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# ON THE DIVISIBILITY OF THE RANK OF APPEARANCE OF A LUCAS SEQUENCE

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ABSTRACT. Let  $U = (U_n)_{n \geq 0}$  be a Lucas sequence and, for every prime number  $p$ , let  $\rho_U(p)$  be the rank of appearance of  $p$  in  $U$ , that is, the smallest positive integer  $k$  such that  $p$  divides  $U_k$ , whenever it exists. Furthermore, let  $d$  be an odd positive integer. Under some mild hypotheses, we prove an asymptotic formula for the number of primes  $p \leq x$  such that  $d$  divides  $\rho_U(p)$ , as  $x \rightarrow +\infty$ .

## 1. INTRODUCTION

Let  $(U_n)_{n \geq 0}$  be a Lucas sequence, that is, a sequence of integers satisfying  $U_0 = 0$ ,  $U_1 = 1$ , and  $U_n = a_1 U_{n-1} + a_2 U_{n-2}$  for every integer  $n \geq 2$ , where  $a_1, a_2$  are fixed nonzero integers. The *rank of appearance* of a prime number  $p$ , denoted by  $\rho_U(p)$ , is the smallest positive integer  $k$  such that  $p \mid U_k$ . It can be easily seen that  $\rho_U(p)$  exists whenever  $p \nmid a_2$ . Define

$$\mathcal{R}_U(d; x) := \#\{p \leq x : p \nmid a_2, d \mid \rho_U(p)\},$$

for every positive integer  $d$  and for every  $x > 1$ .

Let  $(F_n)_{n \geq 0}$  be the Lucas sequence of Fibonacci numbers, corresponding to  $a_1 = a_2 = 1$ . In 1985, Lagarias [5] (see [6] for a correction and [8, 10] for generalizations) showed that  $\mathcal{R}_F(2; x) \sim \frac{2}{3}x$ , as  $x \rightarrow +\infty$ . More recently, Cubre and Rouse [2], settling a conjecture of Bruckman and Anderson [1], proved that  $\mathcal{R}_F(d; x) \sim c(d) d^{-1} \prod_{p \mid d} (1 - p^{-2})^{-1}$ , as  $x \rightarrow +\infty$ , for every positive integer  $d$ , where  $c(d)$  is equal to 1,  $\frac{5}{4}$ , or  $\frac{1}{2}$ , whenever  $10 \nmid d$ ,  $d \equiv 10 \pmod{20}$ , or  $20 \mid d$ , respectively.

Let  $\alpha, \beta$  be the roots of the characteristic polynomial  $f_U(X) := X^2 - a_1 X - a_2$ , and assume that  $\gamma := \alpha/\beta$  is not a root of unity. Let  $\Delta := a_1^2 + 4a_2$  be the discriminant of  $f_U(X)$ , and let  $\Delta_0$  be the squarefree part of  $\Delta$ . Assume that  $\Delta$  is not a square, so that  $K := \mathbb{Q}(\sqrt{\Delta})$  is a quadratic number field. Let  $h$  be the greatest positive integer such that  $\gamma$  is a  $h$ th power in  $K$ .

Our result is the following:

**Theorem 1.1.** *Let  $d$  be an odd positive integer with  $3 \nmid d$  whenever  $\Delta_0 = -3$ . Then, for every  $x > \exp(Be^{8\omega(d)}d^8)$ , we have*

$$\mathcal{R}_U(d; x) = \delta_U(d) \operatorname{Li}(x) + O_U \left( \frac{(\omega(d) + 1)d}{\varphi(d)} \cdot \frac{x (\log \log x)^{\omega(d)}}{(\log x)^{9/8}} \right),$$

where  $B > 0$  is an absolute constant and

$$\delta_U(d) := \frac{1}{d} \left( \frac{1}{(d^\infty, h)} + \eta_U(d) \right) \prod_{p \mid d} \left( 1 - \frac{1}{p^2} \right)^{-1},$$

with  $\eta_U(d) := 0$  if  $\Delta > 0$  or  $\Delta_0 \not\equiv 1 \pmod{4}$  or  $\Delta_0 \nmid d^\infty$ ; and

$$\eta_U(d) := \frac{(d^\infty, h)}{[(d^\infty, h), \Delta_0/(d, \Delta_0)]^2}$$

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otherwise.

Cubre and Rouse's proof of the asymptotic formula for  $\mathcal{R}_F(d; x)$  relies on the study of the algebraic group  $G : x^2 - 5y^2 = 1$  and relates  $\rho_F(p)$  with the order of  $(3/2, 1/2) \in G(\mathbb{F}_p)$ . Instead, our proof of Theorem 1.1 is an adaptation of the methods that Moree [9] used to prove an asymptotic formula for the number of primes  $p \leq x$  such that the multiplicative order of  $g$  modulo  $p$  is divisible by  $d$ , where  $g \notin \{-1, 0, +1\}$  is a fixed rational number.

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## 3. NOTATION

We employ the Landau–Bachmann “Big Oh” notation  $O$ , as well as the associated Vinogradov symbol  $\ll$ . Any dependence of the implied constants is explicitly stated or indicated with subscripts. In particular, notations like  $O_U$  and  $\ll_U$  are shortcuts for  $O_{a_1, a_2}$  and  $\ll_{a_1, a_2}$ , respectively. For  $x \geq 2$  we let  $\text{Li}(x) := \int_2^x \frac{dt}{\log t}$  denote the logarithmic integral. We reserve the letter  $p$  for prime numbers. Given an integer  $d$ , we let  $d^\infty$  denote the supernatural number  $\prod_{p|d} p^\infty$ . Given a field  $F$  and a positive integer  $n$ , we write  $F^n$  for the set of  $n$ th powers of elements of  $F$ . Given a Galois extension  $E/F$  of number fields and a prime ideal  $P$  of  $\mathcal{O}_E$  lying above an unramified prime ideal  $\mathfrak{p}$  of  $\mathcal{O}_F$ , we write  $\left[\frac{E/F}{P}\right]$  for the Frobenius automorphism corresponding to  $P/\mathfrak{p}$ , that is, the unique element  $\sigma$  of the Galois group  $\text{Gal}(E/F)$  that satisfies  $\sigma(a) \equiv a^{N(\mathfrak{p})} \pmod{P}$  for every  $a \in \mathcal{O}_E$ , where  $N(\mathfrak{p})$  denotes the norm of  $\mathfrak{p}$ . Moreover, we let  $\left[\frac{E/F}{\mathfrak{p}}\right]$  be the set of all  $\left[\frac{E/F}{P}\right]$  with  $P$  prime ideal of  $\mathcal{O}_E$  lying over  $\mathfrak{p}$ . We write  $\Delta_{E/F}$  for the relative discriminant of  $E/F$ , and  $\Delta_E := \Delta_{E/\mathbb{Q}}$  for the absolute discriminant of  $E$ . For every integer  $d$  and for every prime number  $p$  we let  $\left(\frac{d}{p}\right)$  be the Legendre symbol. For every positive integer  $n$ , we let  $\zeta_n := e^{2\pi i/n}$  be a primitive  $n$ th root of unity. We write  $\omega(n)$ ,  $\varphi(n)$ ,  $\mu(n)$ , and  $\tau(n)$ , for the number of prime factors, the totient function, the Möbius function, and the number of divisors of a positive integer  $n$ , respectively.

## 4. GENERAL PRELIMINARIES

**Lemma 4.1.** *Let  $n$  be a positive integer, let  $p$  be a prime number not dividing  $n$ , and let  $P$  be a prime ideal of  $\mathcal{O}_{\mathbb{Q}(\zeta_n)}$  lying over  $p$ . Then  $\zeta_n$  has multiplicative order modulo  $P$  equal to  $n$ .*

*Proof.* Let  $k$  be the multiplicative order of  $\zeta_n$  modulo  $P$ , that is,  $k$  is the least positive integer such that  $\zeta_n^k \equiv 1 \pmod{P}$ . On the one hand, we have that  $p \mid N(P) \mid N(\zeta_n^k - 1)$ . On the other hand, since  $\zeta_n^n \equiv 1 \pmod{P}$ , we have that  $k \mid n$ , and consequently  $\zeta_n^k$  is a  $m$ th primitive root of unity, where  $m := n/k$ . If  $k < n$  then  $m > 1$  and  $N(\zeta_n^k - 1)$  is either 1 or a prime factor of  $m$ , but both cases are impossible since  $p \nmid n$ . Hence,  $k = n$ .  $\square$

**Lemma 4.2.** *Let  $F$  be a field, let  $a \in F$ , and let  $n$  be a positive integer. Then  $X^n - a$  is irreducible over  $F$  if and only if  $a \notin F^p$  for each prime  $p$  dividing  $n$  and  $a \notin -4F^4$  whenever  $4 \mid n$ .*

*Proof.* See [4, Chapter 8, Theorem 1.6].  $\square$

**Lemma 4.3.** *Let  $F$  be a field, let  $n$  be a positive integer not divisible by the characteristic of  $F$ , and let  $m$  be the number of  $n$ th roots of unity contained in  $F$ . Then, for every  $a \in F$ , the extension  $F(\zeta_n, a^{1/n})/F$  is abelian if and only if  $a^m \in F^n$ .*

*Proof.* See [4, Chapter 8, Theorem 3.2].  $\square$

**Lemma 4.4.** *Let  $n$  be an odd positive integer and let  $d$  be a squarefree integer. Then  $\sqrt{d} \in \mathbb{Q}(\zeta_n)$  if and only if  $d \mid n$  and  $d \equiv 1 \pmod{4}$ .*

*Proof.* See [12, Lemma 3].  $\square$

We need the following form of the Chebotarev Density Theorem.

**Theorem 4.5.** *Let  $E/F$  be a Galois extension of number fields with Galois group  $G$ , and let  $C$  be the union of  $k$  conjugacy classes of  $G$ . Then*

$$\begin{aligned} & \#\left\{ \mathfrak{p} \text{ prime ideal of } \mathcal{O}_F \text{ non-ramifying in } E : N_{F/\mathbb{Q}}(\mathfrak{p}) \leq x, \left[ \frac{E/F}{\mathfrak{p}} \right] \subseteq C \right\} \\ &= \frac{\#C}{\#G} \cdot \text{Li}(x) + O\left( kx \exp\left(-c_1(\log x/n_E)^{1/2}\right) \right) \end{aligned}$$

for every

$$x \geq \exp\left(c_2 \max\left(n_E(\log |\Delta_E|)^2, |\Delta_E|^{2/n_E}/n_E\right)\right),$$

where  $n_E := [E : \mathbb{Q}]$  and  $c_1, c_2 > 0$  are absolute constants.

*Proof.* The result follows from the effective form of the Chebotarev Density Theorem given by Lagarias and Odlyzko [7, Theorem 1.3] and from the bounds for the exceptional zero of the Dedekind zeta function  $\zeta_E$  given by Stark [13, Lemma 8 and 11].  $\square$

## 5. PRELIMINARIES TO THE PROOF OF THEOREM 1.1

Recalling that  $h$  is the greatest positive integer such that  $\gamma$  is an  $h$ th power in  $K$ , write  $\gamma = \gamma_0^h$  for some  $\gamma_0 \in K$ . Also, let  $\sigma_K \in \text{Gal}(K/\mathbb{Q})$  be the nontrivial automorphism, which satisfies  $\sigma_K(\sqrt{\Delta}) = -\sqrt{\Delta}$ . Note that, since  $\gamma = \alpha/\beta$  and  $\sigma_K$  swaps  $\alpha$  and  $\beta$ , we have that  $\sigma_K(\gamma) = \gamma^{-1}$ . For all positive integers  $d, n$  such that  $d \mid n$ , let  $K_{n,d} := K(\zeta_n, \gamma^{1/d})$ .

**Lemma 5.1.** *Let  $p$  be a prime number not dividing  $a_2\Delta$  and let  $\pi$  be a prime ideal of  $\mathcal{O}_K$  lying over  $p$ . Then  $\rho_U(p)$  is equal to the multiplicative order of  $\gamma$  modulo  $\pi$ . Moreover,  $\rho_U(p)$  divides  $p - \left(\frac{\Delta}{p}\right)$ .*

*Proof.* First, note that  $p \nmid a_2$  ensures that  $\beta$  is invertible modulo  $\pi$ , and consequently it makes sense to consider the multiplicative order of  $\gamma = \alpha/\beta$  modulo  $\pi$ . Also,  $p \nmid \Delta$  implies that  $p$  does not ramify in  $K$  and that  $\alpha \not\equiv \beta \pmod{\pi}$ .

We shall prove that  $p \mid U_n$  if and only if  $\gamma^n \equiv 1 \pmod{\pi}$ , for every positive integer  $n$ . Then the claim on  $\rho_U(p)$  follows easily. It is well known that the Binet's formula

$$(1) \quad U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$$

holds for every positive integer  $n$ . On the one hand, if  $p \mid U_n$  then, since  $p\mathcal{O}_K \subseteq \pi$  and (1), we have  $\alpha^n \equiv \beta^n \pmod{\pi}$ , and consequently  $\gamma^n \equiv 1 \pmod{\pi}$ . On the other hand, if  $\gamma^n \equiv 1 \pmod{\pi}$  then by (1) we get  $U_n \equiv 0 \pmod{\pi}$ . If  $p$  is inert in  $K$ , then  $p\mathcal{O}_K = \pi$  and so  $p \mid U_n$ . If  $p$  splits in  $K$ , then  $p\mathcal{O}_K = \pi \cap \sigma_K(\pi)$ . Thus  $U_n \equiv 0 \pmod{\pi}$  and  $U_n \equiv \sigma_K(U_n) \equiv 0 \pmod{\sigma_K(\pi)}$  imply that  $p \mid U_n$ .

Let  $\sigma := \left[\frac{K/\mathbb{Q}}{\pi}\right]$ . On the one hand, if  $\left(\frac{\Delta}{p}\right) = -1$  then  $\sigma = \sigma_K$  and  $\gamma^{p+1} \equiv \sigma_K(\gamma)\gamma \equiv \gamma^{-1}\gamma \equiv 1 \pmod{\pi}$ , so that  $\rho_U(p) \mid p+1$ . On the other hand, if  $\left(\frac{\Delta}{p}\right) = +1$  then  $\sigma = \text{id}$  and  $\gamma^{p-1} \equiv \gamma\gamma^{-1} \equiv 1 \pmod{\pi}$ , so that  $\rho_U(p) \mid p-1$ .  $\square$

For each prime number  $p$  not dividing  $a_2\Delta$ , let us define the *index of appearance* of  $p$  as

$$\iota_U(p) := \left(p - \left(\frac{\Delta}{p}\right)\right) / \rho_U(p).$$

Note that, in light of Lemma 5.1,  $\iota_U(p)$  is an integer.

**Lemma 5.2.** *Let  $d, n$  be positive integers such that  $d \mid n$ , and let  $p$  be a prime number not dividing  $a_2\Delta$ . Moreover, let  $P$  be a prime ideal of  $\mathcal{O}_{K_{n,d}}$  lying over  $p$  and let  $\sigma := \left[\frac{K_{n,d}/\mathbb{Q}}{P}\right]$ . Then*

$$(2) \quad p \equiv \left(\frac{\Delta}{p}\right) \pmod{n} \quad \text{and} \quad d \mid \iota_U(p)$$

if and only if  $\sigma = \text{id}$  or

$$(3) \quad \sigma(\zeta_n) = \zeta_n^{-1} \quad \text{and} \quad \sigma(\gamma^{1/d}) = \gamma^{-1/d}.$$

*Proof.* First, suppose that  $\left(\frac{\Delta}{p}\right) = -1$ . Let us assume (2). On the one hand, since  $p \equiv -1 \pmod{n}$ , we have

$$(4) \quad \sigma(\zeta_n) \equiv \zeta_n^p \equiv \zeta_n^{-1} \pmod{P}.$$

Since  $\sigma(\zeta_n) = \zeta_n^k$  for some integer  $k$ , and since  $p$  does not divide  $n$ , Lemma 4.1 and (4) yield that  $\sigma(\zeta_n) = \zeta_n^{-1}$ .

On the other hand,  $d \mid \iota_U(p)$  implies that  $\rho_U(p) \mid (p+1)/d$ . Hence, letting  $\pi := P \cap \mathcal{O}_K$ , Lemma 5.1 yields  $\gamma^{(p+1)/d} \equiv 1 \pmod{\pi}$ . Consequently,

$$(5) \quad \sigma(\gamma^{1/d}) \equiv (\gamma^{1/d})^p \equiv \gamma^{(p+1)/d} \cdot \gamma^{-1/d} \equiv \gamma^{-1/d} \pmod{P}.$$

Note that, since  $\left(\frac{\Delta}{p}\right) = -1$ , we have

$$\sigma(\gamma) = \sigma|_K(\gamma) = \left[\frac{K/\mathbb{Q}}{\pi}\right](\gamma) = \sigma_K(\gamma) = \gamma^{-1},$$

so that  $\sigma(\gamma^{1/d}) = \zeta_d^k \gamma^{-1/d}$  for some integer  $k$ . Thus Lemma 4.1 and (5) yield that  $\sigma(\gamma^{1/d}) = \gamma^{-1/d}$ . We have proved (3).

Now let us assume (3). On the one hand, we have

$$\zeta_n^{-1} = \sigma(\zeta_n) = \sigma|_{\mathbb{Q}(\zeta_n)}(\zeta_n) = \left[\frac{\mathbb{Q}(\zeta_n)/\mathbb{Q}}{P \cap \mathcal{O}_{\mathbb{Q}(\zeta_n)}}\right](\zeta_n) = \zeta_n^p,$$

so that  $p \equiv -1 \pmod{n}$ . On the other hand,

$$\gamma^{(p+1)/d} \equiv (\gamma^{1/d})^p \cdot \gamma^{1/d} \equiv \sigma(\gamma^{1/d}) \cdot \gamma^{1/d} \equiv \gamma^{-1/d} \cdot \gamma^{1/d} \equiv 1 \pmod{P},$$

so that  $\gamma^{(p+1)/d} \equiv 1 \pmod{\pi}$ , which, by Lemma 5.1, implies  $d \mid \iota_U(p)$ . We have proved (2).

If  $\left(\frac{\Delta}{p}\right) = +1$  then the proof proceeds similarly to the case  $\left(\frac{\Delta}{p}\right) = -1$ , and yields that (2) is equivalent to  $\sigma(\zeta_n) = \zeta_n$  and  $\sigma(\gamma^{1/d}) = \gamma^{1/d}$ , that is,  $\sigma = \text{id}$ .  $\square$

**Lemma 5.3.** *The roots of unity contained in  $K$  are: the sixth roots of unity, if  $\Delta_0 = -3$ ; the fourth roots of unity, if  $\Delta_0 = -1$ ; or the second roots of unity, if  $\Delta_0 \neq -1, -3$ .*

*Proof.* If  $\zeta_n \in K$  for some positive integer  $n$ , then  $\mathbb{Q}(\zeta_n) \subseteq K$ , so that  $\varphi(n) \leq 2$ , and  $n \in \{1, 2, 3, 4, 6\}$ . Then the claim follows easily since  $\zeta_3 = (-1 + \sqrt{-3})/2$ ,  $\zeta_4 = \sqrt{-1}$ , and  $\zeta_6 = (1 + \sqrt{-3})/2$ .  $\square$

**Lemma 5.4.** *Let  $n$  be an odd positive integer with  $3 \nmid n$  whenever  $\Delta_0 = -3$ , and let  $d$  be a positive integer dividing  $n$ . Then  $a \in K \cap K(\zeta_n)^d$  if and only if  $a \in K^d$ .*

*Proof.* The ‘‘if’’ part is obvious. Let us prove the ‘‘only if’’ part. Note that, by the hypothesis on  $n$  and by Lemma 5.3, the only  $n$ th root of unity in  $K$  is 1. Suppose that  $a \in K \cap K(\zeta_n)^d$ . Hence, there exists  $b \in K(\zeta_n)$  such that  $a = b^d$ . Putting  $a_1 := a^{n/d}$ , we get that  $a_1 = b^n$ . Therefore,  $K(\zeta_n, a_1^{1/n}) = K(\zeta_n, b) = K(\zeta_n)$  is an abelian extension of  $K$ . Consequently, by Lemma 4.3, we have  $a_1 \in K^n$ , that is,  $a_1 = b_1^n$  for some  $b_1 \in K$ . Thus  $a^n = a_1^d = b_1^{dn}$ , so that  $a = \zeta b_1^d$ , where  $\zeta$  is a  $n$ th root of unity in  $K$ . We already noticed that  $\zeta = 1$ , hence  $a \in K^d$ .  $\square$

**Lemma 5.5.** *Let  $n$  be an odd positive integer with  $3 \nmid n$  whenever  $\Delta_0 = -3$ , and let  $d$  be a positive integer dividing  $n$ . Then*

$$(6) \quad [K_{n,d} : \mathbb{Q}] = \frac{\varphi(n)d}{(d, h)} \cdot \begin{cases} 1 & \text{if } \sqrt{\Delta} \in \mathbb{Q}(\zeta_n), \\ 2 & \text{if } \sqrt{\Delta} \notin \mathbb{Q}(\zeta_n), \end{cases}$$

while

$$(7) \quad |\Delta_{K_{n,d}}|^{1/[K_{n,d}:\mathbb{Q}]} \ll_U n^3 \quad \text{and} \quad \log |\Delta_{K_{n,d}}| \ll_U n^2 \log(n+1).$$

Moreover, there exists  $\sigma \in \text{Gal}(K_{n,d}/\mathbb{Q})$  satisfying (3) if and only if  $\sqrt{\Delta} \notin \mathbb{Q}(\zeta_n)$  or  $\Delta < 0$ . In particular, if  $\sigma$  exists then it belongs to the center of  $\text{Gal}(K_{n,d}/\mathbb{Q})$ .

*Proof.* Let  $d_0 := d/(d, h)$ ,  $h_0 := h/(d, h)$ , and  $f(X) = X^{d_0} - \gamma_0^{h_0}$ . Suppose that  $\gamma_0^{h_0} \in K(\zeta_n)^p$  for some prime number  $p$  dividing  $d_0$ . Then, by Lemma 5.4, we have  $\gamma_0^{h_0} \in K^p$ . In turn, by the maximality of  $h$ , it follows that  $p \mid h_0$ , which is impossible, since  $(d_0, h_0) = 1$ . Hence,  $\gamma_0^{h_0} \notin K(\zeta_n)^p$  for every prime number  $p$  dividing  $d_0$ . Consequently, by Lemma 4.2,  $f$  is irreducible over  $K(\zeta_n)$ . Thus  $K_{n,d} \cong K(\zeta_n)[X]/(f(X))$ , so that  $[K_{n,d} : K(\zeta_n)] = d_0$  and  $(\gamma^{1/d})^{d_0} = \gamma_0^{h_0}$ . It is easy to check that  $[K(\zeta_n) : \mathbb{Q}] = \varphi(n)$  if  $\sqrt{\Delta} \in \mathbb{Q}(\zeta_n)$ , and  $[K(\zeta_n) : \mathbb{Q}] = 2\varphi(n)$  otherwise. Hence, (6) follows.

Let  $s$  be a positive integer such that  $s\gamma_0 \in \mathcal{O}_K$ , and put  $g(X) := s^{d_0}f(X/s) = X^{d_0} - s^{d_0}\gamma_0^{h_0}$ . Since  $f$  is the minimal polynomial of  $\gamma^{1/d}$  over  $K(\zeta_n)$ , we get that  $g$  is the minimal polynomial of  $s\gamma^{1/d}$  over  $K(\zeta_n)$ . In particular, since  $g \in \mathcal{O}_K[X]$ , we have that  $s\gamma^{1/d} \in \mathcal{O}_{K_{n,d}}$ . Hence, from  $K_{n,d} = K(\zeta_n)(s\gamma^{1/d})$  it follows that

$$\begin{aligned} \Delta_{K_{n,d}/K(\zeta_n)} &\supseteq \text{disc}(g) \mathcal{O}_{K(\zeta_n)} = \prod_{1 \leq i < j \leq d_0} (s\gamma^{1/d}\zeta_{d_0}^i - s\gamma^{1/d}\zeta_{d_0}^j)^2 \mathcal{O}_{K(\zeta_n)} \\ &= (s\gamma^{1/d})^{d_0(d_0-1)} d_0^{d_0} \mathcal{O}_{K(\zeta_n)} = \gamma_0^{h_0(d_0-1)} (s^{d_0-1}d_0)^{d_0} \mathcal{O}_{K(\zeta_n)}, \end{aligned}$$

and

$$N_{K(\zeta_n)/\mathbb{Q}}(\Delta_{K_{n,d}/K(\zeta_n)}) = N_{K/\mathbb{Q}}(\gamma_0^{h_0})^{(d_0-1)[K(\zeta_n):K]} (s^{d_0-1}d_0)^{d_0[K(\zeta_n):\mathbb{Q}]} | (N_{K/\mathbb{Q}}(\gamma)sn)^\infty.$$

Also, a quick computation shows that  $\Delta_{K(\zeta_n)} \mid (4\Delta n)^\infty$ . Therefore, since

$$\Delta_{K_{n,d}} = \Delta_{K(\zeta_n)}^{[K_{n,d}:K(\zeta_n)]} N_{K(\zeta_n)/\mathbb{Q}}(\Delta_{K_{n,d}/K(\zeta_n)}),$$

we get that every prime factor of  $\Delta_{K_{n,d}}$  divides  $An$ , where  $A := 4\Delta N_{K/\mathbb{Q}}(\gamma)s$ . By Hensel's estimate (see, e.g., [11, comments after Theorem 7.3]), we have that

$$|\Delta_L|^{1/n_L} \leq n_L \prod_{p \mid \Delta_L} p,$$

for every Galois extension  $L/\mathbb{Q}$  of degree  $n_L$ . Consequently,

$$|\Delta_{K_{n,d}}|^{1/[K_{n,d}:\mathbb{Q}]} \leq [K_{n,d}:\mathbb{Q}]An \ll_U \varphi(n)dn \leq n^3,$$

and

$$\log |\Delta_{K_{n,d}}| \leq [K_{n,d}:\mathbb{Q}] (\log(n^3) + O_U(1)) \ll_U \varphi(n)d \log(n+1) \ll n^2 \log(n+1),$$

so that (7) is proved.

Suppose that there exists  $\sigma \in \text{Gal}(K_{n,d}/\mathbb{Q})$  satisfying (3). We shall prove that  $\sqrt{\Delta} \notin \mathbb{Q}(\zeta_n)$  or  $\Delta < 0$ . Assume that  $\sqrt{\Delta} \in \mathbb{Q}(\zeta_n)$ . On the one hand,  $\sigma(\gamma) = \sigma(\gamma^{1/d})^d = \gamma^{-1}$ , and consequently  $\sigma(\sqrt{\Delta}) = -\sqrt{\Delta}$ . On the other hand, since  $\sqrt{\Delta} \in \mathbb{Q}(\zeta_n)$  and  $\sigma(\zeta_n) = \zeta_n^{-1}$ , we have that  $\sigma(\sqrt{\Delta}) = \overline{\sqrt{\Delta}}$ . Therefore,  $\overline{\sqrt{\Delta}} = -\sqrt{\Delta}$  and so  $\Delta < 0$ . Now let us check that  $\sigma$  belongs to the center of  $\text{Gal}(K_{n,d}/\mathbb{Q})$ . Note that  $N_{K/\mathbb{Q}}(\gamma) = \gamma \sigma_K(\gamma) = \gamma\gamma^{-1} = 1$ . Also,  $N_{K/\mathbb{Q}}(\gamma_0^{h_0}) = N_{K/\mathbb{Q}}(\gamma_0^h) = N_{K/\mathbb{Q}}(\gamma) = 1$ , since  $d$  is odd and so  $h_0 \equiv h \pmod{2}$ . Therefore, for every  $\tau \in \text{Gal}(K_{n,d}/\mathbb{Q})$ , we have  $\tau(\gamma_0^{h_0}) = \gamma_0^{h_0}$ , if  $\tau|_K = \text{id}$ , or  $\tau(\gamma_0^{h_0}) = N_{K/\mathbb{Q}}(\gamma_0^{h_0})\gamma_0^{-h_0} = \gamma_0^{-h_0}$  if  $\tau|_K = \sigma_K$ . Consequently, recalling that  $(\gamma^{1/d})^{d_0} = \gamma_0^{h_0}$ , we have that  $\tau(\zeta_n) = \zeta_n^s$  and  $\tau(\gamma^{1/d}) = \zeta_{d_0}^t \gamma^{\pm 1/d}$  for some integers  $s, t$ . At this point, it can be easily checked that  $(\sigma\tau)(\zeta_n) = (\tau\sigma)(\zeta_n)$  and  $(\sigma\tau)(\gamma^{1/d}) = (\tau\sigma)(\gamma^{1/d})$ . Hence,  $\sigma$  belongs to the center of  $\text{Gal}(K_{n,d}/\mathbb{Q})$ .

Suppose that  $\sqrt{\Delta} \notin \mathbb{Q}(\zeta_n)$  or  $\Delta < 0$ . We shall prove the existence of  $\sigma \in \text{Gal}(K_{n,d}/\mathbb{Q})$  satisfying (3). It suffices to show that there exists  $\sigma_1 \in \text{Gal}(K(\zeta_n)/K)$  such that  $\sigma_1(\zeta_n) = \zeta_n^{-1}$

and  $\sigma_1|_K = \sigma_K$ . Indeed, recalling that  $K_{n,d} \cong K(\zeta_n)[X]/(f(X))$ , we can extend  $\sigma_1$  to an automorphism  $\sigma \in \text{Gal}(K_{n,d}/\mathbb{Q})$  that sends the root  $\gamma^{1/d}$  of  $f$  to the root  $\gamma^{-1/d}$  of

$$(\sigma_1 f)(X) = X^{d_0} - \sigma_1(\gamma_0^{h_0}) = X^{d_0} - N_{K/\mathbb{Q}}(\gamma_0^{h_0})\gamma_0^{-h_0} = X^{d_0} - \gamma_0^{-h_0},$$

and so  $\sigma$  satisfies (3). Pick  $\sigma_0 \in \text{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q})$  such that  $\sigma_0(\zeta_n) = \zeta_n^{-1}$ . If  $\sqrt{\Delta} \in \mathbb{Q}(\zeta_n)$  then  $K(\zeta_n) = \mathbb{Q}(\zeta_n)$ ,  $\Delta < 0$ , and  $\sigma_0(\sqrt{\Delta}) = \sqrt{\Delta} = -\sqrt{\Delta}$ , so we let  $\sigma_1 := \sigma_0$ . If  $\sqrt{\Delta} \notin \mathbb{Q}(\zeta_n)$  then  $X^2 - \Delta$  is the minimal polynomial of  $\sqrt{\Delta}$  over  $\mathbb{Q}(\zeta_n)$  and we can extend  $\sigma_0$  to  $\sigma_1 \in \text{Gal}(K(\zeta_n)/\mathbb{Q})$  such that  $\sigma_1(\sqrt{\Delta}) = -\sqrt{\Delta}$ .  $\square$

## 6. PROOF OF THEOREM 1.1

The proof proceeds similarly to [9, Section 2]. For all positive integers  $d, n$  with  $d \mid n$ , and for all  $x > 1$ , let us define

$$\pi_{U,n,d}(x) := \#\{p \leq x : p \nmid a_2\Delta, p \equiv \left(\frac{\Delta}{p}\right) \pmod{n}, d \mid \iota_U(p)\}.$$

In what follows, we will tacitly ignore the finitely many prime numbers dividing  $a_2\Delta$ .

**Lemma 6.1.** *For every positive integer  $d$  and for every  $x > 1$ , we have*

$$(8) \quad \mathcal{R}_U(d; x) = \sum_{v \mid d^\infty} \sum_{a \mid d} \mu(a) \pi_{U,dv,av}(x).$$

*Proof.* Every prime number  $p$  counted by the inner sum of (8) satisfies  $p \leq x$ ,  $p \equiv \left(\frac{\Delta}{p}\right) \pmod{dv}$ , and  $\iota_U(p) = vw$  for some integer  $w$ . Writing  $w = w_1 w_2$ , with  $w_1 := (w, d)$ , we get that the contribution of  $p$  to the inner sum of (8) is equal to  $\sum_{a \mid w_1} \mu(a)$ . Hence,

$$(9) \quad \sum_{a \mid d} \mu(a) \pi_{U,dv,av}(x) = \#\{p \leq x : p \equiv \left(\frac{\Delta}{p}\right) \pmod{dv}, v \mid \iota_U(p), (\iota_U(p)/v, d) = 1\}.$$

Now it suffices to show that

$$(10) \quad \mathcal{R}_U(d; x) = \sum_{v \mid d^\infty} \#\{p \leq x : p \equiv \left(\frac{\Delta}{p}\right) \pmod{dv}, v \mid \iota_U(p), (\iota_U(p)/v, d) = 1\}.$$

On the one hand, let  $p$  be a prime number counted on the right-hand side of (10). Note that this is counted only one, namely for  $v = (\iota_U(p), d^\infty)$ . Then, from  $\rho_U(p)\iota_U(p) = p - \left(\frac{\Delta}{p}\right)$ , it follows that  $d \mid \rho_U(p)$ . Hence,  $p$  is counted on the left-hand side of (10).

On the other hand, let  $p$  be a prime number counted by  $\mathcal{R}_U(d; x)$ . Then  $d \mid \rho_U(p)$  and, by Lemma 5.1,  $p \equiv \left(\frac{\Delta}{p}\right) \pmod{d}$ . Consequently, there is an integer  $v$  such that  $v \mid d^\infty$ ,  $p \equiv \left(\frac{\Delta}{p}\right) \pmod{dv}$ , and  $(\iota_U(p)/v, d) = 1$ . Hence,  $p$  is counted on the right-hand side of (10).  $\square$

**Lemma 6.2.** *Let  $n$  be an odd positive integer with  $3 \nmid n$  whenever  $\Delta_0 = -3$ , and let  $d$  be a positive integer dividing  $n$ . There exist absolute constants  $A, B > 0$  such that*

$$\pi_{U,n,d}(x) = \delta_{U,n,d} \text{Li}(x) + O_U\left(x \exp(-A(\log x)^{1/2}/n)\right)$$

for  $x \geq \exp(Bn^8)$ , where

$$(11) \quad \delta_{U,n,d} := \frac{(d, h)}{\varphi(n)d} \cdot \begin{cases} 1 & \text{if } \Delta > 0 \text{ or } \Delta_0 \not\equiv 1 \pmod{4} \text{ or } \Delta_0 \nmid n, \\ 2 & \text{otherwise.} \end{cases}$$

*Proof.* Put  $E := K_{n,d}$ ,  $F := \mathbb{Q}$ ,  $G := \text{Gal}(E/F)$ , and  $C = \{\text{id}, \sigma\}$  if there exists  $\sigma \in \text{Gal}(K_{n,d}/\mathbb{Q})$  satisfying (3), or  $C = \{\text{id}\}$  otherwise. By Lemma 5.5,  $\sigma$  belongs to the center of  $G$ , so that  $C$  is the union of conjugacy classes of  $G$ . By Lemma 5.2, we have that  $\pi_{U,n,d}(x)$  is the number of primes  $p$  not exceeding  $x$  and such that  $\left[\frac{E/F}{p}\right] \subseteq C$ . Thus, taking into account the bounds for the degree and the discriminant of  $E/F$  given in Lemma 5.5, and considering Lemma 4.4, the asymptotic formula follows by applying Theorem 4.5.  $\square$

**Lemma 6.3.** *Let  $d$  be an odd positive integer with  $3 \nmid d$  whenever  $\Delta_0 = -3$ . If  $x > 1$  and  $e^{\omega(d)} \leq y \leq \log x / \varphi(d)$ , then*

$$(12) \quad \sum_{\substack{v|d^\infty \\ v>y}} \sum_{a|d} \mu(a) \pi_{U,dv,av}(x) \ll \frac{x}{\log x} \cdot \frac{\omega(d)+1}{\varphi(d)} \cdot \frac{(\log y)^{\omega(d)}}{y}$$

and

$$\sum_{\substack{v|d^\infty \\ v>y}} \sum_{a|d} \mu(a) \delta_{U,dv,av} \ll_U \frac{\omega(d)+1}{\varphi(d)} \cdot \frac{(\log y)^{\omega(d)}}{y}.$$

*Proof.* Let  $\pi(m, r; x) := \#\{p \leq x : p \equiv r \pmod{m}\}$ . From (9) it follows that

$$(13) \quad \left| \sum_{a|d} \mu(a) \pi_{U,dv,av}(x) \right| \leq \pi_{U,dv,v}(x) \leq \pi(x; dv, \pm 1).$$

Moreover, letting  $x \rightarrow +\infty$ , Lemma 6.2 and the first inequality of (13) yield

$$(14) \quad \left| \sum_{a|d} \mu(a) \delta_{U,dv,av} \right| \leq \delta_{U,dv,v}.$$

Now we have  $M_d(x) := \#\{v \leq x : v | d^\infty\} \ll (\log x)^{\omega(d)}$ , for every  $x \geq 2$ . Hence, by partial summation and since  $y \geq e^{\omega(d)}$ , we obtain that

$$(15) \quad \sum_{\substack{v|d^\infty \\ v>y}} \frac{1}{v} = \frac{M_d(t)}{t} \Big|_{t=y}^{+\infty} + \int_y^{+\infty} \frac{M_d(t)}{t^2} dt \ll \int_y^{+\infty} \frac{(\log t)^{\omega(d)}}{t^2} dt \leq \frac{(\omega(d)+1)(\log y)^{\omega(d)}}{y}.$$

On the one hand, using the Brun–Titchmarsh inequality [3, Theorem 12.7]

$$\pi(m, r; x) \ll \frac{x}{\varphi(m) \log(x/m)},$$

holding for  $x > m$ , and (15) we get that

$$(16) \quad \sum_{\substack{v|d^\infty \\ v>y, dv \leq x^{2/3}}} \pi(dv, \pm 1; x) \ll \frac{x}{\varphi(d) \log x} \sum_{\substack{v|d^\infty \\ v>y}} \frac{1}{v} \ll \frac{x}{\log x} \cdot \frac{\omega(d)+1}{\varphi(d)} \cdot \frac{(\log y)^{\omega(d)}}{y}.$$

On the other hand, using the trivial bound  $\pi(m, \pm 1; x) \ll x/m$ , holding for  $x \geq 1$ , and (15) again, we find that

$$(17) \quad \sum_{\substack{v|d^\infty \\ dv > x^{2/3}}} \pi(dv, \pm 1; x) \ll \sum_{\substack{v|d^\infty \\ dv > x^{2/3}}} \frac{x}{dv} \leq \sum_{\substack{w|d^\infty \\ w > x^{2/3}}} \frac{x}{w} \ll x^{1/3} (\omega(d)+1) (\log x)^{\omega(d)}.$$

Putting together (16), (17), and (13), taking into account that  $\omega(d) \leq \log y$  and  $\varphi(d)y \leq \log x$ , we obtain (12). Finally, from (14), (11), and (15), we get

$$\sum_{\substack{v|d^\infty \\ v>y}} \sum_{a|d} \mu(a) \delta_{U,dv,av} \leq \sum_{\substack{v|d^\infty \\ v>y}} \delta_{U,dv,v} \ll_U \frac{1}{\varphi(d)} \sum_{\substack{v|d^\infty \\ v>y}} \frac{1}{v^2} \ll \frac{\omega(d)+1}{\varphi(d)} \cdot \frac{(\log y)^{\omega(d)}}{y},$$

as desired.  $\square$

**Lemma 6.4.** *Let  $d$  be an odd positive integer with  $3 \nmid d$  whenever  $\Delta_0 = -3$ . Then*

$$\sum_{v|d^\infty} \sum_{a|d} \mu(a) \delta_{U,dv,av} = \delta_U(d).$$

*Proof.* For every integer  $e$  dividing  $d^\infty$ , define

$$S_{d,e,h} := \sum_{\substack{v|d^\infty \\ e|v}} \sum_{a|d} \frac{\mu(a)(av, h)}{\varphi(dv)av}.$$

The value of  $S_{d,1,h}$  was computed in [9, Lemma 4] and a slight modification of the proof (precisely, replacing  $(h, d^\infty)$  with  $[e, (h, d^\infty)]$  in the last equation) yields

$$S_{d,e,h} = \frac{(d^\infty, h)}{d[(d^\infty, h), e]^2} \prod_{p|d} \left(1 - \frac{1}{p^2}\right)^{-1}.$$

At this point, by (11) and considering that  $\Delta_0 | dv$  if and only if  $e | v$ , where  $e := \Delta_0/(d, \Delta_0)$ , we have

$$\sum_{v|d^\infty} \sum_{a|d} \mu(a) \delta_{U,dv,av} = \begin{cases} S_{d,1,h} & \text{if } \Delta > 0 \text{ or } \Delta_0 \not\equiv 1 \pmod{4} \text{ or } \Delta_0 \nmid d^\infty \\ S_{d,1,h} + S_{d,e,h} & \text{otherwise} \end{cases} = \delta_U(d),$$

as claimed.  $\square$

*Proof of Theorem 1.1.* Let  $A, B > 0$  be the constants of Lemma 6.2. Assume that  $x \geq \exp(Be^{8\omega(d)}d^8)$  and put  $y := (\log x/B)^{1/8}/d$ . Note that  $e^{\omega(d)} \leq y \leq \log x/\varphi(d)$  and  $\log y \leq \log \log x$ , for every  $x \gg_B 1$ . By Lemma 6.1, Lemma 6.2, and Lemma 6.4, we obtain that

$$\begin{aligned} \mathcal{R}_U(d; x) &= \sum_{\substack{v|d^\infty \\ v \leq y}} \sum_{a|d} \mu(a) \pi_{U,dv,av}(x) + O(E_1) \\ &= \sum_{\substack{v|d^\infty \\ v \leq y}} \sum_{a|d} \mu(a) \delta_{U,dv,av} \text{Li}(x) + O(E_1) + O_U(E_2) \\ &= \delta_U(d) \text{Li}(x) + O(E_1) + O_U(E_2) + O(E_3), \end{aligned}$$

where, by Lemma 6.3, we have

$$E_1 := \sum_{\substack{v|d^\infty \\ v > y}} \sum_{a|d} \mu(a) \pi_{U,dv,av}(x) \ll \frac{x}{\log x} \cdot \frac{\omega(d) + 1}{\varphi(d)} \cdot \frac{(\log y)^{\omega(d)}}{y} \ll \frac{(\omega(d) + 1)d}{\varphi(d)} \cdot \frac{x (\log \log x)^{\omega(d)}}{(\log x)^{9/8}}$$

and

$$E_3 := \sum_{\substack{v|d^\infty \\ v > y}} \sum_{a|d} \mu(a) \delta_{U,dv,av} \text{Li}(x) \ll_U \frac{\omega(d) + 1}{\varphi(d)} \cdot \frac{(\log y)^{\omega(d)}}{y} \cdot \text{Li}(x) \ll \frac{(\omega(d) + 1)d}{\varphi(d)} \cdot \frac{x (\log \log x)^{\omega(d)}}{(\log x)^{9/8}},$$

while, also using the inequality  $\tau(d)/d \leq d/\varphi(d)$ , we have

$$\begin{aligned} E_2 &:= \sum_{\substack{v|d^\infty \\ v \leq y}} \sum_{a|d} x \exp(-A(\log x)^{1/2}/(dv)) \ll x \exp(-AB^{1/8}(\log x)^{3/8}) \tau(d) y \\ &\ll x \exp(-AB^{1/8}(\log x)^{3/8}) (\log x)^{1/8} \cdot \frac{\tau(d)}{d} \ll \frac{d}{\varphi(d)} \cdot \frac{x}{(\log x)^{9/8}}. \end{aligned}$$

The result follows.  $\square$

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