Series Expansions and Approximations of the Nakagami-m Sum Probability Density Function

Series Expansions and Approximations of the Nakagami-m Sum Probability Density Function

Giorgio Taricco

***Abstract*—The numerical evaluation of the pdf of a sum of Nakagami-m random variables is considered in this letter. Different methods are proposed to obtain this approximation: $i)$ a power series expansion based on elementary properties of unilateral convolution of power signals; $ii)$ a Laplace approximation stemming from Laplace's method for asymptotic integral approximations; and $iii)$ a Gaussian series expansion capturing the deviation of the wanted pdf from the Gaussian with the same mean and variance. Numerical results are included for validation and comparison with the literature and a critical assessment of the different methods is provided.**

***Index Terms*—Nakagami-m distribution. Power series expansion. Gaussian series expansion. Laplace approximation.**

I. Introduction

The distribution of the sum of *i.n.i.d.* (independent non identically distributed) Nakagami-m random variables (RV's) has received considerable attention in the literature since the introduction of this distribution in [1]. The Nakagami-m distribution is a flexible tool to model the multipath fading gain in a number of different scenarios [1]. The interest in the distribution of the sum of *i.n.i.d.* Nakagami-m RV's derives mainly from the analysis of multi-antenna receivers based on Equal Gain Combining (EGC). In fact, the average bit error probability of BPSK is given by $P_b(e) = \mathbb{E}[Q(\Gamma\sqrt{2E_b/N_0})]$ where Γ represents the sum of a number of Nakagami-m RV's equal to the number of receive antennas used with ECG in a diversity receiver [2, Sec. 9.3.3]. The authors of [2] observed that finding the pdf of Γ requires the convolution of different Nakagami-m pdf's and can be quite difficult to evaluate.

The derivation of approximations and series expansions for the sum of Nakagami-m RV's has received much attention in the literature. Nakagami himself gave an approximation for the sum of *i.i.d.* Nakagami-m RV's [1]. A series expansion has been proposed by [3] for the complementary cumulative distribution function of a sum of *i.n.i.d.* Rayleigh RV's (which represents a special case of the sum of *i.n.i.d.* Nakagami-m RV's, see [4] for a discussion on the convergence of multipath fading to the Rayleigh distribution). More recently, other approximations have been given in [5], [6]. Several works consider special limited cases and provide exact results for the *i.n.i.d.* Nakagami-m RV's [7]–[10].

More recently, a series expansion based on the Lauricella multivariable hypergeometric functions has been suggested in [11]. Unfortunately, Lauricella functions are very difficult to evaluate and are not implemented in standard software programs.

In this work we propose a new power series expansion that presents several advantages with respect to other approaches in the literature. First of all, it does not involve the computation of special functions of any kind, except for the Gamma function, which is widely available in standard computational packages. Secondly, the coefficients of this series expansion can easily be computed by simple recursive relationships. Finally, the accuracy of the result can be made as high as required by taking a sufficient number of terms of the power series. We compare the numerical results obtained against existing literature results and with the Gaussian approximation. The letter is organized as follows. Section II introduces the power series expansion and provides the derivation of the recursive relationships for its coefficients. Section III proposes an approximation based on the multidimensional Laplace's asymptotic method to approximate integrals of the type (12). Section IV illustrates an approach based on the Edgeworth series expansion applied to the Nakagami-m sum distribution [12]. Section V summarizes the implementation of the methods considered and provides numerical results which are compared with those in the literature [11].

II. Power series expansion

In this section we provide our first series expansion of the sum of N *i.n.i.d.* Nakagami-m RV's without resorting to the transform of the pdf. The approach relies on a basic property of signal convolution and its derivation is very straightforward though novel in the literature. We consider the RV

$$\Gamma \triangleq \sum_{k=1}^{N} \Gamma_k, \tag{1}$$

where each Γ_k is a RV with Nakagami-m distribution characterized, for $k = 1, \dots, N$, by the pdf

$$f_{\Gamma_k}(\gamma) = 2\frac{m_k^{m_k}\gamma^{2m_k-1}}{\Omega_k^{m_k}\Gamma(m_k)} \exp\left(-\frac{m_k\gamma^2}{\Omega_k}\right)u(\gamma). \tag{2}$$

Here, $\Gamma(z) \triangleq \int_0^\infty u^{z-1}\mathrm{e}^{-u}du$ is the Gamma function and $u(\gamma)$ is the unit step function defined as $u(\gamma) = 0$ if $\gamma < 0$ and 1 otherwise. The parameter Ω_k represents the average value of the square of Γ_k and the parameter $m_k \geq 0.5$ is called the *fading number*. When $m_k = 1$, the Nakagami distribution reduces to the Rayleigh distribution. When $m_k = (K+1)^2/(2K+1)$, it approximates the Rician distribution with factor K.

In order to obtain the pdf of the RV Γ defined in (1), we need the convolution of all the pdf's of the Γ_k, namely,

$$f_\Gamma(\gamma) = f_{\Gamma_1}(\gamma) * \cdots * f_{\Gamma_N}(\gamma). \tag{3}$$

Now, we note that the pdf's $f_{\Gamma_k}(\gamma)$ admit the series expansion

$$f_{\Gamma_k}(\gamma) = 2\frac{m_k^{m_k}\gamma^{2m_k-1}}{\Omega_k^{m_k}\Gamma(m_k)}\sum_{\ell=0}^{\infty}\frac{1}{\ell!}\bigg(-\frac{m_k\gamma^2}{\Omega_k}\bigg)^{\ell}u(\gamma). \tag{4}$$

This series expansion is absolutely convergent for every γ. Then, we can calculate the series expansion corresponding to the lhs of (3) by term-by-term convolution of the series components, which can be done by resorting to the following elementary result:

$$\frac{x^{m-1}}{\Gamma(m)}u(x) * \frac{x^{n-1}}{\Gamma(n)}u(x) = \frac{x^{m+n-1}}{\Gamma(m+n)}u(x), \tag{5}$$

holding for all $m, n > 0$. We get:

$$\begin{aligned} f_\Gamma(\gamma) &= 2^N\prod_{k=1}^{N}\frac{m_k^{m_k}}{\Omega_k^{m_k}\Gamma(m_k)}\sum_{\ell_1=0}^{\infty}\cdots\sum_{\ell_N=0}^{\infty}\prod_{k=1}^{N}\frac{1}{\ell_k!}\bigg(-\frac{m_k}{\Omega_k}\bigg)^{\ell_k} \\ &\quad\times\Big\{\gamma^{2(m_1+\ell_1)-1}u(\gamma)\Big\} * \cdots * \Big\{\gamma^{2(m_N+\ell_N)-1}u(\gamma)\Big\} \\ &= \sum_{\ell=0}^{\infty}\alpha_\ell\gamma^{2(m+\ell)-1}u(\gamma). \end{aligned} \tag{6}$$

Here, we defined $m \triangleq \sum_{k=1}^{N} m_k$ and

$$\alpha_\ell \triangleq 2^N\prod_{k=1}^{N}\frac{m_k^{m_k}}{\Omega_k^{m_k}\Gamma(m_k)}\times\frac{(-1)^\ell\beta_\ell[N]}{\Gamma(2(m+\ell))} \tag{7}$$

$$\beta_\ell[n] \triangleq \sum_{\substack{\ell_1+\cdots+\ell_n=\ell \\ \ell_1\ge0,\ldots,\ell_n\ge0}}\prod_{k=1}^{n}\frac{\Gamma(2(m_k+\ell_k))}{\ell_k!}\bigg(\frac{m_k}{\Omega_k}\bigg)^{\ell_k} \tag{8}$$

for $n = 1, \ldots, N$. The coefficients $\beta_\ell[n]$ can be calculated recursively as

$$\beta_\ell[n] = \sum_{p=0}^{\ell}\frac{\Gamma(2(m_n+p))}{p!}\bigg(\frac{m_n}{\Omega_n}\bigg)^{p}\beta_{\ell-p}[n-1] \tag{9}$$

with

$$\beta_\ell[1] = \frac{\Gamma(2(m_1+\ell))}{\ell!}\bigg(\frac{m_1}{\Omega_1}\bigg)^{\ell}. \tag{10}$$

We can see that $|\alpha_\ell|$ is decreasing for sufficiently large ℓ and its asymptotic behavior (for $\ell \to \infty$) is given by

$$\ln|\alpha_\ell| \lesssim -\ell\ln\ell + O(\ell). \tag{11}$$

See Appendix A. Hence, the series (6) is absolutely convergent and alternating, so that the truncation error upper bound is given by (11).

III. Laplace approximation

In this section we consider the case of asymptotically large values of $\gamma \to \infty$ and apply Laplace's method to obtain an asymptotic approximation of the pdf $f_\Gamma(\gamma)$. The multidimensional Laplace method is summarized in the following equation (see [13, Sec. 27]) holding for $x \to \infty$:

$$\begin{aligned} f(x) &\triangleq \int g(\boldsymbol{t})\mathrm{e}^{-xh(\boldsymbol{t})}d\boldsymbol{t} \\ &\sim \det(\boldsymbol{H}_h(\boldsymbol{t}_0)/(2\pi))^{-1/2}g(\boldsymbol{t}_0)e^{-xh(\boldsymbol{t}_0)}. \end{aligned} \tag{12}$$

Here, $\boldsymbol{t}_0$ is the asymptotic (for $x \to \infty$) global maximum of $h(\boldsymbol{t})$ and $\boldsymbol{H}_h(\boldsymbol{t})$ is the Hessian matrix of $h(\boldsymbol{t})$. We start by writing the N-fold convolution of the pdf's of Γ_k as the following integral:

$$\begin{aligned} f_\Gamma(\gamma) &= \int_{\mathbb{R}^{N-1}} f_{\Gamma_1}(\gamma_1)\cdots f_{\Gamma_{N-1}}(\gamma_{N-1}) \\ &\quad\times f_{\Gamma_N}(\gamma-\gamma_1-\ldots-\gamma_{N-1})d\gamma_1\ldots d\gamma_{N-1} \\ &= \gamma^{N-1}\int_{\mathbb{R}^{N-1}} f_{\Gamma_1}(\gamma t_1)\cdots f_{\Gamma_{N-1}}(\gamma t_{N-1}) \\ &\quad\times f_{\Gamma_N}(\gamma(1-t_1-\ldots-t_{N-1}))dt_1\ldots dt_{N-1} \\ &= \kappa_1\cdots\kappa_N\gamma^{\mu_1+\ldots+\mu_N+N-1} \\ &\quad\times\int_{\mathcal{S}_{N-1}} t_1^{\mu_1}\cdots t_{N-1}^{\mu_{N-1}}(1-t_1-\ldots-t_{N-1})^{\mu_N} \\ &\quad\times\exp\{-\gamma^2[t_1^2/\rho_1+\ldots+t_{N-1}^2/\rho_{N-1} \\ &\qquad+(1-t_1-\ldots-t_{N-1})^2/\rho_N]\}dt_1\ldots dt_{N-1}. \end{aligned} \tag{13}$$

$\mathcal{S}_n \triangleq \{(t_1,\ldots,t_n) : t_k \ge 0, k = 1,\ldots,n, \sum_{k=1}^{n} t_k \le 1\}$ is the n-dimensional simplex region. Here, we set, as shorthand notation, $\kappa_k \triangleq 2m_k^{m_k}/(\Omega_k^{m_k}\Gamma(m_k))$, $\mu_k \triangleq 2m_k - 1$, and $\rho_k \triangleq \Omega_k/m_k$, so that the Nakagami-$m$ pdf's from (2) becomes $f_{\Gamma_k}(\gamma_k) = \kappa_k\gamma^{\mu_k}\mathrm{e}^{-\gamma^2/\rho_k}$ for $k = 1,\ldots,N$. We can easily check that the asymptotic global maximum required by Laplace's method is $\boldsymbol{t}_0 = (\rho_1,\ldots,\rho_{N-1})/(\rho_1+\cdots+\rho_N)$. The Hessian matrix of $h(\boldsymbol{t})$ in $\boldsymbol{t} = \boldsymbol{t}_0$ is $\boldsymbol{H}_h(\boldsymbol{t}_0) = 2\gamma^2(\boldsymbol{I}_{N-1} + \mathbf{1}_{(N-1)\times(N-1)})$.[1] Thus, the asymptotic approximation becomes

$$\begin{aligned} f_\Gamma(\gamma) &\sim \frac{2^N\pi^{(N-1)/2}\rho_1^{m_1-1}\cdots\rho_N^{m_N-1}\gamma^{\mu_1+\ldots+\mu_N}}{\sqrt{N}\Gamma(m_1)\ldots\Gamma(m_N)(\rho_1+\cdots+\rho_N)^{\mu_1+\ldots+\mu_N}} \\ &\quad\times\mathrm{e}^{-\gamma^2/(\rho_1+\cdots+\rho_N)}. \end{aligned} \tag{14}$$

This approximation is a scalar multiple of the Nakagami-m pdf with parameters

$$\tilde{m} = m_1+\ldots+m_N-\frac{N-1}{2}, \quad \tilde{\Omega} = \tilde{m}\sum_{k=1}^{N}\frac{\Omega_k}{m_k}. \tag{15}$$

We shall refer to the Nakagami-m pdf with the above parameters as *Laplace approximation* of the sum of *i.n.i.d.* Nakagami-m RV's. Clearly, the Laplace approximation coincides with the Nakagami-m distribution in the special case $N = 1$.

IV. Gaussian series expansion

The moments of the Nakagami-m distribution are well known [1]:

$$E[\Gamma_k^n] = \rho_k^{n/2}\frac{\Gamma(m_k+\frac{1}{2})}{\Gamma(m_k)} \tag{16}$$

with $\rho_k \triangleq \Omega_k/m_k$. The mean and variance are:

$$\mu_k \triangleq \sqrt{\rho_k}\frac{\Gamma(m_k+\frac{1}{2})}{\Gamma(m_k)}, \tag{17}$$

$$\sigma_k^2 \triangleq \rho_k\bigg(m_k-\frac{\Gamma(m_k+\frac{1}{2})^2}{\Gamma(m_k)^2}\bigg). \tag{18}$$

[1] $\mathbf{1}_{m\times n}$ denotes the $m\times n$ all-1 matrix.

Thus, the Gaussian approximation to Γ_k is $\mathcal{N}(\mu_k, \sigma_k^2)$. The CF of Γ_k is given by:

$$\Phi_{\Gamma_k}(\gamma) = \int_0^\infty f_{\Gamma_k}(\gamma) \mathrm{e}^{-\mathrm{j}\,\omega\gamma} d\gamma \tag{19}$$
$$= {}_1F_1\left(m_k; \frac{1}{2}; -\frac{\rho_k\omega^2}{4}\right) - \mathrm{j}\,\omega\mu_{k1}F_1\left(m_k + \frac{1}{2}; \frac{3}{2}; -\frac{\rho_k\omega^2}{4}\right).$$

The CF of the Gaussian approximation is

$$\Phi_{\Gamma_k}^{(G)}(\gamma) = \mathrm{e}^{-\mathrm{j}\,\omega\mu_k - \frac{1}{2}\omega^2\sigma_k^2}. \tag{20}$$

Then, we can get the following series expansion:

$$\begin{aligned}\frac{\Phi_{\Gamma_k}(\gamma)}{\Phi_{\Gamma_k}^{(G)}(\gamma)} &= 1 + \frac{\mu_k(4\sigma_k^2 - \rho_k)}{12}(\mathrm{j}\,\omega)^3 \\ &+ \frac{-m_k\rho_k^2 + 2(1+2m_k)\rho_k\sigma_k^2 - 6\sigma_k^4}{24}(\mathrm{j}\,\omega)^4 \\ &+ \mu_k\frac{-(3+8m_k)\rho_k^2 + 8(5+4m_k)\rho_k\sigma_k^2 - 96\sigma_k^4}{480}(\mathrm{j}\,\omega)^5 \\ &+ O(\omega^6).\end{aligned} \tag{21}$$

By multiplying these ratios, we obtain, for properly defined coefficients ζ_n,

$$\Phi_\Gamma(\gamma) = \mathrm{e}^{-\mathrm{j}\,\omega\mu - \frac{1}{2}\omega^2\sigma^2} \sum_{n=0}^\infty \zeta_n(\mathrm{j}\,\omega)^n. \tag{22}$$

where μ and σ^2 are defined as

$$\mu = \sum_{k=1}^N \sqrt{\frac{\Omega_k}{m_k}} \frac{\Gamma(m_k + \frac{1}{2})}{\Gamma(m_k)}, \tag{23}$$

$$\sigma^2 = \sum_{k=1}^N \Omega_k\left(1 - \frac{\Gamma(m_k + \frac{1}{2})^2}{m_k\Gamma(m_k)^2}\right). \tag{24}$$

As a result, we can invert the CF, recalling that the multiplication by $\mathrm{j}\,\omega$ corresponds to the derivation of the pdf with respect to γ, and obtain the following result:

$$\begin{aligned}f_\Gamma(\gamma) &= \frac{1}{\sqrt{2\pi\sigma^2}} \sum_{n=0}^\infty \zeta_n \frac{d^n}{d\gamma^n} \mathrm{e}^{-(\gamma-\mu)^2/(2\sigma^2)} \\ &= \frac{1}{\sqrt{2\pi\sigma^2}} \sum_{n=0}^\infty \zeta_n \sigma^{-n} H_n\left(\frac{\gamma-\mu}{\sigma}\right) \mathrm{e}^{-(\gamma-\mu)^2/(2\sigma^2)}.\end{aligned} \tag{25}$$

The functions $H_n(x)$ are the Hermite polynomials defined by

$$H_n(x) \triangleq \mathrm{e}^{x^2/2} \frac{d^n}{dx^n} e^{-x^2/2} \tag{26}$$

satisfying the recursive relationships

$$H_0(x) = 1, \quad H_{n+1}(x) = H_n'(x) - xH_n(x), \quad n = 0, 1, \ldots \tag{27}$$

V. Numerical results

We consider the following four different examples to validate the results obtained in this section.

1) $m_1 = 1, m_2 = 1.1, m_3 = 1.7, \Omega_1 = \Omega_2 = \Omega_3 = 1$.
2) $m_k = \Omega_k = 1, k = 1, \ldots, 10$.
3) $m_k = \Omega_k = 1, k = 1, \ldots, 20$.
4) $m_k = \frac{1}{2}, \Omega_k = 1, k = 1, \ldots, 10$.

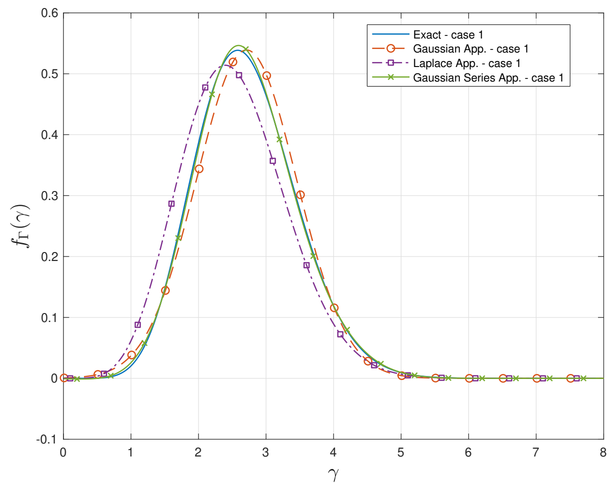

Fig. 1. Plot of the pdf of Γ in Case 1: $m_1 = 1, m_2 = 1.1, m_3 = 1.7, \Omega_1 = \Omega_2 = \Omega_3 = 1$. The exact pdf (power series) and the Gaussian, Laplace, Nakagami, and Gaussian series approximations are reported.

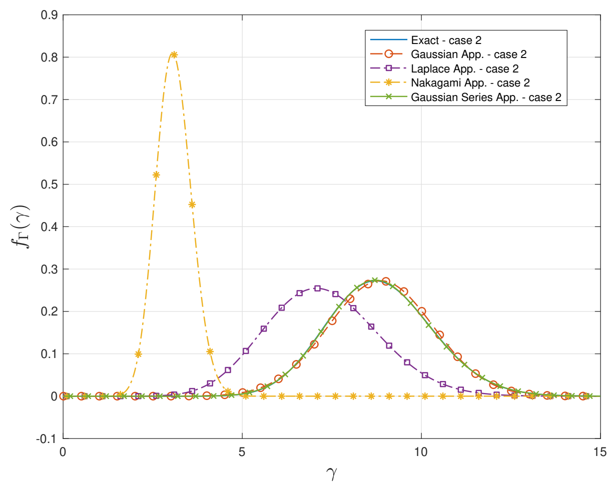

Fig. 2. Same as Fig. 1 for Case 2: $m_k = \Omega_k = 1, k = 1, \ldots, 10$.

The first three ones have been considered in [11] and we use them to assess the numerical correctness of the method. We report the following approximations:

- The power series expansion developed in Section II.
- The Gaussian approximation $\mathcal{N}(\mu, \sigma^2)$ with μ and σ^2 defined in (23) and (24).
- The Laplace approximation developed in Section III.
- The Nakagami approximation reported in [1, eqs. (82-83)], applicable only in the case of constant m_k.
- The Gaussian series approximation developed in Section IV truncated to a maximum degree equal to 10.

Figs. 1 to 4 report numerical results for the three cases considered. We can see (Figs. 1 to 3) that the power series and Gaussian series methods provide numerical values in agreement with those reported in [11]. The Gaussian approximation is very good in all cases, even though the Gaussian series provides better results. On the contrary, the Laplace approximation is barely acceptable in Case 1 but very coarse in the other cases. The Nakagami approximation does not provide acceptable results in the cases considered.

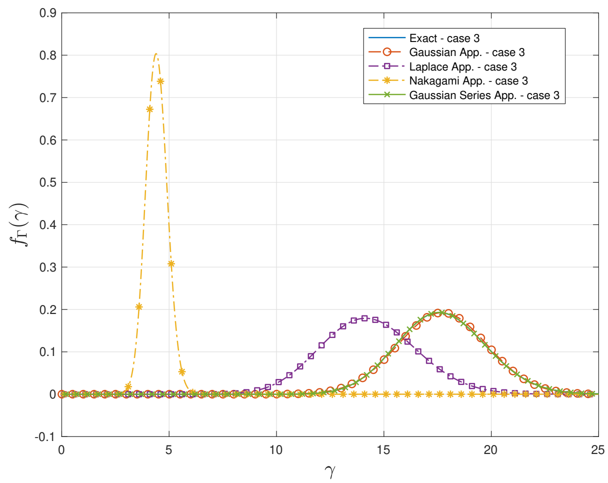

Fig. 3. Same as Fig. 1 for Case 3: $m_k = \Omega_k = 1, k = 1, \ldots, 20$.

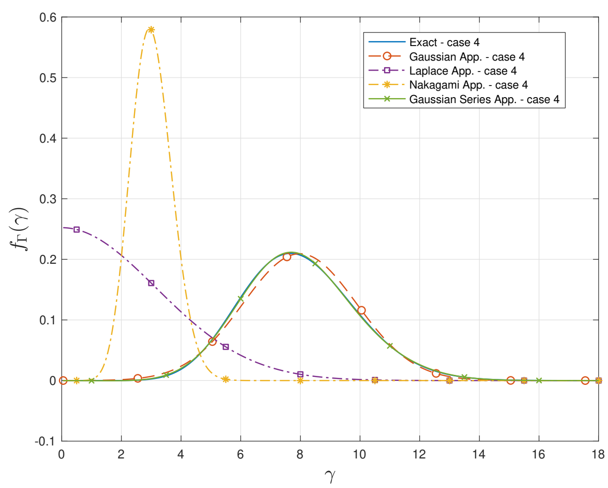

Fig. 4. Same as Fig. 1 for Case 4: $m_k = \frac{1}{2}, \Omega_k = 1, k = 1, \ldots, 10$.

A. Critical comments on the power series method

The main drawback of the power series method is that it requires the use of high-precision arithmetic, due to the large dynamic range of the terms in the alternating series (6). This is illustrated in Fig. 5 for Case 3 ($m_k = \Omega_k = 1, k = 1, \ldots, 20$). The diagram shows that in order to obtain a pdf value as low as 10^{-6} with a few significant digits, in consideration of the large number of recursion steps required to calculate the β coefficients, we have to calculate terms in (6) as low as 10^{-15}. If we consider $\gamma = 25$ (which is the approximate upper limit of this pdf), Fig. 5 shows that the number of terms required is ≈ 1600 and the number of precision digits required is around 240. Both the number of terms and the number of precision digits increase as γ increases. Therefore, this approach is more convenient for lower values of γ and may become demanding for large values. Nevertheless, the computational time on a standard notebook is on the order of a few minutes (using Mathematica). The complexity is lower in Case 1.

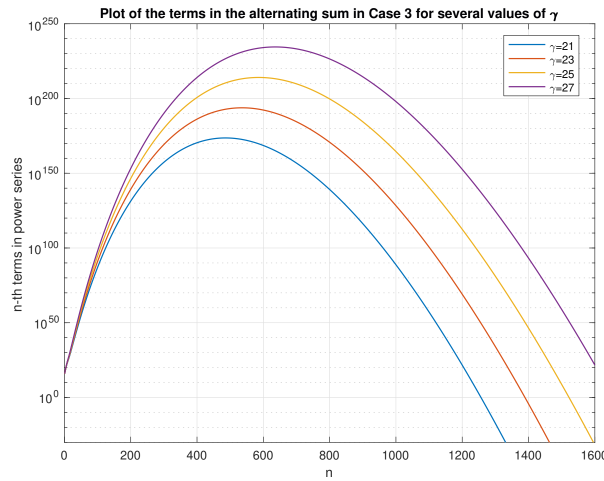

Fig. 5. Plot of the absolute values of the terms in the alternating sum (6) for Case 3 ($m_k = \Omega_k = 1, k = 1, \ldots, 20$) and $\gamma = 21, 23, 25, 27$.

```
function zet = calc_zeta(mv,Omegav,maxdeg)
N    = numel(mv);
rov  = Omegav./mv;
muv  = sqrt(rov).*gamma(mv+0.5)./gamma(mv);
sgv  = Omegav-muv.^2;
syms x
Phi  = 1;
for k=1:N
  Phi  = expand(series(...
    Phi*(hypergeom(mv(k),1/2,rov(k)*x^2/4)...
    -muv(k)*x...
    *hypergeom(mv(k)+1/2,3/2,rov(k)*x^2/4))...
    *exp(muv(k)*x-sgv(k)*x^2/2),x,...
    'Order',maxdeg+1));
end
zet  = double(coeffs(expand(Phi),x,'all'));
zet  = fliplr(zet);
```

Fig. 6. Matlab code for the calculation of the coefficients ζ_n in (25) (based on Symbolic Toolbox).

B. Critical comments on the Gaussian series method

There are no major drawbacks with the Gaussian series method in all cases where the pdf is close to a Gaussian distribution. This applies to both Cases 2 and 3, while it is slightly less applicable to Case 1. The key problem here is the computations of the coefficients ζ_n in (25). A possible implementation of the algorithm is reported in Fig. 6 based on Matlab with the Symbolic Toolbox.

VI. Conclusions

This letter presented two series expansions and two approximations to the Nakagami-m sum pdf. The Laplace approximation has been shown to be coarser than the Gaussian approximation. The power series expansion has been shown to be applicable when the pdf is considerably far from the Gaussian but may be computationally demanding in some cases. The Gaussian series expansion has been shown to be applicable whenever the pdf is close to Gaussian and it is less computationally demanding than the power series expansion.

Appendix A
Truncation error in series (6)

First of all, we can see that

$$\prod_{k=1}^{N}\frac{(m_k/\Omega_k)^{\ell_k}}{\ell_k!}\leq\frac{(\rho_{\max})^{\ell}}{[\Gamma(1+\ell/N)]^N}\tag{28}$$

after defining

$$\rho_{\max}\triangleq\max_{1\leq k\leq N}(m_k/\Omega_k).\tag{29}$$

Applying Jensen's inequality [14] to the convex function $\ln\Gamma(1+x)$, yields:

$$\frac{1}{N}\sum_{k=1}^{N}\ln\Gamma(1+\ell_k)\geq\ln\Gamma\left(1+\frac{1}{N}\sum_{k=1}^{N}\ell_k\right)=\ln\Gamma\left(1+\frac{\ell}{N}\right).\tag{30}$$

Hence, we obtain:

$$\prod_{k=1}^{N}\ell_k!\geq[\Gamma(1+\ell/N)]^N.\tag{31}$$

Inserting this result in the definition of $\beta_\ell[N]$, we get the following upper bound:

$$\frac{\beta_\ell[N]}{\Gamma(2(m+\ell))}\leq\sum_{\substack{\ell_1+\cdots+\ell_N=\ell\\ \ell_1\geq0,\ldots,\ell_N\geq0}}\frac{\prod_{k=1}^{N}\Gamma(2(m_k+\ell_k))}{\Gamma(2(m+\ell))}\times\frac{(\rho_{\max})^{\ell}}{[\Gamma(1+\ell/N)]^N}.\tag{32}$$

In order to upper bound the term $\frac{\prod_{k=1}^{N}\Gamma(2(m_k+\ell_k))}{\Gamma(2(m+\ell))}$ in the sum, we note that, if $m_1\geq m_2,\ell_1>0$, we have:

$$\frac{\Gamma(2(m_1+\ell_1))\Gamma(2(m_2+\ell_2))}{\Gamma(2(m_1+\ell_1+\ell_2))\Gamma(2m_2)}=\frac{(2m_2+2\ell_2-1)\cdots(2m_2)}{(2m_1+2\ell_1+2\ell_2-1)\cdots(2m_1+2\ell_1)}<1.\tag{33}$$

By repeatedly applying the above inequality we get

$$\prod_{k=1}^{N}\Gamma(2(m_k+\ell_k))<\Gamma(2(m_{\max}+\ell))\prod_{k=1,k\neq k_{\max}}^{N}\Gamma(2m_k)<\Gamma(2(m+\ell))\prod_{k=1,k\neq k_{\max}}^{N}\Gamma(2m_k),\tag{34}$$

assuming that $m_{\max}\triangleq\max_{1\leq k\leq N}m_k=m_{k_{\max}}$.[2] Now, we should count the number of terms present in the sum of the upper bound (32). Let

$$\sigma_n(\ell)\triangleq\sum_{\substack{\ell_1+\cdots+\ell_n=\ell\\ \ell_1\geq0,\ldots,\ell_n\geq0}}1.\tag{35}$$

We can see that

$$\sigma_n(\ell)=\sum_{p=0}^{\ell}\sigma_{n-1}(\ell-p)=\sum_{p=0}^{\ell}\sigma_{n-1}(p).\tag{36}$$

[2] $k_{\max}$ can be taken as any of the indexes k such that $m_k=m_{\max}$.

Since $\sigma_1(\ell)=\ell+1$, the subsequent $\sigma_n(\ell)$ can be calculated exactly by the recursion (36) but, for our purposes, the following approximation is sufficient:

$$\sigma_N(\ell)\approx\frac{\ell^{N-1}}{(N-1)!}.\tag{37}$$

Thus, we obtain the following approximate upper bound:

$$\frac{\beta_\ell[N]}{\Gamma(2(m+\ell))}\lesssim\frac{\ell^{N-1}}{(N-1)!}\prod_{k=1,k\neq k_{\max}}^{N}\Gamma(2m_k)\frac{(\rho_{\max})^{\ell}}{[\Gamma(1+\ell/N)]^N}.\tag{38}$$

Finally, applying the asymptotic Stirling approximation $\ln\Gamma(1+x)=x\ln x-x+O(\ln x)$, we get

$$\ln|\alpha_\ell|\leq-\ell\ln\ell+\ell\ln(\rho_{\max}N\mathrm{e})+O(\ln\ell),\tag{39}$$

which can be simplified as in (11).

References

[1] M. Nakagami, "The m-distribution – A general formula for intensity distribution of rapid fading," in *Statistical Methods in Radio Wave Propagation*, W. G. Hoffman, editor. Oxford, UK: Pergamon, 1960.

[2] M. K. Simon and M. S. Alouini, *Digital Communications over Fading Channels (2nd Ed.)*. John Wiley & Sons, 2005.

[3] N.C. Beaulieu, "An infinite series for the computation of the complementary probability distribution function of a sum of independent random variables and its application to the sum of Rayleigh random variables," *IEEE Trans. Commun.*, vol. 38, no. 9, pp. 1463-1474, Sep. 1990.

[4] G. Taricco, "On the convergence of multipath fading channel gains to the Rayleigh distribution," *IEEE Wireless Communications Letters*, vol. 4, no. 5, pp. 549–552, Oct. 2015.

[5] J. Reig and N. Cardona, "Nakagami-m approximate distribution of sum of two Nakagami-m correlated variables," *Electron. Lett.*, vol. 36, no. 11, pp. 978-980, May, 2000.

[6] J.C.S.S. Filho and M.D. Yacoub, "Nakagami-m approximation to the sum of M non-identical independent Nakagami-m variates," *Electron. Lett.*, vol. 40, no. 15, pp. 951-952, July 2004.

[7] M. D. Yacoub, C. R. C. M. da Silva, and J. E. V. Bautista, "Second-order statistics for diversity-combining techniques in Nakagami-fading channels," *IEEE Trans. Veh. Technol.*, vol. 50, no. 6, pp. 1464-1470, Nov. 2001.

[8] C.-D. Iskandar and P. T. Mathiopoulos, "Analytical level crossing rates and average fade durations for diversity techniques in Nakagami fading channels," *IEEE Trans. Commun.*, vol. 50, no. 8, pp. 1301-1309, Aug. 2002.

[9] P. Dharmawansa, R. Rajatheva, and K. Ahmed, "On the distribution of the sum of Nakagami-m random variables," *IEEE Trans. Commun.*, vol. 55, no. 7, pp. 1407-1416, July 2007.

[10] C.-D. Iskandar and P. T. Mathiopoulos, "Exact performance analysis of dual-branch coherent equal-gain combining in Nakagami-m, Rician and Hoyt fading," *IEEE Trans. Veh. Technol.*, vol. 57, no. 2, pp. 921-931, Mar. 2008.

[11] M.A. Rahman and H. Harada, "New exact closed-form PDF of the sum of Nakagami-m random variables with applications," *IEEE Trans. Commun.*, vol. 59, no. 2, pp. 395–401, Feb. 2011.

[12] J. E. Kolassa, *Series Approximation Methods in Statistics (3rd ed)*. Springer, 2006.

[13] D.J.C. MacKay, *Information Theory, Inference and Learning Algorithms*. Cambridge: Cambridge University Press. 2003.

[14] T.M. Cover and J.A. Thomas, *Elements of Information Theory (2nd ed.)*. New York: Wiley, 2006.