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NONLINEAR VIBRATION CORRELATION AND BUCKLING ANALYSIS OF FLAT PLATES AND SHELLS

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ABSTRACT

The employment of nondestructive techniques in aerospace industries is rising thanks to advances in technologies and analysis. This part of the aerospace testing industry is essential to design and validate the new structures' methodology and safety. Therefore, robust and reliable nondestructive methods have been extensively studied for decades in order to reduce safety problems and maintenance cost.

One of the most important and employed nondestructive methods to compute large-scale aerospace structures' critical buckling load is the Vibration Correlation Technique (VCT). This methodology allows to obtain the buckling load and equivalent boundary conditions by interpolating the natural frequencies of the structures for progressively increasing loadings without considering instabilities. VCT has been successfully investigated and employed for many structures, but it is still under development for composite shell structures.

The present work provides a numerical model for carrying out virtual VCT to predict the buckling load, to characterize the natural frequencies variation with progressive higher loadings, and to provide an efficient means for verifying the experimental VCT results.

The proposed nonlinear methodology is based on the well-established Carrera Unified Formulation (CUF). CUF represents a hierarchical formulation in which the structural model's order is considered the analysis's input. According to CUF, any theory is degenerated into generalized kinematics and is compactly handled. By adopting this formulation, the nonlinear governing equations and the relative FE arrays of the two-dimensional (2D) theories are written in terms of Fundamental Nuclei (FNs). FNs represent the basic building blocks of the proposed formulation. In order to investigate far nonlinear regimes, the full Green-Lagrange strain tensor is considered. Furthermore, the geometrical nonlinear equations are written in a total Lagrangian framework and solved with an opportune Newton-Raphson method.

For an assessment of the robustness of the virtual VCT, several flat plate and shell structures are studied and compared with the solutions found in the available VCT literature. The results prove that the proposed approach provides results with an excellent correlation with the experimental ones, allowing to investigate the buckling load and the natural frequencies variation in the nonlinear regime with high reliability.

INTRODUCTION

In the industrial application, one of the most used nondestructive methods for buckling tests is the Vibration Correlation Technique (VCT). VCT is able to compute the buckling load of structures by monitoring the natural frequencies variation for progressively increasing applied loads. Furthermore, in the industrial application, the VCT is also adopted to evaluate the actual in situ boundary conditions of structures to enhance the ac-

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curacy of the numerical buckling prediction. Therefore, the VCT can be divided in two groups: 1) those for determining of the actual boundary conditions and 2) those for the direct buckling load prediction. Readers can find a detailed description of this experimental method in [1, 2, 3].

In view of its importance and potential, several experimental tests and studies were carried out for decades. For instance, Lurie [4] conducted experimental tests on columns and plates in order to determine the critical load by frequency measurements. Nonlinear studies on imperfect curved panels were provided by Massonet [5]. A modified vibration correlation method was presented by Souza and Assaid [6]. Recently, Abramovich *et al.* [7] adopted the VCT to evaluate the buckling load of metallic and laminated structures. Jansen *et al.* [8] presented the capability of analysis tools for supporting and improving the accuracy of the VCT results obtained through semi-empirical methods. The analytical and theoretical verification of the VCT results of imperfect metallic cylindrical shell structures was carried out by Franzoni *et al.* [9]. Readers are referred to [10, 11, 12] for further detailed investigations.

The goal of this research is to create a numerical tool for performing virtual VCT in order to determine the buckling load, to evaluate the natural frequencies variation with progressively higher loadings, and to provide a verification of the experimental VCT results. One of the advantages of the present formulation with respect to the others, often based on linear approaches, is to consider the geometrical nonlinearities that allow to guarantee a remarkable accuracy of the results. The nonlinear methodology presented in this article is conducted in the Carrera Unified Formulation (CUF) domain [13, 14]. The main advantage of the CUF is to be able to consider the structural model order as an input of the analysis. In this way, the refined generic models do not need specific formulations. According to CUF, any theory has degenerated into generalized kinematics and is compactly handled. The nonlinear governing equations and the relative finite element (FE) arrays of the two-dimensional (2D) theories are written in terms of Fundamental Nuclei (FNs). FNs represent the basic building blocks of the presented formulation. CUF has already been used successfully in many fields [12, 15, 16, 17, 18, 19, 20], and in the present research it is adopted to study nonlinear vibrations.

LINEARIZED FREE VIBRATIONS OF STRUCTURES

The presented method is carried out as follows: 1) First, nonlinear equilibrium curves of the considered structures are obtained by performing the static geometrical nonlinear problem adopting the Newton-Raphson method [21, 22, 23] with a path-following approach based on the arc-length constraint; 2) then, in each state of interest of the equilibrium path the tangent stiffness matrix, $\boldsymbol{K}_T$, is computed; 3) Finally, natural frequencies and relative mode shapes are calculated through a linear eigenvalue problem, which is obtained by simplifying the equations of motion by considering harmonic displacements around non-trivial equilibrium states. Therefore, the form of the equations of motion for free vibrations is a linear eigenvalue problem:

$$(\boldsymbol{K}_T - \omega^2 \boldsymbol{M})\boldsymbol{q} = 0 \tag{1}$$

where $\boldsymbol{M}$ represents the mass matrix, ω indicates the natural frequencies and $\boldsymbol{q}$ stands for the eigenvectors. Instead, the buckling load is calculated using the following eigenvalue problem:

$$(\boldsymbol{K} + \lambda \boldsymbol{K}_\sigma)\boldsymbol{q} = 0 \tag{2}$$

where the $\boldsymbol{K}$ is the linear stiffness matrix, λ denotes the buckling load factor and $\boldsymbol{K}_\sigma$ represents the geometric stiffness matrix. Interested readers are referred to [24, 25] for the complete formulation.

The $\boldsymbol{K}$, $\boldsymbol{K}_\sigma$, $\boldsymbol{K}_T$ and $\boldsymbol{M}$ are formulated in the framework of the well-established CUF. According to CUF, the three-dimensional (3D) displacement field $\boldsymbol{u}(\alpha,\beta,z)$ for a shell structure, represented using an orthogonal curvilinear reference system (α,β,z), is:

$$\boldsymbol{u}(\alpha,\beta,z) = F_\tau(z)\boldsymbol{u}_\tau(\alpha,\beta) \qquad \tau = 1,...,N \tag{3}$$

in which $\boldsymbol{u}_\tau(\alpha,\beta)$ represents the generalized in-plane displacement vector, F_τ is the expansion functions of the thickness coordinate z and N stands for the order of expansion in the z direction. In this research, Lagrange polynomials (LE) are adopted for the expansion functions F_τ. The acronym LDN, used in the following analyses, denotes the LE of order N assumed in the z direction. The reader can refer to the book [14] for a complete description of the mathematical derivation of the 2D FE formulation in the framework of CUF.

In this work, using the total Lagrangian formulation, the full Green-Lagrange nonlinear strain vector is considered.

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}_l + \boldsymbol{\varepsilon}_{nl} = (\boldsymbol{b}_l + \boldsymbol{b}_{nl})\boldsymbol{u} \tag{4}$$

in which $\boldsymbol{b}_l$ and $\boldsymbol{b}_{nl}$ represent the linear and nonlinear differential operators. Readers can find these matrices in [26].

The generalized displacement vector is derived through the finite element method employing the shape functions $N_i(\alpha,\beta)$.

$$\boldsymbol{u}_\tau(\alpha,\beta) = N_i(\alpha,\beta)\boldsymbol{q}_{\tau i} \qquad i = 1,2,...,n_{el} \tag{5}$$

in which $\boldsymbol{q}_{\tau i}$ represent the unknown nodal variables, n_{el} denotes the number of nodes per element and the i stands for summation. By considering the Eq. 3 and Eq. 5, the strain vector is formulated in algebraic form as:

$$\boldsymbol{\varepsilon} = (\boldsymbol{B}_l^{\tau i} + \boldsymbol{B}_{nl}^{\tau i})\boldsymbol{q}_{\tau i} \tag{6}$$

where $\boldsymbol{B}_l^{\tau i}$ and $\boldsymbol{B}_{nl}^{\tau i}$ indicate the linear and nonlinear algebraic matrices.

The governing equations for free vibration analysis are obtained via the principle of virtual work.

$$\delta L_{int} = -\delta L_{ine} \tag{7}$$

where δL_{int} and δL_{ine} stand for the virtual variation of the nonlinear strain energy and inertia loads. Readers can refer to [12] for further details.

The $\boldsymbol{K}_T$ is computed by linearizing the virtual variation of the nonlinear strain energy.

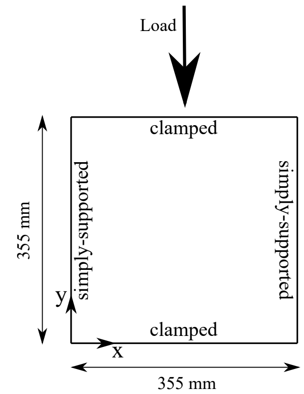
$$\begin{aligned}\delta(\delta L_{int}) &= \delta\boldsymbol{q}_{sj}^T(\boldsymbol{K}_0^{ij\tau s} + \boldsymbol{K}_{T1}^{ij\tau s})\boldsymbol{q}_{\tau i} + \delta\boldsymbol{q}_{sj}^T\boldsymbol{K}_\sigma^{ij\tau s}\boldsymbol{q}_{\tau i} \\ &= \delta\boldsymbol{q}_{sj}^T\boldsymbol{K}_T^{ij\tau s}\boldsymbol{q}_{\tau i}\end{aligned} \tag{8}$$

where $\boldsymbol{K}_T^{ij\tau s}$ represents the FN of the tangent stiffness matrix, $\boldsymbol{K}_0^{ij\tau s}$ stands for the linear contribution of $\boldsymbol{K}_T$, $\boldsymbol{K}_{T1}^{ij\tau s} = 2\boldsymbol{K}_{lnl}^{ij\tau s} + \boldsymbol{K}_{nll}^{ij\tau s} + 2\boldsymbol{K}_{nlnl}^{ij\tau s}$ indicates the nonlinear component, and $\boldsymbol{K}_\sigma^{ij\tau s}$ denotes the the so-called geometric stiffness matrix. The superscripts i, j, τ, s are the four indexes exploited to assemble the matrices.

NUMERICAL RESULTS

For the representative purpose, a flat aluminium plate subjected to axial compressive load is considered This benchmark case have the following geometrical and material data: a (width)= 355 mm, b (height)= 355 mm, t (thickness)= 2 mm and Young modulus (E)= 70 GPa, Poisson's ratio (ν)= 0.33, density (ρ)= 2780 kg/m^3. Figure 1 depicts the geometry and boundary conditions of this structure. In particular, clamped-clamped-simply supported- simply supported constraints are employed. For the following discussions, the convergent model is obtained by adopting at least 10$\times$10Q9 for the in-plane mesh approximation and only one LD2 in the thickness direction.

The equilibrium curve of the aluminium flat plate subjected to compressive load is illustrated in Fig. 2. A small load defect d= 0.01 N was applied at the center of the aluminium plate to enforce the unstable solution branches. The linear variation of the first four natural frequencies for progressively higher loadings is reported in Fig. 3. For completeness, the characteristics first four vibration mode shapes are provided in Fig. 4. Instead,



(a) Plate 1

FIGURE 1: GEOMETRY AND BOUNDARY CONDITIONS OF THE ALUMINIUM FLAT PLATE.

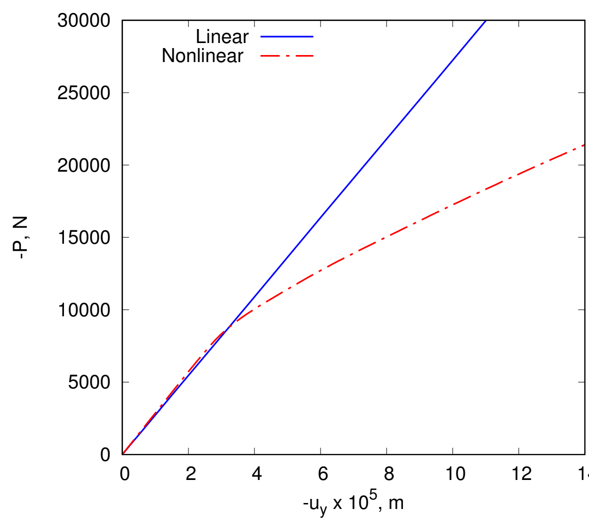


FIGURE 2: LINEAR AND NONLINEAR LOAD-DEFLECTION CURVE. ALUMINIUM FLAT PLATE SUBJECTED TO COMPRESSIVE LOAD.

Fig. 5 illustrates the comparison between the linear, nonlinear variation of the natural frequencies and experimental results [27] for progressively increasing loadings. The results suggest that the linear approach allows one to evaluate the benchmark case's frequency variation at lower levels of the compressive load. On the other hand, for higher loadings, the deviation of the linear results from the nonlinear and experimental ones becomes evident. The nonlinear and experimental results show a minimum value near the critical buckling load, and after the buckling, frequencies start to increase. The change in slope of the natural frequencies represents a criterion to evaluate the buckling. The presented

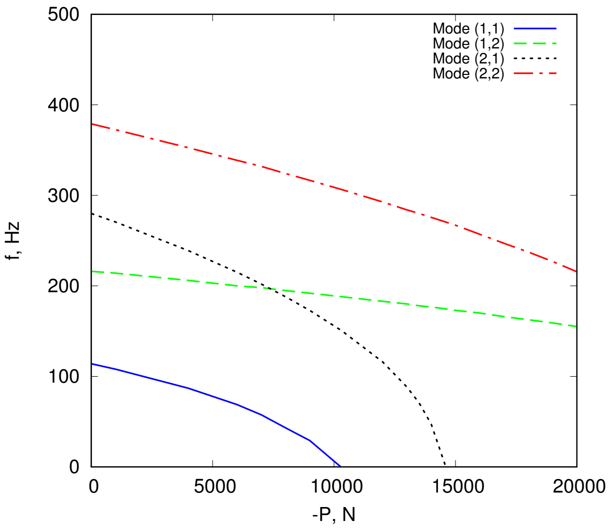
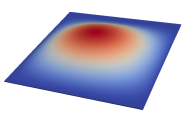
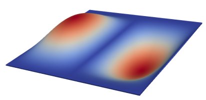


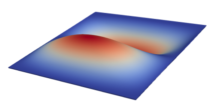
FIGURE 3: LINEAR VARIATION OF THE FIRST FOUR NATURAL FREQUENCIES OF VIBRATION FOR THE ALUMINIUM FLAT PLATE UNDER COMPRESSIVE LOAD.



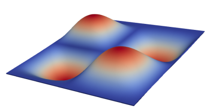
(a) Mode (1,1)



(b) Mode (1,2)



(c) Mode (2,1)



(d) Mode (2,2)

FIGURE 4: CHARACTERISTICS FIRST FOUR VIBRATION MODE SHAPES FOR THE ALUMINIUM FLAT PLATE. *P*= 0 N.

nonlinear virtual VCT method is able to obtain results with an excellent correlation with the experimental solutions. Discrepancies between numerical and experimental results are probably due to variations between the real boundary conditions and the numerical ones and initial geometric imperfections.

CONCLUSIONS

In this work, a novel virtual Vibration Correlation Technique (VCT) approach to predict the buckling load and to characterize the natural frequencies variation for progressively increasing compressive loadings is presented. The method has been successfully assessed through the study of benchmark problems available in the VCT literature. The analysis were conducted in

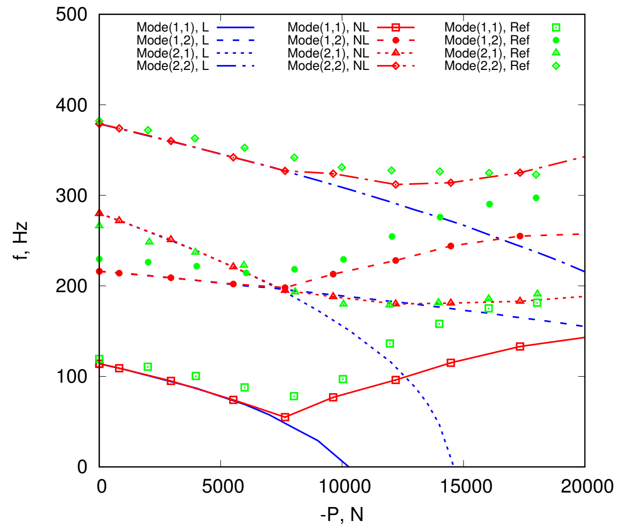


FIGURE 5: COMPARISON BETWEEN LINEAR, NONLINEAR AND EXPERIMENTAL VARIATION OF THE NATURAL FREQUENCIES OF THE ALUMINIUM FLAT PLATE FOR PROGRESSIVELY INCREASING LOADINGS.

the domain of the two-dimensional (2D) Carrera Unified Formulation (CUF), which allows us to perform the numerous analyses with high accuracy. Isotropic flat plate and shell structures have been investigated. For the sake of brevity, the shell structures will be shown in the presentation during the conference. The results suggest that the presented nonlinear methodology provide accurate solutions compared with the experimental ones and a good confidence for future studies in this topic.

REFERENCES

[1] Singer, J., Arbocz, J., and Weller, T., 1998. *Buckling Experiments: Experimental Methods in Buckling of Thin-Walled Structures, Vol.1: Basic Concepts, Columns, Beams and Plate*. John Wiley & Sons, Inc., New York, USA.

[2] Singer, J., Arbocz, J., and Weller, T., 2002. *Buckling Experiments: Experimental Methods in Buckling of Thin-Walled Structures, Vol.2: Shells, Built-up Structures and Additional Topics*. John Wiley & Sons, Inc., New York, USA.

[3] Abramovich, H., 2017. *Stability and vibrations of thin-walled composite structures*. Woodhead Publishing.

[4] Lurie, H., 1952. "Lateral vibrations as related to structural stability". *Journal of Applied Mechanics, ASME*, ***19***, pp. 195–204.

[5] Massonnet, C., 1948. "Le voilement des plaques planes sollicitées dans leur plan". *Final Report of the Third Congress of the International Association for Bridge and Structural Engineering*, pp. 291–300.

[6] Souza, M., and Assaid, L., 1991. "A new technique for the prediction of buckling loads from nondestructive vibration tests". *Experimental Mechanics,* ***31***(2), pp. 93–97.

[7] Abramovich, H., Govich, D., and Grunwald, A., 2015. "Buckling prediction of panels using the vibration correlation technique". *Progress in Aerospace Sciences,* ***78***, pp. 62–73.

[8] Jansen, E., Abramovich, H., and Rolfes, R., 2014. "The direct prediction of buckling loads of shells under axial compression using VCT-towards an upgraded approach". In 29th Congress on the International Council of the Aeronutical Science, pp. 1–9.

[9] Franzoni, F., Degenhardt, R., Albus, J., and Arbelo, M., 2019. "Vibration Correlation Technique for predicting the buckling load of imperfection-sensitive isotropic cylindrical shells: An analytical and numerical verification". *Thin-Walled Structures,* ***140***, pp. 236–247.

[10] Franzoni, F., Odermann, F., Lanbans, E., Bisagni, C., Arbelo, M., and Degenhardt, R., 2019. "Experimental validation of the Vibration Correlation Technique robustness to predict buckling of unstiffened composite cylindrical shells". *Composite Structures,* ***224***, p. 111107.

[11] Skukis, E., Ozolins, O., Andersons, J., Kalnins, K., and Arbelo, M., 2017. "Applicability of the Vibration Correlation Technique for estimation of the buckling load in axial compression of cylindrical isotropic shells with and without circular cutouts". *Shock and Vibration,* ***2017***.

[12] Carrera, E., Pagani, A., Azzara, R., and Augello, R., 2020. "Vibration of metallic and composite shells in geometrical nonlinear equilibrium states". *Thin-Walled Structures,* ***157***, p. 107131.

[13] Carrera, E., Giunta, G., and Petrolo, M., 2011. *Beam Structures: Classical and Advanced Theories*. John Wiley & Sons.

[14] Carrera, E., Cinefra, M., Petrolo, M., and Zappino, E., 2014. *Finite element analysis of structures through unified formulation*. John Wiley & Sons.

[15] Carrera, E., Petrolo, M., and Varello, A., 2012. "Advanced beam formulations for free vibration analysis of conventional and joined wings". *Journal of Aerospace Engineering,* ***25***(2), pp. 282–293.

[16] Azzara, R., Carrera, E., Filippi, M., and Pagani, A., 2020. "Time response stress analysis of solid and reinforced thin-walled structures by component-wise models". *International Journal of Structural Stability and Dynamics*, p. 2043010.

[17] Pagani, A., Augello, R., and Carrera, E., 2018. "Frequency and mode change in the large deflection and post-buckling of compact and thin-walled beams". *Journal of Sound and Vibration,* ***432***, pp. 88–104.

[18] Pagani, A., Azzara, R., Augello, R., Carrera, E., and Wu, B., 2020. "Accurate through-the-thickness stress distributions in thin-walled metallic structures subjected to large displacements and large rotations". *Vietnam Journal of Mechanics,* ***42***(3), pp. 239–254.

[19] Pagani, A., Azzara, R., Augello, R., and Carrera, E., 2021. "Stress states in highly flexible thin-walled composite structures by unified shell model". *AIAA Journal*, pp. 1–14.

[20] Miglioretti, F., Carrera, E., and Petrolo, M., 2014. "Variable kinematic beam elements for electro-mechanical analysis". *Smart Structures and Systems,* ***13***(4), pp. 517–546.

[21] Reddy, J., 2014. *An Introduction to Nonlinear Finite Element Analysis: with applications to heat transfer, fluid mechanics, and solid mechanics*. Oxford University Press, Oxford.

[22] Carrera, E., 1994. "A study on arc-length-type methods and their operation failures illustrated by a simple model". *Computers & Structures,* ***50***(2), pp. 217–229.

[23] Crisfield, M., 1981. "A fast incremental/iterative solution procedure that handles "snap-through"". *Computers & Structures,* ***13***(1), pp. 55–62.

[24] D'Ottavio, M., and Carrera, E., 2010. "Variable-kinematics approach for linearized buckling analysis of laminated plates and shells". *AIAA Journal,* ***48***(9), pp. 1987–1996.

[25] Nali, P., Carrera, E., and Lecca, S., 2011. "Assessments of refined theories for buckling analysis of laminated plates". *Composite Structures,* ***93***(2), pp. 456–464.

[26] Carrera, E., Pagani, A., Augello, R., and Wu, B., 2020. "Popular benchmarks of nonlinear shell analysis solved by 1D and 2D CUF-based finite elements". *Mechanics of Advanced Materials and Structures*, pp. 1–12.

[27] Arbelo, M., de Almeida, S., Donadon, M., Rett, S., Degenhardt, R., Castro, S., Kalnins, K., and Ozoliņš, O., 2014. "Vibration Correlation Technique for the estimation of real boundary conditions and buckling load of unstiffened plates and cylindrical shells". *Thin-Walled Structures,* ***79***, pp. 119–128.