Zeckendorf representation of multiplicative inverses modulo a Fibonacci number

Monatshefte für Mathematik (2023) 201:1–9
https://doi.org/10.1007/s00605-022-01724-y

Zeckendorf representation of multiplicative inverses modulo a Fibonacci number

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Received: 21 March 2022 / Accepted: 18 May 2022 / Published online: 10 June 2022

Abstract
Prempreesuk, Noppakaew, and Pongsriiam determined the Zeckendorf representation of the multiplicative inverse of 2 modulo F_n, for every positive integer n not divisible by 3, where F_n denotes the nth Fibonacci number. We determine the Zeckendorf representation of the multiplicative inverse of a modulo F_n, for every fixed integer $a \geq 3$ and for all positive integers n with $\gcd(a, F_n) = 1$. Our proof makes use of the so-called base-φ expansion of real numbers.

Keywords Base-φ expansion · Fibonacci number · Multiplicative inverse · Zeckendorf representation

Mathematics Subject Classification Primary 11B39 · Secondary 11A67, 11A99

1 Introduction

Let $(F_n)_{n\geq 1}$ be the sequence of Fibonacci numbers, which is defined by the initial conditions $F_1 = F_2 = 1$ and by the linear recurrence $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$. It is well known [22] that every positive integer n can be written as a sum of distinct non-

Communicated by Adrian Constantin.

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consecutive Fibonacci numbers, that is, $n = \sum_{i=1}^{m} d_i F_i$, where $m \in \mathbb{N}$, $d_i \in \{0, 1\}$, and $d_i d_{i+1} = 0$ for all $i \in \{1, \ldots, m-1\}$. This is called the *Zeckendorf representation* of n and, apart from the equivalent use of F_1 instead of F_2 or vice versa, is unique.

The Zeckendorf representation of integer sequences has been studied in several works. For instance, Filipponi and Freitag [6, 7] studied the Zeckendorf representation of numbers of the form F_{kn}/F_n, F_n^2/d and L_n^2/d, where L_n are the Lucas numbers and d is a Lucas or Fibonacci number. Filipponi, Hart, and Sanchis [8, 13, 14] analyzed the Zeckendorf representation of numbers of the form mF_n. Filipponi [8] determined the Zeckendorf representation of mF_nF_{n+k} and mL_nL_{n+k} for $m \in \{1, 2, 3, 4\}$. Bugeaud [3] studied the Zeckendorf representation of smooth numbers. The study of Zeckendorf representations has been also approached from a combinatorial point of view [1, 9, 12, 21]. Moreover, generalizations of the Zeckendorf representation to linear recurrences other than the sequence of Fibonacci numbers have been considered [4, 5, 10, 11, 16].

For all integers a and $m \geq 1$ with $\gcd(a, m) = 1$, let $(a^{-1} \bmod m)$ denote the least positive multiplicative inverse of a modulo m, that is, the unique $b \in \{1, \ldots, m\}$ such that $ab \equiv 1 \pmod{m}$. Prempreesuk, Noppakaew, and Pongsriiam [17] determined the Zeckendorf representation of $(2^{-1} \bmod F_n)$, for every positive integer n that is not divisible by 3. (The condition $3 \nmid n$ is necessary and sufficient to have $\gcd(2, F_n) = 1$.) In particular, they showed [17, Theorem 3.2] that

$$(2^{-1} \bmod F_n) = \begin{cases} \sum_{k=0}^{(n-7)/2} F_{n-3k-2} + F_3 & \text{if } n \equiv 1 \mod 3; \\ \sum_{k=0}^{(n-8)/2} F_{n-3k-2} + F_4 & \text{if } n \equiv 2 \mod 3; \end{cases}$$

for every integer $n \geq 8$. We extend their result by determining the Zeckendorf representation of the multiplicative inverse of a modulo F_n, for every fixed integer $a \geq 3$ and every positive integer n with $\gcd(a, F_n) = 1$. Precisely, we prove the following result.

Theorem 1.1 *Let $a \geq 3$ be an integer. Then there exist integers $M, n_0, i_0 \geq 1$ and periodic sequences $\boldsymbol{z}^{(0)}, \ldots, \boldsymbol{z}^{(M-1)}$ and $\boldsymbol{w}^{(1)}, \ldots, \boldsymbol{w}^{(i_0)}$ with values in $\{0, 1\}$ such that, for all integers $n \geq n_0$ with $\gcd(a, F_n) = 1$, the Zeckendorf representation of $(a^{-1} \bmod F_n)$ is given by*

$$(a^{-1} \bmod F_n) = \sum_{i=i_0}^{n-1} z_{n-i}^{(n \bmod M)} F_i + \sum_{i=1}^{i_0-1} w_n^{(i)} F_i.$$

From the proof of Theorem 1.1 it follows that M, n_0, i_0, $\boldsymbol{z}^{(0)}, \ldots, \boldsymbol{z}^{(M-1)}$, and $\boldsymbol{w}^{(1)}, \ldots, \boldsymbol{w}^{(i_0)}$ can be computed from a (see also Remark 4.1 at the end of the paper).

2 Preliminaries on Fibonacci numbers

Let us recall that for every integer $n \geq 1$ it holds the *Binet formula*

$$F_n = \frac{\varphi^n - \overline{\varphi}^n}{\sqrt{5}},$$

where $\varphi := (1 + \sqrt{5})/2$ is the Golden ratio and $\overline{\varphi} := (1 - \sqrt{5})/2$ is its algebraic conjugate. Furthermore, it is well known that for every integer $m \geq 1$ the Fibonacci sequence $(F_n)_{n \geq 1}$ is (purely) periodic modulo m. Let $\pi(m)$ denote its period length, or the so-called *Pisano period*.

The next lemma gives a formula for the inverse of a modulo F_n.

Lemma 2.1 *For all integers $a \geq 1$ and $n \geq 3$ with $\gcd(a, F_n) = 1$, we have that*

$$(a^{-1} \bmod F_n) = \frac{bF_n + 1}{a},$$

where $b := (-F_r^{-1} \bmod a)$ and $r := (n \bmod \pi(a))$.

Proof Since $r \equiv n \pmod{\pi(a)}$, we have that $F_r \equiv F_n \pmod{a}$. In particular, it follows that $\gcd(a, F_r) = \gcd(a, F_n) = 1$. Hence, F_r is invertible modulo a, and consequently b is well defined. Moreover, we have that

$$bF_n + 1 \equiv -F_r^{-1} F_r + 1 \equiv 0 \pmod{a},$$

and thus $c := (bF_n + 1)/a$ is an integer. On the one hand, we have that

$$ac \equiv bF_n + 1 \equiv 1 \pmod{F_n}.$$

On the other hand, since $b \leq a - 1$ and $n \geq 3$, we have that

$$0 \leq c \leq \frac{(a-1)F_n + 1}{a} = F_n - \frac{F_n - 1}{a} < F_n.$$

Therefore, we get that $c = (a^{-1} \bmod F_n)$, as desired.

3 Preliminaries on base-φ expansion

We need some basic results regarding the so-called *base-φ expansion* of real numbers, which was introduced by Bergman [2] in 1957 (see also [19]), and which is a particular case of non-integer base expansion (see, e.g., [15, 18]). Let $\mathfrak{D}$ be the set of sequences in $\{0, 1\}$ that have no two consecutive terms equal to 1, and that are not ultimately equal to the periodic sequence $0, 1, 0, 1, \ldots$. Then for every $x \in [0, 1)$ there exists a unique sequence $\boldsymbol{\delta}(x) = (\delta_i(x))_{i \in \mathbb{N}}$ in $\mathfrak{D}$ such that $x = \sum_{i=1}^{\infty} \delta_i(x) \varphi^{-i}$. Precisely, $\delta_i(x) = \lfloor T^{(i)}(x) \rfloor$ for every $i \in \mathbb{N}$, where $T^{(i)}$ denotes the ith iterate of the map

$T : [0, 1) \to [0, 1)$ defined by $T(\hat{x}) := (\varphi\hat{x} \bmod 1)$ for every $\hat{x} \in [0, 1)$. Furthermore, letting $\mathcal{F} := \mathbb{Q}(\varphi) \cap [0, 1)$, if $x \in \mathcal{F}$ then $\boldsymbol{\delta}(x)$ is ultimately periodic. In particular, if $x \in \mathcal{F}$ is given as $x = x_1 + x_2\varphi$, where $x_1, x_2 \in \mathbb{Q}$, then the preperiod and the period of $\boldsymbol{\delta}(x)$ can be effectively computed by finding the smallest $i \in \mathbb{N}$ such that $T^{(i)}(x) = T^{(j)}(x)$ for some $j \in \mathbb{N}$ with $j < i$. Conversely, for every ultimately periodic sequence $\boldsymbol{d} = (d_i)_{i\in\mathbb{N}}$ in $\mathfrak{D}$ we have that the number $x = \sum_{i=1}^{\infty} d_i\varphi^{-i}$ belongs to $\mathcal{F}$, and $x_1, x_2 \in \mathbb{Q}$ such that $x = x_1 + x_2\varphi$ can be effectively computed in terms of the preperiod and period of $\boldsymbol{d}$ by using the formula for the sum of the geometric series. Moreover, in the case that x is a rational number in $[0, 1)$ then $\boldsymbol{\delta}(x)$ is purely periodic [20].

The next lemma collects two easy inequalities for sums involving sequences in $\mathfrak{D}$.

Lemma 3.1 *For every sequence $(d_i)_{i\in\mathbb{N}}$ in $\mathfrak{D}$ and for every $m \in \mathbb{N} \cup \{\infty\}$, we have:*
(1) $\sum_{i=1}^{m} d_i\varphi^{-i} \in [0, 1)$ *and*
(2) $\sum_{i=1}^{m} d_i(-\varphi)^{-i} \in (-1, \varphi^{-1})$.

Proof Since $(d_i)_{i\in\mathbb{N}}$ belongs to $\mathfrak{D}$, there exists $k \in \mathbb{N}$ such that $d_k = d_{k+1} = 0$. Let k be the minimum integer with such property. Then

$$\sum_{i=1}^{\infty} d_i\varphi^{-i} = \sum_{i=1}^{k-1} d_i\varphi^{-i} + \sum_{i=k+2}^{\infty} d_i\varphi^{-i} < \sum_{j=1}^{\lfloor k/2\rfloor} \varphi^{-(2j-1)} + \sum_{i=k+2}^{\infty} \varphi^{-i}$$
$$= \left(1 - \varphi^{-2\lfloor k/2\rfloor}\right) + \varphi^{-k} \le 1,$$

and (1) is proved. Let us prove (2). On the one hand, we have

$$\sum_{i=1}^{m} d_i(-\varphi)^{-i} \le \sum_{j=1}^{m} d_{2j}\varphi^{-2j} < \sum_{j=1}^{\infty} \varphi^{-2j} = \varphi^{-1},$$

where the second inequality is strict because $\mathfrak{D}$ does not contain sequences that are ultimately equal to $(0, 1, 0, 1, \dots)$. On the other hand, similarly, we have

$$\sum_{i=1}^{m} d_i(-\varphi)^{-i} \ge -\sum_{j=1}^{m} d_{2j-1}\varphi^{-(2j-1)} > -\sum_{j=1}^{\infty} \varphi^{-(2j-1)} = -1.$$

Thus (2) is proved.

The following lemma relates base-φ expansion and Zeckendorf representation.

Lemma 3.2 *Let N be a positive integer and write $N = x\varphi^m/\sqrt{5}$ for some $x \in \mathcal{F}$ and some integer $m \ge 2$. Then the Zeckendorf representation of N is given by*

$$N = \sum_{i=1}^{m-1} \delta_{m-i}(x)F_i.$$

Moreover, we have $\delta_m(x) = 0$.

Proof Let $R := N - \sum_{i=1}^{m-1} \delta_{m-i}(x)F_i$. We have to prove that $R = 0$. Since R is an integer, it suffices to show that $|R| < 1$. We have

$$\begin{aligned}\sqrt{5}N = x\varphi^m &= \sum_{i=1}^{\infty} \delta_i(x)\varphi^{m-i} = \sum_{i=1}^{m} \delta_i(x)\varphi^{m-i} + \sum_{i=m+1}^{\infty} \delta_i(x)\varphi^{m-i} \\ &= \sum_{i=0}^{m-1} \delta_{m-i}(x)\varphi^i + \sum_{i=1}^{\infty} \delta_{i+m}(x)\varphi^{-i} \\ &= \sum_{i=0}^{m-1} \delta_{m-i}(x)(\varphi^i - \overline{\varphi}^i) + \sum_{i=0}^{m-1} \delta_{m-i}(x)\overline{\varphi}^i + \sum_{i=1}^{\infty} \delta_{i+m}(x)\varphi^{-i} \\ &= \sqrt{5}\sum_{i=1}^{m-1} \delta_{m-i}(x)F_i + \sum_{i=0}^{m-1} \delta_{m-i}(x)(-\varphi)^{-i} + \sum_{i=1}^{\infty} \delta_{i+m}(x)\varphi^{-i}.\end{aligned}$$

Hence, we get that

$$\sqrt{5}R = \sum_{i=0}^{m-1} \delta_{m-i}(x)(-\varphi)^{-i} + \sum_{i=1}^{\infty} \delta_{i+m}(x)\varphi^{-i}.$$

For the sake of contradiction, suppose that $\delta_m(x) = 1$. Then $\delta_{m+1}(x) = 0$ and, by Lemma 3.1, it follows that

$$\sqrt{5}R = 1 + \sum_{i=1}^{m-1} \delta_{m-i}(x)(-\varphi)^{-i} + \sum_{i=2}^{\infty} \delta_{i+m}(x)\varphi^{-i} \in (1-1+0,\, 1+\varphi^{-1}+\varphi^{-1}) = (0, \sqrt{5}),$$

which is a contradiction, since R is an integer.

Therefore, $\delta_m(x) = 0$ and, again by Lemma 3.1, we have

$$\sqrt{5}R = \sum_{i=1}^{m-1} \delta_{m-i}(x)(-\varphi)^{-i} + \sum_{i=1}^{\infty} \delta_{i+m}(x)\varphi^{-i} \in (-1+0,\, \varphi^{-1}+1) \subseteq (-\sqrt{5}, \sqrt{5}),$$

so that $|R| < 1$, as desired.

The next lemma regards the base-φ expansions of the sum of two numbers.

Lemma 3.3 *Let $x, y \in [0, 1)$, $m \in \mathbb{N}$, and put $v := x + y\varphi^{-m}$. Suppose that there exists $\lambda \in \mathbb{N}$ such that $\lambda + 2 \le m$ and $\delta_\lambda(x) = \delta_{\lambda+1}(x) = 0$. Then, putting*

$$w := \sum_{i=\lambda+2}^{\infty} \delta_i(x)\varphi^{-i} + \sum_{i=m+1}^{\infty} \delta_{i-m}(y)\varphi^{-i},$$

we have that $v, w \in [0, 1)$ *and*

$$\delta_i(v) = \begin{cases} \delta_i(x) & \text{if } i \le \lambda, \\ \delta_i(w) & \text{if } i > \lambda, \end{cases} \tag{1}$$

for every $i \in \mathbb{N}$.

Proof From Lemma 3.1(1), we have that

$$0 \le w < \varphi^{-(\lambda+1)} + \varphi^{-m} < \varphi^{-(\lambda+1)} + \varphi^{-(\lambda+2)} = \varphi^{-\lambda}.$$

Hence, $w \in [0, \varphi^{-\lambda}) \subseteq [0, 1)$ and so $w = \sum_{i=\lambda+1}^{\infty} \delta_i(w)\varphi^{-i}$. Therefore, recalling that $\delta_{\lambda+1}(x) = 0$, we get that

$$\begin{aligned} v = x + y\varphi^{-m} &= \sum_{i=1}^{\infty} \delta_i(x)\varphi^{-i} + \sum_{i=1}^{\infty} \delta_i(y)\varphi^{-i-m} = \sum_{i=1}^{\infty} \delta_i(x)\varphi^{-i} + \sum_{i=m+1}^{\infty} \delta_{i-m}(y)\varphi^{-i} \\ &= \sum_{i=1}^{\lambda} \delta_i(x)\varphi^{-i} + w = \sum_{i=1}^{\lambda} \delta_i(x)\varphi^{-i} + \sum_{i=\lambda+1}^{\infty} \delta_i(w)\varphi^{-i}, \end{aligned}$$

which is the base-φ expansion of v. (Note that $\delta_\lambda(x) = 0$.) In particular, by Lemma 3.1(1), we have that $v \in [0, 1)$. Thus (1) follows.

4 Proof of Theorem 1.1

Fix an integer $a \ge 3$. Let us begin by defining M, n_0, i_0, and $\boldsymbol{z}^{(0)}, \ldots, \boldsymbol{z}^{(M-1)}$. Put $M := \pi(a)$. For each $r \in \{0, \ldots, M-1\}$ with $\gcd(a, F_r) = 1$, let $b_r := (-F_r^{-1} \bmod a)$, $x_r := b_r/a$, and $\boldsymbol{z}^{(r)} := \boldsymbol{\delta}(x_r)$. Note that $x_r \in (0, 1)$. Since x_r is a positive rational number, we have that $\boldsymbol{z}^{(r)}$ is a (purely) periodic sequence belonging to $\mathfrak{D}$. Let ℓ be the least common multiple of the period lengths of $\boldsymbol{z}^{(0)}, \ldots, \boldsymbol{z}^{(M-1)}$, and put $i_0 := \ell + 3$. Finally, let $n_0 := \max\{i_0 + 1, \lceil \log(2a)/\log\varphi \rceil\}$.

Pick an integer $n \ge n_0$ with $\gcd(a, F_n) = 1$ and, for the sake of brevity, put $r := (n \bmod M)$. From Lemma 2.1 and Binet's formula (2), we get that

$$(a^{-1} \bmod F_n) = \frac{b_r F_n + 1}{a} = \frac{b_r(\varphi^n - \overline{\varphi}^n)}{\sqrt{5}a} + \frac{1}{a} = (x_r + y_n\varphi^{-n})\frac{\varphi^n}{\sqrt{5}}, \tag{2}$$

where

$$y_n := \frac{\sqrt{5}}{a} - x_r(-\varphi)^{-n}.$$

Since $n \geq n_0$, it follows that $y_n \in (0, 1)$ and $x_r + y_n\varphi^{-n} \in (0, 1)$. Therefore, from (2) and Lemma 3.2, we get that

$$(a^{-1} \bmod F_n) = \sum_{i=1}^{n-1} \delta_{n-i}(x_r + y_n\varphi^{-n})F_i.$$

Since $\boldsymbol{\delta}(x_r)$ is (purely) periodic and belongs to $\mathfrak{D}$, we have that $\boldsymbol{\delta}(x_r)$ contains infinitely many pairs of consecutive zeros. Furthermore, since the period length of $\boldsymbol{\delta}(x_r)$ is at most ℓ, we have that among every $\ell + 1$ consecutive terms of $\boldsymbol{\delta}(x_r)$ there are two consecutive zero. In particular, there exists $\lambda = \lambda(r)$ such that $n-\ell-3 \leq \lambda \leq n-2$ and $\delta_\lambda(x_r) = \delta_{\lambda+1}(x_r) = 0$. Consequently, by Lemma 3.3, we get that $\delta_i(x_r + y_n\varphi^{-n}) = \delta_i(x_r)$ for each positive integer $i \leq \lambda$ and, a fortiori, for each positive integer $i \leq n-i_0$. Therefore, we have that

$$\begin{aligned}(a^{-1} \bmod F_n) &= \sum_{i=i_0}^{n-1} \delta_{n-i}(x_r)F_i + \sum_{i=1}^{i_0-1} \delta_{n-i}(x_r + y_n\varphi^{-n})F_i \\ &= \sum_{i=i_0}^{n-1} z_{n-i}^{(r)}F_i + \sum_{i=1}^{i_0-1} w_n^{(i)}F_i,\end{aligned} \tag{3}$$

where $\boldsymbol{w}^{(1)}, \cdots, \boldsymbol{w}^{(i_0)}$ are the sequences defined by $w_n^{(i)} := \delta_{n-i}(x_r + y_n\varphi^{-n})$. Note that, by construction,

$$z_1^{(r)}, z_2^{(r)}, \ldots, z_{n-i_0}^{(r)}, w_n^{(i_0-1)}, w_n^{(i_0-2)}, \ldots, w_n^{(1)}$$

is a string in $\{0, 1\}$ with no consecutive zeros. Hence, (3) is the Zeckendorf representation of $(a^{-1} \bmod F_n)$.

It remains only to prove that $\boldsymbol{w}^{(1)}, \cdots, \boldsymbol{w}^{(i_0)}$ are periodic. By (3) and the uniqueness of the Zeckendorf representation, it suffices to prove that

$$R(n) := (a^{-1} \bmod F_n) - \sum_{i=i_0}^{n-1} z_{n-i}^{(r)}F_i = \sum_{i=1}^{i_0-1} w_n^{(i)}F_i \tag{4}$$

is a periodic function of n. From the last equality in (4), we have that $0 \leq R(n) < \sum_{i=1}^{i_0-1} F_i$. (Actually, one can prove that $0 \leq R(n) < F_{i_0}$, but this is not necessary for our proof.) Fix a prime number $p > \max\{a, \sum_{i=1}^{i_0-1} F_i\}$. It suffices to prove that $R(n)$ is periodic modulo p. Recalling that $(a^{-1} \bmod F_n) = (b_r F_n + 1)/a$ and that the sequence of Fibonacci numbers is periodic modulo p, it follows that $(a^{-1} \bmod F_n)$ is periodic modulo p. Hence, it suffices to prove that $R'(n) := \sum_{i=i_0}^{n-1} z_{n-i}^{(r)}F_i$ is periodic modulo p. Using that $\boldsymbol{z}^{(r)}$ has period length dividing ℓ, we get that

$$
\begin{aligned}
R'(n+\ell M) - R'(n) &= \sum_{i=i_0}^{n+\ell M-1} z_{n+\ell M-i}^{((n+\ell M) \bmod M)} F_i - \sum_{i=i_0}^{n-1} z_{n-i}^{(r)} F_i \\
&= \sum_{i=i_0}^{n+\ell M-1} z_{n+\ell M-i}^{(r)} F_i - \sum_{i=i_0}^{n-1} z_{n-i}^{(r)} F_i \\
&= \sum_{i=n}^{n+\ell M-1} z_{n+\ell M-i}^{(r)} F_i + \sum_{i=i_0}^{n-1} (z_{n+\ell M-i}^{(r)} - z_{n-i}^{(r)}) F_i \\
&= \sum_{j=1}^{\ell M} z_j^{(r)} F_{n+\ell M-j},
\end{aligned}
$$

which is a linear combination of sequences that are periodic modulo p. Hence $R'(n)$ is periodic modulo p. The proof is complete.

Remark 4.1 The proof of Theorem 1.1 provides a way to compute the positive integers M, i_0, n_0 and the periods of the periodic sequences $\boldsymbol{z}^{(0)}, \ldots, \boldsymbol{z}^{(M-1)}$ and $\boldsymbol{w}^{(1)}, \ldots, \boldsymbol{w}^{(i_0)}$. Indeed, going through the proof, we have that: $M = \pi(a)$ is the Pisano period of a, which can be computed in an obvious way; $\boldsymbol{z}^{(r)} = \boldsymbol{\delta}\big((-F_r^{-1} \bmod a)/a\big)$ and so the period of $\boldsymbol{z}^{(r)}$ can be computed as explained at the beginning of Section 3; i_0 and n_0 have simple formulas in terms of ℓ, which is the least common multiple of the period lengths of $\boldsymbol{z}^{(0)}, \ldots, \boldsymbol{z}^{(M-1)}$. Finally, the periods of $\boldsymbol{w}^{(1)}, \ldots, \boldsymbol{w}^{(i_0)}$ can be computed from (4) and the fact that $R(n)$ is periodic with period length at most $\pi(p)^2 \ell M$, which follows from the arguments after (4). However, note that proceeding in this way might be impractical, since ℓ might be exponential in M, and thus p might be double exponential in M; making the search for the periods of $\boldsymbol{w}^{(1)}, \ldots, \boldsymbol{w}^{(i_0)}$ extremely long.

Acknowledgements The authors are members of GNSAGA of INdAM and of CrypTO, the group of Cryptography and Number Theory of Politecnico di Torino.

Funding Open access funding provided by Università degli Studi di Trento within the CRUI-CARE Agreement.

Data Availability Statement Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

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