\mathrm{R}^{-1}\,\mathtt{F}^{-1}\,\mathcal{A}^{-1}. \tag{18}$$

This implies that $\mathtt{F}\mathrm{R}\hat{\mathtt{F}}^{-1}$ takes values in the orthogonal group $\mathrm{O}(\mathfrak{C}^5, g_{\mathfrak{C}})$. Imposing the orthogonality conditions, we obtain, letting $\varepsilon = \pm 1$,

$$\begin{cases} r_0 = \varepsilon\hat{a}^2, \\ r_1 = 2\varepsilon\hat{a}\hat{a}', \\ r_2 = \frac{\varepsilon}{7}(5(\hat{a}^3\hat{b} - \hat{a}^2 b) + 29\hat{a}'^2 + 4\hat{a}\hat{a}''), \\ r_3 = \frac{\varepsilon}{42\hat{a}}\left(14\hat{a}\hat{a}''' + 132\hat{a}\hat{a}'\hat{a}'' + (390\hat{a}'^2 - 200\hat{a}^2 b + 235\hat{a}^3\hat{b})\hat{a}' - 35\hat{a}^3 b' + 35\hat{a}^4\hat{b}'\right), \\ r_4 = \frac{\varepsilon}{294\hat{a}^2}\Big(1372\hat{a}^2\hat{a}'\hat{a}''' + (1266\hat{a}^4\hat{b} + 480\hat{a}\hat{a}'^2 - 720\hat{a}^3 b + 624\hat{a}^2\hat{a}'')\hat{a}'' + \\ \qquad (2548\hat{a}^4\hat{b}' - 1960\hat{a}^3 b')\hat{a}' + (8342\hat{a}^3\hat{b} - 6760\hat{a}^2 b)\hat{a}'^2 + 9195\hat{a}'^4 + \\ \qquad 588\hat{a}^8 + (294\hat{b}'' - 900 b\hat{b})\hat{a}^5 - (294(2 + \hat{b}'') + 513 b^2)\hat{a}^4 + 387\hat{a}^6\hat{b}^2\Big), \end{cases} \tag{19}$$

and, in addition,

$$\hat{b} = \frac{1}{\hat{a}} b + 2\frac{\hat{a}''}{\hat{a}^2} - 3\frac{\hat{a}'^2}{\hat{a}^3}. \tag{20}$$

This proves that $\gamma_{\hat{f}} = \gamma_f + \mathfrak{s}$. Conversely, if $\gamma_{\hat{f}} = \gamma_f + \mathfrak{s}$, then $\hat{b}$ is as in (20). Define $r_0, \dots, r_4$, R, and $\mathcal{D}$ as in (19), (17) and (18). Then $\mathcal{D}$ is $\mathrm{O}(\mathfrak{C}^5, g_{\mathfrak{C}})$-valued. From (18) it follows that $\mathcal{A}_1$ and $\hat{\mathcal{A}}_1$ satisfy (16). This implies that $\hat{f}$ is a fourth order deformation of f. □

As a consequence of the Lemma 14, it follows that an isotropic curve of general type is deformable of order four and that its local deformations depend on one arbitrary holomorphic function. We conclude the proof by showing that an isotropic curve of general type is rigid to order five. Let $\hat{f}$ be a fifth order deformation of f. Since $\hat{f}$ is a fourth order deformation of f, then $\hat{b}$ is as in (20) and the lifts $\mathcal{A}_1$, $\hat{\mathcal{A}}_1$ of ψ and $\hat{\psi}$ satisfy (16), where R and $\mathcal{D}$ are as in (17), (19) and (18). Since $\hat{f}$ is a fifth order deformation, there exists a holomorphic function r_5, such that

$$\frac{d^5\hat{\mathcal{A}}_1}{dz^5} = \mathcal{D} \cdot \left(J^{(4)}(\mathcal{A}_1)(r_5\mathbf{b}_1 + 5r_4\mathbf{b}_2 + 10r_3\mathbf{b}_3 + 10r_2\mathbf{b}_4 + 5r_1\mathbf{b}_5) + r_0\frac{d^5\mathcal{A}_1}{dz^5} \right),$$

where $(\mathbf{b}_1, \dots, \mathbf{b}_5)$ is the standard basis of $\mathbb{C}^5$. Taking into account that

$$\frac{d^5\mathcal{A}_1}{dz^5} = \mathcal{A}\mathtt{F}_{(5)}, \quad \frac{d^5\hat{\mathcal{A}}_1}{dz^5} = \hat{\mathcal{A}}\hat{\mathtt{F}}_{(5)},$$

we obtain

$$\hat{\mathcal{A}}\hat{\mathtt{F}}_{(5)} = \mathcal{D}\mathcal{A}(\mathtt{F}(r_5\mathbf{b}_1 + 5r_4\mathbf{b}_2 + 10r_3\mathbf{b}_3 + 10r_2\mathbf{b}_4) + r_0\mathtt{F}_{(5)}).$$

Using (18), we have

$$\hat{\mathtt{F}}_{(5)} = \hat{\mathtt{F}}\mathrm{R}^{-1}((r_5\mathbf{b}_1 + 5r_4\mathbf{b}_2 + 10r_3\mathbf{b}_3 + 10r_2\mathbf{b}_4) + r_0\mathtt{F}^{-1}\mathtt{F}_{(5)}),$$

which, in turn, implies

$$\mathtt{F}\mathrm{R}\hat{\mathtt{F}}^{-1}\hat{\mathtt{F}}_{(5)} = \mathtt{F}(r_5\mathbf{b}_1 + 5r_4\mathbf{b}_2 + 10r_3\mathbf{b}_3 + 10r_2\mathbf{b}_4) + r_0\mathtt{F}_{(5)}. \tag{21}$$

The second scalar equation of the system (21) implies $6\hat{a}(1-\hat{a}^4) = 0$. Then $\hat{a}$ is a fourth root of unity. Possibly acting on the right of $\hat{A}$ with an element of Z_8, we may assume $\hat{a} = 1$. From (20), we have $\hat{b} = b$. Then, (11) and (12) imply that $\hat{A} = X \cdot A$, for a unique $X \in \mathrm{Sp}(2,\mathbb{C})$. This proves that f and $\hat{f}$ are conformally congruent to each other. □

5. Isotropic curves with constant bending

5.1. **Isotropic W-curves.** Let $q = m/n \in \mathbb{Q}$ be a nonzero rational number, $|q| \neq 1$. Put $\varepsilon_q = m + n \pmod 2$ and define $\mathtt{v}_q, \mathtt{w}_q : \mathbb{C} \to \mathbb{C}^4$ by

$$\begin{cases} \mathtt{v}_q(z) = (m-n)\mathbf{e}_1 - (m+n)z^{(1+\varepsilon_q)m}\mathbf{e}_3 - 2i\sqrt{mn}z^{\frac{(m+n)(1+\varepsilon_q)}{2}}\mathbf{e}_4, \\ \mathtt{w}_q(z) = (m-n)\mathbf{e}_2 - 2i\sqrt{mn}z^{\frac{(m+n)(1+\varepsilon_q)}{2}}\mathbf{e}_3 + (m+n)z^{n(1+\varepsilon_q)}\mathbf{e}_4. \end{cases} \tag{22}$$

Let $f_q : \mathbb{C} \cup \{\infty\} \to \mathbb{Q}_3$ be the one-to-one isotropic curve defined by $f_q(z) = [\mathtt{v}_q(z) \wedge \mathtt{w}_q(z)]$, $f_q(\infty) = [\mathbf{e}_3 \wedge \mathbf{e}_4]$. We call f_q the *isotropic W-curve* with parameter q. The Legendre associate of f_q is computed to be

$$f_q^\sharp(z) = [i\sqrt{n}(\mathbf{e}_1 + iz^{(1+\varepsilon_q)m}\mathbf{e}_3) + \sqrt{m}(z^{\frac{(1+\varepsilon_q)(m-n)}{2}}\mathbf{e}_2 - z^{\frac{(1+\varepsilon_q)(m+n)}{2}}\mathbf{e}_4)], \tag{23}$$

which is a W-curve in the classical sense (cf. [1, 13]). This motivates the terminology. If $q = \pm 3, \pm 1/3$, f_q is a conformal cycle. For all other values of q, the isotropic W-curve is of general type. Assuming $q \neq \pm 3, \pm 1/3$, one can explicitly build a global meromorphic section of the Z_8-bundle $\widetilde{\mathcal{F}}$, holomorphic on $\dot{\mathbb{C}}$, with poles at 0 and ∞. Consequently, one can compute the meromorphic differentials to obtain

$$\delta_q = -\frac{9m^4 - 82m^2n^2 + 9n^4}{100z^4}dz^4, \quad \gamma_q = \frac{2(m^2+n^2)}{5z^2}dz^2.$$

Thus, f_q has constant bending

$$\kappa_q = -16(1+q^2)^2(9q^4 - 82q^2 + 9)^{-1}, \quad q \in \mathbb{Q} \setminus \{\pm 3, \pm 1/3\}. \tag{24}$$

In particular, f_q, $f_{q^{-1}}$, f_{-q} and $f_{-q^{-1}}$ have the same differentials. By Theorem C, from the viewpoint of conformal geometry, these curves are then equivalent to each other. Consequently, we may assume $q > 1$. From [8, 9], using (22) and (23), it follows that the degree d_{f_q} and the ramification degree r_{f_q} of f_q are $r_{f_q} = 2((1+\varepsilon_q)n - 1)$ and $d_{f_q} = (1+\varepsilon_q)(m+n)$. In particular, if $q = 2p+1$ is an odd integer, f_q is an isotropic immersion of degree $2(p+1)$. Referring again to [8, 9], conformal cycles exhaust the class of isotropic immersions of degree 4 and each isotropic immersion of degree 6 is conformally equivalent to f_5.

Remark 8. If $m+n$ is even, the W-curve $f_q^\sharp$ is congruent to the Legendrian curve with Bryant's potentials $g = iz^{(n-m)/2}$ and $f = (m-n)z^n/n$. If $m+n$ is odd, $f_q^\sharp$ is congruent to the Legendrian curve with Bryant's potentials $g = -i\sqrt{m/n}z^{n-m}$ and $f = (m-n)m^{n/(m-n)}n^{m/(n-m)})z^{2n}$.

Example 15. We now briefly describe the surfaces tamed by isotropic W-curves, with the exclusion of the cycles (i.e., $q = \pm 3, \pm 1/3$), already considered in Example 7. Let $q = m/n \in \mathbb{Q} \setminus \{\pm 1, \pm 3, \pm 1/3\}$.

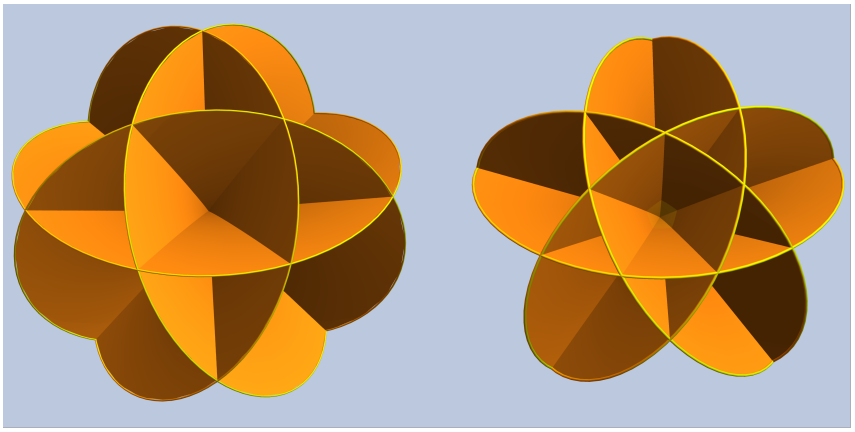

FIGURE 4. The minimal surfaces tamed by $f_{5/3}$ (left) and by $f_{3/2}$ (right).

(1) The minimal surface tamed by f_q is a conformal Goursat transform of the branched minimal immersion with Weierstrass data $(z^{h_1(q)}, z^{h_2(q)}dz)$, where $h_1(q) = \frac{1}{2}(n-m)(1+\varepsilon_q)$ and $h_2(q) = \frac{1}{2}(m+n)(1+\varepsilon_q) - 1$. In particular, if $m = 2\hat{m}+1$, $n = 1$, then $h_1 = h_2 = \hat{m}$ and, if $m = 2\hat{m}+1$ and $n = 1$, we have $h_1 = \hat{m}+1$ and $h_2 = \hat{m}-1$. The minimal surfaces with this Weierstrass data are the *Enneper surface of order* $\hat{m}$ and the A_4 *surface of order* $\hat{m}$, respectively (cf. [15, pp. 202-204]). These are conformal Goursat transforms of each other. The Enneper surface of order $\hat{m}$ possesses one non-embedded end of order $2\hat{m}+1$ at infinity. The A_4-surface of order $\hat{m}$ has one non-embedded end of order $2\hat{m}+1$ at infinity and one planar embedded end at the origin. Then, they are not classical Goursat transforms of each other. This shows that the (conformal) Goursat transform can modify the behaviour of the ends, even in the case of minimal surfaces tamed by rational isotropic curves. Figure 4 reproduces the minimal surfaces tamed by the isotropic W-curves $f_{5/3}$ and $f_{3/2}$, respectively.

(2) The CMC 1 surfaces of hyperbolic 3-space tamed by isotropic W-curves are Goursat transforms of the Catenoid cousin with rational parameter $\mu = (q-1)/2$ considered in [7]. CMC 1 surfaces in hyperbolic space with and arbitrary number $\hat{m}$ of smooth ends and a rotational symmetry group of order $\hat{m}-1$ can be constructed from the CMC 1 surface tamed by $f_{2\hat{m}+1}$, via Goursat transformation [4]. Figure 5 depicts a Goursat transform of the Catenoid cousin with parameter 5, with seven ends. It is invariant by the group of order six generated by the rotation of angle $2\pi/6$ around the z-axis. This shows that the Catenoid cousin with rational parameter $(q-1)/2$ is a conformal Goursat transform of the minimal surfaces with Weierstrass data $h_1(q)$ and $h_2(q)$.

(3) The flat front of $\mathcal{H}^3$ tamed by f_q is a Goursat transform of the flat fronts of revolution with rational parameter $\mu = (q-1)/(q+1)$ examined in [29]. In particular, the flat front of revolution with rational parameter μ is a conformal Goursat transform of the Catenoid cousin with parameter $\mu/(\mu-1)$.

5.2. **Isotropic curves with constant bending.** Let f_κ be an isotropic curve with constant bending κ. We say that f is of *regular type* if $\kappa \neq 1, -16/9$ and

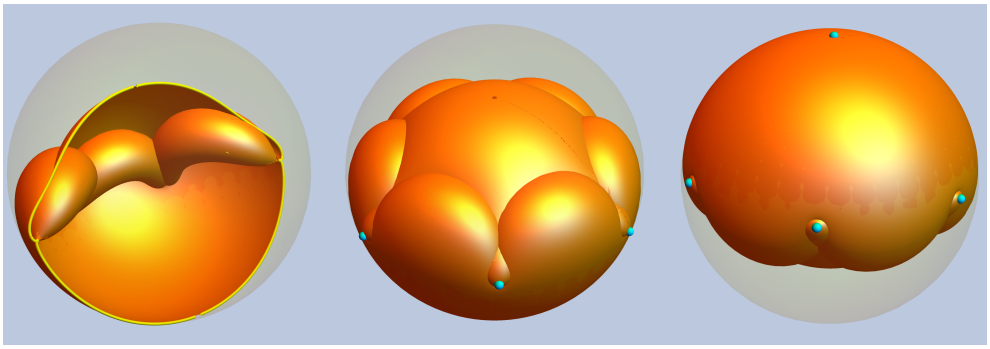

FIGURE 5. A CMC 1 surface with seven smooth ends and symmetry group of order six: Goursat transform of the Bryant surface tamed by f_{11}. The points coloured in cyan are the ends.

of *exceptional type*, otherwise. Let $\mathcal{D}_4$ be the dihedral transformation group of $\dot{\mathbb{C}}$ generated by $z \to 1/z$ and $z \to -z$. Let $|[z]|$ be the equivalence class of $z \in \dot{\mathbb{C}}$ in $\dot{\mathbb{C}}/\mathcal{D}_4$. Given $\kappa \in \mathbb{C} \setminus \{1, -16/9\}$, we choose c such that $c^2 = \kappa$. The equivalence class of $\sqrt{(5c - 4\sqrt{c^2-1})(5c + 4\sqrt{c^2-1})^{-1}}$ does not depend on the choice of c and of the square roots. This originates a map

$$\mathbf{r} : \mathbb{C} \setminus \{1, -16/9\} \ni k \mapsto r_k = |[\sqrt{(5c - 4\sqrt{c^2-1})(5c + 4\sqrt{c^2-1})^{-1}}]| \in \dot{\mathbb{C}}/\mathcal{D}_4.$$

The range of $\mathbf{r}$ is $\dot{\mathbb{C}}/\mathcal{D}_4 \setminus \{|[3]|\}$. For instance, $\mathbf{r}_{\kappa_q} = |[q]|$, as can be easily seen from (24). Note that $\dot{\mathbb{Q}}$ is $\mathcal{D}_4$-stable. Its projection to $\dot{\mathbb{C}}/\mathcal{D}_4$ is denoted by $|[\dot{\mathbb{Q}}]|$.

The main result of the section is the following.

Theorem E. *Let $f : S \to \mathbb{Q}_3$ be a compact isotropic curve with constant bending κ. Then, f is of general type, $\mathbf{r}(\kappa)$ belongs to $|[\dot{\mathbb{Q}}]|$, and f is dominated by f_q, where q is the unique element of $\mathbf{r}(\kappa)$ strictly greater than 1.*

The proof of Theorem E is based on the following three lemmas.

Lemma 16. *Let $f : S \to \mathbb{Q}_3$ be a compact isotropic curve of general type with constant bending κ. If $\mathbf{r}_\kappa \in |[\mathbb{Q}]|$, then f is dominated by the isotropic W-curve f_q, where q is the unique element of $\mathbf{r}_\kappa$ strictly greater than 1.*

Proof of Lemma 16. We start by recalling the following two facts about isotropic W-curves.

- $f_q : \mathbb{CP}^1 \to \mathbb{Q}_3$ is injective with at most two branch points at 0 and ∞, with $f_q(0) = [\mathbf{e}_1 \wedge \mathbf{e}_2]$ and $f_q(\infty) = [\mathbf{e}_3 \wedge \mathbf{e}_4]$.
- There exists a holomorphic frame field $A_q : \mathbb{CP}^1 \setminus \{0, \infty\} \to \mathrm{Sp}(2, \mathbb{C})$, such that

$$A_q^{-1} dA_q = \frac{i}{\sqrt{10}}(s(\mathbf{e}_2^1 + \mathbf{e}_4^2 + \mathbf{e}_1^3 - \mathbf{e}_3^4) - 4(m^2 + n^2)s^{-1}(\mathbf{e}_2^4 + \frac{3}{4}\mathbf{e}_1^2 - \frac{3}{4}\mathbf{e}_4^3))\frac{dz}{z}, \tag{25}$$

  where s is a fourth root of $-9m^4 + 82m^2n^2 - 9n^4$.

Next, we prove that there exists $X \in \mathrm{Sp}(2, \mathbb{C})$ such that $f(S) \subseteq X \cdot f_q(\mathbb{CP}^1)$. Put $c = -4i(m^2 + n^2)s^{-2}$. Notice that $c^2 = \kappa$. Let $S_\sharp = S \setminus |\Delta_f| \cup |\Delta_{f^\sharp}| \cup |\Delta_{\delta_f}|$.

Consider the universal covering $\mathtt{p} : \hat{S}_\sharp \to S_\star$. By construction, $\hat{S}_\sharp \ncong \mathbb{CP}^1$. Hence, $\hat{S}_\sharp$ is contractible. From Lemma 10, $f \circ \mathtt{p}$ admits a lift $A : \hat{S}_\sharp \to \mathrm{Sp}(2,\mathbb{C})$ such that

$$\begin{cases} A^{-1}dA = \big((\mathbf{e}_2^1 + \mathbf{e}_4^2 + \mathbf{e}_1^3 - \mathbf{e}_3^4) - c(\mathbf{e}_2^4 + \frac{3}{4}\mathbf{e}_1^2 - \frac{3}{4}\mathbf{e}_4^3)\big)\zeta, & 1 < q < 3, \\ A^{-1}dA = i\big((\mathbf{e}_2^1 + \mathbf{e}_4^2 + \mathbf{e}_1^3 - \mathbf{e}_3^4) - c(\mathbf{e}_2^4 + \frac{3}{4}\mathbf{e}_1^2 - \frac{3}{4}\mathbf{e}_4^3)\big)\zeta, & q > 3, \end{cases} \tag{26}$$

where ζ is a nonzero holomorphic 1-form. Let $w : \hat{S}_\sharp \to \dot{\mathbb{C}}$ be defined by

$$w = \exp\left((-i)^{\frac{1-\mathrm{sign}(q-3)}{2}} 10^{1/2} s^{-1} \int \zeta\right).$$

From (25) and (26), we have $A^{-1}dA = w^*(A_q^{-1}dA_q)$. Then, there exists $X \in \mathrm{Sp}(2,\mathbb{C})$, such that $A = X \cdot A_q \circ w$. This implies $f \circ \mathtt{p} = X \cdot f_q \circ w$. Hence, $f(S_\sharp) \subseteq X \cdot f_q(\mathbb{CP}^1)$. By continuity, $f(S) \subseteq X \cdot f_q(\mathbb{CP}^1)$. Possibly replacing f with $X^{-1}f$, we can assume $f(S) \subseteq f_q(\mathbb{CP}^1)$. Since f_q is injective, there exists a unique map $h : S \to \mathbb{CP}^1$, such that $f = f_q \circ h$. Let $S_\star = S \backslash f^{-1}([\mathbf{e}_1 \wedge \mathbf{e}_2]) \cup f^{-1}([\mathbf{e}_3 \wedge \mathbf{e}_4])$. Taking into account that $f_q|_{\mathbb{CP}^1 \backslash \{0,\infty\}}$ is immersive, then $h|_{S_\star}$ is holomorphic. Consider a point p_0 in $f^{-1}([\mathbf{e}_1 \wedge \mathbf{e}_2])$. In an open neighborhood U of p_0, we have

$$f|_U = [(\mathbf{e}_1 + f_1^3\mathbf{e}_3 + f_1^4\mathbf{e}_4) \wedge (\mathbf{e}_2 + f_1^4\mathbf{e}_3 + f_2^4\mathbf{e}_4)],$$

where f_j^i are holomorphic on U and $f_j^i(p_0) = 0$. From (22), we have

$$f_1^3 = -\frac{m+n}{m-n} h^{(1+\varepsilon_q)m}, \quad f_1^4 = -\frac{2i\sqrt{mn}}{m-n} h^{\frac{(1+\varepsilon_q)(m+n)}{2}}, \quad f_2^4 = \frac{m+n}{m-n} h^{(1+\varepsilon_q)n}.$$

Then h is holomorphic on U. The same argument shows that h is holomorphic also on a neighborhood of every point of $f^{-1}([\mathbf{e}_3 \wedge \mathbf{e}_4])$. This concludes the proof of Lemma 16. □

Lemma 17. *There are no compact isotropic curve of general type with constant bending κ such that $\mathbf{r}_\kappa \notin |[\mathbb{Q}]|$.*

Proof. Consider the maximal abelian subgroup $\mathbb{T}^2 \subset \mathrm{Sp}(2,\mathbb{C})$,

$$\mathbb{T}^2 = \{u^{-1}\mathbf{e}_1^1 + v^{-1}\mathbf{e}_2^2 + u\mathbf{e}_3^3 + v\mathbf{e}_4^4 \mid u, v \in \dot{\mathbb{C}}\}.$$

Let $\hat{\mathcal{U}} \subset \mathcal{U}$ be the subset defined by

$$\hat{\mathcal{U}} = \{[(\mathbf{e}_1 + m_1^3\mathbf{e}_3 + m_1^4\mathbf{e}_4) \wedge (\mathbf{e}_2 + m_1^4\mathbf{e}_3 + m_2^4\mathbf{e}_4)] \mid m_1^3, m_1^4, m_2^4 \neq 0,\ m_1^3 m_2^4 \neq (m_1^4)^2\}.$$

Then, $\hat{\mathcal{U}}$ is $\mathbb{T}^2$-stable, the action of $\mathbb{T}^2$ has no fixed points and

$$\Sigma = \{P_r \mid P_r = [(\mathbf{e}_1 + \mathbf{e}_3 + \mathbf{e}_4) \wedge (\mathbf{e}_2 + \mathbf{e}_3 + r\mathbf{e}_4)],\ r \neq 0, 1\}$$

is a slice. Let $\mathcal{O}_r$ be the $\mathbb{T}^2$-orbit through P_r. The orbits are the integral manifolds of the holomorphic completely integrable plane field distribution generated by the two fundamental vector fields of the action. In addition, $\mathcal{O}_r$ inherits a conformal structure from $\mathbb{Q}_3$. Let $\mathcal{N}_r^+$ and $\mathcal{N}_r^+$ the two null distributions of $\mathcal{O}_r$. They can be viewed as completely integrable real distributions of rank 2.

Let $\kappa \in \mathbb{C}$, $\kappa \neq 1, -16/9$. Choose c such that $c^2 = \kappa$. Define λ_1 and λ_2 by $\lambda_1 = \frac{1}{2}(5c - 4(c^2-1)^{1/2})^{1/2}$ and $\lambda_2 = \frac{1}{2}(5c + 4(c^2-1)^{1/2})^{1/2}$. Observe that $\lambda_1 \neq \lambda_2$

and $|[\lambda_1/\lambda_2]| = \mathbf{r}_\kappa$. Let $\mathbf{v}_\kappa, \mathbf{w}_\kappa : \mathbb{C} \to \mathbb{C}^4$ be defined by

$$\mathbf{v}_\kappa = \mathbf{e}_1 - \frac{\lambda_1+\lambda_2}{\lambda_1-\lambda_2} e^{2\lambda_1 z}\mathbf{e}_3 - \frac{2i\lambda_1^{1/2}\lambda_2^{1/2}}{\lambda_1-\lambda_2} e^{(\lambda_1+\lambda_2)z}\mathbf{e}_4,$$
$$\mathbf{w}_\kappa = \mathbf{e}_2 - \frac{2i\lambda_1^{1/2}\lambda_2^{1/2}}{\lambda_1-\lambda_2} e^{(\lambda_1+\lambda_2)z}\mathbf{e}_3 + \frac{\lambda_1+\lambda_2}{\lambda_1-\lambda_2} e^{2\lambda_2 z}\mathbf{e}_4.$$

Then,

$$\hat{f}_\kappa = [\mathbf{v}_\kappa \wedge \mathbf{w}_\kappa] : \mathbb{C} \to \hat{\mathcal{U}} \tag{27}$$

is an isotropic curve with constant bending κ. If λ_1/λ_2 is not rational, $\hat{f}_\kappa$ is injective and $\hat{f}_\kappa(\mathbb{C})$ is contained in the orbit $\mathcal{O}_r$, $r = (\lambda_1+\lambda_2)^2/4\lambda_1\lambda_2$. Thus, $\hat{f}_\kappa$ is an integral manifold of one of the two null distributions of $\mathcal{O}_r$.

We are now in a position to conclude the proof of Lemma 17. By contradiction, let $f : S \to \mathbb{Q}_3$ be a compact isotropic curve with constant bending κ, with $\kappa \neq 1, -16/9$ and $\mathbf{r}_\kappa \notin |[\mathbb{Q}]|$. Consider $S_\star = S \setminus D$, $D = |\Delta_f| \cup |\Delta_{f^\sharp}| \cup |\Delta_\delta|$. Since f and $\hat{f}_\kappa$ have the same constant bending, proceeding as in the proof of Lemma 16, there exists $X \in \mathrm{Sp}(2,\mathbb{C})$, such that $f(S_\star) \subseteq X \cdot f_\kappa(\mathbb{C})$. Possibly replacing f with $X^{-1}f$, we assume $f(S_\star) \subseteq f_\kappa(\mathbb{C})$. If $\mathbf{r}_\kappa \notin |[\mathbb{Q}]|$, f_κ is an integral manifold of a completely integrable distribution of $\mathcal{O}_r$. Thus (cf. [42, Theorem 1.62, p. 47]) there exists a differentiable map $h : S_\star \to \mathbb{C}$, such that $f|_{S_\star} = \hat{f}_\kappa \circ h$. Since f and f_κ are holomorphic, also h is holomorphic. Note that $f(S) \cap (\mathbb{Q}_3 \setminus \mathcal{U}) \neq \emptyset$. Otherwise, $f(S) \subset \mathcal{U}$ and $\mathcal{U}$ is biholomorphically equivalent to $\mathbb{C}^3$. Hence, by the maximum principle, f would be constant. Next, pick $p_0 \in S$ such that $f(p_0) \notin \mathcal{U}$. The point p_0 belongs to D. In fact, if $p_0 \notin D$, we would have $f(p_0) \in f(S_\star) \subseteq f_\kappa(\mathbb{C}) \subset \mathcal{U}$. Choose a complex chart (U, z) centered at p_0, such that $D \cap U = \{p_0\}$. The next argument is of a local nature, so we can think of U as an open disk of the complex plane and take $p_0 = 0$. There exist a positive integer ℓ and holomorphic functions m_1^3, m_1^4, $m_2^3 : U \to \mathbb{C}$, not all vanishing at the origin and not identically 0 on U, such that

$$f|_{\dot{U}} = [(\mathbf{e}_1 + z^{-\ell}(m_1^3\mathbf{e}_3 + m_1^4\mathbf{e}_4)) \wedge (\mathbf{e}_2 + z^{-\ell}(m_1^4\mathbf{e}_3 + m_2^4\mathbf{e}_4))],$$

where $\dot{U} = U \setminus \{p_0\}$. Therefore, on $\dot{U}$ we have

$$\begin{cases} z^{-\ell}m_1^3 = -\frac{\lambda_1+\lambda_2}{\lambda_1-\lambda_2} e^{2\lambda_1 h(z)}, \\ z^{-\ell}m_1^4 = -\frac{2i\lambda_1^{1/2}\lambda_2^{1/2}}{\lambda_1-\lambda_2} e^{(\lambda_1+\lambda_2)h(z)}, \\ z^{-\ell}m_2^4 = \frac{\lambda_1+\lambda_2}{\lambda_1-\lambda_2} e^{2\lambda_2 h(z)}. \end{cases} \tag{28}$$

Next, we use (28) to get a contradiction. At least one of the functions m_j^i is nonzero at the origin. Suppose $m_1^3(0) \neq 0$. Possibly taking a smaller neighborhood, we assume that m_1^3 is never zero. So we can write $m_1^3 = e^{\alpha_1^3 + i\beta_1^3}$ and $h = a + ib$, where α_1^3, β_1^3, a, b are real valued. Consider a small circle $\mathrm{C}_\rho = \{e^{\rho+it} \mid t \in \mathbb{R}\}$, such that the functions m_1^4 and m_2^4 are never zero on C_ρ. Hence, we have

$$\begin{aligned} &m_1^3(e^{\rho+it}) = e^{\alpha_1^3(t)+i\beta_1^3(t)}, \quad m_1^4(e^{\rho+it}) = e^{\alpha_1^4(t)+i\beta_1^4(t)}, \\ &m_2^4(e^{\rho+it}) = e^{\alpha_2^4(t)+i\beta_2^4(t)}, \quad h(e^{\rho+it}) = e^{\alpha(t)+i\beta(t)}, \end{aligned} \tag{29}$$

where $\alpha, \alpha_1^3, \alpha_1^4, \alpha_1^4, \beta_1^3, \beta$ are periodic function of period 2π and

$$\beta_1^4(t+2\pi) = \beta_1^4 + 2\pi n_1, \quad \beta_2^4(t+2\pi) = \beta_1^4 + 2\pi n_2, \quad n_1, n_2 \in \mathbb{Z}.$$

From (28) and (29), we have

$$
\begin{aligned}
2\lambda_1(\alpha(t)+i\beta(t)) &= \ell\rho+\alpha_1^3(t)+i(\ell t+\beta_1^3(t)+2\pi\hat{n}_1-\pi)-c_1,\\
2\lambda_2(\alpha(t)+i\beta(t)) &= \ell\rho+\alpha_2^4(t)+i(\ell t+\beta_2^4(t)+2\pi\hat{n}_2-\pi)-c_2,
\end{aligned}
\tag{30}
$$

where $\hat{n}_1$, $\hat{n}_2$ are two integers and c_1, c_2 are the constants defined by $e^{c_1} = (\lambda_1+\lambda_2)(\lambda_1-\lambda_2)^{-1}$ and $e^{c_2} = \sqrt{\lambda_1}\sqrt{\lambda_2}(\lambda_1-\lambda_2)^{-1}$. Since $|[\lambda_2/\lambda_1]| \in \mathbf{r}_\kappa \notin |[\mathbb{Q}]|$, we have $\lambda_2 = r\lambda_1$, $r \notin \mathbb{Q}$. From (30), we get

$$
\begin{aligned}
&\ell\rho+\alpha_2^4(t)+i(\ell t+\beta_2^4(t)+2\pi\hat{n}_2-\pi)-c_2 = \\
&\qquad\qquad r(\ell\rho+\alpha_1^3(t)+i(\ell t+\beta_1^3(t)+2\pi\hat{n}_1-\pi)-c_1).
\end{aligned}
\tag{31}
$$

Taking into account the periodicity and the quasi-periodicity of the functions α_j^i and β_j^i and using (31), we conclude that $2\pi i(\ell+n_2) = 2\pi i\ell r$, which is a contradiction. With the same argument, one can show that assuming $m_2^4(0) \neq 0$, or $m_1^4(0) \neq 0$, would yield to a contradiction as well. □

Lemma 18. *There are no compact isotropic curve of exceptional type with constant bending.*

Proof. We shall give the proof in the case $\kappa = 1$. The same argument can be used in the other case. The general structure of the reasoning is quite similar to that of the previous proof. First, observe that

$$
\hat{f} : \mathbb{C} \ni z \mapsto [(\mathbf{e}_1 + e^z\mathbf{e}_3 + z\mathbf{e}_4) \wedge (\mathbf{e}_2 + z\mathbf{e}_3 - e^{-z}\mathbf{e}_4)] \in \mathbb{Q}_3 \tag{32}
$$

is an embedded isotropic curve with constant bending $\kappa = 1$. By contradiction, let $f : S \to \mathbb{Q}_3$ be a compact isotropic curve with bending 1. Consider the exceptional locus $D = |\Delta_f| \cup |\Delta_{f^\sharp}| \cup |\Delta_\delta|$ and put $S_\star = S \setminus D$. In analogy with Lemma 16, one can prove that $f(S_\star) \subseteq X \cdot \hat{f}(\mathbb{C})$, for some $X \in \mathrm{Sp}(2,\mathbb{C})$. Replacing f with $X^{-1}f$, we may assume $f(S_\star) \subseteq \hat{f}(\mathbb{C})$. Since $\hat{f}$ is an embedding, there exists a holomorphic map $h : S_\star \to \mathbb{C}$ such that $\hat{f} \circ h = f|_{S_\star}$. Choose $p_0 \in S$, such that $f(p_0) \notin \mathcal{U}$. As in Lemma 17, $p_0 \in D$. Consider a coordinate system (U, z) centered at p_0, such that $U \cap D = \{p_0\}$. The next argument is of a local nature, so we assume that U is an open disk on the complex plane and take $p_0 = 0$. There exist a positive integer ℓ and holomorphic functions $m_1^3, m_1^4, m_2^3 : U \to \mathbb{C}$, not all vanishing at the origin and not identically zero on U, such that

$$
f|_{\dot{U}} = [(\mathbf{e}_1 + z^{-\ell}(m_1^3\mathbf{e}_3 + m_1^4\mathbf{e}_4)) \wedge (\mathbf{e}_2 + z^{-\ell}(m_1^4\mathbf{e}_3 + m_2^4\mathbf{e}_4))], \tag{33}
$$

where $\dot{U} = U \setminus \{p_0\}$. Possibly shrinking U, there exist integers $0 \leq k_1^3, k_1^4, k_2^4 \leq \ell$, at least one of them zero, and holomorphic functions $\mu_1^3, \mu_1^4, \mu_2^4$, such that $m_1^3 = z^{k_1^3}e^{\mu_1^3}$, $m_1^4 = z^{k_1^4}e^{\mu_1^4}$, $m_2^4 = z^{k_2^4}e^{\mu_2^4}$. Comparing (32) and (33), we have

$$
h = z^{k_1^4-\ell}e^{\mu_1^4}, \quad e^h = z^{k_1^3-\ell}e^{\mu_1^3}, \quad e^{-h} = -z^{k_2^4-\ell}e^{\mu_2^4}.
$$

These equalities hold on $\dot{U}$. We claim that $k_1^3 - \ell = 0$. Consider a circle $\{e^{r+it} \mid t \in \mathbb{R}\}$ contained in U. Then, $h(e^{r+it}) = a(t) + ib(t)$ and $\mu_1^3(e^{r+it}) = a_1^3(t) + ib_1^3(t)$, where a, b, a_1^3, b_1^3 are periodic, of period 2π. The equality $e^h = z^{k_1^3-\ell}e^{\mu_1^3}$ gives $a(t)+ib(t) = (k_1^3-\ell)(r+it)+a_1^3(t)+ib_1^3(t)+2\pi\hat{n}i$, for some $\hat{n} \in \mathbb{Z}$. The periodicity of a, b, a_1^3 and b_1^3 implies $k_1^3 - \ell = 0$. The same argument shows that $k_2^4 - \ell = 0$. Hence $k_1^4 = 0$ and $h = z^{-\ell}e^{\mu_1^4}$, $e^h = e^{\mu_1^3}$, $e^{-h} = -e^{\mu_2^4}$ on $\dot{U}$. Thus, $e^{\mu_1^3} = e^{z^{-\ell}e^{\mu_1^4}}$. In

turn, this implies $\mu_1^3 = z^{-\ell} e^{\mu_1^4} + 2\pi\check{n}i$, $\check{n} \in \mathbb{Z}$. This gives the seeked contradiction, since the left hand side is bounded near 0 and the right hand side is unbounded. □

Example 19. It is easy to show that the Catenoid cousins of $\mathcal{H}^3$ (cf. [7]) with parameters μ and $\mu(1-2\mu)^{-1}$ are Goursat transforms of each other. So, from our perspective, it suffices to examine Catenoid cousins with $\mu \in (-1/2, 0)$. We exclude the case $\mu = -1/3$ corresponding to a pseudo-Cyclide. The Catenoid cousin with parameter $\mu \in (-1/2, 0)$, $\mu \neq -1/3$, is tamed by the standard isotropic curve (27) with constant bending

$$\kappa_\mu = \frac{4(1+2\mu+2\mu^2)^2}{4+16\mu-7\mu^2-18\mu^3-9\mu^4} \in (-\infty, -16/9) \cup (1, +\infty).$$

Two rotationally invariant flat fonts (cf. [29]) of $\mathcal{H}^3$ with parameters m and $1/m$ are Goursat transforms of each other. So, we may assume $m \in (0, 1)$. If $m = 1/2$, the flat front is a pseudo-Cyclide. Excluding this case, a rotationally invariant flat front with parameter $m \in (0, 1)$, $m \neq 1/2$, is tamed by the standard isotropic curve with constant bending

$$\kappa_m = \frac{4(1+m)^2}{4m^4 - 17m^2 + 4}.$$

In particular, the Catenoid cousin with parameter $\mu \in (-1/2, 0)$ is a Goursat transform of the rotationally invariant flat front with parameter $m = -\mu/(1+\mu) \in (0, 1)$. The Catenoid cousins or the rotationally invariant flat fronts tamed by W-curves are those with rational parameters.

Example 20. Consider the isotropic curve

$$\hat{f}(z) = [(\mathbf{e}_1 + e^z \mathbf{e}_3 + z\mathbf{e}_4) \wedge (\mathbf{e}_2 + z\mathbf{e}_3 - e^{-z}\mathbf{e}_4)]$$

with conformal bending $\kappa = 1$. The minimal surface tamed by $\hat{f}$ is the Catenoid. Consider

$$\hat{f}_b = \big(\cosh(\frac{b}{2}) I_4 + \sinh(\frac{b}{2})(\mathbf{e}_2^1 + \mathbf{e}_1^2 - \mathbf{e}_4^3 - \mathbf{e}_3^4)\big)\hat{f}, \quad b \in \mathbb{R}.$$

The 1-parameter family $\{\hat{\phi}_b\}_{b\in\mathbb{R}}$ of minimal surfaces tamed by $\{\hat{f}_b\}_{b\in\mathbb{R}}$ is the *Bonnet deformation of the Catenoid* [5, 24, 33]. With the exception of the Enneper surface and planes, the Bonnet family exhaust (up to similarity transformations of $\mathbb{R}^3$) the class of minimal surfaces with plane line of curvature. Note that all Bonnet minimal surfaces are conformal Goursat transforms of the catenoid. The minimal surface $\check{\phi}_b$ tamed by $\check{f}_b = \big((-1)^{1/4}(\mathbf{e}_1^1 + \mathbf{e}_2^2) - (-1)^{-1/4}(\mathbf{e}_3^3 + \mathbf{e}_4^4)\big)\hat{f}_b$ is the associate of $\hat{\phi}_b$ (for $b = 0$, we have the Helicoid). Then, $\{\check{\phi}_b\}_{b\in\mathbb{R}}$ is the *Thomsen deformation* of the Helicoid [40]. The Thomsen surfaces, the Enneper surface and the plane are (up to a similarity of $\mathbb{R}^3$) the only minimal surfaces which are affine minimal as well [2]. By construction, the Thomsen surfaces are conformal Goursat transforms of the Catenoid.

Example 21. Up to a linear change of the independent variable, the CMC 1 surfaces of $\mathcal{H}^3$ and $\mathcal{H}^{2,1}$ tamed by the isotropic curve with constant bending $-16/9$ are the Enneper cousin and the spacelike Enneper cousin considered in [7, 31]. They are conformal Goursat transforms of each other.

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