Mathematics and Financial Economics
https://doi.org/10.1007/s11579-022-00321-9

Multivariate tempered stable additive subordination for financial models

Patrizia Semeraro[1]

Received: 15 November 2021 / Accepted: 29 June 2022

Abstract
We study a class of multivariate tempered stable distributions and introduce the associated class of tempered stable Sato subordinators. These Sato subordinators are used to build additive inhomogeneous processes by subordination of a multiparameter Brownian motion. The resulting process is additive and time inhomogeneous and it is a generalization of multivariate Lévy processes with good fit properties on financial data. We specify the model to have unit time normal inverse Gaussian distribution and we discuss the ability of the model to fit time inhomogeneous correlations on real data.

Keywords Tempered stable distributions · Sato processes · Multivariate additive subordination · Multivariate asset modeling

Mathematics Subject Classification 60G51 · 60E07

Introduction

Additive processes with independent but inhomogeneous increments have been proposed to model asset returns. Carr et al. [6] showed that these processes can synthesize the surface of option prices and Eberlein and Madan [8] empirically analyzed the use of Sato processes in the evaluation of equity structured products.

Sato processes are a class of additive processes with inhomogeneous increments used in finance to model asset returns. Sato [32] showed that given a self decomposable law μ an additive process $Y(t)$ always exists such that $Y(1) \sim \mu$ and the time t distribution is the law of $t^q Y(1)$. Eberlein and Madan [8] termed this additive process the Sato process. Sato processes exhibit a moment term structure which is in line with the term structure observed in financial markets, see Boen and Guillaume [3].

A possible approach to include time inhomogeneity in financial models is to use additive subordination (see Mendoza-Arriaga and Linetsky [26], Li et al. [18] and Kokholm and Nicolato [15]). In this case, inhomogeneity comes from the subordinator. The success of

✉ Patrizia Semeraro
patrizia.semeraro@polito.it

1 Department of Mathematical Science, Politecnico di Torino, c.so Duca degli Abruzzi, 24, Torino, Italy

Published online: 13 July 2022

subordination in finance stems from many factors. Price processes under no arbitrage are semimartingales and these can be represented as time-changed Brownian motions. Furthermore, time change models economic time: the more intense the market activity, the faster economic time runs compared to calendar time. When the change of time is a subordinator, the resulting process belongs to the (pure jump) Lévy class, a class of analytically tractable processes. In the one-dimensional case, the most famous subordinated Brownian motions, such as the variance gamma and the normal inverse Gaussian processes, have unit time self-decomposable distributions. Therefore, it is possible to define the corresponding Sato processes. Unfortunately, in multivariate subordination this is not true.

In Takano [36], the author provided conditions for a multivariate subordinated Brownian motion to have self-decomposable unit time distribution. The author also considered the subcase of a one-dimensional generalized gamma subordinator. Even in this case he found that if the Brownian motion has non-zero drift the subordinated process distribution at unit time is not self-decomposable. As a consequence, multivariate subordinated Brownian motions are not good candidates to construct multivariate Sato processes. However, it is possible to consider multivariate self-decomposable distributions to define multivariate Sato subordinators. Sato subordinators associated with self-decomposable distributions are easy to construct, introduce time inhomogeneity and perform well on financial data (Sun et al. [35]). Therefore, it is possible to use Sato subordination of multivariate Brownian motions to obtain additive processes with inhomogeneous increments to model asset returns. Other additive subordinators could be considered to introduce time inhomogeneity (see Mendoza-Arriaga and Linetsky [26]), but the advantage of Sato subordinators is that they are parsimonious in terms of parameters. In fact, there is only one parameter that drives time inhomogeneity.

In this paper we introduce and study a self-decomposable class of multivariate exponential tempered distributions and their associated multivariate Sato subordinators. This class is defined as a particular case of the tempered distributions in Rosiński [30] and it is a generalization of the multivariate gamma distribution in Pérez-Abreu and Stelzer [27]. Exponential tempered stable distributions are a multivariate version of the self-decomposable distributions most used in finance, such as the well known CGMY (Carr et al. [5]), the variance gamma (Madan and Seneta [23]), the bilateral gamma (Küchler and Tappe [16]), the gamma and the inverse Gaussian distributions. We study some properties of multivariate exponential tempered stable distributions, for example Proposition 2.2 provides the existence conditions for their moments. We then characterize multivariate Sato subordinators by providing their time t Lévy measure in Theorem 3.1 and we focus on a specific dependence structure widely used in finance to include correlations in multivariate models.

Finally, we build a multivariate additive process using additive multivariate subordination of a Brownian motion. The construction is designed to obtain a multivariate process with the same unit time distribution as the factor-based $\rho\alpha$-model in Luciano and Semeraro [21] and with time varying correlations. For simplicity, correlations are assumed to be constant in many financial models, however this is not a realistic assumption, as discussed e.g. in Tóth and Kertész [38], Teng et al. [37] and Lundin et al. [22]. A calibration on financial data shows the ability of our model to fit correlations of asset returns at different time horizons.

The paper is organized as follows. Section 1 introduces tempered stable distributions. Exponential tempered stable distributions are studied in Sect. 2, while Sect. 3 defines the corresponding Sato subordinators. Section 4 discusses multivariate Sato subordination of a multiparameter Brownian motion. The specification of an asset return model and its empirical investigation on financial data is developed in Sect. 5. Section 6 provides concluding remarks and outlines future research.

1 Tempered stable distributions

This section introduces multivariate stable and tempered stable distributions.

Let μ be an infinitely divisible distribution on $\mathbb{R}^d$ without Gaussian component and ν its Lévy measure. The following proposition provides the polar decomposition of the Lévy measure ν (see e.g. Maejima et al. [24] and Rosinski [29]).

Proposition 1.1 *Let ν be a Lévy measure. Then there exists a measure λ on S^{d-1}, where $S^{d-1} = \{\boldsymbol{x} \in \mathbb{R}^d : ||\boldsymbol{x}|| = 1\}$, with $0 < \lambda(S^{d-1}) \leq \infty$ and a family $\{\nu_{\boldsymbol{w}} : \boldsymbol{w} \in S^{d-1}\}$ of measures on $(0, \infty)$, $0 < \nu_{\boldsymbol{w}}(\mathbb{R}_+) \leq \infty$, such that $\nu_{\boldsymbol{w}}(E)$ is measurable in $\boldsymbol{w}$ for any $E \in \mathcal{B}((0, \infty))$ and it is σ-finite for any $\boldsymbol{w} \in S^{d-1}$,*

$$\int_0^1 r^2 \nu_{\boldsymbol{w}}(dr) < \infty \tag{1.1}$$

and

$$\nu(E) = \int_{S^{d-1}} \lambda(d\boldsymbol{w}) \int_{\mathbb{R}_+} \mathbf{1}_E(s\boldsymbol{w}) \nu_{\boldsymbol{w}}(ds),$$

where λ and $\nu_{\boldsymbol{w}}$ are uniquely determined up to multiplication of measurable functions $0 < c(\boldsymbol{w}) < \infty$ and $\frac{1}{c(\boldsymbol{w})}$, respectively.

We say that ν has polar decomposition $(\lambda, \nu_{\boldsymbol{w}})$, where λ and $\nu_{\boldsymbol{w}}$ are the spherical and the radial components of ν, respectively. We write $\nu = (\lambda, \nu_{\boldsymbol{w}})$. Condition (1.1) guarantees that $\nu_{\boldsymbol{w}}$ is a one-dimensional Lévy measure for any $\boldsymbol{w}$. The radial component of μ is the real valued infinitely divisible (i.d.) distribution $\mu_{\boldsymbol{w}}$ - without Gaussian component - whose Lévy measure is $\nu_{\boldsymbol{w}}$.

The measure ν is said radially absolutely continuous (Sato [32]) if there exists a nonnegative measurable function $f(\boldsymbol{w}, r)$ such that

$$\nu(E) = \int_{S^{d-1}} \lambda(d\boldsymbol{w}) \int_{\mathbb{R}_+} \mathbf{1}_E(s\boldsymbol{w}) f(\boldsymbol{w}, s) ds.$$

From integration in polar coordinates (see e.g. Folland [10]), we have that if $\nu(d\boldsymbol{x})$ is a Lévy measure with Lévy density $f(\boldsymbol{x})$, then there exists a unique Borel measure σ on S^{d-1} such that

$$\nu(B) = \int_{\mathbb{R}^d} \mathbf{1}_B(\boldsymbol{x}) f(\boldsymbol{x}) d\boldsymbol{x} = \int_{S^{d-1}} \int_{\mathbb{R}^+} \mathbf{1}_B(r\boldsymbol{w}) f(\boldsymbol{w}r) r^{d-1} dr \sigma(d\boldsymbol{w}).$$

We call $f_{\boldsymbol{w}}(r) := f(\boldsymbol{w}r) r^{d-1}$ the radial component of the density of ν.

If ν is absolutely continuous then ν is also radially absolutely continuous. The other implication does not hold. It suffices to choose λ with finite support and ν is not absolutely continuous.

Tempered stable distributions are obtained by tempering the radial component of the Lévy measure of an α- stable distribution, $\alpha \in (0, 2)$. It is well known that the Lévy measure ν_0 of an α-stable, $\alpha \in (0, 2)$, measure μ_0 is of the form

$$\nu_0(E) = \int_{S^{d-1}} \int_{\mathbb{R}_+} \mathbf{1}_E(r\boldsymbol{w}) \frac{1}{r^{\alpha+1}} dr \lambda(d\boldsymbol{w}), \tag{1.2}$$

where λ is a finite measure on S^{d-1}. Tempered stable distributions are formally defined as follows (Rosiński [30]).

Definition 1.1 A probability measure μ on $\mathbb{R}^d$ is called tempered α-stable (abbreviated as TαS) if it is infinitely divisible without the Gaussian part and it has Lévy measure ν of the following form

$$\nu(dr, d\boldsymbol{w}) = \frac{q(r, \boldsymbol{w})}{r^{\alpha+1}} dr\lambda(d\boldsymbol{w}), \tag{1.3}$$

where $\alpha \in (0, 2)$, λ is a finite measure on S^{d-1} and $q : (0, \infty) \to (0, \infty)$ is a Borel function such that $q(\cdot, \boldsymbol{w})$ is completely monotone with $\lim_{r\to\infty} q(r, \boldsymbol{w}) = 0$. The measure μ is called a proper TαS distribution if, in addition to the above, $q(0+, \boldsymbol{w}) = 1$ for each $\boldsymbol{w} \in S^{d-1}$.

The complete monotonicity of $q(r, \boldsymbol{w})$ means that $(-1)^n \frac{\partial^n}{\partial r^n} q(r, \boldsymbol{w}) > 0$ for all $r > 0$, $\boldsymbol{w} \in S^{d-1}$, and $n = 0, 1, 2, \ldots$. In particular $q(r, \boldsymbol{w})$ is strictly increasing and convex, see Rosiński [30]. $T\alpha S$ distributions are radially absolutely continuous and belong to the extended Thorin class of infinitely divisible distributions introduced in Grigelionis [11]: $T^{1-\alpha}(\mathbb{R}^d)$ for $\alpha \in (0, 1)$ and $T^{\chi}(\mathbb{R}^d)$ for all $\chi > 0$ and $1 \leq \alpha < 2$. The radial component $\mu_{\boldsymbol{w}}$ of a $T\alpha S$ distributions has Lévy measure $\nu_{\boldsymbol{w}}(dr) = \frac{q(r,\boldsymbol{w})}{r^{\alpha+1}} dr$.

Stable and tempered stable distributions are clearly self-decomposable, see Eqs. (1.2), (1.3) and (A.1). The notion of self-decomposability is recalled in Appendix A.

2 Exponential tempered stable distributions

This section introduces tempered stable distributions with an exponential tempering function $q(r, \boldsymbol{w})$.

Definition 2.1 An infinitely divisible probability measure μ on $\mathbb{R}^d$ is called exponential TαS ($\mathcal{E}T\alpha S$) if it is without the Gaussian part and it has Lévy measure ν on $\mathbb{R}^d$ of the form

$$\nu(E) = \int_{S^{d-1}} \int_{\mathbb{R}_+} \mathbf{1}_E(r\boldsymbol{w}) \frac{e^{-\beta(\boldsymbol{w})r}}{r^{\alpha+1}} dr\lambda(d\boldsymbol{w}),\ E \in \mathcal{B}(\mathbb{R}^d) \tag{2.1}$$

where $\alpha \in [0, 2)$, λ is a finite measure on S^{d-1} and $\beta : S^{d-1} \to (0, \infty)$ is a Borel-measurable function.

If μ has Lévy measure (2.1) we write $\mu \sim \mathcal{E}T\alpha S(\alpha, \beta, \lambda)$. We focus on $\alpha \in (0, 2)$ and refer to Pérez-Abreu and Stelzer [27] for the case $\alpha = 0$. The following proposition gives conditions for a $\mathcal{E}T\alpha S$ distribution to exist and characterizes its Lévy measure.

Proposition 2.1 *Equation (2.1) defines a Lévy measure and therefore there exists an $\mathcal{E}T\alpha S$ distribution.*

1. *if $\alpha \in [0, 1)$, it holds $\int_{||x||\leq 1} ||\boldsymbol{x}||\nu(d\boldsymbol{x}) < \infty$ and $\int_{\mathbb{R}^d} \nu(d\boldsymbol{x}) = \infty$;*
2. *if $\alpha \in [1, 2)$, it holds $\int_{||x||\leq 1} ||\boldsymbol{x}||\nu(d\boldsymbol{x}) = \infty$.*

Proof Let $\nu = (\lambda, \nu_{\boldsymbol{w}})$ be the Lévy measure in (2.1) and let $B = \{\boldsymbol{x} \in \mathbb{R}^d : ||\boldsymbol{x}|| \leq 1\}$. Since

$$\int_B ||\boldsymbol{x}||^2 \nu(d\boldsymbol{x}) = \int_{S^{d-1}} \int_0^1 \frac{r^2 e^{-\beta(\boldsymbol{w})r}}{r^{\alpha+1}} dr\lambda(d\boldsymbol{w}) \leq \int_{S^{d-1}} \int_0^1 r^{1-\alpha} dr\lambda(d\boldsymbol{w})$$
$$= \int_{S^{d-1}} \frac{1}{2-\alpha} \lambda(d\boldsymbol{w}) = \frac{1}{2-\alpha} \lambda(S^{d-1}) < \infty,$$

and

$$\int_{S^{d-1}}\int_1^\infty \frac{e^{-\beta(\boldsymbol{w})r}}{r^{\alpha+1}}dr\lambda(d\boldsymbol{w}) \le \int_{S^{d-1}}\int_1^\infty \frac{1}{r^{\alpha+1}}dr\lambda(d\boldsymbol{w})$$
$$= \int_{S^{d-1}}\frac{1}{\alpha}\lambda(d\boldsymbol{w}) = \frac{1}{\alpha}\lambda(S^{d-1}) < \infty,$$

we have $\int_{\mathbb{R}^d}(||\boldsymbol{x}||^2 \wedge 1)\nu(d\boldsymbol{x}) < \infty$ and ν is a Lévy measure.

1. See Pérez-Abreu and Stelzer [27] for $\alpha = 0$ and let $\alpha \in (0,1)$. It holds:

$$\int_B ||\boldsymbol{x}||\nu(d\boldsymbol{x}) = \int_{S^{d-1}}\int_0^1 r\frac{e^{-\beta(\boldsymbol{w})r}}{r^{\alpha+1}}dr\lambda(d\boldsymbol{w})$$
$$\le \int_{S^{d-1}}\int_0^1 \frac{1}{r^\alpha}dr\lambda(d\boldsymbol{w}) = \int_{S^{d-1}}\frac{1}{1-\alpha}\lambda(d\boldsymbol{w}) = \frac{1}{1-\alpha}\lambda(S^{d-1}).$$

The infinite activity of ν follows from the infinite activity of its radial component. it holds:

$$\int_B ||\boldsymbol{x}||\nu(d\boldsymbol{x}) = \int_{S^{d-1}}\int_0^1 r\frac{e^{-\beta(\boldsymbol{w})r}}{r^{\alpha+1}}dr\lambda(d\boldsymbol{w})$$
$$= \int_{S^{d-1}}\int_0^1 \frac{e^{-\beta(\boldsymbol{w})r}}{r^{\alpha}}dr\lambda(d\boldsymbol{w}) \ge \int_{S^{d-1}} e^{-\beta(\boldsymbol{w})}\int_0^1 \frac{1}{r^\alpha}dr\lambda(d\boldsymbol{w}),$$

and $\int_0^1 \frac{1}{r^\alpha}dr$ diverges if $\alpha \ge 1$. Therefore $\int_B ||\boldsymbol{x}||\nu(d\boldsymbol{x}) = \infty$.

□

Remark 1 The $\mathcal{E}T\alpha S$ distributions are proper $T\alpha S$ distributions (see Definition 1.1), they are self-decomposable and they are radially absolutely continuous.

If $\mu \sim \mathcal{E}T\alpha S(\alpha, \beta, \lambda)$ and β is constant we say that the measure μ and its Lévy measure ν are homogeneous.

Corollary 2.1 *The characteristic function of an $\mathcal{E}T\alpha S$ distribution has the form:*

$$\hat{\mu}(z) = \exp\{i\boldsymbol{\gamma}\cdot z + \int_{\mathbb{R}^d}(e^{i\langle z,\boldsymbol{x}\rangle} - 1 - i\langle z,\boldsymbol{x}\rangle \mathbf{1}_{||\boldsymbol{x}||\le 1}(\boldsymbol{x}))\frac{e^{-\beta(\frac{\boldsymbol{x}}{||\boldsymbol{x}||})||\boldsymbol{x}||}}{||\boldsymbol{x}||^{\alpha+1}}\tilde{\lambda}(d\boldsymbol{x})\}, \quad (2.2)$$

where $\tilde{\lambda}$ has the form of (A.2).

Proof From (2.1) and (A.1) μ is self-decomposable with $k_{\boldsymbol{w}}(r) = \frac{e^{-\beta(\boldsymbol{w})r}}{r^\alpha}$, thus it is self-decomposable with h function given by $h(\boldsymbol{x}) = k_{\frac{\boldsymbol{x}}{||\boldsymbol{x}||}}(||\boldsymbol{x}||) = \frac{e^{-\beta(\frac{\boldsymbol{x}}{||\boldsymbol{x}||})||\boldsymbol{x}||}}{||\boldsymbol{x}||^\alpha}$, thus (2.2) follows. □

If ν is a one-dimensional $\mathcal{E}T\alpha S$ Lévy measure from Eq. (2.2) we have

$$\nu(dx) = (\mathbf{1}_{(-\infty,0)}(x)\frac{e^{-\beta^+|x|}}{|x|^{\alpha+1}}\lambda^+ + \mathbf{1}_{(0,\infty)}(x)\frac{e^{-\beta^- x}}{x^{\alpha+1}}\lambda^-)dx,$$

where $\beta^+ = \beta(1), \beta^- = \beta(-1), \lambda^+ = \lambda(\{1\})$ and $\lambda^- = \lambda(\{-1\})$. Thus, $\mathcal{E}T\alpha S$ distributions are multivariate versions of the tempered stable distributions studied in Küchler and Tappe [17], with the restriction that α is constant. By properly specifying the parameters, we find multivariate versions of well known tempered stable distributions as the CGMY, the variance gamma, the inverse Gaussian and the gamma ($\alpha = 0$) distributions. We also observe that the one-dimensional radial component $\mu_{\boldsymbol{w}}$ of an $\mathcal{E}T\alpha S$ distribution belongs to the class of one sided tempered stable distributions studied in Küchler and Tappe [17].

Tempered stable distributions have been characterized in Rosiński [30] in terms of their spectral measure R. We recall the definition of spectral measure and then we find it for $\mathcal{E}T\alpha S$ distributions.

Let $\mu \in \mathcal{E}T\alpha S(\alpha, \beta, \lambda)$. The tempering function of μ can be represented as

$$e^{-\beta(\boldsymbol{w})r} = \int_{\mathbb{R}_+} e^{-rs} Q(ds|\boldsymbol{w}),$$

where $Q(ds|\boldsymbol{w}) = \delta_{\beta(\boldsymbol{w})}(s)ds$ - $\mathcal{E}T\alpha S$ are proper tempered stable distributions. Let us introduce the measure Q:

$$Q(A) = \int_{S^{d-1}} \int_{\mathbb{R}_+} \mathbf{1}_A(r\boldsymbol{w}) Q(dr|\boldsymbol{w}) \lambda(d\boldsymbol{w}).$$

Clearly in this case the measure Q becomes

$$\begin{aligned} Q(A) &= \int_{S^{d-1}} \int_{\mathbb{R}_+} \mathbf{1}_A(r\boldsymbol{w}) Q(dr|\boldsymbol{w}) \lambda(d\boldsymbol{w}) = \int_{S^{d-1}} \mathbf{1}_A(\beta(\boldsymbol{w})\boldsymbol{w}) \lambda(d\boldsymbol{w}) \\ &= \int_{S^{d-1}} \mathbf{1}_{A_\beta}(\boldsymbol{w}) \lambda(d\boldsymbol{w}), \end{aligned} \tag{2.3}$$

where $A_\beta = \{\boldsymbol{w} \in S^{d-1} : \beta(\boldsymbol{w})\boldsymbol{w} \in A\}$. The spectral measure R is defined from Q as

$$R(A) := \int_{\mathbb{R}^d} \mathbf{1}_A \left(\frac{\boldsymbol{x}}{||\boldsymbol{x}||^2} \right) ||\boldsymbol{x}||^\alpha Q(d\boldsymbol{x}),$$

in this case we have

$$R(A) = \int_{S^{d-1}} \mathbf{1}_A \left(\frac{\boldsymbol{w}}{\beta(\boldsymbol{w})} \right) \beta(\boldsymbol{w})^\alpha \lambda(d\boldsymbol{w}). \tag{2.4}$$

Therefore

$$R(\mathbb{R}^d) = \int_{S^{d-1}} \beta(\boldsymbol{w})^\alpha \lambda(d\boldsymbol{w}).$$

Furthermore, from (2.3) we have $Q(\mathbb{R}^d) = \lambda(S^{d-1})$. We also have

$$\int_{\mathbb{R}^d} ||\boldsymbol{x}||^\alpha R(d\boldsymbol{x}) = \int_{S^{d-1}} ||\frac{\boldsymbol{w}}{\beta(\boldsymbol{w})}||^\alpha \beta(\boldsymbol{w})^\alpha \lambda(d\boldsymbol{w}) = \int_{S^{d-1}} \lambda(d\boldsymbol{w}) = \lambda(S^{d-1}). \tag{2.5}$$

Finally,

$$\lambda(S^{d-1}) = Q(\mathbb{R}^d) = \int_{\mathbb{R}^d} ||\boldsymbol{x}||^\alpha R(d\boldsymbol{x}).$$

The existence of the moments of tempered stable distributions depends on their spectral measure, as proved in Proposition 2.7, Rosiński [30]. Therefore, by (2.4), the existence of the moments of an $\mathcal{E}T\alpha S$ distributions depends on the function β and on the spherical components λ of its Lévy measure, as stated in the following proposition.

Proposition 2.2 *Let* $\mu \sim \mathcal{E}T\alpha S(\alpha, \beta, \lambda)$, $\alpha \in (0, 2)$ *then we have*

1. $\int_{\mathbb{R}^d} ||\boldsymbol{x}||^k d\mu(\boldsymbol{x}) < \infty$ *for* $k \in (0, \alpha)$*;*

2. $\int_{\mathbb{R}^d} ||\boldsymbol{x}||^\alpha d\mu(\boldsymbol{x}) < \infty$ *if and only if* $\int_{\{\boldsymbol{w}:\beta(\boldsymbol{w})>1\}} \beta(\boldsymbol{w})^\alpha \log(\beta(\boldsymbol{w}))\lambda(d\boldsymbol{w}) < \infty$*;*
3. $\int_{\mathbb{R}^d} ||\boldsymbol{x}||^k d\mu(\boldsymbol{x}) < \infty$ *if and only if* $\int_{S^{d-1}} \beta(\boldsymbol{w})^{-k-\alpha}\lambda(d\boldsymbol{w}) < \infty$*, for* $k > \alpha$*.*

Proof 1. Follows from Proposition 2.7, Rosiński [30].

2. Clearly

$$\int_{||\boldsymbol{x}||>1} ||\boldsymbol{x}||^\alpha \log(||\boldsymbol{x}||)R(dx) = \int_{S^{d-1}} 1_{\{\beta(\boldsymbol{w})>1\}}\beta(\boldsymbol{w})^\alpha \log(\beta(\boldsymbol{w}))\lambda(d\boldsymbol{w}).$$

and then Proposition 2.7 Rosiński [30] applies.

3. The condition $\int_{\mathbb{R}^d} ||\boldsymbol{x}||^k d\mu(\boldsymbol{x})$ is equivalent to $\int_{\{||\boldsymbol{x}||>1\}} ||\boldsymbol{x}||^k \nu(d\boldsymbol{x}) < \infty$ (Sato [32] p.159) that is equivalent to

$$\int_{\{||\boldsymbol{x}||>1\}} ||\boldsymbol{x}||^k \nu(d\boldsymbol{x}) = \int_{S^{d-1}} \int_1^\infty e^{-\beta(\boldsymbol{w})r} r^{k-\alpha-1} dr\lambda(d\boldsymbol{w}) < \infty.$$

The latter is equivalent to $\int_{S^{d-1}} \beta(\boldsymbol{w})^{-k-\alpha}\lambda(d\boldsymbol{w}) < \infty$, as proved in Pérez-Abreu and Stelzer [27], Proposition 3.12.

□

Notice that, in particular, if λ has finite support all its moments exist. The next proposition provides the characteristic function of $\mathcal{E}T\alpha S$ distributions.

Proposition 2.3 *The characteristic function* $\hat{\mu}$ *of* $\mu \sim \mathcal{E}T\alpha S(\alpha, \lambda, \beta)$ *is:*
for $\alpha \in (0, 1)$

$$\hat{\mu}(z) = \exp\{\Gamma(-\alpha) \int_{S^{d-1}} [(\beta(\boldsymbol{w}) - i\langle \boldsymbol{w}, z\rangle)^\alpha - \beta(\boldsymbol{w})^\alpha]\lambda(d\boldsymbol{w})\}, \quad \forall z \in \mathbb{R}^d, \tag{2.6}$$

for $\alpha \in (1, 2)$

$$\begin{aligned}\hat{\mu}(z) = \exp\{\Gamma(-\alpha) \int_{S^{d-1}} &[(\beta(\boldsymbol{w}) - i\langle \boldsymbol{w}, z\rangle)^\alpha - \beta(\boldsymbol{w})^\alpha \\ &+ i\langle \boldsymbol{w}, z\rangle\alpha\beta(\boldsymbol{w})^{\alpha-1}]\lambda(d\boldsymbol{w})\}, \quad \forall z \in \mathbb{R}^d,\end{aligned} \tag{2.7}$$

and for $\alpha = 1$

$$\hat{\mu}(z) = \exp\{\int_{S^{d-1}} [(\beta(\boldsymbol{w}) - i\langle \boldsymbol{w}, z\rangle) \log(\beta(\boldsymbol{w}) - i\langle \boldsymbol{w}, z\rangle) + i\langle \boldsymbol{w}, z\rangle]\lambda(d\boldsymbol{w})\}, \quad \forall z \in \mathbb{R}^d. \tag{2.8}$$

Proof The Lévy Khintchine formula in polar coordinates with (2.1) gives

$$\hat{\mu}(z) = \exp\{\int_{S^{d-1}} \phi_{\boldsymbol{w}}(\langle \boldsymbol{w}, z\rangle)\lambda(d\boldsymbol{w})\}, \quad \forall z \in \mathbb{R}^d, \tag{2.9}$$

where if $\alpha \in (0, 1)$ we have

$$\phi_{\boldsymbol{w}}(\langle \boldsymbol{w}, z\rangle) := \int_{\mathbb{R}_+} (e^{ir\langle \boldsymbol{w}, z\rangle} - 1)\frac{e^{-\beta(\boldsymbol{w})r}}{r^{\alpha+1}} dr,$$

because by Proposition 2.1 it holds $\int_{|x|\le 1} ||\boldsymbol{x}||\nu(dx) < \infty$. If $\alpha \in (1, 2)$ we have

$$\phi_{\boldsymbol{w}}(\langle \boldsymbol{w}, z\rangle) := \int_{\mathbb{R}_+} (e^{ir\langle \boldsymbol{w}, z\rangle} - 1 - ir\mathbf{1}_{|r|<1}(r)\langle \boldsymbol{w}, z\rangle)\frac{e^{-\beta(\boldsymbol{w})r}}{r^{\alpha+1}} dr.$$

Therefore for each $\boldsymbol{w}$, $\phi_{\boldsymbol{w}}(\langle \boldsymbol{w}, z\rangle)$ is the characteristic exponent of the one sided $\mathcal{E}T\alpha S$ distribution $\mu_{\boldsymbol{w}}$ that are provided in Küchler and Tappe [17].

If $\alpha \in (0, 1)$, for each $\boldsymbol{w}$ the characteristic function of the radial component $\mu_{\boldsymbol{w}}$ of μ is:

$$\phi_{\boldsymbol{w}}(\langle \boldsymbol{w}, z\rangle) = \Gamma(-\alpha)[(\beta(\boldsymbol{w}) - i\langle \boldsymbol{w}, z\rangle)^{\alpha} - \beta(\boldsymbol{w})^{\alpha}]$$

and (2.6) follows. If $\alpha \in (1, 2)$, the characteristic exponent of the radial component $\mu_{\boldsymbol{w}}$ of μ is:

$$\phi_{\boldsymbol{w}}(\langle \boldsymbol{w}, z\rangle) = \Gamma(-\alpha)[(\beta(\boldsymbol{w}) - i\langle \boldsymbol{w}, z\rangle)^{\alpha} - \beta(\boldsymbol{w})^{\alpha} + i\langle \boldsymbol{w}, z\rangle\alpha\beta(\boldsymbol{w})^{\alpha-1}]$$

and (2.7) follows.

If $\alpha = 1$, from Theorem 2.9 in Rosiński [30] we have:

$$\hat{\mu}(z) = \exp\left\{\int_{\mathbb{R}^d} \Phi(\langle \boldsymbol{x}, z\rangle)R(d\boldsymbol{x})\right\}, \quad \forall z \in \mathbb{R}^d,$$

where

$$\Phi(\langle \boldsymbol{x}, z\rangle) = (1 - i\langle \boldsymbol{x}, z\rangle)\log(1 - i\langle \boldsymbol{x}, z\rangle) + i\langle \boldsymbol{x}, z\rangle, \quad \forall z \in \mathbb{R}^d,$$

From (2.4) it follows

$$\hat{\mu}(z) = \exp\left\{\int_{S^{d-1}} \Phi(\langle \frac{\boldsymbol{w}}{\beta(\boldsymbol{w})}, z\rangle)\lambda(d\boldsymbol{w})\right\}, \quad \forall z \in \mathbb{R}^d,$$

that gives (2.8). □

The following propositions allow us to construct multivariate $\mathcal{E}T\alpha S$ distributions useful in applications, as we do in the next section.

Proposition 2.4 *A measure $\mu \sim \mathcal{E}T\alpha S(\alpha, \beta, \lambda)$ is absolutely continuous if and only if the support of λ contains d linearly independent vectors $\boldsymbol{w}_j$, $j = 1, \ldots, d$. Let $\boldsymbol{X} \sim \mu$, $\boldsymbol{X}$ has independent components in and only if λ has support on $\boldsymbol{e}_i$, $i = 1, \ldots, d$ in $\mathbb{R}^d$.*

Proof If the support of λ contains d linearly independent vectors, then the support of ν is full dimension. Therefore μ is a genuinely d-dimensional self-decomposable distribution, then μ is absolutely continuous (Sato [31]). Let $\boldsymbol{X} \sim \mu$. Then $\boldsymbol{X}$ has independent components if and only if ν has support on the coordinate axes (see e.g. Sato [32], E 12.10).

□

Proposition 2.5 *Let $\alpha \in (0, 1)$. Let $\boldsymbol{X}_1 \sim \mathcal{E}T\alpha S$ $(\alpha, \beta, \lambda_1)$, $\boldsymbol{X}_2 \sim \mathcal{E}T\alpha S$ $(\alpha, \beta, \lambda_2)$ and let them be independent. Then*

1. *$\boldsymbol{X}_1 + \boldsymbol{X}_2 \sim \mathcal{E}T\alpha S$ $(\alpha, \beta, \lambda_1 + \lambda_2)$;*
2. *For a constant $c > 0$, $c\boldsymbol{X}_1 \sim \mathcal{E}T\alpha S$ $(\alpha, \frac{\beta}{c}, c^{\alpha}\lambda_1)$.*

Proof 1. By independence

$$\hat{\mu}_{X_1+X_2}(z) = \hat{\mu}_{X_1}(z)\hat{\mu}_{X_2}(z). = \exp\{\Gamma(-\alpha)\int_{S^{d-1}} \phi_{\boldsymbol{w}}(\langle \boldsymbol{w}, z\rangle)(\lambda_1 + \lambda_2)(d\boldsymbol{w})\}$$

2. Since (see Küchler and Tappe [17])

$$\begin{aligned}\phi_{\boldsymbol{w}}(\langle \boldsymbol{w}, cz\rangle) &= \Gamma(-\alpha)[(\beta(\boldsymbol{w}) - i\langle \boldsymbol{w}, cz\rangle)^{\alpha} - \beta(\boldsymbol{w})^{\alpha}]\\ &= \Gamma(-\alpha)c^{\alpha}[(\frac{\beta(\boldsymbol{w})}{c} - i\langle \boldsymbol{w}, cz\rangle)^{\alpha} - (\frac{\beta(\boldsymbol{w})}{c})^{\alpha}],\end{aligned}$$

from (2.9) we have the assertion.

□

Proposition 2.6 *Let* $\alpha \in (0, 1)$. *The mean vector* $\boldsymbol{m}$ *and the covariance matrix* Σ *of* $\mu \sim \mathcal{E}T\alpha S(\alpha, \beta, \lambda)$ *are*

$$\boldsymbol{m} = \int_{S^{d-1}} \Gamma(1-\alpha)\beta(\boldsymbol{w})^{\alpha-1}\boldsymbol{w}\lambda(d\boldsymbol{w})$$

and

$$\Sigma = \int_{S^{d-1}} \Gamma(2-\alpha)\beta(\boldsymbol{w})^{\alpha-2}\boldsymbol{w}\boldsymbol{w}^T\lambda(d\boldsymbol{w})$$

Proof The cumulant generating function exists on $\{z \in \mathbb{R}^n : \langle \boldsymbol{w}, z\rangle \le \beta(\boldsymbol{w})\}$ and it is

$$k(z) = \int_{S^{d-1}} \Gamma(-\alpha)[(\beta(\boldsymbol{w}) - \langle \boldsymbol{w}, z\rangle)^\alpha - \beta(\boldsymbol{w})^\alpha]\lambda(d\boldsymbol{w})$$

We have $m_j = \frac{\partial}{\partial z_j} k(z)|_{z=0}$ and $\Sigma_{ij} = \frac{\partial}{\partial z_i \partial z_j} k(z)|_{z=0}$, the thesis follows by inverting integration and differentiation. □

The higher order moments can be found with similar arguments. We report the third μ_{ijk} and fourth μ_{ijkl} cross moments because they are linked to co-skewness and co-kurtosis:

$$\mu_{ijk} = \int_{S^{d-1}} \Gamma(3-\alpha)\beta(\boldsymbol{w})^{\alpha-3}\boldsymbol{w}_i\boldsymbol{w}_j\boldsymbol{w}_k\lambda(d\boldsymbol{w})$$

and

$$\mu_{ijkl} = \int_{S^{d-1}} \Gamma(4-\alpha)\beta(\boldsymbol{w})^{\alpha-4}\boldsymbol{w}_i\boldsymbol{w}_j\boldsymbol{w}_k\boldsymbol{w}_l\lambda(d\boldsymbol{w}).$$

2.1 Specifications

By properly choosing the parameters we have the following multivariate distributions.

1. **Multivariate CGMY distribution.** A Multivariate $CGMY(C, \beta, \alpha)$ distribution is a distribution μ with Lévy measure in (2.1), where $\lambda(d\boldsymbol{w}) = C\sigma(d\boldsymbol{w})$ and σ is the unique measure induced on S^{d-1} from the Lebesgue measure on $\mathbb{R}^d$. In this case if μ is one-dimensional we have

$$\nu(dx) = C(\mathbf{1}_{(-\infty,0)}(x)\frac{e^{-G|x|}}{|x|^{\alpha+1}} + \mathbf{1}_{(0,\infty)}(x)\frac{e^{-Mx}}{x^{\alpha+1}})dx,$$

   and μ is a CGMY distribution. We have $G = \beta(-1)$, $M = \beta(1)$ and α is the parameter Y, where $CGMY$ are the original parameters in Carr et al. [5].
2. **Multivariate gamma, bilateral gamma and variance gamma distributions.** The case $\alpha = 0$ has been introduced and studied in Pérez-Abreu and Stelzer [27] under the name of multivariate Gamma distribution. It includes multivariate versions of the bilateral gamma distribution introduced in Küchler and Tappe [16] and of the famous variance gamma distribution Madan and Seneta [23]. With the further condition $\lambda(S^d_-) = 0$ we have a multivariate gamma distribution in a narrow sense, see also Semeraro [33].
3. **Multivariate inverse Gaussian distribution.** In the application on financial data we focus on the inverse Gaussian distribution. A d-dimensional inverse Gaussian distribution μ with parameters λ and β, denoted $IG(\lambda, \beta)$, is a $\mathcal{E}T\alpha S$ distribution with the following Lévy measure:

$$\nu_{IG}(E) = \int_{S^{d-1}} \lambda(d\boldsymbol{w}) \int_{\mathbb{R}_+} \mathbf{1}_E(r\boldsymbol{w})\frac{e^{-\beta(\boldsymbol{w})r}}{r^{3/2}}dr,$$

where $\forall \boldsymbol{w} \in S^{d-1}$, $\nu_{\boldsymbol{w}}(dr) = \frac{e^{-\beta(\boldsymbol{w})r}}{r^{3/2}}dr$ is the Lévy measure of a one-dimensional IG process with parameters $(1, \beta(\boldsymbol{w}))$. If λ has support in S_+^{d-1} then ν_{IG} has support on $\mathbb{R}_+^d$ and we have a multivariate version of the one dimensional IG distribution. We also provide its characteristic function that is easily derived from of Proposition 2.3.

$$\hat{\mu}(z) = \exp\{-2\sqrt{\pi}\int_{S^{d-1}}[\sqrt{\beta(\boldsymbol{w}) - i\langle \boldsymbol{w}, z\rangle} - \sqrt{\beta(\boldsymbol{w})}]\lambda(d\boldsymbol{w})\}, \quad \forall z \in \mathbb{R}^d.$$

As in the one-dimensional case if $\lambda(d\boldsymbol{w}) = C\sigma(d\boldsymbol{w})$ and σ is the unique measure induced on S^{d-1} from the Lebesgue measure on $\mathbb{R}^d$, the inverse Gaussian distribution is a special case of $CGMY$ distribution with $\alpha = \frac{1}{2}$.

3 Multivariate $\mathcal{E}T\alpha S$ Sato subordinators

This section introduces time inhomogeneous additive subordinators with unit time $\mathcal{E}T\alpha S$ distribution. An additive subordinator is an increasing process with non stationary independent increments. The characteristic function of an additive subordinator $\boldsymbol{S}(t)$ is (Mendoza-Arriaga and Linetsky [26])

$$\hat{\mu}_t(z) = \exp\{i\langle \boldsymbol{\gamma}(t), z\rangle + \int_0^t\int_{\mathbb{R}_+^d}(e^{i\langle \boldsymbol{x}, z\rangle} - 1)g(d\boldsymbol{x}, u)du\}, \tag{3.1}$$

where $\boldsymbol{\gamma}(t) \in \mathbb{R}_+^d$ is the time dependent drift and $g(d\boldsymbol{x}, u)$ is a time-dependent measure so that $\int_0^1 ||\boldsymbol{x}||g(d\boldsymbol{x}, u) < \infty$ for almost all u. We assume $\boldsymbol{\gamma}(t) = 0$ and we call g the differential Lévy measure of $\boldsymbol{S}(t)$ according to Li et al. [18].

Sato processes are additive processes associated to self- decomposable distributions (see Appendix A). Since a measure $\mu \sim \mathcal{E}T\alpha S(\alpha, \beta, \lambda)$ is self-decomposable, we can define a Sato process associated with μ, that we call $\mathcal{E}T\alpha S$ Sato process.

Definition 3.1 A d-dimensional Sato subordinator is a Sato process with positive and increasing trajectories for each coordinate.

We study multivariate Sato subordinators $\boldsymbol{S}(t)$ with unit time $\mathcal{E}T\alpha S$ distributions μ, i.e. $\boldsymbol{S}(1) \sim \mu$. The next proposition gives condition for an $\mathcal{E}T\alpha S$ distribution to be the self-decomposable distribution associated with a Sato subordinator.

Proposition 3.1 *The Lévy measure in (2.1) is the Lévy measure of an $\mathcal{E}T\alpha S$ Sato subordinator $\boldsymbol{S}(t)$ if and only if $\alpha \in [0, 1)$ and λ has support on S_+^{d-1}.*

Proof For any $\boldsymbol{w} \in S^{d-1}$ let $S_{\boldsymbol{w}}(t)$ be a one-dimensional Sato process with Lévy measure $\nu_{\boldsymbol{w}}$. The process $S_{\boldsymbol{w}}(t)$ only has positive jumps because $\nu_{\boldsymbol{w}}$ is a positive Lévy measure. Therefore $S_{\boldsymbol{w}}(t)$ is an increasing process if and only if $\int_{(0,1]} x\nu_{\boldsymbol{w}}(dx) < \infty$, see Sato [32], thus if and only if $\alpha \in [0, 1)$. Finally, the Lévy measure ν is positive if and only if λ has support on S_+^{d-1}.

$\square$

Theorem 3.1 *Let $\mu \sim \mathcal{E}T\alpha S(\alpha, \beta, \lambda)$, the associated Sato subordinator $\boldsymbol{S}(t)$ has time t Lévy measure given by*

$$\nu(E, t) = \int_{S_+^{d-1}}\int_{\mathbb{R}_+}\mathbf{1}_E(r\boldsymbol{w})\frac{e^{-\beta(\boldsymbol{w})rt^{-q}}}{r^{\alpha+1}}t^{\alpha q}dr\lambda(d\boldsymbol{w}) \tag{3.2}$$

and it has characteristic function (3.1) *with differential Lévy measure*

$$g(E,u)=\int_{S_+^{d-1}}\int_{\mathbb{R}_+}\mathbf{1}_E(r\boldsymbol{w})e^{-\beta(\boldsymbol{w})ru^{-q}}qu^{\alpha q-1}\frac{\beta(\boldsymbol{w})ru^{-q}+\alpha}{r^{\alpha+1}}dr\lambda(d\boldsymbol{w}). \tag{3.3}$$

Proof The time t characteristic function of an $\mathcal{E}T\alpha S$-Sato subordinator with zero drift is given by (2.6), therefore

$$\begin{aligned}\hat{\mu}_t(z)=\hat{\mu}(t^q z)&=\exp\left\{\int_{S^{d-1}}\int_{\mathbb{R}_+}(e^{ir\langle\boldsymbol{w},t^q z\rangle}-1)\frac{e^{-\beta(\boldsymbol{w})r}}{r^{\alpha+1}}dr\lambda(d\boldsymbol{w})\right\}\\&=\exp\left\{\int_{S^{d-1}}\int_{\mathbb{R}_+}(e^{irt^q\langle\boldsymbol{w},z\rangle}-1)\frac{e^{-\beta(\boldsymbol{w})r}}{r^{\alpha+1}}dr\lambda(d\boldsymbol{w})\right\}\\&=\exp\left\{\int_{S^{d-1}}\int_{\mathbb{R}_+}(e^{iu\langle\boldsymbol{w},z\rangle}-1)\frac{e^{-\beta(\boldsymbol{w})ut^{-q}}}{u^{\alpha+1}t^{-\alpha q}}du\lambda(d\boldsymbol{w})\right\},\quad\forall z\in\mathbb{R}^d.\end{aligned}$$

and the time t Lévy measure is (3.2). Let now

$$g_{\boldsymbol{w}}(r,u)=\frac{\partial\nu_{\boldsymbol{w}}(r,u)}{\partial u}=e^{-\beta(\boldsymbol{w})ru^{-q}}qu^{\alpha q-1}\frac{\beta(\boldsymbol{w})ru^{-q}+\alpha}{r^{\alpha+1}}.$$

We have

$$\hat{\mu}_t(z)=\exp\left\{\int_{S^{d-1}}\int_{\mathbb{R}_+}(e^{ir\langle\boldsymbol{w},t^q z\rangle}-1)\int_0^t e^{-\beta(\boldsymbol{w})ru^{-q}}qu^{\alpha q-1}\frac{\beta(\boldsymbol{w})ru^{-q}+\alpha}{r^{\alpha+1}}dudr\lambda(d\boldsymbol{w})\right\}.$$

Fubini-Tonelli applies and we have

$$\hat{\mu}_t(z)=\exp\left\{\int_0^t\int_{S^{d-1}}\int_{\mathbb{R}_+}(e^{ir\langle\boldsymbol{w},t^q z\rangle}-1)e^{-\beta(\boldsymbol{w})ru^{-q}}qu^{\alpha q-1}\frac{\beta(\boldsymbol{w})ru^{-q}+\alpha}{r^{\alpha+1}}dr\lambda(d\boldsymbol{w})du\right\},$$

that with (3.1) gives that $g_{\boldsymbol{w}}(r,u)$ is the density of the radial component of the differential Lévy measure, therefore (3.3) follows. □

The parameter q in (3.2) drives time inhomogeneity of the increments of the subordinator $\boldsymbol{S}(t)$ and it is called Sato exponent.

We conclude this section with the specification of the Sato-inverse gamma subordinator used in the financial application. A Sato-inverse gamma (S-IG) subordinator is a Sato process $\boldsymbol{S}(t)$ such that $\boldsymbol{S}(1)\sim\mathcal{E}T\alpha S(\frac{1}{2},\beta,\lambda)$ and λ has support on S_+^{d-1}. We write $\boldsymbol{S}(t)\sim$ S-IG(β,λ). The time t Lévy measure of a S-IG subordinator is

$$\nu_{IG}(E,t)=\int_{S_+^{d-1}}\int_{\mathbb{R}_+}\mathbf{1}_E(r\boldsymbol{w})\frac{t^{\frac{q}{2}}e^{-\beta(\boldsymbol{w})rt^{-q}}}{r^{\frac{3}{2}}}dr\lambda(d\boldsymbol{w}).$$

The S-IG subordinator is used in Sect. 5 for our application to finance. Therefore we report its time t characteristic function. Let $\boldsymbol{S}(t)\sim$ S-IG(β,λ), then its characteristic function is

$$\hat{\mu}_t(z)=\hat{\mu}(t^q z)=\exp\left\{-2\sqrt{\pi}\int_{S_+^{d-1}}\left[\sqrt{b^2(\boldsymbol{w})-i\langle\boldsymbol{w},t^q z\rangle}-b(\boldsymbol{w})\right]\lambda(d\boldsymbol{w})\right\},\quad\forall z\in\mathbb{R}^d,$$

where $b(\boldsymbol{w})=\sqrt{\beta(\boldsymbol{w})}$.

If λ has support on S^{d-1} we have a multivariate Sato inverse Gaussian process.

4 Sato-$\mathcal{E}T\alpha S$ subordinated Brownian motion

In this section we build a multivariate additive process by subordinating a multiparameter Brownian motion with a multivariate $\mathcal{E}T\alpha S$ Sato subordinator. For the formal definition of multiparameter (Lévy) process we refer to Barndorff-Nielsen et al. [2]. The Sato subordinator is assumed to have zero drift, i.e. $\gamma(0) = 0$ in (3.1). This assumption allows us to avoid the introduction of regularized Sato $\mathcal{E}T\alpha S$ subordinators (see Li et al. [18]).

Let $\boldsymbol{B}_i(t)$ be independent Brownian motions on $\mathbb{R}^{n_i}$ with drift $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma_i}$, and let $\boldsymbol{B} = \{\boldsymbol{B}(\boldsymbol{s}), \boldsymbol{s} \in \mathbb{R}^d_+\}$, where $\boldsymbol{B}(\boldsymbol{s}) := (\boldsymbol{B}_1(s_1), \ldots, \boldsymbol{B}_d(s_d))^T$, be the associated multiparameter Lévy process. Let $\boldsymbol{A}_i \in \mathcal{M}_{n\times n_i}(\mathbb{R})$. We can define the process $\boldsymbol{B_A} = \{\boldsymbol{B_A}(\boldsymbol{s}), \boldsymbol{s} \in \mathbb{R}^d_+\}$ as

$$\boldsymbol{B_A}(\boldsymbol{s}) = \boldsymbol{A}_1\boldsymbol{B}_1(s_1) + \ldots + \boldsymbol{A}_d\boldsymbol{B}_d(s_d), \quad \boldsymbol{s} \in \mathbb{R}^d_+. \tag{4.1}$$

The process $\boldsymbol{B_A}$ is a multiparameter Lévy process on $\mathbb{R}^n$, see Example 4.4 in Barndorff-Nielsen et al. [2]. We call the multiparameter Lévy process $\boldsymbol{B_A}(\boldsymbol{s})$ in (4.1) *multiparameter Brownian motion*.

Definition 4.1 A process $\boldsymbol{Y}$ defined by

$$\boldsymbol{Y}(t) := \boldsymbol{B_A}(\boldsymbol{S}(t)), \tag{4.2}$$

where $\boldsymbol{B_A}(\boldsymbol{s})$ is the multiparameter process in (4.1) and $\boldsymbol{S}(t)$ is a multivariate Sato subordinator independent of $\boldsymbol{B_A}(\boldsymbol{s})$, is a Sato subordinated multiparameter Brownian motion.

The process $\boldsymbol{Y}(t)$ is defined without a separate drift term that controls the mean. The drift term is added in the application. We now provide the characteristic function of $\boldsymbol{Y}(t)$ in (4.2).

Theorem 4.1 *The Sato subordinated Brownian motion $\boldsymbol{Y}(t)$ in (4.1) is an additive pure jump process with time t characteristic function:*

$$\hat{\mu}_t(z) = \exp\left\{\Gamma(-\alpha)\int_{S^{d-1}}[(\beta(\boldsymbol{w}) - t^q\langle\log(\hat{\phi}_{\boldsymbol{A}}(z)), \boldsymbol{w}\rangle)^\alpha - \beta(\boldsymbol{w})^\alpha]\lambda(d\boldsymbol{w})\right\}, \quad \forall z \in \mathbb{R}^d, \tag{4.3}$$

where $\log(\hat{\phi}_{\boldsymbol{A}}(z)) = (\log\hat{\phi}_1(z), \ldots, \log\hat{\phi}_d(z))$*,* $\hat{\phi}_l(z)$ *is the characteristic function of* $\boldsymbol{A}_l\boldsymbol{B}_l(1)$*,* $l = 1, \ldots, d$*.*

Proof Let $\boldsymbol{A}_i \in \mathcal{M}_{n\times n_i}(\mathbb{R})$ and let the process $\boldsymbol{B_A}$ be defined as in (4.1). The process $\tilde{\boldsymbol{B}}(s_l) = \boldsymbol{A}_l\boldsymbol{B}_l(s_l)$ is a n-dimensional Brownian motion with parameters $\boldsymbol{\mu_A} = \boldsymbol{A}_l\boldsymbol{\mu}_l$ and $\boldsymbol{\Sigma}_l = \boldsymbol{A}_l\boldsymbol{\Sigma}_l\boldsymbol{A}_l^T$. We have

$$\boldsymbol{B_A}(\boldsymbol{\delta}_j) = \boldsymbol{B_A}(0, \ldots, \underbrace{1}_{\text{j-th}}, \ldots, 0) = \boldsymbol{A}_j\boldsymbol{B}_j(1).$$

Thus

$$\hat{\phi}_j(z) = \mathbb{E}[\exp\{i\langle\boldsymbol{A}_j\boldsymbol{B}_j(1), z\rangle\}] = \mathbb{E}[\exp\{i\langle\boldsymbol{B_A}(\boldsymbol{\delta}_j), z\rangle\}]$$

and

$$\begin{aligned}\hat{\mu}_t(z) &= \mathbb{E}[\exp\{i\langle\boldsymbol{Y}(t), z\rangle\}] = \mathbb{E}[\mathbb{E}[\exp\{i\langle\boldsymbol{B_A}(\boldsymbol{s}), z\rangle\}|\boldsymbol{S}(t) = \boldsymbol{s}]] \\ &= \mathbb{E}[\exp\{\langle\log(\hat{\phi}_{\boldsymbol{A}}(z)), \boldsymbol{S}(t)\rangle\}] = \hat{\mu}(-it^q\log(\phi_{\boldsymbol{A}}(z))),\end{aligned} \tag{4.4}$$

where the second equality follows from Theorem 4.7 in Barndorff-Nielsen et al. [2] and last equality follows because $\boldsymbol{S}(t)$ is a Sato process. Equation (4.4) with (2.6), gives (4.3). $\square$

Remark 2 Subordination of multiparameter processes has been introduced in Barndorff-Nielsen et al. [2], where the authors consider the case of a Lévy subordinator. In Jevtić et al. [14], the authors consider the case where the multiparameter process is the multiparameter Brownian motion in 4.1. The Sato subordinated Brownian motion in (4.2) has the same unit time distribution of a subordinated multiparameter Brownian motion, as one can see from the unit time characteristic function.

5 Application to asset returns modeling

This section specifies a Sato-subordinated Brownian motion to model asset returns and presents a first calibration on real data. We start with the introduction of factor-based $\mathcal{E}T\alpha S$ distributions.

Proposition 5.1 *Let $\alpha \in [0, 1)$. Let us consider a random vector $\boldsymbol{S} = (S_1, \dots, S_n) \sim \mu$, such that*

$$S_j = X_j + a_j Z, \quad j = 1, \dots, d, \tag{5.1}$$

where $X_i \sim \mathcal{E}T\alpha S(\alpha, \beta_i, \lambda_i)$ are independent of each other and they are independent of $Z \sim \mathcal{E}T\alpha S(\alpha, \beta_Z, \lambda_Z)$. Then $\boldsymbol{S} \sim \mathcal{E}T\alpha S(\alpha, \beta, \lambda)$, where λ has support $Supp(\lambda) = \{\boldsymbol{w}, \boldsymbol{e}_j, j = 1, \dots, d\}$, $\{\boldsymbol{e}_j, j = 1, \dots, d\}$ is the canonical $\mathbb{R}^d$ basis, $\boldsymbol{w} = \frac{\boldsymbol{a}}{||\boldsymbol{a}||}$ and $\beta : Supp(\lambda) \to \mathbb{R}$, is defined by $\beta(\boldsymbol{e}_i) = \beta_i$ and $\beta(\boldsymbol{w}) = \frac{\beta_Z}{||\boldsymbol{a}||}$.

Proof Let $\beta : Supp(\lambda) \to \mathbb{R}_+$, such that $\beta(\boldsymbol{w}) = \beta_j$ if $\boldsymbol{w} = \boldsymbol{e}_j$ and $\beta(\boldsymbol{w}) = \frac{\beta_Z}{||\boldsymbol{a}||}$. The vector $\boldsymbol{a}Z := (a_1 Z, \dots, a_d Z)^T$ has $\mathcal{E}T\alpha S$ distribution with parameters α, β and λ_Z, where λ_Z is a finite measure with support on the point $\boldsymbol{w} = \frac{\boldsymbol{a}}{||\boldsymbol{a}||}$, see Lemma 2.1 in Semeraro [33].

Since $\boldsymbol{X} = (X_1, \dots, X_d)$ has independent components, by Proposition 2.4, its Lévy measure has support on $\{\boldsymbol{e}_j, j = 1, \dots, d\}$. Thus

$$\nu_{\boldsymbol{X}}(E) = \sum_{j=1}^{n} \nu_{\boldsymbol{X}}(E_j) = \sum_{i=1}^{d} \mathbf{1}_{E_i}(r\boldsymbol{e}_i)\frac{e^{-\beta_i r}}{r^{\alpha+1}}\lambda_i = \sum_{i=1}^{d} \mathbf{1}_{E_i}(r\boldsymbol{e}_i)\frac{e^{-\beta(\boldsymbol{e}_i) r}}{r^{\alpha+1}}\lambda_{\boldsymbol{X}}(\{\boldsymbol{e}_i\}),$$

where $E \in \mathcal{B}(\mathbb{R}^n \setminus \{0\})$, $E_j = E \cap A_j$ and $A_j = \{x \in \mathbb{R}^n : x_k = 0, k \neq j, k = 1, ..., n\}$. Thus $\boldsymbol{X} \sim \mathcal{E}T\alpha S(\alpha, \beta, \lambda_{\boldsymbol{X}})$ with $\lambda_{\boldsymbol{X}}(\{\boldsymbol{w}\}) = \lambda_i$ if $\boldsymbol{w} = \boldsymbol{e}_i$ and $\lambda_{\boldsymbol{X}}(\{\boldsymbol{w}\}) = 0$ otherwise.

Since $\boldsymbol{a}Z$ and $\boldsymbol{X}$ are independent, by Proposition 2.5 $\boldsymbol{X} + \boldsymbol{a}Z \sim \mathcal{E}T\alpha S(\alpha, \beta, \lambda)$, where $\lambda = \lambda_{\boldsymbol{X}} + \lambda_Z$ and thus it has support on $\{\boldsymbol{w}; \boldsymbol{e}_j, j = 1, \dots, d\}$. $\square$

We say that $\boldsymbol{S}$ in Proposition 5.1 has a factor-based distribution $\mu \sim \mathcal{E}T\alpha S(\alpha, \beta, \lambda)$.

Corollary 5.1 *A factor-based measure $\mu \sim \mathcal{E}T\alpha S(\alpha, \beta, \lambda)$ has characteristic function the form*

$$\begin{aligned} \hat{\mu}(z) = \prod_{j=1}^{d} \exp\{\Gamma(-\alpha)[(\beta_j - i z_j)^\alpha - \beta_j^\alpha]\lambda_j\} \cdot \\ \cdot \exp\{\Gamma(-\alpha)[(\beta_Z - i(\sum_{k=1}^{d} a_k z_k))^\alpha - \beta_Z^\alpha]\lambda_Z\}, \quad \forall z \in \mathbb{R}^d, \end{aligned} \tag{5.2}$$

where $\beta_j, \beta_Z \in \mathbb{R}_+$ *and* $\lambda_j, \lambda_Z \in \mathbb{R}_+$.

Proof It is sufficient to notice that if λ has finite support $\{\boldsymbol{w}_j,\ j = 1, \dots d\}$, then its characteristic function becomes

$$\hat{\mu}(z) = \prod_{j=1}^{d} \exp\{\Gamma(-\alpha)\lambda_j[(\beta_j - i\langle \boldsymbol{w}_j, z\rangle)^{\alpha} - \beta_j^{\alpha}]\}, \quad \forall z \in \mathbb{R}^d,$$

where $\beta(\boldsymbol{w}_j) = \beta_j$ and $\lambda(\{\boldsymbol{w}_j\}) = \lambda_j$. The assertion follows by choosing λ and β as in Proposition 5.1. □

Notice that, if λ has finite support, the one-dimensional marginal distributions μ_j of $\mu \sim \mathcal{E}T\alpha S(\alpha, \beta, \lambda)$ are convolutions of one-dimensional $\mathcal{E}T\alpha S$ distributions.

Definition 5.1 A factor-based Sato subordinator $\boldsymbol{S}(t)$ is a Sato process such that $\boldsymbol{S}(1)$ has the factor-based $\mathcal{E}T\alpha S$ distribution in Proposition 5.1.

If $\boldsymbol{S}(1) \sim \mathcal{E}T\alpha S(\alpha, \beta, \lambda)$, the time t characteristic function of the Sato subordinator $\boldsymbol{S}(t)$ is

$$\hat{\mu}_t(z) = \hat{\mu}(t^q z),$$

where $\hat{\mu}$ is the characteristic function in (5.2). If $\boldsymbol{S}(t)$ is the Sato subordinator associated with $\boldsymbol{S}$ we write $\boldsymbol{S}(t) \sim \mathcal{E}T\alpha S(\alpha, \beta, \lambda, q)$.

Proposition 5.2 *If* $\alpha \in [0, 1/2)$*, it holds* $\int_{||x|| \le 1} ||\boldsymbol{x}|| \nu_{\boldsymbol{Y}}(d\boldsymbol{x}) < \infty$ *and* $\int_{\mathbb{R}^d} \nu_{\boldsymbol{Y}}(d\boldsymbol{x}) = \infty$. *The process* $\boldsymbol{Y}(t)$ *is of bounded variations.*

Proof Let $\nu = (\lambda, \nu_{\boldsymbol{w}})$ be the time one Lévy measure in (3.2) and let $B = \{\boldsymbol{x} \in \mathbb{R}^d :\ ||\boldsymbol{x}|| \le 1\}$. It is sufficient to show that

$$\int_B ||\boldsymbol{x}||^{1/2} \nu(\boldsymbol{x}) < \infty, \tag{5.3}$$

where ν is the time one Lévy measure of $\boldsymbol{S}(t)$. Then the assertion follows from Theorems 3.3 and 4.7 in Barndorff-Nielsen et al. [2]. The proof that (5.3) holds if and only if $\alpha \in [0, 1/2)$ is similar to the proof of Proposition 2.1 and therefore omitted. □

5.1 Factor-based Sato subordinated Brownian motion

We propose here a model with the same dependence structure of the factor-Based subordinated Brownian motion in Luciano and Semeraro [21]. Our aim is to keep the flexibility of their dependence structure, to have one-dimensional unit time distributions in given classes and to include time inhomogeneous increments.

Definition 5.2 Let $B_j(t)$, $j = 1, \dots, d$ be independent Brownian motions with drift μ_j and diffusion σ_j. Let $\boldsymbol{B}^{\rho}(t)$ be a correlated d-dimensional Brownian motion, with correlations ρ_{ij}, marginal drifts $\boldsymbol{\mu}_j^{\rho} = \mu_j a_j$ and diffusion matrix $\Sigma^{\rho} := (\rho_{ij}\sigma_i\sigma_j\sqrt{a_i}\sqrt{a_j})_{ij}$, $i, j = 1, \dots d$. The $\mathbb{R}^d$-valued subordinated process $\boldsymbol{Y} = \{\boldsymbol{Y}^{\rho}(t), t > 0\}$ defined by

$$\boldsymbol{Y}^{\rho}(t) = \begin{pmatrix} B_1(X_1(t)) + B_1^{\rho}(Z(t)) \\ \dots. \\ B_d(X_d(t)) + B_d^{\rho}(Z(t)) \end{pmatrix}, \tag{5.4}$$

where $X_j(t)$ and $Z(t)$ are the independent Sato subordinators with unit time distribution of X_j and Z in Proposition 5.1, independent from $\boldsymbol{B}(t)$ and $\boldsymbol{B}^\rho(t)$ is a ρ-factor-based Sato subordinated Brownian motion.

The unit time dependence structure of ρ-factor-based Sato subordinated Brownian motions is driven by the Brownian motion correlations and the common component of the Sato subordinator. The unit time dependence structure is widely discussed in Luciano and Semeraro [21]. It should be noticed that the process may have dependent marginals also in the symmetric case - Brownian motions with zero drift - if $\boldsymbol{B}^\rho(t)$ has independent components. In this case all the -nonlinear- dependence derives from the component Z. The Sato exponent of the subordinator drives time inhomogeneity and time varying correlations.

Proposition 5.3 *Let $\boldsymbol{Y}^\rho(t)$ be a ρ-factor-based Sato subordinated Brownian motion in (5.4). Then $\boldsymbol{Y}^\rho(t)$ belong to the class of Sato subordinated Brownian motions in Definition 4.1.*

Proof Let us consider the independent Brownian motions $B_i(t),\ i = 1, \ldots d$ and $\boldsymbol{B}^\rho(t)$ 5.4 and let

$$\boldsymbol{B}_{\boldsymbol{A}}(\boldsymbol{s}) = \sum_{i=1}^{d} A_i B_i(s_i) + \boldsymbol{A}_{d+1}\boldsymbol{B}^\rho(s_{d+1}),$$

where $A_i \in \mathcal{M}(d \times 1), i = 1, \ldots, d$ and $\boldsymbol{A} \in \mathcal{M}_{d\times(d)}$ such that $A_i = (0, \ldots, 1, \ldots, 0)$, $i_1, \ldots, d$ and $\boldsymbol{A}_{d+1} = \boldsymbol{I}_d$. Let now $\boldsymbol{S}(t) = (X_1(t), \ldots, X_d(t), Z(t))$ and $\mathcal{E}T\alpha S$ Sato subordinator with independent components. Let $\boldsymbol{Y}^\rho(t)$ be the process in (5.4) we have $\boldsymbol{Y}(t)^\rho = \boldsymbol{B}_{\boldsymbol{A}}(\boldsymbol{S}(t))$. □

Corollary 5.2 *If $\boldsymbol{S}(t) \sim \mathcal{E}T\alpha S(\alpha, \beta, \lambda, q)$ is the Sato subordinator in (5.1) the subordinated process $\boldsymbol{Y}(t)$ in (5.4) has characteristic function*

$$\begin{aligned}\hat{\mu}_t(\boldsymbol{z}) = \prod_{j=1}^{d} &\exp\{\Gamma(-\alpha)[(\beta_j - t^q(i\mu_j z_j - \frac{1}{2}\sigma^2 z_j^2))^\alpha - \beta_j^\alpha]\lambda_j\}\cdot \\ &\exp\{\Gamma(-\alpha)[(\beta_Z - t^q(i\boldsymbol{u}^T\boldsymbol{\mu}^\rho - \frac{1}{2}\boldsymbol{z}^T\boldsymbol{\Sigma}^\rho\boldsymbol{z}))^\alpha - \beta_Z^\alpha]\lambda_Z\},\quad \forall \boldsymbol{z} \in \mathbb{R}^d,\end{aligned} \tag{5.5}$$

where $\beta_j, \beta_Z, \lambda_j$ and λ_Z are the parameters in Proposition 5.1.

Corollary 5.3 *If the Brownian motions in Definition 5.2 have zero drift the subordinated process $\boldsymbol{Y}^\rho(t)$ in (5.4) is a Sato process.*

Proof Since $\boldsymbol{Y}^\rho(t) = \boldsymbol{B}_A(\boldsymbol{S}(t))$, it is the sum of the independent subordinated Brownian motions $B_i(S_i(t)),\ i = 1, \ldots, d$ and $\boldsymbol{B}^\rho(S_{d+1}(t))$. It is sufficient to observe that if Brownian motions in Definition 5.2 have zero drift the unit time distribution of $B_i(S_i(t)),\ i = 1, \ldots, d$ and $\boldsymbol{B}^\rho(S_{d+1}(t))$ are self-decomposable (Takano [36]), therefore the unit time distribution of and $\boldsymbol{Y}^\rho(t)$ is. Furthermore, $t^{2q}\log(\hat{\phi}_{\boldsymbol{A}}(\boldsymbol{z})) = \log(\hat{\phi}_{\boldsymbol{A}}(t^q\boldsymbol{z}))$. □

Proposition 5.4 *If all the parameters ρ_{ij} in (5.4) collapse to 0 across different components, i.e. $\rho_{ij} = 0$, for $i \neq j$, $\rho_{ij} = 1$, for $i = j$, the factor-based Sato subordinated Brownian motion $\boldsymbol{Y}^\rho(t)$ in (5.4) becomes:*

$$\boldsymbol{Y}(t) = \boldsymbol{B}(\boldsymbol{S}(t)) = (B_1(S_1(t)), \ldots, B_d(S_d(t))), \tag{5.6}$$

where $\boldsymbol{S}(t)$ is the factor-based Sato subordinator in Definition 5.1 and $\boldsymbol{B}(\boldsymbol{s})$ is a d-dimensional multiparameter Brownian motion with independent components $B_j(s_j)$ that have mean μ_j and diffusion σ_j.

Proof By substituting $\rho_{ij} = 0$ in (5.4) the characteristic function of $\boldsymbol{Y}^{\rho}(t)$ becomes

$$\begin{aligned}\hat{\mu}_t(z) &= \prod_{j=1}^{d} \exp\{\Gamma(-\alpha)[(\beta_j - t^q(i\mu_j z_j - \frac{1}{2}\sigma^2 z_j^2))^{\alpha} - \beta_j^{\alpha}]\lambda_j\}\cdot \\ &\quad \cdot \exp\{\Gamma(-\alpha)[(\beta_Z - it^q \sum_{k=1}^{d} a_k(\mu_k z_k - \frac{1}{2}\sigma^2 z_k^2))^{\alpha} - \beta_Z^{\alpha}]\lambda_Z\} \\ &= \exp\{\psi_{\boldsymbol{S}}(t^q \log(\hat{\phi}_{\boldsymbol{I}}(z))\}, \quad \forall z \in \mathbb{R}^d,\end{aligned}$$

where $\hat{\phi}_{\boldsymbol{I}}(z)$ is the characteristic function of the multivariate Brownian motion in (5.6). From Theorem 4.1 it follows that $\hat{\mu}_t(z) = \exp\{\psi_{\boldsymbol{S}}(t^q \log(\hat{\phi}_{\boldsymbol{I}}(z))\}$ is the characteristic function of $\boldsymbol{Y}(t)$ in (5.6). □

A very useful result for applications and calibration is the following, that can be easily proved using characteristic functions, see Theorem 5.1, Luciano and Semeraro [20].

Proposition 5.5 *The processes in* (5.4) *and in* (5.6) *have the same one-dimensional marginal processes in law, that are one-dimensional Sato subordinated Brownian motions.*

As a consequence the processes (5.4) and (5.6) clearly have one-dimensional marginal processes that are themselves Sato-subordinated Brownian motions.

Linear correlation functions are usually used to calibrate the dependence structure in multivariate Lévy models, see e.g. Guillaume [12] and Luciano et al. [19]. By the scaling properties of Lévy processes that have independent and stationary increments, the correlation function of a Lévy process is constant over time, i.e. $\rho_{\boldsymbol{Y}(t)}(h, j) = \rho_{\boldsymbol{Y}(1)}(h, j), \quad t \geq 0$, see Appendix C. However, this is not a realistic assumption, see e.g. Tóth and Kertész [38], Teng et al. [37] and Lundin et al. [22]. We now show that linear correlation functions of the factor-based Sato subordinated Brownian motion change over time and have a simple analytical formula. Standard computations give the mean $\mathbb{E}[Y_j^{\rho}(t)]$, the variance $\mathbb{V}[Y_j^{\rho}(t)]$ and the correlation $\rho_{\boldsymbol{Y}^{\rho}(t)}(h, j)$ functions of $\boldsymbol{Y}^{\rho}(t)$:

$$\mathbb{E}[Y_j^{\rho}(t)] = \mu t^q \mathbb{E}[S_j]; \quad \mathbb{V}[Y_j^{\rho}(t)] = \sigma^2 t^q \mathbb{E}[S_j] + \mu^2 t^{2q} \mathbb{V}[S_j]$$

and

$$\rho_{\boldsymbol{Y}^{\rho}(t)}(h, j) = \frac{\rho_{hj}\sigma_h\sigma_j\sqrt{a_h}\sqrt{a_j}t^q\mathbb{E}[Z] + \mu_j\mu_h a_j a_h t^{2q}\mathbb{V}(Z)}{\sqrt{(\sigma_j^2 t^q \mathbb{E}[S_j] + \mu_j^2 t^{2q}\mathbb{V}[S_j])(\sigma_h^2 t^q \mathbb{E}[S_h] + \mu_h^2 t^{2q}\mathbb{V}[S_h])}}, \tag{5.7}$$

where $Z = Z(1)$ and $S_j = S_j(1)$. Consider the factor-based subordinated Brownian motion in (B.2), where $X_j(t)$ and $Z(t)$ are the independent subordinators with the same unit time distribution of the Sato subordinators in (5.4). The correlation functions of the factor-based Sato subordinated Brownian motion at unit time are equal to the correlations of the factor-based subordinated Brownian motion $\boldsymbol{Y}^L(t)$ defined in (B.2), see Eq. (C.1).

The linear correlation coefficients of $\boldsymbol{Y}(t)$ in (5.6) are obtained by assuming $\rho_{ij} = 0$. Furthermore,

$$\lim_{t\to\infty} \rho_{\boldsymbol{Y}^{\rho}(t)}(l, j) = \rho_{\boldsymbol{S}}(l, j)$$

and

$$\lim_{t\to 0} \rho_{\boldsymbol{Y}^{\rho}(t)}(l, j) = \frac{\rho_{hj}\sqrt{a_1 a_j}\mathbb{E}[Z]}{\sqrt{\mathbb{E}[S_l]\mathbb{E}[S_j]}}.$$

Since $\mathbb{E}[S_j] = \mathbb{E}[X_j] + a_j\mathbb{E}[Z]$, $\lim_{t\to 0} \rho_{\mathbf{Y}^\rho(t)}(l, j) = 1$ if we have the limit values $\rho_{ij} = 1$ and $\mathbb{E}[X_j] = 0$, $i, j = 1, \ldots, d$.

In the next section, we specify a distribution for the factor-based Sato subordinated Brownian motion.

5.2 Normal inverse Gaussian case

We specify the factor-based Sato subordinator to have the same unit time distribution of the factor-based $\rho\alpha$-NIG in Luciano and Semeraro [21]. Let $\boldsymbol{S} = (S_1, \ldots, S_d)$ as in (5.1) and let X_j and Z have inverse Gaussian distribution (IG) with parameters:

$$X_j \sim IG\left(1 - a\sqrt{a_j}, \frac{1}{\sqrt{a_j}}\right),\ j = 1, ..., n \quad \text{and} \quad Z \sim IG(a, 1), \tag{5.8}$$

where $0 < a < \frac{1}{\sqrt{a_j}}, \quad j = 1, ..., n$. Let now $\boldsymbol{S}(t)$ be the factor-based Sato subordinator with unit time distribution of $\boldsymbol{S}$, its time t characteristic function $\hat{\mu}_{\boldsymbol{S}(t)}(z)$ has the form

$$\hat{\mu}_{\boldsymbol{S}(t)}(z) = \prod_{j=1}^{d} \exp\left\{(1 - a\sqrt{a_j})\left[\sqrt{\frac{1}{a_j} - 2it^q z_j} - \frac{1}{\sqrt{a_j}}\right]\right\} \cdot \\ \cdot \exp\left\{-a\left[\sqrt{1 - 2i(t^q \sum_{k=1}^{d} a_k z_k)} - 1\right]\right\}, \quad \forall z \in \mathbb{R}^d.$$

The distribution of $\boldsymbol{S}$ has one-dimensional $IG(1, \frac{1}{\sqrt{a_j}})$ marginal distributions that are self-decomposable. Therefore the processes $S_j(t)$ are one-dimensional Sato processes. Let now $\gamma_j, \beta_j, \delta_j$ be such that $\gamma_j > 0, -\gamma_j < \beta_j < \gamma_j, \delta_j > 0$.

Definition 5.3 Let $\boldsymbol{Y}^\rho(t)$ be the process defined in (5.4). If $X_j(t),\ i = 1, \ldots, d$ and $Z(t)$ are Sato subordinators with unit time distributions in (5.8). If we set $\mu_j = \beta_j\delta_j^2$ and $\sigma_j = \delta_j$, $\frac{1}{\sqrt{a_j}} = \delta_j\sqrt{\gamma_j^2 - \beta_j^2}$, the process $\boldsymbol{Y}^\rho(t)$ is called factor-based Sato-IG subordinated Brownian motion.

From Proposition 5.5 the process $\boldsymbol{Y}^\rho(t)$ in Definition 5.3 has the following one marginal processes

$$Y_j^\rho(t) = \beta_j\delta_j^2 S_j(t) + \delta_j W(S_j(t)), \tag{5.9}$$

where $W(t)$ is a standard Brownian motion and equality is in distribution. Therefore the unit time distributions of $\boldsymbol{Y}_j^\rho(t)$ in (5.9) are normal inverse Gaussian distributions with parameters $(\gamma_j, \beta_j, \delta_j)$, that are self-decomposable. The one-dimensional processes $Y_j^\rho(t)$, $j = 1, \ldots, d$ are Sato subordinated Brownian motions. Therefore in a multivariate asset model the process provides time varying correlations and time inhomogeneity also for single assets. The process $\boldsymbol{Y}^\rho(t)$ has a total of $2 + 3n + \frac{n(n-1)}{2}$ parameters: a and q are common parameters; $\gamma_j, \beta_j, \delta_j,\ j = 1, ..., n$ are marginal parameters and $\rho_{ij}, i, j = 1, ..., n$, are the $\boldsymbol{B}^\rho$ correlations.

From (5.5), the time t characteristic function $\hat{\mu}_t^{\rho}$ of $\boldsymbol{Y}^{\rho}(t)$, is

$$\hat{\mu}_t^{\rho}(\boldsymbol{u}) = -\prod_{j=1}^{d} \exp\left\{\left(1 - \frac{a}{\zeta_j}\right)\left(\sqrt{\zeta_j^2 - 2t^q(i\beta_j\delta_j^2 u_j + \frac{1}{2}\delta_j^2 u_j^2)} - \zeta_j\right)\right.$$
$$\left. -a\left(\sqrt{1 - 2t^q(i\boldsymbol{u}^T\boldsymbol{\mu}^{\rho} - \frac{1}{2}\boldsymbol{u}^T\boldsymbol{\Sigma}^{\rho}\boldsymbol{u})} - 1\right)\right\}$$

where $\zeta_j = \delta_j\sqrt{\gamma_j^2 - \beta_j^2}$. The linear correlations are given in (5.7) with $\sigma_j = \delta_j$, $\mu_j = \beta\delta_j^2$, $\sqrt{a_j} = \frac{1}{\zeta_j}$, $E[Z] = a$ and $V[Z] = a$ where $a \leq min\{\frac{1}{\sqrt{a_j}}\}$. The $\rho\alpha$-NIG in Luciano and Semeraro [21] is the Lévy process with unit time characteristic function $\hat{\mu}_1^{\rho}(z)$. A $\rho\alpha$-NIG process $\boldsymbol{Y}^L(t)$ has correlations constant over time and $\rho_{\boldsymbol{Y}^L(t)}(i,j) = \rho_{\boldsymbol{Y}^{\rho}(1)}(i,j)$, for each $t > 0$. The $\rho\alpha$-NIG process is recalled in Appendix B.

Recently, Boen and Guillaume [3] proposed a multivariate process with marginal Sato processes to model asset returns. They model asset log returns at unit time as linear combinations of independent self-decomposable random variables, obtaining the same dependence structure as the multivariate Lévy process introduced in Ballotta and Bonfiglioli [1]. In fact, these two processes have the same unit time distribution. Differences between the unit time distribution in Boen and Guillaume [3] and ours are both in the one dimensional marginal distributions and in the dependence structure. The marginal distributions in Boen and Guillaume [3] do not belong in general to a given class. The restrictions needed to specify the marginal distributions become challenging from a numerical point of view in the calibration of the model as the number of assets increases. The one dimensional unit time distributions of the ρ-factor-based subordinated Brownian motion can easily be specified to belong to given classes with good fit properties, e.g. the NIG distribution. This is an advantage for calibration, where a simple two step procedure can be performed and easy calibration is possible also with a large number of assets (Luciano et al. [19]). Regarding the dependence structure, Boen and Guillaume [3] propose a linear combination of independent random variables and in their model, if the correlation is zero, the marginals are independent. In our model, the dependence that arises from the subordinator makes zero correlation and dependent marginals possible, as discussed in Luciano and Semeraro [20]. Nevertheless, the main difference between the two approaches is in time inhomogeneity. The multivariate Sato process in Boen and Guillaume [3] has time inhomogeneous increments, but their log return correlation is constant over time. Our idea to include time homogeneity with a Sato subordinator allows us to model time varying correlation, and this is the main feature of this model. Subordination allows us to take advantage of the scaling properties of the Brownian motion to obtain time dependent correlations. Other multivariate processes with marginal Sato processes have been introduced before, e.g Guillaume [13], Boen and Guillaume [4] and Marena et al. [25]. The multivariate process in Marena et al. [25] has the same $\rho\alpha - NIG$ unit time distribution. However, in all these models correlation is constant over time.

5.3 Calibration and fit

The calibration of the process on financial data requires to estimate a multivariate time-inhomogeneous process. The calibration of multivariate time-inhomogeneous processes is out of the aim of this work. Some results of ML estimation of time inhomogeneous processes can be found in the literature, see for example Egorov et al. [9], where the authors considered a one dimensional process.

This section aims at showing that log returns increments do not scale as stationary increments do and that the factor-based S-IG subordinated Brownian motion can be a good and parsimonious - in terms of number of parameters - extension of the $\rho\alpha$-NIG Lévy model. For this reason, the parameters common to both the models are calibrated on the Lévy process and kept as realistic parameters for the Sato extension. Then the potentiality of adding a parameter that drives time-inhomogeneity is shown.

Define a bidimensional price process, $\boldsymbol{S} = \{\boldsymbol{S}(t),\ t \geq 0\}$, by

$$\boldsymbol{S}(t) = \boldsymbol{S}(0)\exp(\boldsymbol{c}t + \boldsymbol{Y}(t)),\ \boldsymbol{c} \in \mathbb{R}^2, \tag{5.10}$$

where $\boldsymbol{c}$ is the drift term (equivalently, $S_j(t) = S_j(0)\exp(c_j t + Y_j(t)),\ t \geq 0,\ j = 1, 2.$). The actual horizon specific return for asset i on day t is evaluated as

$$Y_i(h) = \log\left(\frac{S_i(h)}{S_i(0)}\right) - c_i h. \tag{5.11}$$

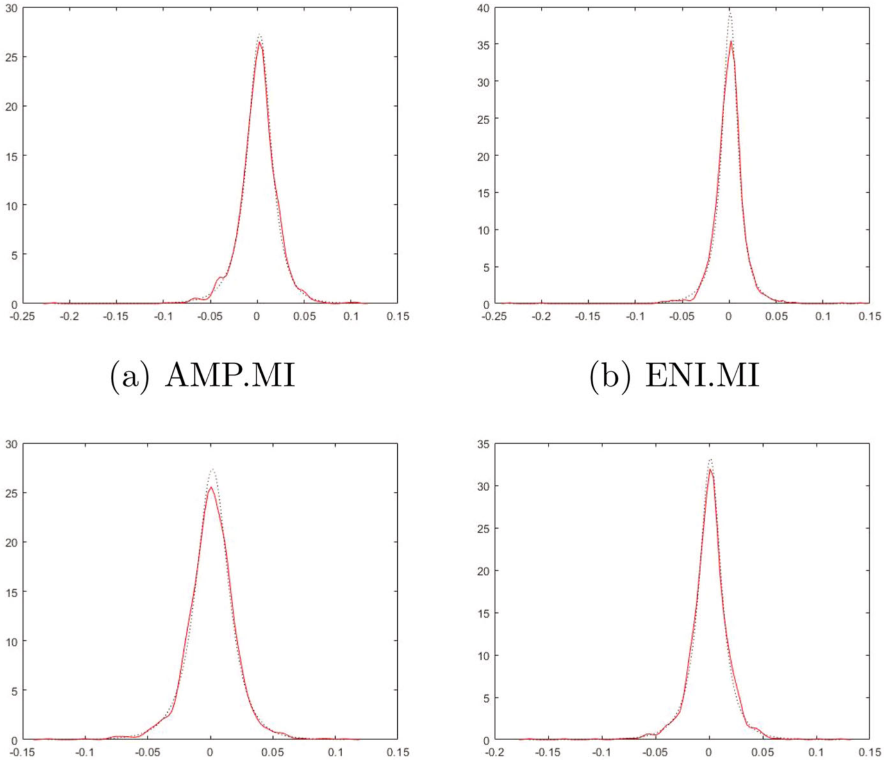
Suppose that the daily return is a random variable $\boldsymbol{Y}$ with a $\rho\alpha$-NIG distribution. The $\rho\alpha$-NIG distribution is associated with a Lévy process. However, this law is also associated to the factor-based S-IG subordinated Brownian motion at unit time. This latter process has independent increments that are not identically distributed. The scaling parameter q drives time-inhomogeneity that, in this first application, we observe by focusing only the correlation function.

5.4 Sensitivity analysis

The parameters of the factor-based S-IG subordinated Brownian motion are: the marginal parameters $(c, \gamma, \beta, \delta)$, the correlation parameters a, ρ and the parameter q. To have realistic parameters we calibrated the $\rho\alpha$-NIG Lévy process on daily returns. In this way, we calibrate the marginal parameters $(c, \gamma, \beta, \delta)$ and the correlation parameters a, ρ. Since the $\rho\alpha$-NIG process has the same distribution of the factor-based S-IG Brownian motion at unit time, the parameter found can be assumed to be realistic parameters also for the Sato extension. Then, we kept the calibrated parameters fixed we moved the parameter q of the Sato extension to show how q drives the correlation function $\rho_{\boldsymbol{Y}^{\rho}(t)}(1, 2) = \rho_{\boldsymbol{Y}^{\rho}(t)}$. We considered log returns on four assets of one of the main Italian indices, FTSE MIB, from September 29, 2016 to September 28, 2021. We looked at the following assets Amplifon (AMP.MI), Eni (ENI.MI), Mediobanca (MB.IM) and FinecoBank (FBK.MI), thus we had six pairs of assets. The assets were chosen to have different levels of pairwise correlation. For example MB.IM and FBK.IM belong to the same sector, while Amplifon is expected to have a low correlation with them.

Calibration on daily returns was performed in two steps, following Luciano et al. [19]. We fitted the marginal parameters from marginal daily return data; then we selected the common parameters by matching daily historical return correlations for each pair. We used MLE to estimate the marginal return distribution on each stock individually. The density function was recovered by applying the Fractional Fast Fourier Transform (FRFT)[1]. Initial conditions

[1] Following the notation in Chourdakis [7] the FRFT setup is: $N = 2^{17}$, $\delta = 0.25$, where δ is the $\boldsymbol{u}$ grid spacing. Let b the absolute value of the maximum observed log return and let $\lambda = 2b/N$; $x_k = -b + k\lambda$ and $\alpha = \delta\lambda/(2\pi)$.



(a) AMP.MI (b) ENI.MI

(c) FBK.MI (d) MB.MI

Fig. 1 Model (dotted line) vs kernel (solid line) densities

were chosen according to the method of moments (see Seneta et al. [34] and Prause [28] for details).

The goodness of fit was evaluated by the KS test. For all assets, we could not reject the null hypothesis that the sample daily returns came from the model distribution, at the 5% level of significance. Figure 1 provides the plots for the density fit of each marginal return.

Following Luciano et al. [19], given the marginal parameters, for each pair (i, j) we jointly calibrate the common parameters a and ρ_{ij} by fitting their sample return correlation. Specifically, for each pair (i, j), we minimized the root-mean-squared error between the empirical and the $\rho\alpha$-model return correlation. Since all the marginal parameters were fixed from step one, the correlation coefficient ρ_{ij}^{Y} between pair i and j depends only on a and on ρ_{ij}, this allowed us to have a very good fit in the bivariate case. As discussed in Marena et al. [25] we underline that the same correlation can be fitted with different pairs (a, ρ_{ij}). In fact the parameter a also drives nonlinear correlation and makes it possible to achieve zero correlation and nonlinear dependence (see also Luciano and Semeraro [20]). The estimated marginal parameters $\gamma_j, \beta_j, \delta_j$ are reported in Table 1. The common parameters a and ρ_{ij} for each pair and the daily correlations are reported in Table 2.

We shall now focus on the factor-based S-IG subordinated Brownian model. We left the same parameters calibrated on daily returns and we addressed long horizon correlations by moving q. We firstly considered daily returns (1d) as a benchmark, then we considered returns

Table 1 Maximum likelihood estimates of marginal NIG return distributions for the $\rho\alpha$NIG distribution

Asset	c	γ	β	δ
AMP.MI	0.0032	41.8689	−4.6962	0.0174
ENI.MI	0.0013	40.1467	−5.2611	0.0107
FBK.MI	0.0021	49.4146	−3.2715	0.0185
MB.MI	0.0015	39.2550	−3.0975	0.0132

Table 2 Estimates of a and ρ, model correlations ρ_A. Asset correlations are equal to model correlation, because the error is lower that 10^{-5}

Pair of assets	a	ρ	ρ_A
AMP.MI; ENI.MI	0.3823	0.3297	0.2522
AMP.MI ;FBK.MI	0.6074	0.3665	0.2745
AMP.MI; MB.MI	0.4711	0.3449	0.2763
ENI.MI; FBK.MI	0.3963	0.7647	0.4809
ENI.MI; MB.MI	0.4191	0.5802	0.5178
FBK.MI;MB.MI	0.5836	0.4414	0.3813

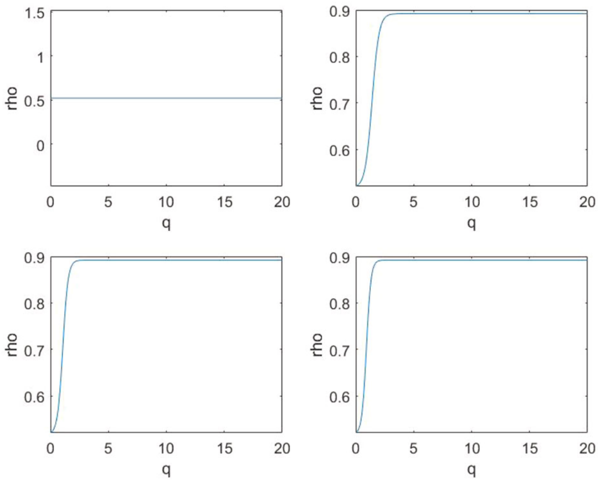


Fig. 2 Correlation q-dynamics of Assets ENI.MI, MB.MI for the four choices of t - from left top to right down: 1d, 1m, 3m, 6m

at different time horizons: a month (1m), a quarter (3m) and half a year (6m). All the pairwise model correlations found with the estimated parameters have similar evolutions as functions of q and time. We shall therefore focus on the pair ENI.MI, MB.MI -the one with the highest daily correlation. Figure 2 illustrates the correlation as a function of parameter q for the four different time horizons. It is evident that the correlation is increasing in q.

Figure 3 illustrates the correlation as a function of time horizon t for the four different values of q considered, the correlation increases in the time scale too.

Finally, Fig. 4 shows the correlation surface and its level curves.

The above figures show that -with our estimates of the other parameters- correlations are increasing functions of q and t.

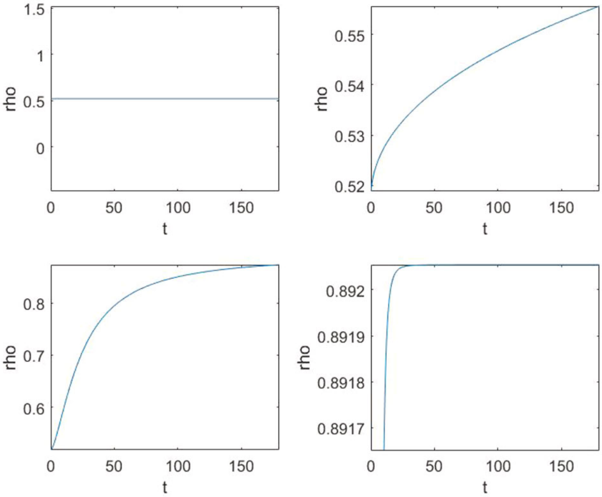
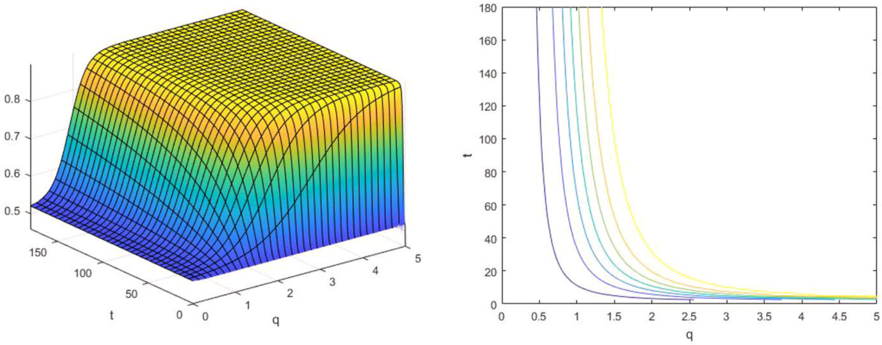


Fig. 3 Correlation t-dynamics of Assets ENI.MI, MB.MI for the four choices of q - from left top to right down: 0, 0.5, 1.5, 5



(a) ENI.MI;MB.MI (b) ENI.MI;MB.MI

Fig. 4 Correlation surface and level curves

5.5 Correlation function fit

Firstly, we show that the Lévy model needs to be extended by showing that sample correlations do not scale as they do in stationary processes, where they are not affected by the time scale, as discussed in Appendix C. Then, we look for a value of q able to fit the observed sample correlations at different time scales: a day (benchmark), a month, a quarter and half a year.

For each pair of stocks we computed empirical correlations using returns at different time scales over the same time period: we use daily increments for daily returns, monthly increments for monthly returns, etc... . If increments were i.i.d. the empirical correlation should not be affected by the time scale. Table 4 shows that empirical correlations are significantly different, being 0.52 for daily returns and 0.82 for 6m returns over the same time period, therefore empirical returns do not scale as stationary processes do. We then proceeded by looking for the value of q that minimizes the distance between the four sample correlations and the corresponding model correlations.

Table 3 Estimates of q for $t = 1d, 1m, 3m, 6m$

Pair of assets	q
AMP.MI; ENI.MI	1.0836
AMP.MI ;FBK.MI	0.9704
AMP.MI; MB.MI	1.1134
ENI.MI; FBK.MI	0.9939
ENI.MI; MB.MI	1.0925
FBK.MI;MB.MI	1.2010

Table 4 Model and empirical correlations for the pair ENI.MI; MB.MI and $t = 1d, 1m, 3m, 6m$

t	ρ_M	ρ_A
1d	0.5207	0.5178
1m	0.6411	0.6791
3m	0.7583	0.7627
6m	0.8186	0.7785

Formally for each pair of assets we minimized:

$$RMSE\,(q) = \sqrt{\frac{1}{n}\sum_{i=1}^{n}\left(\rho_{\boldsymbol{Y}}^{emp}(t_i) - \rho_{\boldsymbol{Y}}(t_i, q)\right)^2},$$

where $\rho_{\boldsymbol{Y}}^{emp}(t_i)$ and $\rho_{\boldsymbol{Y}}(t_i, q)$ are the sample and model return correlations at horizon t_i, respectively. These correlations depend only on the scale parameter q. Table 3 exhibits q estimates, the calibration error order is always 10^{-2} or lower.

Since the results are similar for each pair, we focus on the pair ENI.MI; MB.MI, that gave the highest daily correlations. In Table 4 we report the model and empirical correlations. The parameter q found allowed us to fit correlations of returns at different horizons.

Although this first example does not provide a calibration of the process on real data, its aim is twofold. Firstly, it shows that Lévy processes need to be extended to model time inhomogeneity. Secondly, it shows that the correlation function of the factor-based S-IG subordinated Brownian motion is able to fit sample correlations at different scales with only one parameter and therefore this model is a good candidate for asset return modelling.

We conclude our analysis by showing (Table 5) the model limit correlations for $t \to 0$ and $t \to \infty$ and the half a year empirical correlations. Notice the daily correlations (Table 2) was close to the minimal correlation allowed by the model and the maximal correlation was not been reached over six months. This is a possible consequence of our choice to set a day as the unit time.

Remark 3 In a $T\alpha S$ process, the regularity of the sample paths can be described in terms of the single parameter α, as stated in Proposition 2.1. In our multivariate asset models we use $\mathcal{E}T\alpha S$ subordinators, for which the parameter α lies in $[0, 1)$, as stated in Proposition 3.1. In fact, subordinators have bounded variations. Nevertheless, moving the parameter α, the subordinated process can exhibit bounded or unbounded variations - see Proposition 5.2. For example, a Sato subordinated Brownian motion with a gamma subordinator has time one distribution of variance gamma type. Thus, it has bounded variations, while a Sato subordinated Brownian motion with an IG subordinator has time one distribution of NIG type and unbounded variations. In a multi-asset setting, the sample paths of different asset prices

Table 5 Correlation bounds for the pair ENI.MI; MB.MI

Pair	$\lim_{t\to 0} \rho_{\mathbf{Y}^{\rho}(t)}(l, j)$	$\rho_{\mathbf{Y}(6m)}$	$\lim_{t\to\infty} \rho_{\mathbf{Y}^{\rho}(t)}(l, j) = \rho_{\mathbf{S}}$
AMP.MI; ENI.MI	0.2265	0.5963	0.6869
AMP.MI ;FBK.MI	0.2733	0.5087	0.7458
AMP.MI; MB.MI	0.2654	0.6325	0.7694
ENI.MI; FBK.MI	0.4849	0.5439	0.6341
ENI.MI; MB.MI	0.5175	0.8186	0.8921
FBK.MI;MB.M	0.4316	0.7329	0.7395

can exhibit different levels of local regularity. Multivariate tempered stable distributions, as defined by Rosinski [29], have a unique parameter α. As a consequence, the trajectories of each coordinate have the same parameter α. Our construction relies on subordinators belonging to this class, therefore the sample paths of different asset prices exhibit the same level of local regularity. Although the extension of the class of $\mathcal{E}T\alpha S$ subordinators is beyond the scope of this paper, this is an important issue. A first step to extend the class of $\mathcal{E}T\alpha S$ subordinators to have a vector parameter α could be to allow $X_j(t)$ and $Z(t)$ in (5.1) to have different parameters α. For example in a bivariate case if $X_1(t)$ and $Z(t)$ have $\alpha = 0$ (gamma distributed) and $X_2(t)$ has $\alpha = \frac{1}{2}$ (IG distributed), the subordinated process in (5.4) has marginals $Y_1(t)$ of variance gamma type and $Y_2(t)$ with the component $B_1(X_1(t))$ of NIG type. Therefore, the sample paths of $Y_1(t)$ have bounded variations and the sample paths of $Y_2(t)$ do not.

6 Conclusion

We began by introducing and characterizing a self-decomposable class of multivariate exponential tempered distributions and the associated multivariate Sato subordinators. Then we constructed a multivariate additive process which is able to incorporate time inhomogeneity by additive subordination of a multivariate Brownian motion. The main contribution of Sato subordination is the provision of time varying correlations, while remaining parsimonious in the number of parameters and providing a good fit on financial data. Although the empirical investigation of Sato subordination is beyond the purpose of this paper with a calibration on financial data we have shown that the model is capable of fitting correlations at different time scales for given unit time marginal parameters. In fact, we first calibrated the unit time distribution -not affected by parameter q - in two steps, following Luciano et al. [19]. Then we performed a sensitivity analysis on q and then calibrated q to fit time correlations of returns at different time horizons. This way, we showed the features of the model and the role played by the parameters. A joint multivariate calibration, for example with MLE, could improve the fit of the model.

A full multivariate calibration of factor-based Sato subordinated Brownian motions on financial data is one of our future research directions. A second research direction will be to weaken the assumption of a Sato exponent common to all assets. In the present work, we built on self-decomposability of multivariate $\mathcal{E}T\alpha S$ distributions and considered a multivariate Sato subordinator with scaling parameter q common to all the one-dimensional processes. Since the factor-based S-IG subordinator has self-decomposable marginal distributions, a natural extension would be to allow each marginal process to have its own Sato exponent. To this aim a further step will be to study operator self-decomposability of multivariate $\mathcal{E}T\alpha S$-distributions and the associated Sato subordinators. Finally, path regularity of $\mathcal{E}T\alpha S$

subordinated Brownian motions are linked to the parameter α, that is common to all assets. A third future research direction will be to investigate the possibility of extending the class of multivariate $\mathcal{E}T\alpha S$ distributions to allow different asset returns to exhibit different levels of path regularity.

Acknowledgements The author wishes to thank Marina Marena and Andrea Romeo for the helpful discussions she had with them. The author also gratefully acknowledge financial support from the Italian Ministry of Education, University and Research (MIUR), "Dipartimenti di Eccellenza" grant 2018-2022. The author also thank the referees for their valuable suggestions.

Funding Open access funding provided by Politecnico di Torino within the CRUI-CARE Agreement.

Appendices

A Self-decomposability and Sato processes

The Lévy measure of a self-decomposable distribution μ is of the form (Sato [32], Theorem 15.10):

$$\nu(E) = \int_{S^{d-1}} \int_0^\infty \mathbf{1}_E(r\boldsymbol{w}) \frac{k_{\boldsymbol{w}}(r)}{r} dr \lambda(d\boldsymbol{w}) \tag{A.1}$$

where λ is a finite measure on S^{d-1}, $k_{\boldsymbol{w}}(r) \geq 0$ is measurable in $\boldsymbol{w}$, decreasing in $r > 0$. The Lévy measure of a self-decomposable distribution on $\mathbb{R}$ has the form:

$$\nu(E) = \int_{\mathbb{R}} \mathbf{1}_E(x) \frac{k(x)}{||x||} dx$$

where $k(x) \geq 0$ is increasing on $(-\infty, 0)$ and decreasing on $(0, \infty)$. Therefore a multivariate measure $\mu \in L(\mathbb{R}^d)$ is self-decomposable if and only if its radial component is self-decomposable. The characteristic function of a d-dimensional self-decomposable distribution μ in Cartesian coordinates has the form

$$\hat{\mu}(z) = \exp\{i\boldsymbol{\gamma} \cdot z + \int_{\mathbb{R}^d} (e^{i\langle z, \boldsymbol{x}\rangle} - 1 - i\langle z, \boldsymbol{x}\rangle \mathbf{1}_{||\boldsymbol{x}|| \leq 1}(\boldsymbol{x})) \frac{h(\boldsymbol{x})}{||\boldsymbol{x}||} \tilde{\lambda}(d\boldsymbol{x})\},$$

where

$$\tilde{\lambda}(E) = \int_{S^{d-1}} \int_0^\infty \mathbf{1}_E(r\boldsymbol{w}) dr \lambda(d\boldsymbol{w}), \tag{A.2}$$

and $h(\boldsymbol{x}) = k_{\frac{\boldsymbol{x}}{||\boldsymbol{x}||}}(||\boldsymbol{x}||)$. Thus

$$\nu(E) = \int_{S^{d-1}} \int_0^\infty \mathbf{1}_E(r\boldsymbol{w}) \frac{k_{\boldsymbol{w}}(r)}{r} dr \lambda(d\boldsymbol{w})$$
$$= \int_{\mathbb{R}^d} \mathbf{1}_E(\boldsymbol{x}) \frac{k_{\frac{\boldsymbol{x}}{||\boldsymbol{x}||}}(||\boldsymbol{x}||)}{||\boldsymbol{x}||} \tilde{\lambda}(d\boldsymbol{x}).$$

If λ is the measure σ induced on S^{d-1} by the Lebesgue measure on $\mathbb{R}^d$ we have

$$\hat{\mu}(z) = \exp\{i\boldsymbol{\gamma} \cdot z + \int_{\mathbb{R}^d} (e^{i\langle z, \boldsymbol{x}\rangle} - 1 - i\langle z, \boldsymbol{x}\rangle \mathbf{1}_{||\boldsymbol{x}|| \le 1}) \frac{h(\boldsymbol{x})}{||\boldsymbol{x}||^d} d\boldsymbol{x}\}, \tag{A.3}$$

A distribution is self-decomposable if and only if for any fixed $q > 0$ it is the distribution of $\boldsymbol{Y}(1)$ for some additive process $\{\boldsymbol{Y}(t),\ t > 0\}$ which is q-self-similar (see Sato [32] as a standard reference for self-similar processes). A process $\boldsymbol{Y}(t)$ is q-self-similar if for each $a > 0$

$$\boldsymbol{Y}(at) \stackrel{\mathcal{L}}{=} a^q \boldsymbol{Y}(t),$$

where $\stackrel{\mathcal{L}}{=}$ denotes equality in distribution of processes. The probability law of a Sato process at time t is obtained by scaling a self-decomposable law μ (see Carr et al. [6]). If $\boldsymbol{Y}(t)$ is a Sato process, we have :

$$\boldsymbol{Y}(t) \stackrel{\mathcal{L}}{=} t^q \boldsymbol{Y},$$

where $\boldsymbol{Y} \sim \mu$ and q is the self-similar exponent. The time t characteristic function of $\boldsymbol{Y}(t) \sim \mu_t$ is given by

$$\hat{\mu}_t(z) = \hat{\mu}(t^q z), \quad \forall z \in \mathbb{R}^d.$$

B Factor-based $\rho\alpha$models

We first introduce the Lévy class of factor-based multivariate subordinators used to construct the $\mathbb{R}^n$-valued asset return process $\{\boldsymbol{Y}(t),\ t \ge 0\}$. A multidimensional factor-based subordinator $\{\boldsymbol{G}(t), t \ge 0\}$ is defined as follows

$$\boldsymbol{G}(t) = (X_1(t) + \alpha_1 Z(t), ..., X_n(t) + \alpha_n Z(t)), \quad \alpha_j > 0,\ j = 1, ..., n, \tag{B.1}$$

where $\boldsymbol{X}(t) = \{(X_1(t), ..., X_n(t)), t \ge 0\}$ and $\{Z(t), t \ge 0\}$ are independent subordinators with zero drift, and $\boldsymbol{X}(t)$ has independent components. Let $\boldsymbol{B}(\boldsymbol{s})$ and $\boldsymbol{B}^\rho(t) = (B_1^\rho(t), ..., B_n^\rho(t))$ be the multivariate Brownian motions in Definition 5.2, independent of $\boldsymbol{B}(t)$. The $\mathbb{R}^n$-valued subordinated process $\{\boldsymbol{Y}^L(t), t > 0\}$ defined by

$$\boldsymbol{Y}^L(t) = \begin{pmatrix} B_1(X_1(t)) + B_1^\rho(Z(t)) \\ .... \\ B_n(X_n(t)) + B_n^\rho(Z(t)) \end{pmatrix}, \tag{B.2}$$

where $X_j(t)$ and $Z(t)$ are independent subordinators, independent of $\boldsymbol{B}(t)$ and $\boldsymbol{B}^\rho(t)$ is a factor-based subordinated Brownian motion, called $\rho\alpha$-model. Clearly if $\boldsymbol{G}(1)$ has the same unit time distribution of $\boldsymbol{S}(1)$ in Definition 5.1, also $\boldsymbol{Y}^L(1)$ and $\boldsymbol{Y}^\rho(1)$ have the same distribution.

Let now $\boldsymbol{Y}^L(t)$ be the process defined in (B.2). If $X(t)$ and $Z(t)$ are Lévy subordinators with unit time distribution in (5.8) and if we set $\mu_j = \beta_j \delta_j^2$ and $\sigma_j = \delta_j$ the process

$\boldsymbol{Y}^L(t)$ is the $\rho\alpha$-NIG process in Luciano and Semeraro [21]. Obviously $\boldsymbol{Y}^L(t)$ and $\boldsymbol{Y}^\rho(1)$ in Definition 5.3 have the same distribution at unit time.

C Lévy vs Sato correlation functions

The correlation functions of a subordinated Lévy process $\boldsymbol{Y}^L(t)$ defined in (B.2) are

$$\rho_{\mathbf{Y}^L(t)}(h, j) = \frac{\rho_{hj}\sigma_h\sigma_j\sqrt{a_h}\sqrt{a_j}\mathbb{E}[Z] + \mu_j\mu_h a_j a_h \mathbb{V}(Z)}{\sqrt{(\sigma_j^2\mathbb{E}[G_j] + \mu_j^2\mathbb{V}[G_j])(\sigma_h^2\mathbb{E}[G_h] + \mu_h^2\mathbb{V}[G_h])}}, \quad 0 \leq h < j \leq n. \tag{C.1}$$

They are constant over time and - if $\boldsymbol{G}(1)$ has the same unit time distribution of $\boldsymbol{S}(1)$ in Definition 5.1 - equal to $\rho_{\mathbf{Y}^\rho(1)}(h, j)$ in (5.7). This is a consequence of the Lévy process stationarity. If increments are i.i.d. the covariance functions scale as the variance functions, specifically:

$$V[Y_h^L(t)] = tV[Y_h^L(1)] \text{ and } cov(Y_h^L(t), Y_j^L(t)) = tcov(Y_h^L(1), Y_j^L(1)). \tag{C.2}$$

Consequently correlation functions are constant over time.

References

1. Ballotta, L., Bonfiglioli, E.: Multivariate asset models using Lévy processes and applications. The European Journal of Finance **22**(13), 1320–1350 (2016)
2. Barndorff-Nielsen, O.E., Pedersen, J., Sato, K.-I.: Multivariate subordination, self-decomposability and stability. Adv. Appl. Probab. **33**(1), 160–187 (2001)
3. Boen, L., Guillaume, F.: Building multivariate Sato models with linear dependence. Quantitative Finance **19**(4), 619–645 (2019)
4. Boen, L., Guillaume, F.: Towards a δ-gamma sato multivariate model. Rev. Deriv. Res. **23**(1), 1–39 (2020)
5. Carr, P., Geman, H., Madan, D.B., Yor, M.: The fine structure of asset returns: An empirical investigation. The Journal of Business **75**(2), 305–332 (2002)
6. Carr, P., Geman, H., Madan, D.B., Yor, M.: Self-decomposability and option pricing. Math. Financ. **17**(1), 31–57 (2007)
7. Chourdakis, K.: Option pricing using the fractional fft. Journal of Computational Finance **8**(2), 1–18 (2004)
8. Eberlein, E., Madan, D.B.: Sato processes and the valuation of structured products. Quantitative Finance **9**(1), 27–42 (2009)
9. Egorov, A.V., Li, H., Xu, Y.: Maximum likelihood estimation of time-inhomogeneous diffusions. Journal of Econometrics **114**(1), 107–139 (2003)
10. Folland, G.B.: Real analysis: modern techniques and their applications. John Wiley & Sons (2013)
11. Grigelionis, B.: Thorin classes of Lévy processes and their transforms. Lith. Math. J. **48**(3), 294–315 (2008)
12. Guillaume, F.: The αVG model for multivariate asset pricing: calibration and extension. Rev. Deriv. Res. **16**(1), 25–52 (2013)
13. Guillaume, F.: Multivariate option pricing models with Lévy and sato vg marginal processes. International Journal of Theoretical and Applied Finance **21**(02), 1850007 (2018)
14. Jevtić, P., Marena, M., Semeraro, P.: Multivariate marked Poisson processes and market related multidimensional information flows. International Journal of Theoretical and Applied Finance **22**(02), 1850058 (2019)
15. Kokholm, T., Nicolato, E.: Sato processes in default modelling. Applied Mathematical Finance **17**(5), 377–397 (2010)
16. Küchler, U., Tappe, S.: Bilateral gamma distributions and processes in financial mathematics. Stochastic Processes and their Applications **118**(2), 261–283 (2008)

17. Küchler, U., Tappe, S.: Tempered stable distributions and processes. Stochastic Processes and their Applications **123**(12), 4256–4293 (2013)
18. Li, J., Li, L., Mendoza-Arriaga, R.: Additive subordination and its applications in finance. Finance Stochast. **20**(3), 589–634 (2016)
19. Luciano, E., Marena, M., Semeraro, P.: Dependence calibration and portfolio fit with factor-based subordinators. Quantitative Finance **16**(7), 1037–1052 (2016)
20. Luciano, E., Semeraro, P.: A generalized normal mean-variance mixture for return processes in finance. International Journal of Theoretical and Applied Finance **13**(03), 415–440 (2010)
21. Luciano, E., Semeraro, P.: Multivariate time changes for Lévy asset models: Characterization and calibration. J. Comput. Appl. Math. **233**(8), 1937–1953 (2010)
22. Lundin, M. C., Dacorogna, M. M., and Müller, U. A.: Correlation of high frequency financial time series (1998). Available at SSRN 79848
23. Madan, D. B., Seneta, E.: The variance gamma (vg) model for share market returns. Journal of Business, pages 511–524 (1990)
24. Maejima, M., Nakahara, G., et al.: A note on new classes of infinitely divisible distributions on $\mathbb{R}^d$. Electron. Commun. Probab. **14**, 358–371 (2009)
25. Marena, M., Romeo, A., Semeraro, P.: Multivariate factor-based processes with Sato margins. International Journal of Theoretical and Applied Finance **21**(01), 1850005 (2018)
26. Mendoza-Arriaga, R., Linetsky, V.: Multivariate subordination of Markov processes with financial applications. Math. Financ. **26**(4), 699–747 (2016)
27. Pérez-Abreu, V., Stelzer, R.: Infinitely divisible multivariate and matrix gamma distributions. J. Multivar. Anal. **130**, 155–175 (2014)
28. Prause, K. (1999). The generalized hyperbolic model: Estimation, financial derivatives, and risk measures. PhD thesis, University of Freiburg
29. Rosinski, J.: On series representations of infinitely divisible random vectors. Ann. Probab. **18**(1), 405–430 (1990)
30. Rosiński, J.: Tempering stable processes. Stochastic Processes and their Applications **117**(6), 677–707 (2007)
31. Sato, K.-I.: Absolute continuity of multivariate distributions of class l. J. Multivar. Anal. **12**(1), 89–94 (1982)
32. Sato, K.-I.: Lévy processes and infinitely divisible distributions. Cambridge University Press (1999)
33. Semeraro, P.: A note on the multivariate generalized asymmetric Laplace motion. Communications in Statistics-Theory and Methods **49**(10), 2339–2355 (2020)
34. Seneta, E., et al.: Fitting the variance-gamma model to financial data. J. Appl. Probab. **41**, 177–187 (2004)
35. Sun, Y., Mendoza-Arriaga, R., Linetsky, V.: Marshall-olkin distributions, subordinators, efficient simulation, and applications to credit risk. Advances in Applied Probability **49**, 481–514 (2017)
36. Takano, K.: On mixtures of the normal distribution by the generalized gamma convolutions. Bulletin of the Faculty of Science, Ibaraki University. Series A, Mathematics **21**, 29–41 (1989)
37. Teng, L., Ehrhardt, M., Günther, M.: The dynamic correlation model and its application to the Heston model. In: Innovations in Derivatives Markets, pp. 437–449. Springer, Cham (2016)
38. Tóth, B., Kertész, J.: Increasing market efficiency: Evolution of cross-correlations of stock returns. Physica A **360**(2), 505–515 (2006)