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ON THE DIVISIBILITY OF THE RANK OF APPEARANCE OF A LUCAS SEQUENCE

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ABSTRACT. Let $U = (U_n)_{n \geq 0}$ be a Lucas sequence and, for every prime number p , let $\rho_U(p)$ be the rank of appearance of p in U , that is, the smallest positive integer k such that p divides U_k , whenever it exists. Furthermore, let d be an odd positive integer. Under some mild hypotheses, we prove an asymptotic formula for the number of primes $p \leq x$ such that d divides $\rho_U(p)$, as $x \rightarrow +\infty$.

1. INTRODUCTION

Let $(U_n)_{n \geq 0}$ be a Lucas sequence, that is, a sequence of integers satisfying $U_0 = 0$, $U_1 = 1$, and $U_n = a_1 U_{n-1} + a_2 U_{n-2}$ for every integer $n \geq 2$, where a_1, a_2 are fixed nonzero integers. The *rank of appearance* of a prime number p , denoted by $\rho_U(p)$, is the smallest positive integer k such that $p \mid U_k$. It can be easily seen that $\rho_U(p)$ exists whenever $p \nmid a_2$. Define

$$\mathcal{R}_U(d; x) := \#\{p \leq x : p \nmid a_2, d \mid \rho_U(p)\},$$

for every positive integer d and for every $x > 1$.

Let $(F_n)_{n \geq 0}$ be the Lucas sequence of Fibonacci numbers, corresponding to $a_1 = a_2 = 1$. In 1985, Lagarias [5] (see [6] for a correction and [8, 10] for generalizations) showed that $\mathcal{R}_F(2; x) \sim \frac{2}{3}x$, as $x \rightarrow +\infty$. More recently, Cubre and Rouse [2], settling a conjecture of Bruckman and Anderson [1], proved that $\mathcal{R}_F(d; x) \sim c(d) d^{-1} \prod_{p \mid d} (1 - p^{-2})^{-1}$, as $x \rightarrow +\infty$, for every positive integer d , where $c(d)$ is equal to 1, $\frac{5}{4}$, or $\frac{1}{2}$, whenever $10 \nmid d$, $d \equiv 10 \pmod{20}$, or $20 \mid d$, respectively.

Let α, β be the roots of the characteristic polynomial $f_U(X) := X^2 - a_1 X - a_2$, and assume that $\gamma := \alpha/\beta$ is not a root of unity. Let $\Delta := a_1^2 + 4a_2$ be the discriminant of $f_U(X)$, and let Δ_0 be the squarefree part of Δ . Assume that Δ is not a square, so that $K := \mathbb{Q}(\sqrt{\Delta})$ is a quadratic number field. Let h be the greatest positive integer such that γ is a h th power in K .

Our result is the following:

Theorem 1.1. *Let d be an odd positive integer with $3 \nmid d$ whenever $\Delta_0 = -3$. Then, for every $x > \exp(Be^{8\omega(d)}d^8)$, we have*

$$\mathcal{R}_U(d; x) = \delta_U(d) \operatorname{Li}(x) + O_U \left(\frac{(\omega(d) + 1)d}{\varphi(d)} \cdot \frac{x (\log \log x)^{\omega(d)}}{(\log x)^{9/8}} \right),$$

where $B > 0$ is an absolute constant and

$$\delta_U(d) := \frac{1}{d} \left(\frac{1}{(d^\infty, h)} + \eta_U(d) \right) \prod_{p \mid d} \left(1 - \frac{1}{p^2} \right)^{-1},$$

with $\eta_U(d) := 0$ if $\Delta > 0$ or $\Delta_0 \not\equiv 1 \pmod{4}$ or $\Delta_0 \nmid d^\infty$; and

$$\eta_U(d) := \frac{(d^\infty, h)}{[(d^\infty, h), \Delta_0/(d, \Delta_0)]^2}$$

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otherwise.

Cubre and Rouse's proof of the asymptotic formula for $\mathcal{R}_F(d; x)$ relies on the study of the algebraic group $G : x^2 - 5y^2 = 1$ and relates $\rho_F(p)$ with the order of $(3/2, 1/2) \in G(\mathbb{F}_p)$. Instead, our proof of Theorem 1.1 is an adaptation of the methods that Moree [9] used to prove an asymptotic formula for the number of primes $p \leq x$ such that the multiplicative order of g modulo p is divisible by d , where $g \notin \{-1, 0, +1\}$ is a fixed rational number.

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3. NOTATION

We employ the Landau–Bachmann “Big Oh” notation O , as well as the associated Vinogradov symbol \ll . Any dependence of the implied constants is explicitly stated or indicated with subscripts. In particular, notations like O_U and \ll_U are shortcuts for O_{a_1, a_2} and \ll_{a_1, a_2} , respectively. For $x \geq 2$ we let $\text{Li}(x) := \int_2^x \frac{dt}{\log t}$ denote the logarithmic integral. We reserve the letter p for prime numbers. Given an integer d , we let d^∞ denote the supernatural number $\prod_{p|d} p^\infty$. Given a field F and a positive integer n , we write F^n for the set of n th powers of elements of F . Given a Galois extension E/F of number fields and a prime ideal P of \mathcal{O}_E lying above an unramified prime ideal \mathfrak{p} of \mathcal{O}_F , we write $\left[\frac{E/F}{P}\right]$ for the Frobenius automorphism corresponding to P/\mathfrak{p} , that is, the unique element σ of the Galois group $\text{Gal}(E/F)$ that satisfies $\sigma(a) \equiv a^{N(\mathfrak{p})} \pmod{P}$ for every $a \in \mathcal{O}_E$, where $N(\mathfrak{p})$ denotes the norm of \mathfrak{p} . Moreover, we let $\left[\frac{E/F}{\mathfrak{p}}\right]$ be the set of all $\left[\frac{E/F}{P}\right]$ with P prime ideal of \mathcal{O}_E lying over \mathfrak{p} . We write $\Delta_{E/F}$ for the relative discriminant of E/F , and $\Delta_E := \Delta_{E/\mathbb{Q}}$ for the absolute discriminant of E . For every integer d and for every prime number p we let $\left(\frac{d}{p}\right)$ be the Legendre symbol. For every positive integer n , we let $\zeta_n := e^{2\pi i/n}$ be a primitive n th root of unity. We write $\omega(n)$, $\varphi(n)$, $\mu(n)$, and $\tau(n)$, for the number of prime factors, the totient function, the Möbius function, and the number of divisors of a positive integer n , respectively.

4. GENERAL PRELIMINARIES

Lemma 4.1. *Let n be a positive integer, let p be a prime number not dividing n , and let P be a prime ideal of $\mathcal{O}_{\mathbb{Q}(\zeta_n)}$ lying over p . Then ζ_n has multiplicative order modulo P equal to n .*

Proof. Let k be the multiplicative order of ζ_n modulo P , that is, k is the least positive integer such that $\zeta_n^k \equiv 1 \pmod{P}$. On the one hand, we have that $p \mid N(P) \mid N(\zeta_n^k - 1)$. On the other hand, since $\zeta_n^n \equiv 1 \pmod{P}$, we have that $k \mid n$, and consequently ζ_n^k is a m th primitive root of unity, where $m := n/k$. If $k < n$ then $m > 1$ and $N(\zeta_n^k - 1)$ is either 1 or a prime factor of m , but both cases are impossible since $p \nmid n$. Hence, $k = n$. \square

Lemma 4.2. *Let F be a field, let $a \in F$, and let n be a positive integer. Then $X^n - a$ is irreducible over F if and only if $a \notin F^p$ for each prime p dividing n and $a \notin -4F^4$ whenever $4 \mid n$.*

Proof. See [4, Chapter 8, Theorem 1.6]. \square

Lemma 4.3. *Let F be a field, let n be a positive integer not divisible by the characteristic of F , and let m be the number of n th roots of unity contained in F . Then, for every $a \in F$, the extension $F(\zeta_n, a^{1/n})/F$ is abelian if and only if $a^m \in F^n$.*

Proof. See [4, Chapter 8, Theorem 3.2]. \square

Lemma 4.4. *Let n be an odd positive integer and let d be a squarefree integer. Then $\sqrt{d} \in \mathbb{Q}(\zeta_n)$ if and only if $d \mid n$ and $d \equiv 1 \pmod{4}$.*

Proof. See [12, Lemma 3]. \square

We need the following form of the Chebotarev Density Theorem.

Theorem 4.5. *Let E/F be a Galois extension of number fields with Galois group G , and let C be the union of k conjugacy classes of G . Then*

$$\begin{aligned} & \#\left\{ \mathfrak{p} \text{ prime ideal of } \mathcal{O}_F \text{ non-ramifying in } E : N_{F/\mathbb{Q}}(\mathfrak{p}) \leq x, \left[\frac{E/F}{\mathfrak{p}} \right] \subseteq C \right\} \\ &= \frac{\#C}{\#G} \cdot \text{Li}(x) + O\left(kx \exp\left(-c_1(\log x/n_E)^{1/2}\right) \right) \end{aligned}$$

for every

$$x \geq \exp\left(c_2 \max\left(n_E(\log |\Delta_E|)^2, |\Delta_E|^{2/n_E}/n_E\right)\right),$$

where $n_E := [E : \mathbb{Q}]$ and $c_1, c_2 > 0$ are absolute constants.

Proof. The result follows from the effective form of the Chebotarev Density Theorem given by Lagarias and Odlyzko [7, Theorem 1.3] and from the bounds for the exceptional zero of the Dedekind zeta function ζ_E given by Stark [13, Lemma 8 and 11]. \square

5. PRELIMINARIES TO THE PROOF OF THEOREM 1.1

Recalling that h is the greatest positive integer such that γ is an h th power in K , write $\gamma = \gamma_0^h$ for some $\gamma_0 \in K$. Also, let $\sigma_K \in \text{Gal}(K/\mathbb{Q})$ be the nontrivial automorphism, which satisfies $\sigma_K(\sqrt{\Delta}) = -\sqrt{\Delta}$. Note that, since $\gamma = \alpha/\beta$ and σ_K swaps α and β , we have that $\sigma_K(\gamma) = \gamma^{-1}$. For all positive integers d, n such that $d \mid n$, let $K_{n,d} := K(\zeta_n, \gamma^{1/d})$.

Lemma 5.1. *Let p be a prime number not dividing $a_2\Delta$ and let π be a prime ideal of \mathcal{O}_K lying over p . Then $\rho_U(p)$ is equal to the multiplicative order of γ modulo π . Moreover, $\rho_U(p)$ divides $p - \left(\frac{\Delta}{p}\right)$.*

Proof. First, note that $p \nmid a_2$ ensures that β is invertible modulo π , and consequently it makes sense to consider the multiplicative order of $\gamma = \alpha/\beta$ modulo π . Also, $p \nmid \Delta$ implies that p does not ramify in K and that $\alpha \not\equiv \beta \pmod{\pi}$.

We shall prove that $p \mid U_n$ if and only if $\gamma^n \equiv 1 \pmod{\pi}$, for every positive integer n . Then the claim on $\rho_U(p)$ follows easily. It is well known that the Binet's formula

$$(1) \quad U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$$

holds for every positive integer n . On the one hand, if $p \mid U_n$ then, since $p\mathcal{O}_K \subseteq \pi$ and (1), we have $\alpha^n \equiv \beta^n \pmod{\pi}$, and consequently $\gamma^n \equiv 1 \pmod{\pi}$. On the other hand, if $\gamma^n \equiv 1 \pmod{\pi}$ then by (1) we get $U_n \equiv 0 \pmod{\pi}$. If p is inert in K , then $p\mathcal{O}_K = \pi$ and so $p \mid U_n$. If p splits in K , then $p\mathcal{O}_K = \pi \cap \sigma_K(\pi)$. Thus $U_n \equiv 0 \pmod{\pi}$ and $U_n \equiv \sigma_K(U_n) \equiv 0 \pmod{\sigma_K(\pi)}$ imply that $p \mid U_n$.

Let $\sigma := \left[\frac{K/\mathbb{Q}}{\pi}\right]$. On the one hand, if $\left(\frac{\Delta}{p}\right) = -1$ then $\sigma = \sigma_K$ and $\gamma^{p+1} \equiv \sigma_K(\gamma)\gamma \equiv \gamma^{-1}\gamma \equiv 1 \pmod{\pi}$, so that $\rho_U(p) \mid p+1$. On the other hand, if $\left(\frac{\Delta}{p}\right) = +1$ then $\sigma = \text{id}$ and $\gamma^{p-1} \equiv \gamma\gamma^{-1} \equiv 1 \pmod{\pi}$, so that $\rho_U(p) \mid p-1$. \square

For each prime number p not dividing $a_2\Delta$, let us define the *index of appearance* of p as

$$\iota_U(p) := \left(p - \left(\frac{\Delta}{p}\right)\right) / \rho_U(p).$$

Note that, in light of Lemma 5.1, $\iota_U(p)$ is an integer.

Lemma 5.2. *Let d, n be positive integers such that $d \mid n$, and let p be a prime number not dividing $a_2\Delta$. Moreover, let P be a prime ideal of $\mathcal{O}_{K_{n,d}}$ lying over p and let $\sigma := \left[\frac{K_{n,d}/\mathbb{Q}}{P}\right]$. Then*

$$(2) \quad p \equiv \left(\frac{\Delta}{p}\right) \pmod{n} \quad \text{and} \quad d \mid \iota_U(p)$$

if and only if $\sigma = \text{id}$ or

$$(3) \quad \sigma(\zeta_n) = \zeta_n^{-1} \quad \text{and} \quad \sigma(\gamma^{1/d}) = \gamma^{-1/d}.$$

Proof. First, suppose that $\left(\frac{\Delta}{p}\right) = -1$. Let us assume (2). On the one hand, since $p \equiv -1 \pmod{n}$, we have

$$(4) \quad \sigma(\zeta_n) \equiv \zeta_n^p \equiv \zeta_n^{-1} \pmod{P}.$$

Since $\sigma(\zeta_n) = \zeta_n^k$ for some integer k , and since p does not divide n , Lemma 4.1 and (4) yield that $\sigma(\zeta_n) = \zeta_n^{-1}$.

On the other hand, $d \mid \iota_U(p)$ implies that $\rho_U(p) \mid (p+1)/d$. Hence, letting $\pi := P \cap \mathcal{O}_K$, Lemma 5.1 yields $\gamma^{(p+1)/d} \equiv 1 \pmod{\pi}$. Consequently,

$$(5) \quad \sigma(\gamma^{1/d}) \equiv (\gamma^{1/d})^p \equiv \gamma^{(p+1)/d} \cdot \gamma^{-1/d} \equiv \gamma^{-1/d} \pmod{P}.$$

Note that, since $\left(\frac{\Delta}{p}\right) = -1$, we have

$$\sigma(\gamma) = \sigma|_K(\gamma) = \left[\frac{K/\mathbb{Q}}{\pi}\right](\gamma) = \sigma_K(\gamma) = \gamma^{-1},$$

so that $\sigma(\gamma^{1/d}) = \zeta_d^k \gamma^{-1/d}$ for some integer k . Thus Lemma 4.1 and (5) yield that $\sigma(\gamma^{1/d}) = \gamma^{-1/d}$. We have proved (3).

Now let us assume (3). On the one hand, we have

$$\zeta_n^{-1} = \sigma(\zeta_n) = \sigma|_{\mathbb{Q}(\zeta_n)}(\zeta_n) = \left[\frac{\mathbb{Q}(\zeta_n)/\mathbb{Q}}{P \cap \mathcal{O}_{\mathbb{Q}(\zeta_n)}}\right](\zeta_n) = \zeta_n^p,$$

so that $p \equiv -1 \pmod{n}$. On the other hand,

$$\gamma^{(p+1)/d} \equiv (\gamma^{1/d})^p \cdot \gamma^{1/d} \equiv \sigma(\gamma^{1/d}) \cdot \gamma^{1/d} \equiv \gamma^{-1/d} \cdot \gamma^{1/d} \equiv 1 \pmod{P},$$

so that $\gamma^{(p+1)/d} \equiv 1 \pmod{\pi}$, which, by Lemma 5.1, implies $d \mid \iota_U(p)$. We have proved (2).

If $\left(\frac{\Delta}{p}\right) = +1$ then the proof proceeds similarly to the case $\left(\frac{\Delta}{p}\right) = -1$, and yields that (2) is equivalent to $\sigma(\zeta_n) = \zeta_n$ and $\sigma(\gamma^{1/d}) = \gamma^{1/d}$, that is, $\sigma = \text{id}$. \square

Lemma 5.3. *The roots of unity contained in K are: the sixth roots of unity, if $\Delta_0 = -3$; the fourth roots of unity, if $\Delta_0 = -1$; or the second roots of unity, if $\Delta_0 \neq -1, -3$.*

Proof. If $\zeta_n \in K$ for some positive integer n , then $\mathbb{Q}(\zeta_n) \subseteq K$, so that $\varphi(n) \leq 2$, and $n \in \{1, 2, 3, 4, 6\}$. Then the claim follows easily since $\zeta_3 = (-1 + \sqrt{-3})/2$, $\zeta_4 = \sqrt{-1}$, and $\zeta_6 = (1 + \sqrt{-3})/2$. \square

Lemma 5.4. *Let n be an odd positive integer with $3 \nmid n$ whenever $\Delta_0 = -3$, and let d be a positive integer dividing n . Then $a \in K \cap K(\zeta_n)^d$ if and only if $a \in K^d$.*

Proof. The ‘‘if’’ part is obvious. Let us prove the ‘‘only if’’ part. Note that, by the hypothesis on n and by Lemma 5.3, the only n th root of unity in K is 1. Suppose that $a \in K \cap K(\zeta_n)^d$. Hence, there exists $b \in K(\zeta_n)$ such that $a = b^d$. Putting $a_1 := a^{n/d}$, we get that $a_1 = b^n$. Therefore, $K(\zeta_n, a_1^{1/n}) = K(\zeta_n, b) = K(\zeta_n)$ is an abelian extension of K . Consequently, by Lemma 4.3, we have $a_1 \in K^n$, that is, $a_1 = b_1^n$ for some $b_1 \in K$. Thus $a^n = a_1^d = b_1^{dn}$, so that $a = \zeta b_1^d$, where ζ is a n th root of unity in K . We already noticed that $\zeta = 1$, hence $a \in K^d$. \square

Lemma 5.5. *Let n be an odd positive integer with $3 \nmid n$ whenever $\Delta_0 = -3$, and let d be a positive integer dividing n . Then*

$$(6) \quad [K_{n,d} : \mathbb{Q}] = \frac{\varphi(n)d}{(d, h)} \cdot \begin{cases} 1 & \text{if } \sqrt{\Delta} \in \mathbb{Q}(\zeta_n), \\ 2 & \text{if } \sqrt{\Delta} \notin \mathbb{Q}(\zeta_n), \end{cases}$$

while

$$(7) \quad |\Delta_{K_{n,d}}|^{1/[K_{n,d}:\mathbb{Q}]} \ll_U n^3 \quad \text{and} \quad \log |\Delta_{K_{n,d}}| \ll_U n^2 \log(n+1).$$

Moreover, there exists $\sigma \in \text{Gal}(K_{n,d}/\mathbb{Q})$ satisfying (3) if and only if $\sqrt{\Delta} \notin \mathbb{Q}(\zeta_n)$ or $\Delta < 0$. In particular, if σ exists then it belongs to the center of $\text{Gal}(K_{n,d}/\mathbb{Q})$.

Proof. Let $d_0 := d/(d, h)$, $h_0 := h/(d, h)$, and $f(X) = X^{d_0} - \gamma_0^{h_0}$. Suppose that $\gamma_0^{h_0} \in K(\zeta_n)^p$ for some prime number p dividing d_0 . Then, by Lemma 5.4, we have $\gamma_0^{h_0} \in K^p$. In turn, by the maximality of h , it follows that $p \mid h_0$, which is impossible, since $(d_0, h_0) = 1$. Hence, $\gamma_0^{h_0} \notin K(\zeta_n)^p$ for every prime number p dividing d_0 . Consequently, by Lemma 4.2, f is irreducible over $K(\zeta_n)$. Thus $K_{n,d} \cong K(\zeta_n)[X]/(f(X))$, so that $[K_{n,d} : K(\zeta_n)] = d_0$ and $(\gamma^{1/d})^{d_0} = \gamma_0^{h_0}$. It is easy to check that $[K(\zeta_n) : \mathbb{Q}] = \varphi(n)$ if $\sqrt{\Delta} \in \mathbb{Q}(\zeta_n)$, and $[K(\zeta_n) : \mathbb{Q}] = 2\varphi(n)$ otherwise. Hence, (6) follows.

Let s be a positive integer such that $s\gamma_0 \in \mathcal{O}_K$, and put $g(X) := s^{d_0}f(X/s) = X^{d_0} - s^{d_0}\gamma_0^{h_0}$. Since f is the minimal polynomial of $\gamma^{1/d}$ over $K(\zeta_n)$, we get that g is the minimal polynomial of $s\gamma^{1/d}$ over $K(\zeta_n)$. In particular, since $g \in \mathcal{O}_K[X]$, we have that $s\gamma^{1/d} \in \mathcal{O}_{K_{n,d}}$. Hence, from $K_{n,d} = K(\zeta_n)(s\gamma^{1/d})$ it follows that

$$\begin{aligned} \Delta_{K_{n,d}/K(\zeta_n)} &\supseteq \text{disc}(g) \mathcal{O}_{K(\zeta_n)} = \prod_{1 \leq i < j \leq d_0} (s\gamma^{1/d}\zeta_{d_0}^i - s\gamma^{1/d}\zeta_{d_0}^j)^2 \mathcal{O}_{K(\zeta_n)} \\ &= (s\gamma^{1/d})^{d_0(d_0-1)} d_0^{d_0} \mathcal{O}_{K(\zeta_n)} = \gamma_0^{h_0(d_0-1)} (s^{d_0-1}d_0)^{d_0} \mathcal{O}_{K(\zeta_n)}, \end{aligned}$$

and

$$N_{K(\zeta_n)/\mathbb{Q}}(\Delta_{K_{n,d}/K(\zeta_n)}) = N_{K/\mathbb{Q}}(\gamma_0^{h_0})^{(d_0-1)[K(\zeta_n):K]} (s^{d_0-1}d_0)^{d_0[K(\zeta_n):\mathbb{Q}]} | (N_{K/\mathbb{Q}}(\gamma)sn)^\infty.$$

Also, a quick computation shows that $\Delta_{K(\zeta_n)} \mid (4\Delta n)^\infty$. Therefore, since

$$\Delta_{K_{n,d}} = \Delta_{K(\zeta_n)}^{[K_{n,d}:K(\zeta_n)]} N_{K(\zeta_n)/\mathbb{Q}}(\Delta_{K_{n,d}/K(\zeta_n)}),$$

we get that every prime factor of $\Delta_{K_{n,d}}$ divides An , where $A := 4\Delta N_{K/\mathbb{Q}}(\gamma)s$. By Hensel's estimate (see, e.g., [11, comments after Theorem 7.3]), we have that

$$|\Delta_L|^{1/n_L} \leq n_L \prod_{p \mid \Delta_L} p,$$

for every Galois extension L/\mathbb{Q} of degree n_L . Consequently,

$$|\Delta_{K_{n,d}}|^{1/[K_{n,d}:\mathbb{Q}]} \leq [K_{n,d}:\mathbb{Q}]An \ll_U \varphi(n)dn \leq n^3,$$

and

$$\log |\Delta_{K_{n,d}}| \leq [K_{n,d}:\mathbb{Q}] (\log(n^3) + O_U(1)) \ll_U \varphi(n)d \log(n+1) \ll n^2 \log(n+1),$$

so that (7) is proved.

Suppose that there exists $\sigma \in \text{Gal}(K_{n,d}/\mathbb{Q})$ satisfying (3). We shall prove that $\sqrt{\Delta} \notin \mathbb{Q}(\zeta_n)$ or $\Delta < 0$. Assume that $\sqrt{\Delta} \in \mathbb{Q}(\zeta_n)$. On the one hand, $\sigma(\gamma) = \sigma(\gamma^{1/d})^d = \gamma^{-1}$, and consequently $\sigma(\sqrt{\Delta}) = -\sqrt{\Delta}$. On the other hand, since $\sqrt{\Delta} \in \mathbb{Q}(\zeta_n)$ and $\sigma(\zeta_n) = \zeta_n^{-1}$, we have that $\sigma(\sqrt{\Delta}) = \overline{\sqrt{\Delta}}$. Therefore, $\overline{\sqrt{\Delta}} = -\sqrt{\Delta}$ and so $\Delta < 0$. Now let us check that σ belongs to the center of $\text{Gal}(K_{n,d}/\mathbb{Q})$. Note that $N_{K/\mathbb{Q}}(\gamma) = \gamma \sigma_K(\gamma) = \gamma\gamma^{-1} = 1$. Also, $N_{K/\mathbb{Q}}(\gamma_0^{h_0}) = N_{K/\mathbb{Q}}(\gamma_0^h) = N_{K/\mathbb{Q}}(\gamma) = 1$, since d is odd and so $h_0 \equiv h \pmod{2}$. Therefore, for every $\tau \in \text{Gal}(K_{n,d}/\mathbb{Q})$, we have $\tau(\gamma_0^{h_0}) = \gamma_0^{h_0}$, if $\tau|_K = \text{id}$, or $\tau(\gamma_0^{h_0}) = N_{K/\mathbb{Q}}(\gamma_0^{h_0})\gamma_0^{-h_0} = \gamma_0^{-h_0}$ if $\tau|_K = \sigma_K$. Consequently, recalling that $(\gamma^{1/d})^{d_0} = \gamma_0^{h_0}$, we have that $\tau(\zeta_n) = \zeta_n^s$ and $\tau(\gamma^{1/d}) = \zeta_{d_0}^t \gamma^{\pm 1/d}$ for some integers s, t . At this point, it can be easily checked that $(\sigma\tau)(\zeta_n) = (\tau\sigma)(\zeta_n)$ and $(\sigma\tau)(\gamma^{1/d}) = (\tau\sigma)(\gamma^{1/d})$. Hence, σ belongs to the center of $\text{Gal}(K_{n,d}/\mathbb{Q})$.

Suppose that $\sqrt{\Delta} \notin \mathbb{Q}(\zeta_n)$ or $\Delta < 0$. We shall prove the existence of $\sigma \in \text{Gal}(K_{n,d}/\mathbb{Q})$ satisfying (3). It suffices to show that there exists $\sigma_1 \in \text{Gal}(K(\zeta_n)/K)$ such that $\sigma_1(\zeta_n) = \zeta_n^{-1}$

and $\sigma_1|_K = \sigma_K$. Indeed, recalling that $K_{n,d} \cong K(\zeta_n)[X]/(f(X))$, we can extend σ_1 to an automorphism $\sigma \in \text{Gal}(K_{n,d}/\mathbb{Q})$ that sends the root $\gamma^{1/d}$ of f to the root $\gamma^{-1/d}$ of

$$(\sigma_1 f)(X) = X^{d_0} - \sigma_1(\gamma_0^{h_0}) = X^{d_0} - N_{K/\mathbb{Q}}(\gamma_0^{h_0})\gamma_0^{-h_0} = X^{d_0} - \gamma_0^{-h_0},$$

and so σ satisfies (3). Pick $\sigma_0 \in \text{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q})$ such that $\sigma_0(\zeta_n) = \zeta_n^{-1}$. If $\sqrt{\Delta} \in \mathbb{Q}(\zeta_n)$ then $K(\zeta_n) = \mathbb{Q}(\zeta_n)$, $\Delta < 0$, and $\sigma_0(\sqrt{\Delta}) = \sqrt{\Delta} = -\sqrt{\Delta}$, so we let $\sigma_1 := \sigma_0$. If $\sqrt{\Delta} \notin \mathbb{Q}(\zeta_n)$ then $X^2 - \Delta$ is the minimal polynomial of $\sqrt{\Delta}$ over $\mathbb{Q}(\zeta_n)$ and we can extend σ_0 to $\sigma_1 \in \text{Gal}(K(\zeta_n)/\mathbb{Q})$ such that $\sigma_1(\sqrt{\Delta}) = -\sqrt{\Delta}$. \square

6. PROOF OF THEOREM 1.1

The proof proceeds similarly to [9, Section 2]. For all positive integers d, n with $d \mid n$, and for all $x > 1$, let us define

$$\pi_{U,n,d}(x) := \#\{p \leq x : p \nmid a_2\Delta, p \equiv \left(\frac{\Delta}{p}\right) \pmod{n}, d \mid \iota_U(p)\}.$$

In what follows, we will tacitly ignore the finitely many prime numbers dividing $a_2\Delta$.

Lemma 6.1. *For every positive integer d and for every $x > 1$, we have*

$$(8) \quad \mathcal{R}_U(d; x) = \sum_{v \mid d^\infty} \sum_{a \mid d} \mu(a) \pi_{U,dv,av}(x).$$

Proof. Every prime number p counted by the inner sum of (8) satisfies $p \leq x$, $p \equiv \left(\frac{\Delta}{p}\right) \pmod{dv}$, and $\iota_U(p) = vw$ for some integer w . Writing $w = w_1 w_2$, with $w_1 := (w, d)$, we get that the contribution of p to the inner sum of (8) is equal to $\sum_{a \mid w_1} \mu(a)$. Hence,

$$(9) \quad \sum_{a \mid d} \mu(a) \pi_{U,dv,av}(x) = \#\{p \leq x : p \equiv \left(\frac{\Delta}{p}\right) \pmod{dv}, v \mid \iota_U(p), (\iota_U(p)/v, d) = 1\}.$$

Now it suffices to show that

$$(10) \quad \mathcal{R}_U(d; x) = \sum_{v \mid d^\infty} \#\{p \leq x : p \equiv \left(\frac{\Delta}{p}\right) \pmod{dv}, v \mid \iota_U(p), (\iota_U(p)/v, d) = 1\}.$$

On the one hand, let p be a prime number counted on the right-hand side of (10). Note that this is counted only one, namely for $v = (\iota_U(p), d^\infty)$. Then, from $\rho_U(p)\iota_U(p) = p - \left(\frac{\Delta}{p}\right)$, it follows that $d \mid \rho_U(p)$. Hence, p is counted on the left-hand side of (10).

On the other hand, let p be a prime number counted by $\mathcal{R}_U(d; x)$. Then $d \mid \rho_U(p)$ and, by Lemma 5.1, $p \equiv \left(\frac{\Delta}{p}\right) \pmod{d}$. Consequently, there is an integer v such that $v \mid d^\infty$, $p \equiv \left(\frac{\Delta}{p}\right) \pmod{dv}$, and $(\iota_U(p)/v, d) = 1$. Hence, p is counted on the right-hand side of (10). \square

Lemma 6.2. *Let n be an odd positive integer with $3 \nmid n$ whenever $\Delta_0 = -3$, and let d be a positive integer dividing n . There exist absolute constants $A, B > 0$ such that*

$$\pi_{U,n,d}(x) = \delta_{U,n,d} \text{Li}(x) + O_U\left(x \exp(-A(\log x)^{1/2}/n)\right)$$

for $x \geq \exp(Bn^8)$, where

$$(11) \quad \delta_{U,n,d} := \frac{(d, h)}{\varphi(n)d} \cdot \begin{cases} 1 & \text{if } \Delta > 0 \text{ or } \Delta_0 \not\equiv 1 \pmod{4} \text{ or } \Delta_0 \nmid n, \\ 2 & \text{otherwise.} \end{cases}$$

Proof. Put $E := K_{n,d}$, $F := \mathbb{Q}$, $G := \text{Gal}(E/F)$, and $C = \{\text{id}, \sigma\}$ if there exists $\sigma \in \text{Gal}(K_{n,d}/\mathbb{Q})$ satisfying (3), or $C = \{\text{id}\}$ otherwise. By Lemma 5.5, σ belongs to the center of G , so that C is the union of conjugacy classes of G . By Lemma 5.2, we have that $\pi_{U,n,d}(x)$ is the number of primes p not exceeding x and such that $\left[\frac{E/F}{p}\right] \subseteq C$. Thus, taking into account the bounds for the degree and the discriminant of E/F given in Lemma 5.5, and considering Lemma 4.4, the asymptotic formula follows by applying Theorem 4.5. \square

Lemma 6.3. *Let d be an odd positive integer with $3 \nmid d$ whenever $\Delta_0 = -3$. If $x > 1$ and $e^{\omega(d)} \leq y \leq \log x / \varphi(d)$, then*

$$(12) \quad \sum_{\substack{v|d^\infty \\ v>y}} \sum_{a|d} \mu(a) \pi_{U,dv,av}(x) \ll \frac{x}{\log x} \cdot \frac{\omega(d)+1}{\varphi(d)} \cdot \frac{(\log y)^{\omega(d)}}{y}$$

and

$$\sum_{\substack{v|d^\infty \\ v>y}} \sum_{a|d} \mu(a) \delta_{U,dv,av} \ll_U \frac{\omega(d)+1}{\varphi(d)} \cdot \frac{(\log y)^{\omega(d)}}{y}.$$

Proof. Let $\pi(m, r; x) := \#\{p \leq x : p \equiv r \pmod{m}\}$. From (9) it follows that

$$(13) \quad \left| \sum_{a|d} \mu(a) \pi_{U,dv,av}(x) \right| \leq \pi_{U,dv,v}(x) \leq \pi(x; dv, \pm 1).$$

Moreover, letting $x \rightarrow +\infty$, Lemma 6.2 and the first inequality of (13) yield

$$(14) \quad \left| \sum_{a|d} \mu(a) \delta_{U,dv,av} \right| \leq \delta_{U,dv,v}.$$

Now we have $M_d(x) := \#\{v \leq x : v \mid d^\infty\} \ll (\log x)^{\omega(d)}$, for every $x \geq 2$. Hence, by partial summation and since $y \geq e^{\omega(d)}$, we obtain that

$$(15) \quad \sum_{\substack{v|d^\infty \\ v>y}} \frac{1}{v} = \frac{M_d(t)}{t} \Big|_{t=y}^{+\infty} + \int_y^{+\infty} \frac{M_d(t)}{t^2} dt \ll \int_y^{+\infty} \frac{(\log t)^{\omega(d)}}{t^2} dt \leq \frac{(\omega(d)+1)(\log y)^{\omega(d)}}{y}.$$

On the one hand, using the Brun–Titchmarsh inequality [3, Theorem 12.7]

$$\pi(m, r; x) \ll \frac{x}{\varphi(m) \log(x/m)},$$

holding for $x > m$, and (15) we get that

$$(16) \quad \sum_{\substack{v|d^\infty \\ v>y, dv \leq x^{2/3}}} \pi(dv, \pm 1; x) \ll \frac{x}{\varphi(d) \log x} \sum_{\substack{v|d^\infty \\ v>y}} \frac{1}{v} \ll \frac{x}{\log x} \cdot \frac{\omega(d)+1}{\varphi(d)} \cdot \frac{(\log y)^{\omega(d)}}{y}.$$

On the other hand, using the trivial bound $\pi(m, \pm 1; x) \ll x/m$, holding for $x \geq 1$, and (15) again, we find that

$$(17) \quad \sum_{\substack{v|d^\infty \\ dv > x^{2/3}}} \pi(dv, \pm 1; x) \ll \sum_{\substack{v|d^\infty \\ dv > x^{2/3}}} \frac{x}{dv} \leq \sum_{\substack{w|d^\infty \\ w > x^{2/3}}} \frac{x}{w} \ll x^{1/3} (\omega(d)+1) (\log x)^{\omega(d)}.$$

Putting together (16), (17), and (13), taking into account that $\omega(d) \leq \log y$ and $\varphi(d)y \leq \log x$, we obtain (12). Finally, from (14), (11), and (15), we get

$$\sum_{\substack{v|d^\infty \\ v>y}} \sum_{a|d} \mu(a) \delta_{U,dv,av} \leq \sum_{\substack{v|d^\infty \\ v>y}} \delta_{U,dv,v} \ll_U \frac{1}{\varphi(d)} \sum_{\substack{v|d^\infty \\ v>y}} \frac{1}{v^2} \ll \frac{\omega(d)+1}{\varphi(d)} \cdot \frac{(\log y)^{\omega(d)}}{y},$$

as desired. \square

Lemma 6.4. *Let d be an odd positive integer with $3 \nmid d$ whenever $\Delta_0 = -3$. Then*

$$\sum_{v|d^\infty} \sum_{a|d} \mu(a) \delta_{U,dv,av} = \delta_U(d).$$

Proof. For every integer e dividing d^∞ , define

$$S_{d,e,h} := \sum_{\substack{v|d^\infty \\ e|v}} \sum_{a|d} \frac{\mu(a)(av, h)}{\varphi(dv)av}.$$

The value of $S_{d,1,h}$ was computed in [9, Lemma 4] and a slight modification of the proof (precisely, replacing (h, d^∞) with $[e, (h, d^\infty)]$ in the last equation) yields

$$S_{d,e,h} = \frac{(d^\infty, h)}{d[(d^\infty, h), e]^2} \prod_{p|d} \left(1 - \frac{1}{p^2}\right)^{-1}.$$

At this point, by (11) and considering that $\Delta_0 | dv$ if and only if $e | v$, where $e := \Delta_0/(d, \Delta_0)$, we have

$$\sum_{v|d^\infty} \sum_{a|d} \mu(a) \delta_{U,dv,av} = \begin{cases} S_{d,1,h} & \text{if } \Delta > 0 \text{ or } \Delta_0 \not\equiv 1 \pmod{4} \text{ or } \Delta_0 \nmid d^\infty \\ S_{d,1,h} + S_{d,e,h} & \text{otherwise} \end{cases} = \delta_U(d),$$

as claimed. \square

Proof of Theorem 1.1. Let $A, B > 0$ be the constants of Lemma 6.2. Assume that $x \geq \exp(Be^{8\omega(d)}d^8)$ and put $y := (\log x/B)^{1/8}/d$. Note that $e^{\omega(d)} \leq y \leq \log x/\varphi(d)$ and $\log y \leq \log \log x$, for every $x \gg_B 1$. By Lemma 6.1, Lemma 6.2, and Lemma 6.4, we obtain that

$$\begin{aligned} \mathcal{R}_U(d; x) &= \sum_{\substack{v|d^\infty \\ v \leq y}} \sum_{a|d} \mu(a) \pi_{U,dv,av}(x) + O(E_1) \\ &= \sum_{\substack{v|d^\infty \\ v \leq y}} \sum_{a|d} \mu(a) \delta_{U,dv,av} \text{Li}(x) + O(E_1) + O_U(E_2) \\ &= \delta_U(d) \text{Li}(x) + O(E_1) + O_U(E_2) + O(E_3), \end{aligned}$$

where, by Lemma 6.3, we have

$$E_1 := \sum_{\substack{v|d^\infty \\ v > y}} \sum_{a|d} \mu(a) \pi_{U,dv,av}(x) \ll \frac{x}{\log x} \cdot \frac{\omega(d) + 1}{\varphi(d)} \cdot \frac{(\log y)^{\omega(d)}}{y} \ll \frac{(\omega(d) + 1)d}{\varphi(d)} \cdot \frac{x (\log \log x)^{\omega(d)}}{(\log x)^{9/8}}$$

and

$$E_3 := \sum_{\substack{v|d^\infty \\ v > y}} \sum_{a|d} \mu(a) \delta_{U,dv,av} \text{Li}(x) \ll_U \frac{\omega(d) + 1}{\varphi(d)} \cdot \frac{(\log y)^{\omega(d)}}{y} \cdot \text{Li}(x) \ll \frac{(\omega(d) + 1)d}{\varphi(d)} \cdot \frac{x (\log \log x)^{\omega(d)}}{(\log x)^{9/8}},$$

while, also using the inequality $\tau(d)/d \leq d/\varphi(d)$, we have

$$\begin{aligned} E_2 &:= \sum_{\substack{v|d^\infty \\ v \leq y}} \sum_{a|d} x \exp(-A(\log x)^{1/2}/(dv)) \ll x \exp(-AB^{1/8}(\log x)^{3/8}) \tau(d) y \\ &\ll x \exp(-AB^{1/8}(\log x)^{3/8}) (\log x)^{1/8} \cdot \frac{\tau(d)}{d} \ll \frac{d}{\varphi(d)} \cdot \frac{x}{(\log x)^{9/8}}. \end{aligned}$$

The result follows. \square

REFERENCES

1. P. S. Bruckman and P. G. Anderson, *Conjectures on the Z-densities of the Fibonacci sequence*, Fibonacci Quart. **36** (1998), no. 3, 263–271.
2. P. Cubre and J. Rouse, *Divisibility properties of the Fibonacci entry point*, Proc. Amer. Math. Soc. **142** (2014), no. 11, 3771–3785.
3. J.-M. De Koninck and F. Luca, *Analytic number theory*, Graduate Studies in Mathematics, vol. 134, American Mathematical Society, Providence, RI, 2012, Exploring the anatomy of integers.

4. G. Karpilovsky, *Topics in field theory*, North-Holland Mathematics Studies, vol. 155, North-Holland Publishing Co., Amsterdam, 1989, Notas de Matemática [Mathematical Notes], 124.
5. J. C. Lagarias, *The set of primes dividing the Lucas numbers has density 2/3*, Pacific J. Math. **118** (1985), no. 2, 449–461.
6. J. C. Lagarias, *Errata to: “The set of primes dividing the Lucas numbers has density 2/3”* [Pacific J. Math. **118** (1985), no. 2, 449–461; MR0789184 (86i:11007)], Pacific J. Math. **162** (1994), no. 2, 393–396.
7. J. C. Lagarias and A. M. Odlyzko, *Effective versions of the Chebotarev density theorem*, Algebraic number fields: L -functions and Galois properties (Proc. Sympos., Univ. Durham, Durham, 1975), pp. 409–464.
8. P. Moree, *On the prime density of Lucas sequences*, J. Théor. Nombres Bordeaux **8** (1996), no. 2, 449–459.
9. P. Moree, *On primes p for which d divides $\text{ord}_p(g)$* , Funct. Approx. Comment. Math. **33** (2005), 85–95.
10. P. Moree and P. Stevenhagen, *Prime divisors of Lucas sequences*, Acta Arith. **82** (1997), no. 4, 403–410.
11. M. R. Murty and V. K. Murty, *Non-vanishing of L -functions and applications*, Progress in Mathematics, vol. 157, Birkhäuser Verlag, Basel, 1997.
12. A. Schinzel, *A refinement of a theorem of Gerst on power residues*, Acta Arith. **17** (1970), 161–168.
13. H. M. Stark, *Some effective cases of the Brauer-Siegel theorem*, Invent. Math. **23** (1974), 135–152.

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