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ON THE DIVISIBILITY OF THE RANK OF APPEARANCE OF A LUCAS SEQUENCE

CARLO SANNA[†]

ABSTRACT. Let $U = (U_n)_{n\geq 0}$ be a Lucas sequence and, for every prime number p, let $\rho_U(p)$ be the rank of appearance of p in U, that is, the smallest positive integer k such that p divides U_k , whenever it exists. Furthermore, let d be an odd positive integer. Under some mild hypotheses, we prove an asymptotic formula for the number of primes $p \leq x$ such that d divides $\rho_U(p)$, as $x \to +\infty$.

1. INTRODUCTION

Let $(U_n)_{n\geq 0}$ be a Lucas sequence, that is, a sequence of integers satisfying $U_0 = 0$, $U_1 = 1$, and $U_n = a_1U_{n-1} + a_2U_{n-2}$ for every integer $n \geq 2$, where a_1, a_2 are fixed nonzero integers. The rank of appearance of a prime number p, denoted by $\rho_U(p)$, is the smallest positive integer k such that $p \mid U_k$. It can be easily seen that $\rho_U(p)$ exists whenever $p \nmid a_2$. Define

$$\mathcal{R}_{U}(d;x) := \# \{ p \le x : p \nmid a_2, d \mid \rho_{U}(p) \},\$$

for every positive integer d and for every x > 1.

Let $(F_n)_{n\geq 0}$ be the Lucas sequence of Fibonacci numbers, corresponding to $a_1 = a_2 = 1$. In 1985, Lagarias [5] (see [6] for a correction and [8, 10] for generalizations) showed that $\mathcal{R}_F(2;x) \sim \frac{2}{3}x$, as $x \to +\infty$. More recently, Cubre and Rouse [2], settling a conjecture of Bruckman and Anderson [1], proved that $\mathcal{R}_F(d;x) \sim c(d) d^{-1} \prod_{p|d} (1-p^{-2})^{-1}$, as $x \to +\infty$, for every positive integer d, where c(d) is equal to $1, \frac{5}{4}$, or $\frac{1}{2}$, whenever $10 \nmid d, d \equiv 10 \pmod{20}$, or $20 \mid d$, respectively.

Let α, β be the roots of the characteristic polynomial $f_U(X) := X^2 - a_1 X - a_2$, and assume that $\gamma := \alpha/\beta$ is not a root of unity. Let $\Delta := a_1^2 + 4a_2$ be the discriminant of $f_U(X)$, and let Δ_0 be the squarefree part of Δ . Assume that Δ is not a square, so that $K := \mathbb{Q}(\sqrt{\Delta})$ is a quadratic number field. Let h be the greatest positive integer such that γ is a hth power in K. Our result is the following:

Theorem 1.1. Let d be an odd positive integer with $3 \nmid d$ whenever $\Delta_0 = -3$. Then, for every $x > \exp(Be^{8\omega(d)}d^8)$, we have

$$\mathcal{R}_U(d;x) = \delta_U(d)\operatorname{Li}(x) + O_U\left(\frac{(\omega(d)+1)d}{\varphi(d)} \cdot \frac{x\,(\log\log x)^{\omega(d)}}{(\log x)^{9/8}}\right),\,$$

where B > 0 is an absolute constant and

$$\delta_U(d) := \frac{1}{d} \left(\frac{1}{(d^{\infty}, h)} + \eta_U(d) \right) \prod_{p \mid d} \left(1 - \frac{1}{p^2} \right)^{-1},$$

with $\eta_U(d) := 0$ if $\Delta > 0$ or $\Delta_0 \not\equiv 1 \pmod{4}$ or $\Delta_0 \nmid d^{\infty}$; and

$$\eta_U(d) := \frac{(d^{\infty}, h)}{\left[(d^{\infty}, h), \Delta_0 / (d, \Delta_0) \right]^2}$$

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otherwise.

Cubre and Rouse's proof of the asymptotic formula for $\mathcal{R}_F(d; x)$ relies on the study of the algebraic group $G: x^2 - 5y^2 = 1$ and relates $\rho_F(p)$ with the order of $(3/2, 1/2) \in G(\mathbb{F}_p)$. Instead, our proof of Theorem 1.1 is an adaptation of the methods that Moree [9] used to prove an asymptotic formula for the number of primes $p \leq x$ such that the multiplicative order of g modulo p is divisible by d, where $g \notin \{-1, 0, +1\}$ is a fixed rational number.

2. Acknowledgements

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3. NOTATION

We employ the Landau–Bachmann "Big Oh" notation O, as well as the associated Vinogradov symbol \ll . Any dependence of the implied constants is explicitly stated or indicated with subscripts. In particular, notations like O_U and \ll_U are shortcuts for O_{a_1,a_2} and \ll_{a_1,a_2} , respectively. For $x \ge 2$ we let $\operatorname{Li}(x) := \int_2^x \frac{\mathrm{d}t}{\log t}$ denote the logarithmic integral. We reserve the letter p for prime numbers. Given an integer d, we let d^{∞} denote the supernatural number $\prod_{n \mid d} p^{\infty}$. Given a field F and a positive integer n, we write F^n for the set of nth powers of elements of F. Given a Galois extension E/F of number fields and a prime ideal P of \mathcal{O}_E lying above an unramified prime ideal \mathfrak{p} of \mathcal{O}_F , we write $\left\lceil \frac{E/F}{P} \right\rceil$ for the Frobenius automorphism corresponding to P/\mathfrak{p} , that is, the unique element σ of the Galois group $\operatorname{Gal}(E/F)$ that satisfies $\sigma(a) \equiv a^{N(\mathfrak{p})} \pmod{P}$ for every $a \in \mathcal{O}_E$, where $N(\mathfrak{p})$ denotes the norm of \mathfrak{p} . Moreover, we let $\left\lceil \frac{E/F}{\mathfrak{p}} \right\rceil$ be the set of all $\left\lceil \frac{E/F}{P} \right\rceil$ with P prime ideal of \mathcal{O}_E lying over \mathfrak{p} . We write $\Delta_{E/F}$ for the relative discriminant of E/F, and $\Delta_E := \Delta_{E/\mathbb{Q}}$ for the absolute discriminant of E. For every integer d and for every prime number p we let $\left(\frac{d}{p}\right)$ be the Legendre symbol. For every positive integer n, we let $\zeta_n := e^{2\pi i/n}$ be a primitive nth root of unity. We write $\omega(n), \varphi(n), \mu(n), \psi(n), \psi$ and $\tau(n)$, for the number of prime factors, the totient function, the Möbius function, and the number of divisors of a positive integer n, respectively.

4. General preliminaries

Lemma 4.1. Let n be a positive integer, let p be a prime number not dividing n, and let P be a prime ideal of $\mathcal{O}_{\mathbb{Q}(\zeta_n)}$ lying over p. Then ζ_n has multiplicative order modulo P equal to n.

Proof. Let k be the multiplicative order of ζ_n modulo P, that is, k is the least positive integer such that $\zeta_n^k \equiv 1 \pmod{P}$. On the one hand, we have that $p \mid N(P) \mid N(\zeta_n^k - 1)$. On the other hand, since $\zeta_n^n \equiv 1 \pmod{P}$, we have that $k \mid n$, and consequently ζ_n^k is a *m*th primitive root of unity, where m := n/k. If k < n then m > 1 and $N(\zeta_n^k - 1)$ is either 1 or a prime factor of m, but both cases are impossible since $p \nmid n$. Hence, k = n.

Lemma 4.2. Let F be a field, let $a \in F$, and let n be a positive integer. Then $X^n - a$ is irreducible over F if and only if $a \notin F^p$ for each prime p dividing n and $a \notin -4F^4$ whenever $4 \mid n$.

Proof. See [4, Chapter 8, Theorem 1.6].

Lemma 4.3. Let F be a field, let n be a positive integer not divisible by the characteristic of F, and let m be the number of nth roots of unity contained in F. Then, for every $a \in F$, the extension $F(\zeta_n, a^{1/n})/F$ is abelian if and only if $a^m \in F^n$.

Proof. See [4, Chapter 8, Theorem 3.2].

Lemma 4.4. Let n be an odd positive integer and let d be a squarefree integer. Then $\sqrt{d} \in \mathbb{Q}(\zeta_n)$ if and only if $d \mid n$ and $d \equiv 1 \pmod{4}$.

Proof. See [12, Lemma 3].

We need the following form of the Chebotarev Density Theorem.

Theorem 4.5. Let E/F be a Galois extension of numbers fields with Galois group G, and let C be the union of k conjugacy classes of G. Then

$$# \Big\{ \mathfrak{p} \text{ prime ideal of } \mathcal{O}_F \text{ non-ramifying in } E : N_{F/\mathbb{Q}}(\mathfrak{p}) \leq x, \ \Big[\frac{E/F}{\mathfrak{p}} \Big] \subseteq C \Big\} \\= \frac{\#C}{\#G} \cdot \operatorname{Li}(x) + O\Big(k \, x \exp\Big(-c_1 \big(\log x/n_E \big)^{1/2} \Big) \Big)$$

for every

$$x \ge \exp\left(c_2 \max\left(n_E (\log |\Delta_E|)^2, |\Delta_E|^{2/n_E}/n_E\right)\right),$$

where $n_E := [E : \mathbb{Q}]$ and $c_1, c_2 > 0$ are absolute constants.

Proof. The result follows from the effective form of the Chebotarev Density Theorem given by Lagarias and Odlyzko [7, Theorem 1.3] and from the bounds for the exceptional zero of the Dedekind zeta function ζ_E given by Stark [13, Lemma 8 and 11].

5. Preliminaries to the proof of Theorem 1.1

Recalling that h is the greatest positive integer such that γ is an hth power in K, write $\gamma = \gamma_0^h$ for some $\gamma_0 \in K$. Also, let $\sigma_K \in \text{Gal}(K/\mathbb{Q})$ be the nontrivial automorphism, which satisfies $\sigma_K(\sqrt{\Delta}) = -\sqrt{\Delta}$. Note that, since $\gamma = \alpha/\beta$ and σ_K swaps α and β , we have that $\sigma_k(\gamma) = \gamma^{-1}$. For all positive integers d, n such that $d \mid n$, let $K_{n,d} := K(\zeta_n, \gamma^{1/d})$.

Lemma 5.1. Let p be a prime number not dividing $a_2\Delta$ and let π be a prime ideal of \mathcal{O}_K lying over p. Then $\rho_U(p)$ is equal to the multiplicative order of γ modulo π . Moreover, $\rho_U(p)$ divides $p - \left(\frac{\Delta}{p}\right)$.

Proof. First, note that $p \nmid a_2$ ensures that β is invertible modulo π , and consequently it makes sense to consider the multiplicative order of $\gamma = \alpha/\beta$ modulo π . Also, $p \nmid \Delta$ implies that p does not ramifies in K and that $\alpha \not\equiv \beta \pmod{\pi}$.

We shall prove that $p \mid U_n$ if and only if $\gamma^n \equiv 1 \pmod{\pi}$, for every positive integer n. Then the claim on $\rho_U(p)$ follows easily. It is well known that the Binet's formula

(1)
$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$$

holds for every positive integer n. On the one hand, if $p \mid U_n$ then, since $p\mathcal{O}_K \subseteq \pi$ and (1), we have $\alpha^n \equiv \beta^n \pmod{\pi}$, and consequently $\gamma^n \equiv 1 \pmod{\pi}$. On the other hand, if $\gamma^n \equiv 1 \pmod{\pi}$ then by (1) we get $U_n \equiv 0 \pmod{\pi}$. If p is inert in K, then $p\mathcal{O}_K = \pi$ and so $p \mid U_n$. If p splits in K, then $p\mathcal{O}_K = \pi \cap \sigma_K(\pi)$. Thus $U_n \equiv 0 \pmod{\pi}$ and $U_n \equiv \sigma_K(U_k) \equiv 0 \pmod{\sigma_K(\pi)}$ imply that $p \mid U_n$.

Let $\sigma := \left[\frac{K/\mathbb{Q}}{\pi}\right]$. On the one hand, if $\left(\frac{\Delta}{p}\right) = -1$ then $\sigma = \sigma_K$ and $\gamma^{p+1} \equiv \sigma_K(\gamma)\gamma \equiv \gamma^{-1}\gamma \equiv 1 \pmod{\pi}$, so that $\rho_U(p) \mid p+1$. On the other hand, if $\left(\frac{\Delta}{p}\right) = +1$ then $\sigma = \text{id}$ and $\gamma^{p-1} \equiv \gamma\gamma^{-1} \equiv 1 \pmod{\pi}$, so that $\rho_U(p) \mid p-1$.

For each prime number p not dividing $a_2\Delta$, let us define the *index of appearance* of p as

$$\iota_U(p) := \left(p - \left(\frac{\Delta}{p}\right)\right) / \rho_U(p).$$

Note that, in light of Lemma 5.1, $\iota_U(p)$ is an integer.

Lemma 5.2. Let d, n be positive integers such that $d \mid n$, and let p be a prime number not dividing $a_2\Delta$. Moreover, let P be a prime ideal of $\mathcal{O}_{K_{n,d}}$ lying over p and let $\sigma := \left[\frac{K_{n,d}/\mathbb{Q}}{P}\right]$. Then

(2)
$$p \equiv \left(\frac{\Delta}{p}\right) \pmod{n} \quad and \quad d \mid \iota_U(p)$$

if and only if $\sigma = id$ or

(3)
$$\sigma(\zeta_n) = \zeta_n^{-1} \quad and \quad \sigma(\gamma^{1/d}) = \gamma^{-1/d}.$$

Proof. First, suppose that $\left(\frac{\Delta}{p}\right) = -1$. Let us assume (2). On the one hand, since $p \equiv -1 \pmod{n}$, we have

(4)
$$\sigma(\zeta_n) \equiv \zeta_n^p \equiv \zeta_n^{-1} \pmod{P}$$

Since $\sigma(\zeta_n) = \zeta_n^k$ for some integer k, and since p does not divide n, Lemma 4.1 and (4) yield that $\sigma(\zeta_n) = \zeta_n^{-1}$.

On the other hand, $d \mid \iota_U(p)$ implies that $\rho_U(p) \mid (p+1)/d$. Hence, letting $\pi := P \cap \mathcal{O}_K$, Lemma 5.1 yields $\gamma^{(p+1)/d} \equiv 1 \pmod{\pi}$. Consequently,

(5)
$$\sigma(\gamma^{1/d}) \equiv (\gamma^{1/d})^p \equiv \gamma^{(p+1)/d} \cdot \gamma^{-1/d} \equiv \gamma^{-1/d} \pmod{P}.$$

Note that, since $\left(\frac{\Delta}{p}\right) = -1$, we have

$$\sigma(\gamma) = \sigma|_K(\gamma) = \left[\frac{K/\mathbb{Q}}{\pi}\right](\gamma) = \sigma_K(\gamma) = \gamma^{-1},$$

so that $\sigma(\gamma^{1/d}) = \zeta_d^k \gamma^{-1/d}$ for some integer k. Thus Lemma 4.1 and (5) yield that $\sigma(\gamma^{1/d}) = \gamma^{-1/d}$. We have proved (3).

Now let us assume (3). On the one hand, we have

$$\zeta_n^{-1} = \sigma(\zeta_n) = \sigma|_{\mathbb{Q}(\zeta_n)}(\zeta_n) = \left[\frac{\mathbb{Q}(\zeta_n)/\mathbb{Q}}{P \cap \mathcal{O}_{\mathbb{Q}(\zeta_n)}}\right](\zeta_n) = \zeta_n^p$$

so that $p \equiv -1 \pmod{n}$. On the other hand,

$$\gamma^{(p+1)/d} \equiv \left(\gamma^{1/d}\right)^p \cdot \gamma^{1/d} \equiv \sigma\left(\gamma^{1/d}\right) \cdot \gamma^{1/d} \equiv \gamma^{-1/d} \cdot \gamma^{1/d} \equiv 1 \pmod{P},$$

so that $\gamma^{(p+1)/d} \equiv 1 \pmod{\pi}$, which, by Lemma 5.1, implies $d \mid \iota_U(p)$. We have proved (2).

If $\left(\frac{\Delta}{p}\right) = +1$ then the proof proceeds similarly to the case $\left(\frac{\Delta}{p}\right) = -1$, and yields that (2) is equivalent to $\sigma(\zeta_n) = \zeta_n$ and $\sigma(\gamma^{1/d}) = \gamma^{1/d}$, that is, $\sigma = \text{id}$.

Lemma 5.3. The roots of unity contained in K are: the sixth roots of unity, if $\Delta_0 = -3$; the forth roots of unity, if $\Delta_0 = -1$; or the second roots of unity, if $\Delta_0 \neq -1, -3$.

Proof. If $\zeta_n \in K$ for some positive integer n, then $\mathbb{Q}(\zeta_n) \subseteq K$, so that $\varphi(n) \leq 2$, and $n \in \{1, 2, 3, 4, 6\}$. Then the claim follows easily since $\zeta_3 = (-1 + \sqrt{-3})/2$, $\zeta_4 = \sqrt{-1}$, and $\zeta_6 = (1 + \sqrt{-3})/2$.

Lemma 5.4. Let n be an odd positive integer with $3 \nmid n$ whenever $\Delta_0 = -3$, and let d be a positive integer dividing n. Then $a \in K \cap K(\zeta_n)^d$ if and only if $a \in K^d$.

Proof. The "if" part if obvious. Let us prove the "only if" part. Note that, by the hypothesis on n and by Lemma 5.3, the only nth root of unity in K is 1. Suppose that $a \in K \cap K(\zeta_n)^d$. Hence, there exists $b \in K(\zeta_n)$ such that $a = b^d$. Putting $a_1 := a^{n/d}$, we get that $a_1 = b^n$. Therefore, $K(\zeta_n, a_1^{1/n}) = K(\zeta_n, b) = K(\zeta_n)$ is an abelian extension of K. Consequently, by Lemma 4.3, we have $a_1 \in K^n$, that is, $a_1 = b_1^n$ for some $b_1 \in K$. Thus $a^n = a_1^d = b_1^{dn}$, so that $a = \zeta b_1^d$, where ζ is a nth root of unity in K. We already noticed that $\zeta = 1$, hence $a \in K^d$. \Box

Lemma 5.5. Let n be an odd positive integer with $3 \nmid n$ whenever $\Delta_0 = -3$, and let d be a positive integer dividing n. Then

(6)
$$[K_{n,d}:\mathbb{Q}] = \frac{\varphi(n)d}{(d,h)} \cdot \begin{cases} 1 & \text{if } \sqrt{\Delta} \in \mathbb{Q}(\zeta_n), \\ 2 & \text{if } \sqrt{\Delta} \notin \mathbb{Q}(\zeta_n), \end{cases}$$

while

(7)
$$|\Delta_{K_{n,d}}|^{1/[K_{n,d}:\mathbb{Q}]} \ll_U n^3$$
 and $\log |\Delta_{K_{n,d}}| \ll_U n^2 \log(n+1).$

Moreover, there exists $\sigma \in \operatorname{Gal}(K_{n,d}/\mathbb{Q})$ satisfying (3) if and only if $\sqrt{\Delta} \notin \mathbb{Q}(\zeta_n)$ or $\Delta < 0$. In particular, if σ exists then it belongs to the center of $\operatorname{Gal}(K_{n,d}/\mathbb{Q})$.

Proof. Let $d_0 := d/(d, h)$, $h_0 := h/(d, h)$, and $f(X) = X^{d_0} - \gamma_0^{h_0}$. Suppose that $\gamma_0^{h_0} \in K(\zeta_n)^p$ for some prime number p dividing d_0 . Then, by Lemma 5.4, we have $\gamma_0^{h_0} \in K^p$. In turn, by the maximality of h, it follows that $p \mid h_0$, which is impossible, since $(d_0, h_0) = 1$. Hence, $\gamma_0^{h_0} \notin K(\zeta_n)^p$ for every prime number p dividing d_0 . Consequently, by Lemma 4.2, f is irreducible over $K(\zeta_n)$. Thus $K_{n,d} \cong K(\zeta_n)[X]/(f(X))$, so that $[K_{n,d} : K(\zeta_n)] = d_0$ and $(\gamma^{1/d})^{d_0} = \gamma_0^{h_0}$. It is easy to check that $[K(\zeta_n) : \mathbb{Q}] = \varphi(n)$ if $\sqrt{\Delta} \in \mathbb{Q}(\zeta_n)$, and $[K(\zeta_n) : \mathbb{Q}] = 2\varphi(n)$ otherwise. Hence, (6) follows.

Let s be a positive integer such that $s\gamma_0 \in \mathcal{O}_K$, and put $g(X) := s^{d_0} f(X/s) = X^{d_0} - s^{d_0} \gamma_0^{h_0}$. Since f is the minimal polynomial of $\gamma^{1/d}$ over $K(\zeta_n)$, we get that g is the minimal polynomial of $s\gamma^{1/d}$ over $K(\zeta_n)$. In particular, since $g \in \mathcal{O}_K[X]$, we have that $s\gamma^{1/d} \in \mathcal{O}_{K_{n,d}}$. Hence, from $K_{n,d} = K(\zeta_n)(s\gamma^{1/d})$ it follows that

$$\Delta_{K_{n,d}/K(\zeta_n)} \supseteq \operatorname{disc}(g) \mathcal{O}_{K(\zeta_n)} = \prod_{1 \le i < j \le d_0} \left(s \gamma^{1/d} \zeta_{d_0}^i - s \gamma^{1/d} \zeta_{d_0}^j \right)^2 \mathcal{O}_{K(\zeta_n)}$$
$$= \left(s \gamma^{1/d} \right)^{d_0(d_0-1)} d_0^{d_0} \mathcal{O}_{K(\zeta_n)} = \gamma_0^{h_0(d_0-1)} \left(s^{d_0-1} d_0 \right)^{d_0} \mathcal{O}_{K(\zeta_n)},$$

and

$$N_{K(\zeta_n)/\mathbb{Q}}\left(\Delta_{K_{n,d}/K(\zeta_n)}\right) = N_{K/\mathbb{Q}}\left(\gamma_0^{h_0}\right)^{(d_0-1)[K(\zeta_n):K]} \left(s^{d_0-1}d_0\right)^{d_0[K(\zeta_n):\mathbb{Q}]} \mid \left(N_{K/\mathbb{Q}}(\gamma)sn\right)^{\infty}$$

Also, a quick computation shows that $\Delta_{K(\zeta_n)} \mid (4\Delta n)^{\infty}$. Therefore, since

$$\Delta_{K_{n,d}} = \Delta_{K(\zeta_n)}^{[K_{n,d}:K(\zeta_n)]} N_{K(\zeta_n)/\mathbb{Q}} \big(\Delta_{K_{n,d}/K(\zeta_n)} \big),$$

we get that every prime factor of $\Delta_{K_{n,d}}$ divides An, where $A := 4\Delta N_{K/\mathbb{Q}}(\gamma)s$. By Hensel's estimate (see, e.g., [11, comments after Theorem 7.3]), we have that

$$|\Delta_L|^{1/n_L} \le n_L \prod_{p \mid \Delta_L} p,$$

for every Galois extension L/\mathbb{Q} of degree n_L . Consequently,

$$|\Delta_{K_{n,d}}|^{1/[K_{n,d}:\mathbb{Q}]} \le [K_{n,d}:\mathbb{Q}]An \ll_U \varphi(n)dn \le n^3,$$

and

$$\log |\Delta_{K_{n,d}}| \le [K_{n,d}:\mathbb{Q}] (\log(n^3) + O_U(1)) \ll_U \varphi(n) d\log(n+1) \ll n^2 \log(n+1),$$

so that (7) is proved.

Suppose that there exists $\sigma \in \operatorname{Gal}(K_{n,d}/\mathbb{Q})$ satisfying (3). We shall prove that $\sqrt{\Delta} \notin \mathbb{Q}(\zeta_n)$ or $\Delta < 0$. Assume that $\sqrt{\Delta} \in \mathbb{Q}(\zeta_n)$. On the one hand, $\sigma(\gamma) = \sigma(\gamma^{1/d})^d = \gamma^{-1}$, and consequently $\sigma(\sqrt{\Delta}) = -\sqrt{\Delta}$. On the other hand, since $\sqrt{\Delta} \in \mathbb{Q}(\zeta_n)$ and $\sigma(\zeta_n) = \zeta_n^{-1}$, we have that $\sigma(\sqrt{\Delta}) = \sqrt{\Delta}$. Therefore, $\sqrt{\Delta} = -\sqrt{\Delta}$ and so $\Delta < 0$. Now let us check that σ belongs to the center of $\operatorname{Gal}(K_{n,d}/\mathbb{Q})$. Note that $N_{K/\mathbb{Q}}(\gamma) = \gamma \sigma_K(\gamma) = \gamma \gamma^{-1} = 1$. Also, $N_{K/\mathbb{Q}}(\gamma_0^{h_0}) = N_{K/\mathbb{Q}}(\gamma_0^h) = N_{K/\mathbb{Q}}(\gamma) = 1$, since d is odd and so $h_0 \equiv h \pmod{2}$. Therefore, for every $\tau \in \operatorname{Gal}(K_{n,q}/\mathbb{Q})$, we have $\tau(\gamma_0^{h_0}) = \gamma_0^{h_0}$, if $\tau|_K = \operatorname{id}$, or $\tau(\gamma_0^{h_0}) = N_{K/\mathbb{Q}}(\gamma_0^h) = \gamma_0^{-h_0}$ if $\tau|_K = \sigma_K$. Consequently, recalling that $(\gamma^{1/d})^{d_0} = \gamma_0^{h_0}$, we have that $\tau(\zeta_n) = \zeta_n^s$ and $\tau(\gamma^{1/d}) = \zeta_{d_0}^t \gamma^{\pm 1/d}$ for some integers s, t. At this point, it can be easily checked that $(\sigma\tau)(\zeta_n) = (\tau\sigma)(\zeta_n)$ and $(\sigma\tau)(\gamma^{1/d}) = (\tau\sigma)(\gamma^{1/d})$. Hence, σ belongs to the center of $\operatorname{Gal}(K_{n,d}/\mathbb{Q})$.

Suppose that $\sqrt{\Delta} \notin \mathbb{Q}(\zeta_n)$ or $\Delta < 0$. We shall prove the existence of $\sigma \in \operatorname{Gal}(K_{n,d}/\mathbb{Q})$ satisfying (3). It suffices to show that there exists $\sigma_1 \in \operatorname{Gal}(K(\zeta_n)/K)$ such that $\sigma_1(\zeta_n) = \zeta_n^{-1}$

and $\sigma_1|_K = \sigma_K$. Indeed, recalling that $K_{n,d} \cong K(\zeta_n)[X]/(f(X))$, we can extend σ_1 to an automorphism $\sigma \in \operatorname{Gal}(K_{n,d}/\mathbb{Q})$ that sends the root $\gamma^{1/d}$ of f to the root $\gamma^{-1/d}$ of

$$(\sigma_1 f)(X) = X^{d_0} - \sigma_1(\gamma_0^{h_0}) = X^{d_0} - N_{K/\mathbb{Q}}(\gamma_0^{h_0})\gamma_0^{-h_0} = X^{d_0} - \gamma_0^{-h_0},$$

and so σ satisfies (3). Pick $\sigma_0 \in \operatorname{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q})$ such that $\sigma_0(\zeta_n) = \zeta_n^{-1}$. If $\sqrt{\Delta} \in \mathbb{Q}(\zeta_n)$ then $K(\zeta_n) = \mathbb{Q}(\zeta_n), \ \Delta < 0$, and $\sigma_0(\sqrt{\Delta}) = \sqrt{\Delta} = -\sqrt{\Delta}$, so we let $\sigma_1 := \sigma_0$. If $\sqrt{\Delta} \notin \mathbb{Q}(\zeta_n)$ then $X^2 - \Delta$ is the minimal polynomial of $\sqrt{\Delta}$ over $\mathbb{Q}(\zeta_n)$ and we can extend σ_0 to $\sigma_1 \in \operatorname{Gal}(K(\zeta_n)/\mathbb{Q})$ such that $\sigma_1(\sqrt{\Delta}) = -\sqrt{\Delta}$.

6. Proof of Theorem 1.1

The proof proceeds similarly to [9, Section 2]. For all positive integers d, n with $d \mid n$, and for all x > 1, let us define

$$\pi_{U,n,d}(x) := \# \left\{ p \le x : p \nmid a_2 \Delta, \, p \equiv \left(\frac{\Delta}{p}\right) \pmod{n}, \, d \mid \iota_U(p) \right\}.$$

In what follows, we will tacitly ignore the finitely many prime numbers dividing $a_2\Delta$.

Lemma 6.1. For every positive integer d and for every x > 1, we have

(8)
$$\mathcal{R}_U(d;x) = \sum_{v \mid d^{\infty}} \sum_{a \mid d} \mu(a) \pi_{U,dv,av}(x).$$

Proof. Every prime number p counted by the inner sum of (8) satisfies $p \leq x$, $p \equiv \left(\frac{\Delta}{p}\right)$ (mod dv), and $\iota_U(p) = vw$ for some integer w. Writing $w = w_1w_2$, with $w_1 := (w, d)$, we get that the contribution of p to the inner sum or (8) is equal to $\sum_{a|w_1} \mu(a)$. Hence,

(9)
$$\sum_{a \mid d} \mu(a) \pi_{U,dv,av}(x) = \# \left\{ p \le x : p \equiv \left(\frac{\Delta}{p}\right) \pmod{dv}, v \mid \iota_U(p), \left(\iota_U(p)/v, d\right) = 1 \right\}.$$

Now it suffices to show that

(10)
$$\mathcal{R}_U(d;x) = \sum_{v \mid d^{\infty}} \# \left\{ p \le x : p \equiv \left(\frac{\Delta}{p}\right) \pmod{dv}, v \mid \iota_U(p), \left(\iota_U(p)/v, d\right) = 1 \right\}.$$

On the one hand, let p be a prime number counted on the right-hand side of (10). Note that this is counted only one, namely for $v = (\iota_U(p), d^{\infty})$. Then, from $\rho_U(p)\iota_U(p) = p - (\frac{\Delta}{p})$, it follows that $d \mid \rho_U(p)$. Hence, p is counted on the left-hand side of (10).

On the other hand, let p be a prime number counted by $\mathcal{R}_U(d; x)$. Then $d \mid \rho_U(p)$ and, by Lemma 5.1, $p \equiv \left(\frac{\Delta}{p}\right) \pmod{d}$. Consequently, there is an integer v such that $v \mid d^{\infty}, p \equiv \left(\frac{\Delta}{p}\right) \pmod{dv}$, and $(\iota_U(p)/v, d) = 1$. Hence, p is counted on the right-hand side of (10).

Lemma 6.2. Let n be an odd positive integer with $3 \nmid n$ whenever $\Delta_0 = -3$, and let d be a positive integer dividing n. There exist absolute constants A, B > 0 such that

$$\pi_{U,n,d}(x) = \delta_{U,n,d} \operatorname{Li}(x) + O_U \left(x \exp\left(-A(\log x)^{1/2} / n \right) \right)$$

for $x \ge \exp(Bn^8)$, where

(11)
$$\delta_{U,n,d} := \frac{(d,h)}{\varphi(n)d} \cdot \begin{cases} 1 & \text{if } \Delta > 0 \text{ or } \Delta_0 \not\equiv 1 \pmod{4} \text{ or } \Delta_0 \nmid n, \\ 2 & \text{otherwise.} \end{cases}$$

Proof. Put $E := K_{n,d}$, $F := \mathbb{Q}$, $G := \operatorname{Gal}(E/F)$, and $C = \{\operatorname{id}, \sigma\}$ if there exists $\sigma \in \operatorname{Gal}(K_{n,d}/\mathbb{Q})$ satisfying (3), or $C = \{\operatorname{id}\}$ otherwise. By Lemma 5.5, σ belongs to the center of G, so that C is the union of conjugacy classes of G. By Lemma 5.2, we have that $\pi_{U,n,d}(x)$ is the number of primes p not exceeding x and such that $\left\lfloor \frac{E/F}{p} \right\rfloor \subseteq C$. Thus, taking into account the bounds for the degree and the discriminant of E/F given in Lemma 5.5, and considering Lemma 4.4, the asymptotic formula follows by applying Theorem 4.5.

Lemma 6.3. Let d be an odd positive integer with $3 \nmid d$ whenever $\Delta_0 = -3$. If x > 1 and $e^{\omega(d)} \leq y \leq \log x/\varphi(d)$, then

(12)
$$\sum_{\substack{v \mid d^{\infty} \\ v > y}} \sum_{a \mid d} \mu(a) \pi_{U,dv,av}(x) \ll \frac{x}{\log x} \cdot \frac{\omega(d) + 1}{\varphi(d)} \cdot \frac{(\log y)^{\omega(d)}}{y}$$

and

$$\sum_{\substack{v \mid d^{\infty} \\ v > y}} \sum_{a \mid d} \mu(a) \delta_{U, dv, av} \ll_U \frac{\omega(d) + 1}{\varphi(d)} \cdot \frac{(\log y)^{\omega(d)}}{y}.$$

Proof. Let $\pi(m,r;x) := \#\{p \le x : p \equiv r \pmod{m}\}$. From (9) it follows that

(13)
$$\left|\sum_{a\mid d} \mu(a)\pi_{U,dv,av}(x)\right| \le \pi_{U,dv,v}(x) \le \pi(x;dv,\pm 1).$$

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Moreover, letting $x \to +\infty$, Lemma 6.2 and the first inequality of (13) yield

(14)
$$\left|\sum_{a \mid d} \mu(a) \delta_{U,dv,av}\right| \leq \delta_{U,dv,v}.$$

Now we have $M_d(x) := \#\{v \le x : v \mid d^\infty\} \ll (\log x)^{\omega(d)}$, for every $x \ge 2$. Hence, by partial summation and since $y \ge e^{\omega(d)}$, we obtain that

1

(15)
$$\sum_{\substack{v \mid d^{\infty} \\ v > y}} \frac{1}{v} = \left. \frac{M_d(t)}{t} \right|_{t=y}^{+\infty} + \int_y^{+\infty} \frac{M_d(t)}{t^2} \, \mathrm{d}t \ll \int_y^{+\infty} \frac{(\log t)^{\omega(d)}}{t^2} \, \mathrm{d}t \le \frac{(\omega(d)+1)(\log y)^{\omega(d)}}{y}.$$

On the one hand, using the Brun–Titchmarsh inequality [3, Theorem 12.7]

$$\pi(m,r;x) \ll \frac{x}{\varphi(m)\log(x/m)}$$

holding for x > m, and (15) we get that

(16)
$$\sum_{\substack{v \mid d^{\infty} \\ v > y, \, dv \le x^{2/3}}} \pi(dv, \pm 1; x) \ll \frac{x}{\varphi(d) \log x} \sum_{\substack{v \mid d^{\infty} \\ v > y}} \frac{1}{v} \ll \frac{x}{\log x} \cdot \frac{\omega(d) + 1}{\varphi(d)} \cdot \frac{(\log y)^{\omega(d)}}{y}.$$

On the other hand, using the trivial bound $\pi(m, \pm 1; x) \ll x/m$, holding for $x \ge 1$, and (15) again, we find that

(17)
$$\sum_{\substack{v \mid d^{\infty} \\ dv > x^{2/3}}} \pi(dv, \pm 1; x) \ll \sum_{\substack{v \mid d^{\infty} \\ dv > x^{2/3}}} \frac{x}{dv} \le \sum_{\substack{w \mid d^{\infty} \\ w > x^{2/3}}} \frac{x}{w} \ll x^{1/3} (\omega(d) + 1) (\log x)^{\omega(d)}.$$

Putting together (16), (17), and (13), taking into account that $\omega(d) \leq \log y$ and $\varphi(d)y \leq \log x$, we obtain (12). Finally, from (14), (11), and (15), we get

$$\sum_{\substack{v \mid d^{\infty} \\ v > y}} \sum_{a \mid d} \mu(a) \delta_{U, dv, av} \leq \sum_{\substack{v \mid d^{\infty} \\ v > y}} \delta_{U, dv, v} \ll_{U} \frac{1}{\varphi(d)} \sum_{\substack{v \mid d^{\infty} \\ v > y}} \frac{1}{v^{2}} \ll \frac{\omega(d) + 1}{\varphi(d)} \cdot \frac{(\log y)^{\omega(d)}}{y},$$

as desired.

Lemma 6.4. Let d be an odd positive integer with $3 \nmid d$ whenever $\Delta_0 = -3$. Then

$$\sum_{v \mid d^{\infty}} \sum_{a \mid d} \mu(a) \delta_{U, dv, av} = \delta_{U}(d)$$

Proof. For every integer e dividing d^{∞} , define

$$S_{d,e,h} := \sum_{\substack{v \mid d^{\infty} \\ e \mid v}} \sum_{\substack{a \mid d}} \frac{\mu(a)(av,h)}{\varphi(dv)av}$$

The value of $S_{d,1,h}$ was computed in [9, Lemma 4] and a slight modification of the proof (precisely, replacing (h, d^{∞}) with $[e, (h, d^{\infty})]$ in the last equation) yields

$$S_{d,e,h} = \frac{(d^{\infty}, h)}{d[(d^{\infty}, h), e]^2} \prod_{p \mid d} \left(1 - \frac{1}{p^2}\right)^{-1}$$

At this point, by (11) and considering that $\Delta_0 \mid dv$ if and only if $e \mid v$, where $e := \Delta_0/(d, \Delta_0)$, we have

$$\sum_{\substack{v \mid d^{\infty} \ a \mid d}} \sum_{\substack{a \mid d}} \mu(a) \delta_{U,dv,av} = \begin{cases} S_{d,1,h} & \text{if } \Delta > 0 \text{ or } \Delta_0 \not\equiv 1 \pmod{4} \text{ or } \Delta_0 \nmid d^{\infty} \\ S_{d,1,h} + S_{d,e,h} & \text{otherwise} \end{cases} = \delta_U(d),$$
c claimed.

as claimed.

Proof of Theorem 1.1. Let A, B > 0 be the constants of Lemma 6.2. Assume that $x \ge 1$ $\exp\left(Be^{8\omega(d)}d^8\right)$ and put $y := (\log x/B)^{1/8}/d$. Note that $e^{\omega(d)} \le y \le \log x/\varphi(d)$ and $\log y \le \log\log x$, for every $x \gg_B 1$. By Lemma 6.1, Lemma 6.2, and Lemma 6.4, we obtain that

$$\mathcal{R}_{U}(d;x) = \sum_{\substack{v \mid d^{\infty} \\ v \leq y}} \sum_{a \mid d} \mu(a) \pi_{U,dv,av}(x) + O(E_{1})$$

=
$$\sum_{\substack{v \mid d^{\infty} \\ v \leq y}} \sum_{a \mid d} \mu(a) \delta_{U,dv,av} \operatorname{Li}(x) + O(E_{1}) + O_{U}(E_{2})$$

=
$$\delta_{U}(d) \operatorname{Li}(x) + O(E_{1}) + O_{U}(E_{2}) + O(E_{3}),$$

where, by Lemma 6.3, we have

$$E_1 := \sum_{\substack{v \mid d^{\infty} \\ v > y}} \sum_{a \mid d} \mu(a) \pi_{U, dv, av}(x) \ll \frac{x}{\log x} \cdot \frac{\omega(d) + 1}{\varphi(d)} \cdot \frac{(\log y)^{\omega(d)}}{y} \ll \frac{(\omega(d) + 1)d}{\varphi(d)} \cdot \frac{x (\log \log x)^{\omega(d)}}{(\log x)^{9/8}}$$

and

$$E_3 := \sum_{\substack{v \mid d^{\infty} \\ v > y}} \sum_{a \mid d} \mu(a) \delta_{U, dv, av} \operatorname{Li}(x) \ll_U \frac{\omega(d) + 1}{\varphi(d)} \cdot \frac{(\log y)^{\omega(d)}}{y} \cdot \operatorname{Li}(x) \ll \frac{(\omega(d) + 1)d}{\varphi(d)} \cdot \frac{x (\log \log x)^{\omega(d)}}{(\log x)^{9/8}}$$

while, also using the inequality $\tau(d)/d \leq d/\varphi(d)$, we have

$$E_{2} := \sum_{\substack{v \mid d^{\infty} \\ v \leq y}} \sum_{a \mid d} x \exp\left(-A(\log x)^{1/2}/(dv)\right) \ll x \exp\left(-AB^{1/8}(\log x)^{3/8}\right)\tau(d)y$$
$$\ll x \exp\left(-AB^{1/8}(\log x)^{3/8}\right)(\log x)^{1/8} \cdot \frac{\tau(d)}{d} \ll \frac{d}{\varphi(d)} \cdot \frac{x}{(\log x)^{9/8}}.$$

The result follows.

References

- 1. P. S. Bruckman and P. G. Anderson, Conjectures on the Z-densities of the Fibonacci sequence, Fibonacci Quart. **36** (1998), no. 3, 263–271.
- 2. P. Cubre and J. Rouse, Divisibility properties of the Fibonacci entry point, Proc. Amer. Math. Soc. 142 (2014), no. 11, 3771–3785.
- 3. J.-M. De Koninck and F. Luca, Analytic number theory, Graduate Studies in Mathematics, vol. 134, American Mathematical Society, Providence, RI, 2012, Exploring the anatomy of integers.

- 4. G. Karpilovsky, *Topics in field theory*, North-Holland Mathematics Studies, vol. 155, North-Holland Publishing Co., Amsterdam, 1989, Notas de Matemática [Mathematical Notes], 124.
- J. C. Lagarias, The set of primes dividing the Lucas numbers has density 2/3, Pacific J. Math. 118 (1985), no. 2, 449–461.
- J. C. Lagarias, Errata to: "The set of primes dividing the Lucas numbers has density 2/3" [Pacific J. Math. 118 (1985), no. 2, 449–461; MR0789184 (86i:11007)], Pacific J. Math. 162 (1994), no. 2, 393–396.
- J. C. Lagarias and A. M. Odlyzko, *Effective versions of the Chebotarev density theorem*, Algebraic number fields: L-functions and Galois properties (Proc. Sympos., Univ. Durham, Durham, 1975), pp. 409–464.
- 8. P. Moree, On the prime density of Lucas sequences, J. Théor. Nombres Bordeaux 8 (1996), no. 2, 449-459.
- 9. P. Moree, On primes p for which d divides $\operatorname{ord}_p(g)$, Funct. Approx. Comment. Math. **33** (2005), 85–95.
- 10. P. Moree and P. Stevenhagen, Prime divisors of Lucas sequences, Acta Arith. 82 (1997), no. 4, 403–410.
- M. R. Murty and V. K. Murty, Non-vanishing of L-functions and applications, Progress in Mathematics, vol. 157, Birkhäuser Verlag, Basel, 1997.
- 12. A. Schinzel, A refinement of a theorem of Gerst on power residues, Acta Arith. 17 (1970), 161–168.
- 13. H. M. Stark, Some effective cases of the Brauer-Siegel theorem, Invent. Math. 23 (1974), 135–152.

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