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# ON THE DIVISIBILITY OF THE RANK OF APPEARANCE OF A LUCAS SEQUENCE 

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#### Abstract

Let $U=\left(U_{n}\right)_{n \geq 0}$ be a Lucas sequence and, for every prime number $p$, let $\rho_{U}(p)$ be the rank of appearance of $p$ in $U$, that is, the smallest positive integer $k$ such that $p$ divides $U_{k}$, whenever it exists. Furthermore, let $d$ be an odd positive integer. Under some mild hypotheses, we prove an asymptotic formula for the number of primes $p \leq x$ such that $d$ divides $\rho_{U}(p)$, as $x \rightarrow+\infty$.


## 1. Introduction

Let $\left(U_{n}\right)_{n \geq 0}$ be a Lucas sequence, that is, a sequence of integers satisfying $U_{0}=0, U_{1}=1$, and $U_{n}=a_{1} U_{n-1}+a_{2} U_{n-2}$ for every integer $n \geq 2$, where $a_{1}, a_{2}$ are fixed nonzero integers. The rank of appearance of a prime number $p$, denoted by $\rho_{U}(p)$, is the smallest positive integer $k$ such that $p \mid U_{k}$. It can be easily seen that $\rho_{U}(p)$ exists whenever $p \nmid a_{2}$. Define

$$
\mathcal{R}_{U}(d ; x):=\#\left\{p \leq x: p \nmid a_{2}, d \mid \rho_{U}(p)\right\},
$$

for every positive integer $d$ and for every $x>1$.
Let $\left(F_{n}\right)_{n \geq 0}$ be the Lucas sequence of Fibonacci numbers, corresponding to $a_{1}=a_{2}=1$. In 1985, Lagarias [5] (see [6] for a correction and [8, 10] for generalizations) showed that $\mathcal{R}_{F}(2 ; x) \sim \frac{2}{3} x$, as $x \rightarrow+\infty$. More recently, Cubre and Rouse [2], settling a conjecture of Bruckman and Anderson [1], proved that $\mathcal{R}_{F}(d ; x) \sim \mathrm{c}(d) d^{-1} \prod_{p \mid d}\left(1-p^{-2}\right)^{-1}$, as $x \rightarrow+\infty$, for every positive integer $d$, where $\mathrm{c}(d)$ is equal to $1, \frac{5}{4}$, or $\frac{1}{2}$, whenever $10 \nmid d, d \equiv 10(\bmod 20)$, or $20 \mid d$, respectively.

Let $\alpha, \beta$ be the roots of the characteristic polynomial $f_{U}(X):=X^{2}-a_{1} X-a_{2}$, and assume that $\gamma:=\alpha / \beta$ is not a root of unity. Let $\Delta:=a_{1}^{2}+4 a_{2}$ be the discriminant of $f_{U}(X)$, and let $\Delta_{0}$ be the squarefree part of $\Delta$. Assume that $\Delta$ is not a square, so that $K:=\mathbb{Q}(\sqrt{\Delta})$ is a quadratic number field. Let $h$ be the greatest positive integer such that $\gamma$ is a $h$ th power in $K$.

Our result is the following:
Theorem 1.1. Let $d$ be an odd positive integer with $3 \nmid d$ whenever $\Delta_{0}=-3$. Then, for every $x>\exp \left(B e^{8 \omega(d)} d^{8}\right)$, we have

$$
\mathcal{R}_{U}(d ; x)=\delta_{U}(d) \operatorname{Li}(x)+O_{U}\left(\frac{(\omega(d)+1) d}{\varphi(d)} \cdot \frac{x(\log \log x)^{\omega(d)}}{(\log x)^{9 / 8}}\right),
$$

where $B>0$ is an absolute constant and

$$
\delta_{U}(d):=\frac{1}{d}\left(\frac{1}{\left(d^{\infty}, h\right)}+\eta_{U}(d)\right) \prod_{p \mid d}\left(1-\frac{1}{p^{2}}\right)^{-1},
$$

with $\eta_{U}(d):=0$ if $\Delta>0$ or $\Delta_{0} \not \equiv 1(\bmod 4)$ or $\Delta_{0} \nmid d^{\infty}$; and

$$
\eta_{U}(d):=\frac{\left(d^{\infty}, h\right)}{\left[\left(d^{\infty}, h\right), \Delta_{0} /\left(d, \Delta_{0}\right)\right]^{2}}
$$

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otherwise.
Cubre and Rouse's proof of the asymptotic formula for $\mathcal{R}_{F}(d ; x)$ relies on the study of the algebraic group $G: x^{2}-5 y^{2}=1$ and relates $\rho_{F}(p)$ with the order of $(3 / 2,1 / 2) \in G\left(\mathbb{F}_{p}\right)$. Instead, our proof of Theorem 1.1 is an adaptation of the methods that Moree [9] used to prove an asymptotic formula for the number of primes $p \leq x$ such that the multiplicative order of $g$ modulo $p$ is divisible by $d$, where $g \notin\{-1,0,+1\}$ is a fixed rational number.

## 2. Acknowledgements

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## 3. Notation

We employ the Landau-Bachmann "Big Oh" notation $O$, as well as the associated Vinogradov symbol $\ll$. Any dependence of the implied constants is explicitly stated or indicated with subscripts. In particular, notations like $O_{U}$ and $<_{U}$ are shortcuts for $O_{a_{1}, a_{2}}$ and $<_{a_{1}, a_{2}}$, respectively. For $x \geq 2$ we let $\operatorname{Li}(x):=\int_{2}^{x} \frac{\mathrm{~d} t}{\log t}$ denote the logarithmic integral. We reserve the letter $p$ for prime numbers. Given an integer $d$, we let $d^{\infty}$ denote the supernatural number $\prod_{p \mid d} p^{\infty}$. Given a field $F$ and a positive integer $n$, we write $F^{n}$ for the set of $n$th powers of elements of $F$. Given a Galois extension $E / F$ of number fields and a prime ideal $P$ of $\mathcal{O}_{E}$ lying above an unramified prime ideal $\mathfrak{p}$ of $\mathcal{O}_{F}$, we write $\left[\frac{E / F}{P}\right]$ for the Frobenius automorphism corresponding to $P / \mathfrak{p}$, that is, the unique element $\sigma$ of the Galois group $\operatorname{Gal}(E / F)$ that satisfies $\sigma(a) \equiv a^{N(\mathfrak{p})}(\bmod P)$ for every $a \in \mathcal{O}_{E}$, where $N(\mathfrak{p})$ denotes the norm of $\mathfrak{p}$. Moreover, we let $\left[\frac{E / F}{\mathfrak{p}}\right]$ be the set of all $\left[\frac{E / F}{P}\right]$ with $P$ prime ideal of $\mathcal{O}_{E}$ lying over $\mathfrak{p}$. We write $\Delta_{E / F}$ for the relative discriminant of $E / F$, and $\Delta_{E}:=\Delta_{E / \mathbb{Q}}$ for the absolute discriminant of $E$. For every integer $d$ and for every prime number $p$ we let $\left(\frac{d}{p}\right)$ be the Legendre symbol. For every positive integer $n$, we let $\zeta_{n}:=\mathrm{e}^{2 \pi \mathrm{i} / n}$ be a primitive $n$th root of unity. We write $\omega(n), \varphi(n), \mu(n)$, and $\tau(n)$, for the number of prime factors, the totient function, the Möbius function, and the number of divisors of a positive integer $n$, respectively.

## 4. General preliminaries

Lemma 4.1. Let $n$ be a positive integer, let $p$ be a prime number not dividing $n$, and let $P$ be a prime ideal of $\mathcal{O}_{\mathbb{Q}\left(\zeta_{n}\right)}$ lying over $p$. Then $\zeta_{n}$ has multiplicative order modulo $P$ equal to $n$.
Proof. Let $k$ be the multiplicative order of $\zeta_{n}$ modulo $P$, that is, $k$ is the least positive integer such that $\zeta_{n}^{k} \equiv 1(\bmod P)$. On the one hand, we have that $p|N(P)| N\left(\zeta_{n}^{k}-1\right)$. On the other hand, since $\zeta_{n}^{n} \equiv 1(\bmod P)$, we have that $k \mid n$, and consequently $\zeta_{n}^{k}$ is a $m$ th primitive root of unity, where $m:=n / k$. If $k<n$ then $m>1$ and $N\left(\zeta_{n}^{k}-1\right)$ is either 1 or a prime factor of $m$, but both cases are impossible since $p \nmid n$. Hence, $k=n$.
Lemma 4.2. Let $F$ be a field, let $a \in F$, and let $n$ be a positive integer. Then $X^{n}-a$ is irreducible over $F$ if and only if $a \notin F^{p}$ for each prime $p$ dividing $n$ and $a \notin-4 F^{4}$ whenever $4 \mid n$.

Proof. See [4, Chapter 8, Theorem 1.6].
Lemma 4.3. Let $F$ be a field, let $n$ be a positive integer not divisible by the characteristic of $F$, and let $m$ be the number of $n$th roots of unity contained in $F$. Then, for every $a \in F$, the extension $F\left(\zeta_{n}, a^{1 / n}\right) / F$ is abelian if and only if $a^{m} \in F^{n}$.
Proof. See [4, Chapter 8, Theorem 3.2].
Lemma 4.4. Let $n$ be an odd positive integer and let $d$ be a squarefree integer. Then $\sqrt{d} \in$ $\mathbb{Q}\left(\zeta_{n}\right)$ if and only if $d \mid n$ and $d \equiv 1(\bmod 4)$.
Proof. See [12, Lemma 3].

We need the following form of the Chebotarev Density Theorem.
Theorem 4.5. Let $E / F$ be a Galois extension of numbers fields with Galois group $G$, and let $C$ be the union of $k$ conjugacy classes of $G$. Then

$$
\begin{gathered}
\#\left\{\mathfrak{p} \text { prime ideal of } \mathcal{O}_{F} \text { non-ramifying in } E: N_{F / \mathbb{Q}}(\mathfrak{p}) \leq x,\left[\frac{E / F}{\mathfrak{p}}\right] \subseteq C\right\} \\
=\frac{\# C}{\# G} \cdot \operatorname{Li}(x)+O\left(k x \exp \left(-c_{1}\left(\log x / n_{E}\right)^{1 / 2}\right)\right)
\end{gathered}
$$

for every

$$
x \geq \exp \left(c_{2} \max \left(n_{E}\left(\log \left|\Delta_{E}\right|\right)^{2},\left|\Delta_{E}\right|^{2 / n_{E}} / n_{E}\right)\right)
$$

where $n_{E}:=[E: \mathbb{Q}]$ and $c_{1}, c_{2}>0$ are absolute constants.
Proof. The result follows from the effective form of the Chebotarev Density Theorem given by Lagarias and Odlyzko [7, Theorem 1.3] and from the bounds for the exceptional zero of the Dedekind zeta function $\zeta_{E}$ given by Stark [13, Lemma 8 and 11].

## 5. Preliminaries to the proof of Theorem 1.1

Recalling that $h$ is the greatest positive integer such that $\gamma$ is an $h$ th power in $K$, write $\gamma=\gamma_{0}^{h}$ for some $\gamma_{0} \in K$. Also, let $\sigma_{K} \in \operatorname{Gal}(K / \mathbb{Q})$ be the nontrivial automorphism, which satisfies $\sigma_{K}(\sqrt{\Delta})=-\sqrt{\Delta}$. Note that, since $\gamma=\alpha / \beta$ and $\sigma_{K}$ swaps $\alpha$ and $\beta$, we have that $\sigma_{k}(\gamma)=\gamma^{-1}$. For all positive integers $d, n$ such that $d \mid n$, let $K_{n, d}:=K\left(\zeta_{n}, \gamma^{1 / d}\right)$.

Lemma 5.1. Let $p$ be a prime number not dividing $a_{2} \Delta$ and let $\pi$ be a prime ideal of $\mathcal{O}_{K}$ lying over $p$. Then $\rho_{U}(p)$ is equal to the multiplicative order of $\gamma$ modulo $\pi$. Moreover, $\rho_{U}(p)$ divides $p-\left(\frac{\Delta}{p}\right)$.
Proof. First, note that $p \nmid a_{2}$ ensures that $\beta$ is invertible modulo $\pi$, and consequently it makes sense to consider the multiplicative order of $\gamma=\alpha / \beta$ modulo $\pi$. Also, $p \nmid \Delta$ implies that $p$ does not ramifies in $K$ and that $\alpha \not \equiv \beta(\bmod \pi)$.

We shall prove that $p \mid U_{n}$ if and only if $\gamma^{n} \equiv 1(\bmod \pi)$, for every positive integer $n$. Then the claim on $\rho_{U}(p)$ follows easily. It is well known that the Binet's formula

$$
\begin{equation*}
U_{n}=\frac{\alpha^{n}-\beta^{n}}{\alpha-\beta} \tag{1}
\end{equation*}
$$

holds for every positive integer $n$. On the one hand, if $p \mid U_{n}$ then, since $p \mathcal{O}_{K} \subseteq \pi$ and (1), we have $\alpha^{n} \equiv \beta^{n}(\bmod \pi)$, and consequently $\gamma^{n} \equiv 1(\bmod \pi)$. On the other hand, if $\gamma^{n} \equiv 1$ $(\bmod \pi)$ then by (1) we get $U_{n} \equiv 0(\bmod \pi)$. If $p$ is inert in $K$, then $p \mathcal{O}_{K}=\pi$ and so $p \mid U_{n}$. If $p$ splits in $K$, then $p \mathcal{O}_{K}=\pi \cap \sigma_{K}(\pi)$. Thus $U_{n} \equiv 0(\bmod \pi)$ and $U_{n} \equiv \sigma_{K}\left(U_{k}\right) \equiv 0$ $\left(\bmod \sigma_{K}(\pi)\right)$ imply that $p \mid U_{n}$.

Let $\sigma:=\left[\frac{K / \mathbb{Q}}{\pi}\right]$. On the one hand, if $\left(\frac{\Delta}{p}\right)=-1$ then $\sigma=\sigma_{K}$ and $\gamma^{p+1} \equiv \sigma_{K}(\gamma) \gamma \equiv$ $\gamma^{-1} \gamma \equiv 1(\bmod \pi)$, so that $\rho_{U}(p) \mid p+1$. On the other hand, if $\left(\frac{\Delta}{p}\right)=+1$ then $\sigma=\mathrm{id}$ and $\gamma^{p-1} \equiv \gamma \gamma^{-1} \equiv 1(\bmod \pi)$, so that $\rho_{U}(p) \mid p-1$.

For each prime number $p$ not dividing $a_{2} \Delta$, let us define the index of appearance of $p$ as

$$
\iota_{U}(p):=\left(p-\left(\frac{\Delta}{p}\right)\right) / \rho_{U}(p) .
$$

Note that, in light of Lemma 5.1, $\iota_{U}(p)$ is an integer.
Lemma 5.2. Let $d, n$ be positive integers such that $d \mid n$, and let $p$ be a prime number not dividing $a_{2} \Delta$. Moreover, let $P$ be a prime ideal of $\mathcal{O}_{K_{n, d}}$ lying over $p$ and let $\sigma:=\left[\frac{K_{n, d} / \mathbb{Q}}{P}\right]$. Then

$$
\begin{equation*}
p \equiv\left(\frac{\Delta}{p}\right)(\bmod n) \quad \text { and } \quad d \mid \iota_{U}(p) \tag{2}
\end{equation*}
$$

if and only if $\sigma=\mathrm{id}$ or

$$
\begin{equation*}
\sigma\left(\zeta_{n}\right)=\zeta_{n}^{-1} \quad \text { and } \quad \sigma\left(\gamma^{1 / d}\right)=\gamma^{-1 / d} \tag{3}
\end{equation*}
$$

Proof. First, suppose that $\left(\frac{\Delta}{p}\right)=-1$. Let us assume (2). On the one hand, since $p \equiv-1$ $(\bmod n)$, we have

$$
\begin{equation*}
\sigma\left(\zeta_{n}\right) \equiv \zeta_{n}^{p} \equiv \zeta_{n}^{-1} \quad(\bmod P) \tag{4}
\end{equation*}
$$

Since $\sigma\left(\zeta_{n}\right)=\zeta_{n}^{k}$ for some integer $k$, and since $p$ does not divide $n$, Lemma 4.1 and (4) yield that $\sigma\left(\zeta_{n}\right)=\zeta_{n}^{-1}$.

On the other hand, $d \mid \iota_{U}(p)$ implies that $\rho_{U}(p) \mid(p+1) / d$. Hence, letting $\pi:=P \cap \mathcal{O}_{K}$, Lemma 5.1 yields $\gamma^{(p+1) / d} \equiv 1(\bmod \pi)$. Consequently,

$$
\begin{equation*}
\sigma\left(\gamma^{1 / d}\right) \equiv\left(\gamma^{1 / d}\right)^{p} \equiv \gamma^{(p+1) / d} \cdot \gamma^{-1 / d} \equiv \gamma^{-1 / d} \quad(\bmod P) . \tag{5}
\end{equation*}
$$

Note that, since $\left(\frac{\Delta}{p}\right)=-1$, we have

$$
\sigma(\gamma)=\left.\sigma\right|_{K}(\gamma)=\left[\frac{K / \mathbb{Q}}{\pi}\right](\gamma)=\sigma_{K}(\gamma)=\gamma^{-1}
$$

so that $\sigma\left(\gamma^{1 / d}\right)=\zeta_{d}^{k} \gamma^{-1 / d}$ for some integer $k$. Thus Lemma 4.1 and (5) yield that $\sigma\left(\gamma^{1 / d}\right)=$ $\gamma^{-1 / d}$. We have proved (3).

Now let us assume (3). On the one hand, we have

$$
\zeta_{n}^{-1}=\sigma\left(\zeta_{n}\right)=\left.\sigma\right|_{\mathbb{Q}\left(\zeta_{n}\right)}\left(\zeta_{n}\right)=\left[\frac{\mathbb{Q}\left(\zeta_{n}\right) / \mathbb{Q}}{P \cap \mathcal{O}_{\mathbb{Q}\left(\zeta_{n}\right)}}\right]\left(\zeta_{n}\right)=\zeta_{n}^{p},
$$

so that $p \equiv-1(\bmod n)$. On the other hand,

$$
\gamma^{(p+1) / d} \equiv\left(\gamma^{1 / d}\right)^{p} \cdot \gamma^{1 / d} \equiv \sigma\left(\gamma^{1 / d}\right) \cdot \gamma^{1 / d} \equiv \gamma^{-1 / d} \cdot \gamma^{1 / d} \equiv 1 \quad(\bmod P),
$$

so that $\gamma^{(p+1) / d} \equiv 1(\bmod \pi)$, which, by Lemma 5.1, implies $d \mid \iota_{U}(p)$. We have proved (2).
If $\left(\frac{\Delta}{p}\right)=+1$ then the proof proceeds similarly to the case $\left(\frac{\Delta}{p}\right)=-1$, and yields that (2) is equivalent to $\sigma\left(\zeta_{n}\right)=\zeta_{n}$ and $\sigma\left(\gamma^{1 / d}\right)=\gamma^{1 / d}$, that is, $\sigma=\mathrm{id}$.
Lemma 5.3. The roots of unity contained in $K$ are: the sixth roots of unity, if $\Delta_{0}=-3$; the forth roots of unity, if $\Delta_{0}=-1$; or the second roots of unity, if $\Delta_{0} \neq-1,-3$.
Proof. If $\zeta_{n} \in K$ for some positive integer $n$, then $\mathbb{Q}\left(\zeta_{n}\right) \subseteq K$, so that $\varphi(n) \leq 2$, and $n \in$ $\{1,2,3,4,6\}$. Then the claim follows easily since $\zeta_{3}=(-1+\sqrt{-3}) / 2, \zeta_{4}=\sqrt{-1}$, and $\zeta_{6}=$ $(1+\sqrt{-3}) / 2$.
Lemma 5.4. Let $n$ be an odd positive integer with $3 \nmid n$ whenever $\Delta_{0}=-3$, and let d be a positive integer dividing $n$. Then $a \in K \cap K\left(\zeta_{n}\right)^{d}$ if and only if $a \in K^{d}$.
Proof. The "if" part if obvious. Let us prove the "only if" part. Note that, by the hypothesis on $n$ and by Lemma 5.3, the only $n$th root of unity in $K$ is 1 . Suppose that $a \in K \cap K\left(\zeta_{n}\right)^{d}$. Hence, there exists $b \in K\left(\zeta_{n}\right)$ such that $a=b^{d}$. Putting $a_{1}:=a^{n / d}$, we get that $a_{1}=b^{n}$. Therefore, $K\left(\zeta_{n}, a_{1}^{1 / n}\right)=K\left(\zeta_{n}, b\right)=K\left(\zeta_{n}\right)$ is an abelian extension of $K$. Consequently, by Lemma 4.3, we have $a_{1} \in K^{n}$, that is, $a_{1}=b_{1}^{n}$ for some $b_{1} \in K$. Thus $a^{n}=a_{1}^{d}=b_{1}^{d n}$, so that $a=\zeta b_{1}^{d}$, where $\zeta$ is a $n$th root of unity in $K$. We already noticed that $\zeta=1$, hence $a \in K^{d}$.

Lemma 5.5. Let $n$ be an odd positive integer with $3 \nmid n$ whenever $\Delta_{0}=-3$, and let $d$ be a positive integer dividing $n$. Then

$$
\left[K_{n, d}: \mathbb{Q}\right]=\frac{\varphi(n) d}{(d, h)} \cdot \begin{cases}1 & \text { if } \sqrt{\Delta} \in \mathbb{Q}\left(\zeta_{n}\right),  \tag{6}\\ 2 & \text { if } \sqrt{\Delta} \notin \mathbb{Q}\left(\zeta_{n}\right),\end{cases}
$$

while

$$
\begin{equation*}
\left|\Delta_{K_{n, d}}\right|^{1 /\left[K_{n, d}: \mathbb{Q}\right]}<_{U} n^{3} \quad \text { and } \quad \log \left|\Delta_{K_{n, d}}\right|<_{U} n^{2} \log (n+1) . \tag{7}
\end{equation*}
$$

Moreover, there exists $\sigma \in \operatorname{Gal}\left(K_{n, d} / \mathbb{Q}\right)$ satisfying (3) if and only if $\sqrt{\Delta} \notin \mathbb{Q}\left(\zeta_{n}\right)$ or $\Delta<0$. In particular, if $\sigma$ exists then it belongs to the center of $\operatorname{Gal}\left(K_{n, d} / \mathbb{Q}\right)$.

Proof. Let $d_{0}:=d /(d, h), h_{0}:=h /(d, h)$, and $f(X)=X^{d_{0}}-\gamma_{0}^{h_{0}}$. Suppose that $\gamma_{0}^{h_{0}} \in K\left(\zeta_{n}\right)^{p}$ for some prime number $p$ dividing $d_{0}$. Then, by Lemma 5.4, we have $\gamma_{0}^{h_{0}} \in K^{p}$. In turn, by the maximality of $h$, it follows that $p \mid h_{0}$, which is impossible, since $\left(d_{0}, h_{0}\right)=1$. Hence, $\gamma_{0}^{h_{0}} \notin$ $K\left(\zeta_{n}\right)^{p}$ for every prime number $p$ dividing $d_{0}$. Consequently, by Lemma 4.2, $f$ is irreducible over $K\left(\zeta_{n}\right)$. Thus $K_{n, d} \cong K\left(\zeta_{n}\right)[X] /(f(X))$, so that $\left[K_{n, d}: K\left(\zeta_{n}\right)\right]=d_{0}$ and $\left(\gamma^{1 / d}\right)^{d_{0}}=\gamma_{0}^{h_{0}}$. It is easy to check that $\left[K\left(\zeta_{n}\right): \mathbb{Q}\right]=\varphi(n)$ if $\sqrt{\Delta} \in \mathbb{Q}\left(\zeta_{n}\right)$, and $\left[K\left(\zeta_{n}\right): \mathbb{Q}\right]=2 \varphi(n)$ otherwise. Hence, (6) follows.

Let $s$ be a positive integer such that $s \gamma_{0} \in \mathcal{O}_{K}$, and put $g(X):=s^{d_{0}} f(X / s)=X^{d_{0}}-s^{d_{0}} \gamma_{0}^{h_{0}}$. Since $f$ is the minimal polynomial of $\gamma^{1 / d}$ over $K\left(\zeta_{n}\right)$, we get that $g$ is the minimal polynomial of $s \gamma^{1 / d}$ over $K\left(\zeta_{n}\right)$. In particular, since $g \in \mathcal{O}_{K}[X]$, we have that $s \gamma^{1 / d} \in \mathcal{O}_{K_{n, d}}$, Hence, from $K_{n, d}=K\left(\zeta_{n}\right)\left(s \gamma^{1 / d}\right)$ it follows that

$$
\begin{aligned}
\Delta_{K_{n, d} / K\left(\zeta_{n}\right)} & \supseteq \operatorname{disc}(g) \mathcal{O}_{K\left(\zeta_{n}\right)}=\prod_{1 \leq i<j \leq d_{0}}\left(s \gamma^{1 / d} \zeta_{d_{0}}^{i}-s \gamma^{1 / d} \zeta_{d_{0}}^{j}\right)^{2} \mathcal{O}_{K\left(\zeta_{n}\right)} \\
& =\left(s \gamma^{1 / d}\right)^{d_{0}\left(d_{0}-1\right)} d_{0}^{d_{0}} \mathcal{O}_{K\left(\zeta_{n}\right)}=\gamma_{0}^{h_{0}\left(d_{0}-1\right)}\left(s^{d_{0}-1} d_{0}\right)^{d_{0}} \mathcal{O}_{K\left(\zeta_{n}\right)},
\end{aligned}
$$

and

$$
N_{K\left(\zeta_{n}\right) / \mathbb{Q}}\left(\Delta_{K_{n, d} / K\left(\zeta_{n}\right)}\right)=N_{K / \mathbb{Q}}\left(\gamma_{0}^{h_{0}}\right)^{\left(d_{0}-1\right)\left[K\left(\zeta_{n}\right): K\right]}\left(s^{d_{0}-1} d_{0}\right)^{d_{0}\left[K\left(\zeta_{n}\right): \mathbb{Q}\right]} \mid\left(N_{K / \mathbb{Q}}(\gamma) s n\right)^{\infty} .
$$

Also, a quick computation shows that $\Delta_{K\left(\zeta_{n}\right)} \mid(4 \Delta n)^{\infty}$. Therefore, since

$$
\Delta_{K_{n, d}}=\Delta_{K\left(\zeta_{n}\right)}^{\left[K_{n, d}: K\left(\zeta_{n}\right)\right]} N_{K\left(\zeta_{n}\right) / \mathbb{Q}}\left(\Delta_{K_{n, d} / K\left(\zeta_{n}\right)}\right),
$$

we get that every prime factor of $\Delta_{K_{n, d}}$ divides $A n$, where $A:=4 \Delta N_{K / \mathbb{Q}}(\gamma) s$. By Hensel's estimate (see, e.g., [11, comments after Theorem 7.3]), we have that

$$
\left|\Delta_{L}\right|^{1 / n_{L}} \leq n_{L} \prod_{p \mid \Delta_{L}} p,
$$

for every Galois extension $L / \mathbb{Q}$ of degree $n_{L}$. Consequently,

$$
\left|\Delta_{K_{n, d}}\right|^{1 /\left[K_{n, d}: \mathbb{Q}\right]} \leq\left[K_{n, d}: \mathbb{Q}\right] A n \lll U \varphi(n) d n \leq n^{3},
$$

and

$$
\log \left|\Delta_{K_{n, d}}\right| \leq\left[K_{n, d}: \mathbb{Q}\right]\left(\log \left(n^{3}\right)+O_{U}(1)\right) \ll_{U} \varphi(n) d \log (n+1) \ll n^{2} \log (n+1),
$$

so that (7) is proved.
Suppose that there exists $\sigma \in \operatorname{Gal}\left(K_{n, d} / \mathbb{Q}\right)$ satisfying (3). We shall prove that $\sqrt{\Delta} \notin \mathbb{Q}\left(\zeta_{n}\right)$ or $\Delta<0$. Assume that $\sqrt{\Delta} \in \mathbb{Q}\left(\zeta_{n}\right)$. On the one hand, $\sigma(\gamma)=\sigma\left(\gamma^{1 / d}\right)^{d}=\gamma^{-1}$, and consequently $\sigma(\sqrt{\Delta})=-\sqrt{\Delta}$. On the other hand, since $\sqrt{\Delta} \in \mathbb{Q}\left(\zeta_{n}\right)$ and $\sigma\left(\zeta_{n}\right)=\zeta_{n}^{-1}$, we have that $\sigma(\sqrt{\Delta})=\overline{\sqrt{\Delta}}$. Therefore, $\overline{\sqrt{\Delta}}=-\sqrt{\Delta}$ and so $\Delta<0$. Now let us check that $\sigma$ belongs to the center of $\operatorname{Gal}\left(K_{n, d} / \mathbb{Q}\right)$. Note that $N_{K / \mathbb{Q}}(\gamma)=\gamma \sigma_{K}(\gamma)=\gamma \gamma^{-1}=1$. Also, $N_{K / \mathbb{Q}}\left(\gamma_{0}^{h_{0}}\right)=N_{K / \mathbb{Q}}\left(\gamma_{0}^{h}\right)=N_{K / \mathbb{Q}}(\gamma)=1$, since $d$ is odd and so $h_{0} \equiv h(\bmod 2)$. Therefore, for every $\tau \in \operatorname{Gal}\left(K_{n, q} / \mathbb{Q}\right)$, we have $\tau\left(\gamma_{0}^{h_{0}}\right)=\gamma_{0}^{h_{0}}$, if $\left.\tau\right|_{K}=\operatorname{id}$, or $\tau\left(\gamma_{0}^{h_{0}}\right)=N_{K / \mathbb{Q}}\left(\gamma_{0}^{h_{0}}\right) \gamma_{0}^{-h_{0}}=\gamma_{0}^{-h_{0}}$ if $\left.\tau\right|_{K}=\sigma_{K}$. Consequently, recalling that $\left(\gamma^{1 / d}\right)^{d_{0}}=\gamma_{0}^{h_{0}}$, we have that $\tau\left(\zeta_{n}\right)=\zeta_{n}^{s}$ and $\tau\left(\gamma^{1 / d}\right)=\zeta_{d_{0}}^{t} \gamma^{ \pm 1 / d}$ for some integers $s, t$. At this point, it can be easily checked that $(\sigma \tau)\left(\zeta_{n}\right)=$ $(\tau \sigma)\left(\zeta_{n}\right)$ and $(\sigma \tau)\left(\gamma^{1 / d}\right)=(\tau \sigma)\left(\gamma^{1 / d}\right)$. Hence, $\sigma$ belongs to the center of $\operatorname{Gal}\left(K_{n, d} / \mathbb{Q}\right)$.

Suppose that $\sqrt{\Delta} \notin \mathbb{Q}\left(\zeta_{n}\right)$ or $\Delta<0$. We shall prove the existence of $\sigma \in \operatorname{Gal}\left(K_{n, d} / \mathbb{Q}\right)$ satisfying (3). It suffices to show that there exists $\sigma_{1} \in \operatorname{Gal}\left(K\left(\zeta_{n}\right) / K\right)$ such that $\sigma_{1}\left(\zeta_{n}\right)=\zeta_{n}^{-1}$
and $\left.\sigma_{1}\right|_{K}=\sigma_{K}$. Indeed, recalling that $K_{n, d} \cong K\left(\zeta_{n}\right)[X] /(f(X))$, we can extend $\sigma_{1}$ to an automorphism $\sigma \in \operatorname{Gal}\left(K_{n, d} / \mathbb{Q}\right)$ that sends the root $\gamma^{1 / d}$ of $f$ to the root $\gamma^{-1 / d}$ of

$$
\left(\sigma_{1} f\right)(X)=X^{d_{0}}-\sigma_{1}\left(\gamma_{0}^{h_{0}}\right)=X^{d_{0}}-N_{K / \mathbb{Q}}\left(\gamma_{0}^{h_{0}}\right) \gamma_{0}^{-h_{0}}=X^{d_{0}}-\gamma_{0}^{-h_{0}}
$$

and so $\sigma$ satisfies (3). Pick $\sigma_{0} \in \operatorname{Gal}\left(\mathbb{Q}\left(\zeta_{n}\right) / \mathbb{Q}\right)$ such that $\sigma_{0}\left(\zeta_{n}\right)=\zeta_{n}^{-1}$. If $\sqrt{\Delta} \in \mathbb{Q}\left(\zeta_{n}\right)$ then $K\left(\zeta_{n}\right)=\mathbb{Q}\left(\zeta_{n}\right), \Delta<0$, and $\sigma_{0}(\sqrt{\Delta})=\overline{\sqrt{\Delta}}=-\sqrt{\Delta}$, so we let $\sigma_{1}:=\sigma_{0}$. If $\sqrt{\Delta} \notin \mathbb{Q}\left(\zeta_{n}\right)$ then $X^{2}-\Delta$ is the minimal polynomial of $\sqrt{\Delta}$ over $\mathbb{Q}\left(\zeta_{n}\right)$ and we can extend $\sigma_{0}$ to $\sigma_{1} \in$ $\operatorname{Gal}\left(K\left(\zeta_{n}\right) / \mathbb{Q}\right)$ such that $\sigma_{1}(\sqrt{\Delta})=-\sqrt{\Delta}$.

## 6. Proof of Theorem 1.1

The proof proceeds similarly to [9, Section 2]. For all positive integers $d, n$ with $d \mid n$, and for all $x>1$, let us define

$$
\pi_{U, n, d}(x):=\#\left\{p \leq x: p \nmid a_{2} \Delta, p \equiv\left(\frac{\Delta}{p}\right)(\bmod n), d \mid \iota_{U}(p)\right\} .
$$

In what follows, we will tacitly ignore the finitely many prime numbers dividing $a_{2} \Delta$.
Lemma 6.1. For every positive integer $d$ and for every $x>1$, we have

$$
\begin{equation*}
\mathcal{R}_{U}(d ; x)=\sum_{v \mid d^{\infty}} \sum_{a \mid d} \mu(a) \pi_{U, d v, a v}(x) . \tag{8}
\end{equation*}
$$

Proof. Every prime number $p$ counted by the inner sum of (8) satisfies $p \leq x, p \equiv\left(\frac{\Delta}{p}\right)$ $(\bmod d v)$, and $\iota_{U}(p)=v w$ for some integer $w$. Writing $w=w_{1} w_{2}$, with $w_{1}:=(w, d)$, we get that the contribution of $p$ to the inner sum or (8) is equal to $\sum_{a \mid w_{1}} \mu(a)$. Hence,

$$
\begin{equation*}
\sum_{a \mid d} \mu(a) \pi_{U, d v, a v}(x)=\#\left\{p \leq x: p \equiv\left(\frac{\Delta}{p}\right)(\bmod d v), v \mid \iota_{U}(p),\left(\iota_{U}(p) / v, d\right)=1\right\} \tag{9}
\end{equation*}
$$

Now it suffices to show that

$$
\begin{equation*}
\mathcal{R}_{U}(d ; x)=\sum_{v \mid d^{\infty}} \#\left\{p \leq x: p \equiv\left(\frac{\Delta}{p}\right)(\bmod d v), v \mid \iota_{U}(p),\left(\iota_{U}(p) / v, d\right)=1\right\} . \tag{10}
\end{equation*}
$$

On the one hand, let $p$ be a prime number counted on the right-hand side of (10). Note that this is counted only one, namely for $v=\left(\iota_{U}(p), d^{\infty}\right)$. Then, from $\rho_{U}(p) \iota_{U}(p)=p-\left(\frac{\Delta}{p}\right)$, it follows that $d \mid \rho_{U}(p)$. Hence, $p$ is counted on the left-hand side of (10).

On the other hand, let $p$ be a prime number counted by $\mathcal{R}_{U}(d ; x)$. Then $d \mid \rho_{U}(p)$ and, by Lemma 5.1, $p \equiv\left(\frac{\Delta}{p}\right)(\bmod d)$. Consequently, there is an integer $v$ such that $v \mid d^{\infty}, p \equiv\left(\frac{\Delta}{p}\right)$ $(\bmod d v)$, and $\left(\iota_{U}(p) / v, d\right)=1$. Hence, $p$ is counted on the right-hand side of (10).
Lemma 6.2. Let $n$ be an odd positive integer with $3 \nmid n$ whenever $\Delta_{0}=-3$, and let $d$ be $a$ positive integer dividing $n$. There exist absolute constants $A, B>0$ such that

$$
\pi_{U, n, d}(x)=\delta_{U, n, d} \operatorname{Li}(x)+O_{U}\left(x \exp \left(-A(\log x)^{1 / 2} / n\right)\right)
$$

for $x \geq \exp \left(B n^{8}\right)$, where

$$
\delta_{U, n, d}:=\frac{(d, h)}{\varphi(n) d} \cdot \begin{cases}1 & \text { if } \Delta>0 \text { or } \Delta_{0} \not \equiv 1(\bmod 4) \text { or } \Delta_{0} \nmid n,  \tag{11}\\ 2 & \text { otherwise } .\end{cases}
$$

Proof. Put $E:=K_{n, d}, F:=\mathbb{Q}, G:=\operatorname{Gal}(E / F)$, and $C=\{\mathrm{id}, \sigma\}$ if there exists $\sigma \in$ $\operatorname{Gal}\left(K_{n, d} / \mathbb{Q}\right)$ satisfying (3), or $C=\{i d\}$ otherwise. By Lemma 5.5, $\sigma$ belongs to the center of $G$, so that $C$ is the union of conjugacy classes of $G$. By Lemma 5.2, we have that $\pi_{U, n, d}(x)$ is the number of primes $p$ not exceeding $x$ and such that $\left[\frac{E / F}{p}\right] \subseteq C$. Thus, taking into account the bounds for the degree and the discriminant of $E / F$ given in Lemma 5.5, and considering Lemma 4.4, the asymptotic formula follows by applying Theorem 4.5.

Lemma 6.3. Let $d$ be an odd positive integer with $3 \nmid d$ whenever $\Delta_{0}=-3$. If $x>1$ and $e^{\omega(d)} \leq y \leq \log x / \varphi(d)$, then

$$
\begin{equation*}
\sum_{\substack{v \mid d^{\infty} \\ v>y}} \sum_{a \mid d} \mu(a) \pi_{U, d v, a v}(x) \ll \frac{x}{\log x} \cdot \frac{\omega(d)+1}{\varphi(d)} \cdot \frac{(\log y)^{\omega(d)}}{y} \tag{12}
\end{equation*}
$$

and

$$
\sum_{\substack{v \mid d^{\infty} \\ v>y}} \sum_{a \mid d} \mu(a) \delta_{U, d v, a v}<_{U} \frac{\omega(d)+1}{\varphi(d)} \cdot \frac{(\log y)^{\omega(d)}}{y} .
$$

Proof. Let $\pi(m, r ; x):=\#\{p \leq x: p \equiv r(\bmod m)\}$. From (9) it follows that

$$
\begin{equation*}
\left|\sum_{a \mid d} \mu(a) \pi_{U, d v, a v}(x)\right| \leq \pi_{U, d v, v}(x) \leq \pi(x ; d v, \pm 1) \tag{13}
\end{equation*}
$$

Moreover, letting $x \rightarrow+\infty$, Lemma 6.2 and the first inequality of (13) yield

$$
\begin{equation*}
\left|\sum_{a \mid d} \mu(a) \delta_{U, d v, a v}\right| \leq \delta_{U, d v, v} \tag{14}
\end{equation*}
$$

Now we have $M_{d}(x):=\#\left\{v \leq x: v \mid d^{\infty}\right\} \ll(\log x)^{\omega(d)}$, for every $x \geq 2$. Hence, by partial summation and since $y \geq e^{\omega(d)}$, we obtain that

$$
\begin{equation*}
\sum_{\substack{v \mid d^{\infty} \\ v>y}} \frac{1}{v}=\left.\frac{M_{d}(t)}{t}\right|_{t=y} ^{+\infty}+\int_{y}^{+\infty} \frac{M_{d}(t)}{t^{2}} \mathrm{~d} t \ll \int_{y}^{+\infty} \frac{(\log t)^{\omega(d)}}{t^{2}} \mathrm{~d} t \leq \frac{(\omega(d)+1)(\log y)^{\omega(d)}}{y} \tag{15}
\end{equation*}
$$

On the one hand, using the Brun-Titchmarsh inequality [3, Theorem 12.7]

$$
\pi(m, r ; x) \ll \frac{x}{\varphi(m) \log (x / m)},
$$

holding for $x>m$, and (15) we get that

$$
\begin{equation*}
\sum_{\substack{v \mid d^{\infty} \\>y, d v \leq x^{2 / 3}}} \pi(d v, \pm 1 ; x) \ll \frac{x}{\varphi(d) \log x} \sum_{\substack{v \mid d^{\infty} \\ v>y}} \frac{1}{v} \ll \frac{x}{\log x} \cdot \frac{\omega(d)+1}{\varphi(d)} \cdot \frac{(\log y)^{\omega(d)}}{y} . \tag{16}
\end{equation*}
$$

On the other hand, using the trivial bound $\pi(m, \pm 1 ; x) \ll x / m$, holding for $x \geq 1$, and (15) again, we find that

$$
\begin{equation*}
\sum_{\substack{v \mid d^{\infty} \\ d v>x^{2 / 3}}} \pi(d v, \pm 1 ; x) \ll \sum_{\substack{v \mid d^{\infty} \\ d v>x^{2 / 3}}} \frac{x}{d v} \leq \sum_{\substack{w \mid d^{\infty} \\ w>x^{2 / 3}}} \frac{x}{w} \ll x^{1 / 3}(\omega(d)+1)(\log x)^{\omega(d)} . \tag{17}
\end{equation*}
$$

Putting together (16), (17), and (13), taking into account that $\omega(d) \leq \log y$ and $\varphi(d) y \leq \log x$, we obtain (12). Finally, from (14), (11), and (15), we get

$$
\sum_{\substack{v \mid d^{\infty} \\ v>y}} \sum_{a \mid d} \mu(a) \delta_{U, d v, a v} \leq \sum_{\substack{v \mid d^{\infty} \\ v>y}} \delta_{U, d v, v} \lll U \frac{1}{\varphi(d)} \sum_{\substack{v \mid d^{\infty} \\ v>y}} \frac{1}{v^{2}} \ll \frac{\omega(d)+1}{\varphi(d)} \cdot \frac{(\log y)^{\omega(d)}}{y},
$$

as desired.
Lemma 6.4. Let $d$ be an odd positive integer with $3 \nmid d$ whenever $\Delta_{0}=-3$. Then

$$
\sum_{v \mid d^{\infty}} \sum_{a \mid d} \mu(a) \delta_{U, d v, a v}=\delta_{U}(d) .
$$

Proof. For every integer $e$ dividing $d^{\infty}$, define

$$
S_{d, e, h}:=\sum_{\substack{v\left|d^{\infty} \\ e\right| v}} \sum_{a \mid d} \frac{\mu(a)(a v, h)}{\varphi(d v) a v}
$$

The value of $S_{d, 1, h}$ was computed in [9, Lemma 4] and a slight modification of the proof (precisely, replacing $\left(h, d^{\infty}\right)$ with $\left[e,\left(h, d^{\infty}\right)\right]$ in the last equation) yields

$$
S_{d, e, h}=\frac{\left(d^{\infty}, h\right)}{d\left[\left(d^{\infty}, h\right), e\right]^{2}} \prod_{p \mid d}\left(1-\frac{1}{p^{2}}\right)^{-1}
$$

At this point, by (11) and considering that $\Delta_{0} \mid d v$ if and only if $e \mid v$, where $e:=\Delta_{0} /\left(d, \Delta_{0}\right)$, we have

$$
\sum_{v \mid d^{\infty}} \sum_{a \mid d} \mu(a) \delta_{U, d v, a v}= \begin{cases}S_{d, 1, h} & \text { if } \Delta>0 \text { or } \Delta_{0} \not \equiv 1(\bmod 4) \text { or } \Delta_{0} \nmid d^{\infty}=\delta_{U}(d) \\ S_{d, 1, h}+S_{d, e, h} & \text { otherwise }\end{cases}
$$

as claimed.
Proof of Theorem 1.1. Let $A, B>0$ be the constants of Lemma 6.2. Assume that $x \geq$ $\exp \left(B e^{8 \omega(d)} d^{8}\right)$ and put $y:=(\log x / B)^{1 / 8} / d$. Note that $e^{\omega(d)} \leq y \leq \log x / \varphi(d)$ and $\log y \leq$ $\log \log x$, for every $x>_{B} 1$. By Lemma 6.1, Lemma 6.2, and Lemma 6.4, we obtain that

$$
\begin{aligned}
\mathcal{R}_{U}(d ; x) & =\sum_{\substack{v \mid d^{\infty} \\
v \leq y}} \sum_{a \mid d} \mu(a) \pi_{U, d v, a v}(x)+O\left(E_{1}\right) \\
& =\sum_{\substack{v \mid d^{\infty} \\
v \leq y}} \sum_{a \mid d} \mu(a) \delta_{U, d v, a v} \operatorname{Li}(x)+O\left(E_{1}\right)+O_{U}\left(E_{2}\right) \\
& =\delta_{U}(d) \operatorname{Li}(x)+O\left(E_{1}\right)+O_{U}\left(E_{2}\right)+O\left(E_{3}\right)
\end{aligned}
$$

where, by Lemma 6.3, we have
$E_{1}:=\sum_{\substack{v \mid d^{\infty} \\ v>y}} \sum_{a \mid d} \mu(a) \pi_{U, d v, a v}(x) \ll \frac{x}{\log x} \cdot \frac{\omega(d)+1}{\varphi(d)} \cdot \frac{(\log y)^{\omega(d)}}{y} \ll \frac{(\omega(d)+1) d}{\varphi(d)} \cdot \frac{x(\log \log x)^{\omega(d)}}{(\log x)^{9 / 8}}$
and
$E_{3}:=\sum_{\substack{v \mid d^{\infty} \\ v>y}} \sum_{a \mid d} \mu(a) \delta_{U, d v, a v} \operatorname{Li}(x) \ll_{U} \frac{\omega(d)+1}{\varphi(d)} \cdot \frac{(\log y)^{\omega(d)}}{y} \cdot \operatorname{Li}(x) \ll \frac{(\omega(d)+1) d}{\varphi(d)} \cdot \frac{x(\log \log x)^{\omega(d)}}{(\log x)^{9 / 8}}$,
while, also using the inequality $\tau(d) / d \leq d / \varphi(d)$, we have

$$
\begin{aligned}
E_{2} & :=\sum_{\substack{v \mid d^{\infty} \\
v \leq y}} \sum_{a \mid d} x \exp \left(-A(\log x)^{1 / 2} /(d v)\right) \ll x \exp \left(-A B^{1 / 8}(\log x)^{3 / 8}\right) \tau(d) y \\
& \ll x \exp \left(-A B^{1 / 8}(\log x)^{3 / 8}\right)(\log x)^{1 / 8} \cdot \frac{\tau(d)}{d} \ll \frac{d}{\varphi(d)} \cdot \frac{x}{(\log x)^{9 / 8}}
\end{aligned}
$$

The result follows.

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