A Minimizing Problem of Distances Between Random Variables with Proportional Reversed Hazard Rate Functions

Original
A Minimizing Problem of Distances Between Random Variables with Proportional Reversed Hazard Rate Functions / Ortega Jimenez, Patricia; Pellerey, Franco; Sordo, Miguel; Suarez-Llorens, Alfonso. - STAMPA. - 1433:(2023), pp. 311-318. (Intervento presentato al convegno 10th International Conference on Soft Methods in Probability and Statistics, SMPS 2022 tenutosi a Valladolid (Spain) nel 14-16 September 2022) [10.1007/978-3-031-15509-3_41].

Availability:
This version is available at: 11583/2970833 since: 2022-09-15T22:51:43Z

Publisher:
Springer

Published
DOI:10.1007/978-3-031-15509-3_41

A minimizing problem of distances between random variables with proportional reversed hazard rate functions.

Abstract Let X be a random variable with distribution function F and let $\mathcal{F}_X$ be the family of proportional reversed hazard rate distribution functions associated to F. Given the random vector (X, Y) with copula C and respective marginal distribution functions F and $G \in \mathcal{F}_X$, we obtain sufficient conditions for the existence of $G \in \mathcal{F}_X$ that minimizes $E_C|X - Y|$.

1 Introduction

Given a random variable X, several location measures can be defined as the argument that minimizes a variability functional of X. Examples of such measures are:

- The expectation, μ_X, which can be defined as the value that minimizes the mean square error of X,

$$\mu_X = \operatorname{argmin}_{t \in \mathbb{R}} E[(X - t)^2].$$

- The median, Me_X, defined as the value that minimizes the mean absolute deviation:

$$Me_X = \operatorname{argmin}_{t \in \mathbb{R}} E|X - t|$$

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- The α-expectile, $\rho_\alpha(X)$, which can be defined as the value that minimizes the following linear combination of expected square excesses (Krätschmer and Zähle (2017)):

$$\rho_\alpha(X) = \text{argmin}_{t\in\mathbb{R}} \left\{\alpha E\left[\left((X-t)^+\right)^2\right] + (1-\alpha)E\left[\left((t-X)^+\right)^2\right]\right\}$$

- Analogously, the α-quantiles, $q_\alpha(X)$, that can be defined in a similar way:

$$q_\alpha(X) = \text{argmin}_{t\in\mathbb{R}} \left\{\alpha E\left[(X-t)^+\right] + (1-\alpha)E\left[(t-X)^+\right]\right\}$$

Therefore, it is natural to wonder whether we can proceed analogously with a different variability functional of X, that is, if the functional can be minimized in order to have a measure that gives information about X.

Let us consider a random variable X with strictly increasing distribution function F and $h : [0,1] \to [0,1]$, a strictly increasing distortion function, that is, an strictly increasing function such that $h(0) = 0$ and $h(1) = 1$. Throughout the paper, we will consider that all variables are absolutely continuous and that all distribution functions and copulas are continuously differentiable. If we now consider (X,Y), a random vector with copula C and marginal distribution functions F and $G = h(F)$ respectively, in Ortega-Jiménez et al. (2021) was shown that $\nu(X) = E_C|X-Y|$, where

$$E_C|X-Y| = \int_{-\infty}^{\infty} (F(x) + G(x) - 2\,C\,(F(x), G(x)))\,dx, \tag{1}$$

is a comonotonic additive measure of variability in the sense of Bickel and Lehmann (1979), that is, it is a measure that satisfies the following properties:

(P0) Law invariance: if X and Y have the same distribution, then $\nu(X) = \nu(Y)$.
(P1) Translation invariance: $\nu(X+k) = \nu(X)$ for all X and all constant k.
(P2) Positive homogeneity: $\nu(0) = 0$ and $\nu(\lambda X) = \lambda\nu(X)$ for all X and all $\lambda > 0$.
(P3) Non-negativity: $\nu(X) \geq 0$ for all X, with $\nu(X) = 0$ if X is degenerated at $c \in \mathbf{R}$.
(P4) Consistency with dispersive order: if $X \leq_{disp} Y$, then $\nu(X) \leq \nu(Y)$.
(P5) Comonotonic additivity: if X and Y are comonotonic, then $\nu(X+Y) = \nu(X) + \nu(Y)$.

Recall that, given two random variables X and Y with distribution functions F and G, respectively, we say that X is smaller than Y in the dispersive order $(X \leq_{disp} Y)$ if $F^{-1}(p) - F^{-1}(q) \leq G^{-1}(p) - G^{-1}(q)$ for all $0 \leq q < p \leq 1$.

Let us consider the distortion given by the power function $h(t) = t^\alpha, \alpha > 0$. This distortion function characterizes the proportional reversed hazard rate (PRHR) model, which has interesting applications in insurance risk (see Psarrakos and Sordo (2019)). A variable satisfies such a model if its reversed hazard rate function ($\tilde{r}(t) = \frac{f(t)}{F(t)}$) is proportional to the baseline reversed hazard rate function. Given a random variable X, we will denote as $\mathcal{F}_X = \{X_\alpha : F_{X_\alpha}(x) = F(x)^\alpha, \alpha > 0\}$ the family of all random variables that satisfy the PRHR model. The interest on considering such distortion function lies in the importance of the PRHR model, which has applications in various areas, such as statistics, reliability engineering, demography, physics or forensic science. The interested reader may consult Gupta and Gupta (2007) for an extensive list of further applications.

Following the approach that initialized the paper, we can now consider the following problem. Fixed the copula C and the marginal distribution function F, we study sufficient conditions for the existence of a distribution function G of $Y \in \mathcal{F}_X$ such that the distance (1) is minimal. The first false intuition may suggest, at least when X and Y have the same support, that the smallest value of $E_C|X-Y|$ is reached when $F = G$, that is, when $Y =_{st} X$. This is not necessarily true, as we can see in the following counterexample. Considering C the independence copula and $X \sim U(0,1)$, it is easy to see that, considering any $\alpha \in (1,2)$, if Y_1 has a distribution function $G(u) = u^\alpha$ for $u \in [0,1]$ and $Y_2 \sim U(0,1)$, then $E_I|X-Y_1| < E_I|X-Y_2|$. The minimum is reached when $\alpha = \sqrt{2}$. It may even happen that there is not a minimizer α for the function. If we consider the independence copula and $X \sim Weibull(k,1)$, we can see that, for some values of k, the minimizer exists and for some others it does not. For $k = 1$ the minimum is reached in $\alpha = 0.390$ and for $k = 1.3$ is reached in $\alpha = 0.714$. Although, it can be checked that, for example, for $k = 0.7$ there is not $\alpha > 0$ that minimizes such functional (Figure 1).

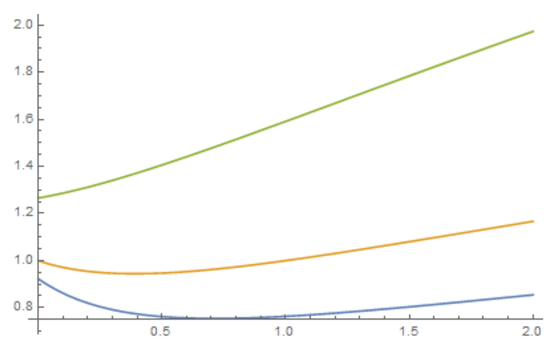

Fig. 1 $E_I|X-X_\alpha|$ in terms of α considering $X \sim Weibull(k,1)$ with $k = 0.7$ (green), $k = 1$ (orange) and $k = 1.3$ (blue).

The rest of the work is organized as follows. Section 2 contains preliminaries. In Section 3, given a random variable X, a copula C and the family of variables that satisfy the PRHR model, $\mathcal{F}_X$, we obtain sufficient conditions for the existence of $Y^* \in \mathcal{F}_X$ that minimizes $E_C|X-Y|$ within all $Y \in \mathcal{F}_X$. Finally, Section 4 includes examples of the result, both for existence and nonexistence of such minimizer. A second step in this research would be to obtain analytically the minimizer α and to provide a plausible interpretation. Such work remains for future research.

2 Preliminaries

Let us consider a random vector $\boldsymbol{X} = (X, Y)$ with marginal distribution functions F and G and joint distribution function K. Throughout the paper, as mentioned in the introduction, we will consider K, F, G continuously differentiable. By the Sklar theorem, the joint distribution K can be written as $K(x, y) = C\,(F(x), G(y))$, where C is the joint distribution function of the vector-copula $(F\,(X)\,, G\,(Y))$. Such C is the copula (Nelsen (2007)) of the vector $\boldsymbol{X}$ and, under the given assumptions, it is unique and continuously differentiable.

We need the following definitions. Here and throughout the paper, the term "increasing" is used for "non-decreasing" and "decreasing" is used for "non-increasing".

Definition 1 Let $\boldsymbol{X} = (X, Y)$ be a random vector with copula C and marginal distribution functions F, G. X is *stochastically increasing* in Y ($X \uparrow_{SI} Y$), if $P\,[X > x \mid Y = y]$ increases in y for all x. A vector is *positively dependent through stochastic ordering* (PDS) if $X \uparrow_{SI} Y$ and $Y \uparrow_{SI} X$.

As $P\,[X > x \mid Y = y] = 1 - \partial_2 C(F(x), G(y))$, $X \uparrow_{SI} Y$ if $\partial_2 C(u, v)$ decreases in v for all u. We will say that the copula C is PDS if the vector is PDS.

Definition 2 A random variable X with distribution function F and density function f is said to have the *increasing failure rate property* (IFR) if its hazard rate function $r(x) = \frac{f\,(x)}{1-F\,(x)}$, is increasing.

3 Determining $\min_{\{Y \in \mathcal{F}_X\}} E_C|X - Y|$ with X and C fixed.

We will see that, under some conditions on the fixed copula, if the variable X is IFR, then $\min_{\{Y \in \mathcal{F}_X\}} E_C|X - Y|$ is reached. We need some results to prove it. From now on, we will denote

$$\varphi_0(u) = \lim_{v \to 0} \partial_2 C(u, v) \quad \text{and} \quad \varphi_1(u) = \lim_{v \to 1} \partial_2 C(u, v).$$

Such limits exist for all $u \in (0, 1)$. When the copula is PDS, $\partial_2 C(u, v)$ decreases in v. Since $\int_0^1 \partial_2 C(u, v) dv = u$, for all $u \in (0, 1)$, it follows that that $\varphi_0(u) \geq u \geq \varphi_1(u)$.

Lemma 1 *Let C be an absolutely continuous copula, and let X be an IFR random variable with F, f and r its respective distribution, density and hazard rate functions. Assume that $r(F^{-1}(0)) \neq 0$. Then:*

$$\lim_{\alpha\to 0}\int_0^1 \frac{1-2\,\partial_2 C(u,u^\alpha)}{f(F^{-1}(u))}u^\alpha \log(u)du = \int_0^1 \frac{1-2\varphi_1(u)}{f(F^{-1}(u))}\log(u)du, \tag{2}$$

$$\lim_{\alpha\to \infty}\int_0^1 \frac{1-2\,\partial_2 C(u,u^\alpha)}{f(F^{-1}(u))}\log(u)du = \int_0^1 \frac{1-2\varphi_0(u)}{f(F^{-1}(u))}\log(u)du. \tag{3}$$

Proof If we define, for all $u \in (0,1), n \in \mathbb{N}$, $g_n(u) = \frac{1-2\,\partial_2 C(u,u^{1/n})}{f(F^{-1}(u))}u^{1/n}\log(u)$, $g(u) = \frac{1-2\varphi_1(u)}{f(F^{-1}(u))}\log(u)$, $h_n(u) = \frac{1-2\,\partial_2 C(u,u^n)}{f(F^{-1}(u))}\log(u)$, $h(u) = \frac{1-2\varphi_0(u)}{f(F^{-1}(u))}\log(u)$, then, for all $u \in (0,1)$ $\lim_{n\to\infty} g_n(u) = g(u)$ and $\lim_{n\to\infty} h_n(u) = h(u)$. As $|g_n(u)|$ and $|h_n(u)|$ are bounded by $\frac{-\log(u)}{f(F^{-1}(u))}$, by the Dominated Convergence Theorem, if $\frac{-\log(u)}{f(F^{-1}(u))}$ is a positive integrable function, (2) and (3) hold. By the Cauchy-Schwarz inequality and considering the hazard rate function r:

$$\int_0^1 \frac{-\log(u)}{f(F^{-1}(u))}du = \int_0^1 \frac{-\log(u)}{1-u}\frac{1}{r(F^{-1}(u))}du \leq$$

$$\leq \left(\int_0^1 \left(\frac{-\log(u)}{1-u}\right)^2 du\right)^{1/2}\left(\int_0^1 \left(\frac{1}{r(F^{-1}(u))}\right)^2 du\right)^{1/2} \tag{4}$$

Note that $\int_0^1 \left(\frac{-\log(u)}{1-u}\right)^2 du = \frac{\pi^2}{3}$ and, since r increases and $r(F^{-1}(0)) \neq 0$, $\frac{1}{r(F^{-1}(u))}$ is bounded. Therefore, (4) is finite and the assertion follows. □

Lemma 2 *Let X be an IFR random variable with F, f and r its respective distribution, density and hazard rate functions. Then:*

$$\int_0^1 \frac{1-2u}{f(F^{-1}(u))}(-\log(u))du \tag{5}$$

is strictly positive and finite.

Proof First we will show that

$$b(s) = \int_0^s \left(\frac{1-2u}{1-u}(-\log(u))\right)du > 0 \text{ for all } s \in [0,1].$$

Since $b'(s) = \frac{1-2s}{1-s}(-\log(s))$, $b(s)$ increases if $s < 1/2$ and decreases if $s > 1/2$. Moreover, $b(0) = 0$ and $b(1) = 2-\pi^2/6 > 0$, therefore $b(s) > 0$ for all $s \in (0,1]$. We can rewrite (5) in the following form:

$$\int_0^1 \left(\frac{1-2u}{1-u}(-\log(u))\right)\frac{1}{r(F^{-1}(u))}du. \tag{6}$$

As X is IFR, $1/r(F^{-1}(u))$ decreases in u. By Lemma 4.7.1 in Barlow and Proschan (1975), if, for all $s \in [0,1]$, $b(s) = \int_0^s \left(\frac{1-2u}{1-u}(-\log(u))\right) du$ is positive, then (6) is also positive. $b'(s) = \frac{1-2s}{1-s}(-\log(s))$, so $b(s)$ increases if $s < 1/2$ and decreases if $s > 1/2$. As $b(0) = 0$ and $b(1) = 2 - \pi^2/6 > 0$, $b(s) > 0$ for all $s \in (0,1]$ and (6) is strictly positive. Also, by Cauchy-Schwarz inequality, (6) is smaller or equal than:

$$\left(\int_0^1 \left(\frac{1-2u}{1-u}(-\log(u))\right)^2 du\right)^{\frac{1}{2}} \left(\int_0^1 \frac{1}{r(F^{-1}(u))^2} du\right)^{\frac{1}{2}}.$$

Integrating by parts and considering the Spence's function, given by $Li_2(u) = \int_0^u \frac{-\log(t)}{1-t} dt$, we can see that $\int_0^1 \left(\frac{-\log(u)(1-2u)}{1-u}\right)^2 du \le 1 + Li_2(1) < 3$ (we can obtain, computationally, that the value is approximately 1.2936). As we already saw in Lemma 1 that the second element is finite, (5) is finite. □

We can now move on to the main result:

Proposition 1 *Let X be a random variable with F and f its respective distribution and density function. Let us consider the family of random variables $\mathcal{F}_X$ described above. Let us consider the random vector $\mathbf{X} = (X, Y)$ with PDS copula C, such that $\partial_2 C(u, u^n)$ increases in u for all $n \in \mathbb{N}$ and* $\lim_{u\to 1} \varphi_1(u) = 1$. *If X is IFR, there exists $\alpha_0 > 0$ such that $X_{\alpha_0} \in \mathcal{F}_X$ and:*

$$E_C|X - X_{\alpha_0}| \le E_C|X - Y| \text{ for all } Y \in \mathcal{F}_X.$$

Proof Given X, we can consider the function $\alpha \mapsto E_C|X - X_\alpha|$, given by:

$$\int_{-\infty}^{\infty} (F(x) + F(x)^\alpha - 2\, C(F(x), F(x)^\alpha))\, dx = \int_0^1 \frac{u + u^\alpha - 2C(u, u^\alpha)}{f(F^{-1}(u))} du$$

This is a continuous and derivable function for $\alpha > 0$, and

$$\partial_\alpha E_C|X - X_\alpha| = \int_0^1 \frac{1 - 2\, \partial_2 C(u, u^\alpha)}{f(F^{-1}(u))} u^\alpha \log(u) du$$

In order to see that there exists α_0 that minimizes $E_C|X - X_\alpha|$, it would be enough to see that

1. $\lim_{\alpha_1 \to 0} \left(\partial_\alpha E_C|X - X_\alpha|_{\alpha=\alpha_1}\right) < 0$, and
2. $\lim_{\alpha_2 \to +\infty} \left(\partial_\alpha E_C|X - X_\alpha|_{\alpha=\alpha_2}\right) \ge 0$.

It would mean that there exists at least one value $\alpha_0 > 0$ where $E_C|X - X_\alpha|$ attains a local minimum, and necessarily one of them will be the global one.

By Lemma 1, $\lim_{\alpha_1 \to 0} \left(\partial_\alpha E_C|X - X_\alpha|_{\alpha=\alpha_1}\right) = -\left(\int_0^1 \frac{1-2\varphi_1(u)}{f(F^{-1}(u))}(-\log(u)) du\right)$ and, as $u \ge \varphi_1(u)$, by Lemma 3,

$$\lim_{\alpha_1\to 0}\left(\partial_\alpha E_C|X-X_\alpha|_{\alpha=\alpha_1}\right)\leq -\left(\int_0^1 \frac{1-2u}{f(F^{-1}(u))}(-\log(u))du\right)<0.$$

In order to study $\lim_{n\to\infty}\partial_\alpha E_C|X-X_\alpha|_{\alpha=n}$, let us note that, as $\partial_2 C(u,u^n)$ increases in u, for each n there exists $c_n\in[0,1)$ such that, for $u<c_n$, $\partial_2 C(u,u^n)\leq\frac{1}{2}$ and for $u>c_n$, $\partial_2 C(u,u^n)\geq\frac{1}{2}$. Note also that $\lim_{n\to\infty}c_n<1$ (such limit exists due to the smoothness of C); otherwise, if $\lim_{n\to\infty}c_n=1$, then $\varphi_1(u)\leq\frac{1}{2}$ for all $u\in(0,1)$, which contradicts the fact that $\lim_{u\to 1}\varphi_1(u)=1$. Taking this into consideration, we have that, for all $n\in\mathbb{N}$:

$$\begin{aligned}
\partial_\alpha E_C|X-X_\alpha|_{\alpha=n} &= \int_0^1\frac{1-2\,\partial_2C(u,u^n)}{f(F^{-1}(u))}u^n\log(u)du=\\
\int_0^{c_n}\frac{1-2\,\partial_2C(u,u^n)}{f(F^{-1}(u))}u^n\log(u)du &+ \int_{c_n}^1\frac{1-2\,\partial_2C(u,u^n)}{f(F^{-1}(u))}u^n\log(u)du>\\
c_n^n\int_0^{c_n}\frac{1-2\,\partial_2C(u,u^n)}{f(F^{-1}(u))}\log(u)du &+ c_n^n\int_{c_n}^1\frac{1-2\,\partial_2C(u,u^n)}{f(F^{-1}(u))}\log(u)du>\\
&>(c_n)^n\int_0^1\frac{1-2\partial_2C(u,u^n)}{f(F^{-1}(u))}\log(u)du
\end{aligned}\tag{7}$$

Taking limits in (7), $\lim_{n\to\infty}(c_n)^n=0$ and by Lemma 2,

$$\lim_{\alpha\to\infty}\int_0^1\frac{1-2\,\partial_2C(u,u^\alpha)}{f(F^{-1}(u))}\log(u)du=\int_0^1\frac{1-2\varphi_0(u)}{f(F^{-1}(u))}\log(u)du$$

Note that, as $u\leq\varphi_0(u)\leq 1$,

$$\int_0^1\frac{1-2u}{f(F^{-1}(u))}\log(u)du\leq\int_0^1\frac{1-2\varphi_0(u)}{f(F^{-1}(u))}\log(u)du\leq\int_0^1\frac{-\log(u)}{f(F^{-1}(u))}du$$

In Lemmas 2 and 3, we saw that both the bounds are finite, so we can conclude that $\lim_{n\to\infty}\left((c_n)^n\int_0^1\frac{1-2\partial_2C(u,u^n)}{f(F^{-1}(u))}(\log(u))du\right)=0$, and, therefore $\lim_{\alpha_2\to+\infty}\left(\partial_\alpha E_C|X-X_\alpha|_{\alpha=\alpha_2}\right)\geq 0$. This concludes the proof. □

4 Examples

Let us give some examples of PDS copulas that satisfy both that $\partial_2C(u,u^n)$ increases in u for all $n\in\mathbb{N}$ and that $\lim_{u\to1}\varphi_1(u)=1$:

1. Independence, $C(u,v)=uv$.
2. Farlie-Gumbel-Morgenstern copula, $C_\theta(u,v)=uv(1+\theta(1-u)(1-v))$, for $\theta\in[0,1]$.
3. Frank copula, $C_\theta(u,v)=-\frac{1}{\theta}\log\left(1+\frac{(e^{-u\theta}-1)(e^{-v\theta}-1)}{e^{-\theta}-1}\right)$, for $\theta>0$.
4. Copula 17 in Table 4.1 in Nelsen (2007), for $\theta>1$,

$$C_\theta(u,v) = \left(1 + \frac{\left[(1+u)^{-\theta} - 1\right]\left[(1+v)^{-\theta} - 1\right]}{2^{-\theta} - 1}\right)^{-1/\theta} - 1$$

Note that there are many PDS copulas that do not verify such conditions. For example, there are copulas that satisfy $\lim_{u\to 1} \varphi_1(u) = 1$ but do not verify that $\partial_2 C(u, u^n)$ increases in u. Examples of this are the following copulas in Table 4.1 in Nelsen (2007): 1 (Clayton), 3 (Ali-Mikhail-Haq), 13 and 19. Also, it can be checked that, for some PDS copulas, $\lim_{u\to 1} \varphi_1(u) = 0$ and therefore, do not satisfy conditions on Proposition 1. Examples of this are the Gaussian copula for $\rho > 0$ or the following copulas in Table 4.1 in Nelsen (2007): 2, 4 (Gumbel-Hougard), 6, 12 and 14.

Let us also recall the importance of the property IFR in X. If (X, Y) is independent and $X \sim Weibull(k, 1)$, the IFR assumption fails for $k < 1$. Figure 1 shows that for $k = 0.7$, $E_I|X - X_\alpha|$ increases in α and the minimizer does not exists.

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