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Zeckendorf representation of multiplicative inverses modulo a Fibonacci number

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Abstract

Prempreesuk, Noppakaew, and Pongsriiam determined the Zeckendorf representation of the multiplicative inverse of 2 modulo F_n , for every positive integer n not divisible by 3, where F_n denotes the n th Fibonacci number. We determine the Zeckendorf representation of the multiplicative inverse of a modulo F_n , for every fixed integer $a \geq 3$ and for all positive integers n with $\gcd(a, F_n) = 1$. Our proof makes use of the so-called base- φ expansion of real numbers.

Keywords Base- φ expansion · Fibonacci number · Multiplicative inverse · Zeckendorf representation

Mathematics Subject Classification Primary 11B39 · Secondary 11A67, 11A99

1 Introduction

Let $(F_n)_{n \geq 1}$ be the sequence of Fibonacci numbers, which is defined by the initial conditions $F_1 = F_2 = 1$ and by the linear recurrence $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$. It is well known [22] that every positive integer n can be written as a sum of distinct non-

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consecutive Fibonacci numbers, that is, $n = \sum_{i=1}^m d_i F_i$, where $m \in \mathbb{N}$, $d_i \in \{0, 1\}$, and $d_i d_{i+1} = 0$ for all $i \in \{1, \dots, m-1\}$. This is called the *Zeckendorf representation* of n and, apart from the equivalent use of F_1 instead of F_2 or vice versa, is unique.

The Zeckendorf representation of integer sequences has been studied in several works. For instance, Filipponi and Freitag [6, 7] studied the Zeckendorf representation of numbers of the form F_{kn}/F_n , F_n^2/d and L_n^2/d , where L_n are the Lucas numbers and d is a Lucas or Fibonacci number. Filipponi, Hart, and Sanchis [8, 13, 14] analyzed the Zeckendorf representation of numbers of the form mF_n . Filipponi [8] determined the Zeckendorf representation of $mF_n F_{n+k}$ and $mL_n L_{n+k}$ for $m \in \{1, 2, 3, 4\}$. Bugeaud [3] studied the Zeckendorf representation of smooth numbers. The study of Zeckendorf representations has been also approached from a combinatorial point of view [1, 9, 12, 21]. Moreover, generalizations of the Zeckendorf representation to linear recurrences other than the sequence of Fibonacci numbers have been considered [4, 5, 10, 11, 16].

For all integers a and $m \geq 1$ with $\gcd(a, m) = 1$, let $(a^{-1} \bmod m)$ denote the least positive multiplicative inverse of a modulo m , that is, the unique $b \in \{1, \dots, m\}$ such that $ab \equiv 1 \pmod{m}$. Prempreesuk, Noppakaew, and Pongsriiam [17] determined the Zeckendorf representation of $(2^{-1} \bmod F_n)$, for every positive integer n that is not divisible by 3. (The condition $3 \nmid n$ is necessary and sufficient to have $\gcd(2, F_n) = 1$.) In particular, they showed [17, Theorem 3.2] that

$$(2^{-1} \bmod F_n) = \begin{cases} \sum_{k=0}^{(n-7)/2} F_{n-3k-2} + F_3 & \text{if } n \equiv 1 \pmod{3}; \\ \sum_{k=0}^{(n-8)/2} F_{n-3k-2} + F_4 & \text{if } n \equiv 2 \pmod{3}; \end{cases}$$

for every integer $n \geq 8$. We extend their result by determining the Zeckendorf representation of the multiplicative inverse of a modulo F_n , for every fixed integer $a \geq 3$ and every positive integer n with $\gcd(a, F_n) = 1$. Precisely, we prove the following result.

Theorem 1.1 *Let $a \geq 3$ be an integer. Then there exist integers $M, n_0, i_0 \geq 1$ and periodic sequences $\mathbf{z}^{(0)}, \dots, \mathbf{z}^{(M-1)}$ and $\mathbf{w}^{(1)}, \dots, \mathbf{w}^{(i_0)}$ with values in $\{0, 1\}$ such that, for all integers $n \geq n_0$ with $\gcd(a, F_n) = 1$, the Zeckendorf representation of $(a^{-1} \bmod F_n)$ is given by*

$$(a^{-1} \bmod F_n) = \sum_{i=i_0}^{n-1} z_{n-i}^{(n \bmod M)} F_i + \sum_{i=1}^{i_0-1} w_n^{(i)} F_i.$$

From the proof of Theorem 1.1 it follows that $M, n_0, i_0, \mathbf{z}^{(0)}, \dots, \mathbf{z}^{(M-1)}$, and $\mathbf{w}^{(1)}, \dots, \mathbf{w}^{(i_0)}$ can be computed from a (see also Remark 4.1 at the end of the paper).

2 Preliminaries on Fibonacci numbers

Let us recall that for every integer $n \geq 1$ it holds the *Binet formula*

$$F_n = \frac{\varphi^n - \bar{\varphi}^n}{\sqrt{5}},$$

where $\varphi := (1 + \sqrt{5})/2$ is the Golden ratio and $\bar{\varphi} := (1 - \sqrt{5})/2$ is its algebraic conjugate. Furthermore, it is well known that for every integer $m \geq 1$ the Fibonacci sequence $(F_n)_{n \geq 1}$ is (purely) periodic modulo m . Let $\pi(m)$ denote its period length, or the so-called *Pisano period*.

The next lemma gives a formula for the inverse of a modulo F_n .

Lemma 2.1 *For all integers $a \geq 1$ and $n \geq 3$ with $\gcd(a, F_n) = 1$, we have that*

$$(a^{-1} \bmod F_n) = \frac{bF_n + 1}{a},$$

where $b := (-F_r^{-1} \bmod a)$ and $r := (n \bmod \pi(a))$.

Proof Since $r \equiv n \pmod{\pi(a)}$, we have that $F_r \equiv F_n \pmod{a}$. In particular, it follows that $\gcd(a, F_r) = \gcd(a, F_n) = 1$. Hence, F_r is invertible modulo a , and consequently b is well defined. Moreover, we have that

$$bF_n + 1 \equiv -F_r^{-1}F_r + 1 \equiv 0 \pmod{a},$$

and thus $c := (bF_n + 1)/a$ is an integer. On the one hand, we have that

$$ac \equiv bF_n + 1 \equiv 1 \pmod{F_n}.$$

On the other hand, since $b \leq a - 1$ and $n \geq 3$, we have that

$$0 \leq c \leq \frac{(a - 1)F_n + 1}{a} = F_n - \frac{F_n - 1}{a} < F_n.$$

Therefore, we get that $c = (a^{-1} \bmod F_n)$, as desired.

3 Preliminaries on base- φ expansion

We need some basic results regarding the so-called *base- φ expansion* of real numbers, which was introduced by Bergman [2] in 1957 (see also [19]), and which is a particular case of non-integer base expansion (see, e.g., [15, 18]). Let \mathfrak{D} be the set of sequences in $\{0, 1\}$ that have no two consecutive terms equal to 1, and that are not ultimately equal to the periodic sequence $0, 1, 0, 1, \dots$. Then for every $x \in [0, 1)$ there exists a unique sequence $\delta(x) = (\delta_i(x))_{i \in \mathbb{N}}$ in \mathfrak{D} such that $x = \sum_{i=1}^{\infty} \delta_i(x)\varphi^{-i}$. Precisely, $\delta_i(x) = \lfloor T^{(i)}(x) \rfloor$ for every $i \in \mathbb{N}$, where $T^{(i)}$ denotes the i th iterate of the map

$T : [0, 1) \rightarrow [0, 1)$ defined by $T(\hat{x}) := (\varphi\hat{x} \bmod 1)$ for every $\hat{x} \in [0, 1)$. Furthermore, letting $\mathcal{F} := \mathbb{Q}(\varphi) \cap [0, 1)$, if $x \in \mathcal{F}$ then $\delta(x)$ is ultimately periodic. In particular, if $x \in \mathcal{F}$ is given as $x = x_1 + x_2\varphi$, where $x_1, x_2 \in \mathbb{Q}$, then the preperiod and the period of $\delta(x)$ can be effectively computed by finding the smallest $i \in \mathbb{N}$ such that $T^{(i)}(x) = T^{(j)}(x)$ for some $j \in \mathbb{N}$ with $j < i$. Conversely, for every ultimately periodic sequence $\mathbf{d} = (d_i)_{i \in \mathbb{N}}$ in \mathfrak{D} we have that the number $x = \sum_{i=1}^{\infty} d_i\varphi^{-i}$ belongs to \mathcal{F} , and $x_1, x_2 \in \mathbb{Q}$ such that $x = x_1 + x_2\varphi$ can be effectively computed in terms of the preperiod and period of \mathbf{d} by using the formula for the sum of the geometric series. Moreover, in the case that x is a rational number in $[0, 1)$ then $\delta(x)$ is purely periodic [20].

The next lemma collects two easy inequalities for sums involving sequences in \mathfrak{D} .

Lemma 3.1 *For every sequence $(d_i)_{i \in \mathbb{N}}$ in \mathfrak{D} and for every $m \in \mathbb{N} \cup \{\infty\}$, we have:*

- (1) $\sum_{i=1}^m d_i\varphi^{-i} \in [0, 1)$ and
- (2) $\sum_{i=1}^m d_i(-\varphi)^{-i} \in (-1, \varphi^{-1})$.

Proof Since $(d_i)_{i \in \mathbb{N}}$ belongs to \mathfrak{D} , there exists $k \in \mathbb{N}$ such that $d_k = d_{k+1} = 0$. Let k be the minimum integer with such property. Then

$$\begin{aligned} \sum_{i=1}^{\infty} d_i\varphi^{-i} &= \sum_{i=1}^{k-1} d_i\varphi^{-i} + \sum_{i=k+2}^{\infty} d_i\varphi^{-i} < \sum_{j=1}^{\lfloor k/2 \rfloor} \varphi^{-(2j-1)} + \sum_{i=k+2}^{\infty} \varphi^{-i} \\ &= \left(1 - \varphi^{-2\lfloor k/2 \rfloor}\right) + \varphi^{-k} \leq 1, \end{aligned}$$

and (1) is proved. Let us prove (2). On the one hand, we have

$$\sum_{i=1}^m d_i(-\varphi)^{-i} \leq \sum_{j=1}^m d_{2j}\varphi^{-2j} < \sum_{j=1}^{\infty} \varphi^{-2j} = \varphi^{-1},$$

where the second inequality is strict because \mathfrak{D} does not contain sequences that are ultimately equal to $(0, 1, 0, 1, \dots)$. On the other hand, similarly, we have

$$\sum_{i=1}^m d_i(-\varphi)^{-i} \geq -\sum_{j=1}^m d_{2j-1}\varphi^{-(2j-1)} > -\sum_{j=1}^{\infty} \varphi^{-(2j-1)} = -1.$$

Thus (2) is proved.

The following lemma relates base- φ expansion and Zeckendorf representation.

Lemma 3.2 *Let N be a positive integer and write $N = x\varphi^m/\sqrt{5}$ for some $x \in \mathcal{F}$ and some integer $m \geq 2$. Then the Zeckendorf representation of N is given by*

$$N = \sum_{i=1}^{m-1} \delta_{m-i}(x)F_i.$$

Moreover, we have $\delta_m(x) = 0$.

Proof Let $R := N - \sum_{i=1}^{m-1} \delta_{m-i}(x)F_i$. We have to prove that $R = 0$. Since R is an integer, it suffices to show that $|R| < 1$. We have

$$\begin{aligned} \sqrt{5}N &= x\varphi^m = \sum_{i=1}^{\infty} \delta_i(x)\varphi^{m-i} = \sum_{i=1}^m \delta_i(x)\varphi^{m-i} + \sum_{i=m+1}^{\infty} \delta_i(x)\varphi^{m-i} \\ &= \sum_{i=0}^{m-1} \delta_{m-i}(x)\varphi^i + \sum_{i=1}^{\infty} \delta_{i+m}(x)\varphi^{-i} \\ &= \sum_{i=0}^{m-1} \delta_{m-i}(x)(\varphi^i - \bar{\varphi}^i) + \sum_{i=0}^{m-1} \delta_{m-i}(x)\bar{\varphi}^i + \sum_{i=1}^{\infty} \delta_{i+m}(x)\varphi^{-i} \\ &= \sqrt{5} \sum_{i=1}^{m-1} \delta_{m-i}(x)F_i + \sum_{i=0}^{m-1} \delta_{m-i}(x)(-\varphi)^{-i} + \sum_{i=1}^{\infty} \delta_{i+m}(x)\varphi^{-i}. \end{aligned}$$

Hence, we get that

$$\sqrt{5}R = \sum_{i=0}^{m-1} \delta_{m-i}(x)(-\varphi)^{-i} + \sum_{i=1}^{\infty} \delta_{i+m}(x)\varphi^{-i}.$$

For the sake of contradiction, suppose that $\delta_m(x) = 1$. Then $\delta_{m+1}(x) = 0$ and, by Lemma 3.1, it follows that

$$\sqrt{5}R = 1 + \sum_{i=1}^{m-1} \delta_{m-i}(x)(-\varphi)^{-i} + \sum_{i=2}^{\infty} \delta_{i+m}(x)\varphi^{-i} \in (1 - 1 + 0, 1 + \varphi^{-1} + \varphi^{-1}) = (0, \sqrt{5}),$$

which is a contradiction, since R is an integer.

Therefore, $\delta_m(x) = 0$ and, again by Lemma 3.1, we have

$$\sqrt{5}R = \sum_{i=1}^{m-1} \delta_{m-i}(x)(-\varphi)^{-i} + \sum_{i=1}^{\infty} \delta_{i+m}(x)\varphi^{-i} \in (-1 + 0, \varphi^{-1} + 1) \subseteq (-\sqrt{5}, \sqrt{5}),$$

so that $|R| < 1$, as desired.

The next lemma regards the base- φ expansions of the sum of two numbers.

Lemma 3.3 *Let $x, y \in [0, 1)$, $m \in \mathbb{N}$, and put $v := x + y\varphi^{-m}$. Suppose that there exists $\lambda \in \mathbb{N}$ such that $\lambda + 2 \leq m$ and $\delta_\lambda(x) = \delta_{\lambda+1}(x) = 0$. Then, putting*

$$w := \sum_{i=\lambda+2}^{\infty} \delta_i(x)\varphi^{-i} + \sum_{i=m+1}^{\infty} \delta_{i-m}(y)\varphi^{-i},$$

we have that $v, w \in [0, 1)$ and

$$\delta_i(v) = \begin{cases} \delta_i(x) & \text{if } i \leq \lambda, \\ \delta_i(w) & \text{if } i > \lambda, \end{cases} \quad (1)$$

for every $i \in \mathbb{N}$.

Proof From Lemma 3.1(1), we have that

$$0 \leq w < \varphi^{-(\lambda+1)} + \varphi^{-m} < \varphi^{-(\lambda+1)} + \varphi^{-(\lambda+2)} = \varphi^{-\lambda}.$$

Hence, $w \in [0, \varphi^{-\lambda}) \subseteq [0, 1)$ and so $w = \sum_{i=\lambda+1}^{\infty} \delta_i(w)\varphi^{-i}$. Therefore, recalling that $\delta_{\lambda+1}(x) = 0$, we get that

$$\begin{aligned} v &= x + y\varphi^{-m} = \sum_{i=1}^{\infty} \delta_i(x)\varphi^{-i} + \sum_{i=1}^{\infty} \delta_i(y)\varphi^{-i-m} = \sum_{i=1}^{\infty} \delta_i(x)\varphi^{-i} + \sum_{i=m+1}^{\infty} \delta_{i-m}(y)\varphi^{-i} \\ &= \sum_{i=1}^{\lambda} \delta_i(x)\varphi^{-i} + w = \sum_{i=1}^{\lambda} \delta_i(x)\varphi^{-i} + \sum_{i=\lambda+1}^{\infty} \delta_i(w)\varphi^{-i}, \end{aligned}$$

which is the base- φ expansion of v . (Note that $\delta_{\lambda}(x) = 0$.) In particular, by Lemma 3.1(1), we have that $v \in [0, 1)$. Thus (1) follows.

4 Proof of Theorem 1.1

Fix an integer $a \geq 3$. Let us begin by defining M, n_0, i_0 , and $z^{(0)}, \dots, z^{(M-1)}$. Put $M := \pi(a)$. For each $r \in \{0, \dots, M-1\}$ with $\gcd(a, F_r) = 1$, let $b_r := (-F_r^{-1} \bmod a)$, $x_r := b_r/a$, and $z^{(r)} := \delta(x_r)$. Note that $x_r \in (0, 1)$. Since x_r is a positive rational number, we have that $z^{(r)}$ is a (purely) periodic sequence belonging to \mathfrak{D} . Let ℓ be the least common multiple of the period lengths of $z^{(0)}, \dots, z^{(M-1)}$, and put $i_0 := \ell + 3$. Finally, let $n_0 := \max\{i_0 + 1, \lceil \log(2a)/\log \varphi \rceil\}$.

Pick an integer $n \geq n_0$ with $\gcd(a, F_n) = 1$ and, for the sake of brevity, put $r := (n \bmod M)$. From Lemma 2.1 and Binet's formula (2), we get that

$$(a^{-1} \bmod F_n) = \frac{b_r F_n + 1}{a} = \frac{b_r(\varphi^n - \bar{\varphi}^n)}{\sqrt{5}a} + \frac{1}{a} = (x_r + y_n \varphi^{-n}) \frac{\varphi^n}{\sqrt{5}}, \quad (2)$$

where

$$y_n := \frac{\sqrt{5}}{a} - x_r(-\varphi)^{-n}.$$

Since $n \geq n_0$, it follows that $y_n \in (0, 1)$ and $x_r + y_n\varphi^{-n} \in (0, 1)$. Therefore, from (2) and Lemma 3.2, we get that

$$(a^{-1} \bmod F_n) = \sum_{i=1}^{n-1} \delta_{n-i}(x_r + y_n\varphi^{-n})F_i.$$

Since $\delta(x_r)$ is (purely) periodic and belongs to \mathfrak{D} , we have that $\delta(x_r)$ contains infinitely many pairs of consecutive zeros. Furthermore, since the period length of $\delta(x_r)$ is at most ℓ , we have that among every $\ell + 1$ consecutive terms of $\delta(x_r)$ there are two consecutive zero. In particular, there exists $\lambda = \lambda(r)$ such that $n - \ell - 3 \leq \lambda \leq n - 2$ and $\delta_\lambda(x_r) = \delta_{\lambda+1}(x_r) = 0$. Consequently, by Lemma 3.3, we get that $\delta_i(x_r + y_n\varphi^{-n}) = \delta_i(x_r)$ for each positive integer $i \leq \lambda$ and, a fortiori, for each positive integer $i \leq n - i_0$. Therefore, we have that

$$\begin{aligned} (a^{-1} \bmod F_n) &= \sum_{i=i_0}^{n-1} \delta_{n-i}(x_r)F_i + \sum_{i=1}^{i_0-1} \delta_{n-i}(x_r + y_n\varphi^{-n})F_i \tag{3} \\ &= \sum_{i=i_0}^{n-1} z_{n-i}^{(r)}F_i + \sum_{i=1}^{i_0-1} w_n^{(i)}F_i, \end{aligned}$$

where $w^{(1)}, \dots, w^{(i_0)}$ are the sequences defined by $w_n^{(i)} := \delta_{n-i}(x_r + y_n\varphi^{-n})$. Note that, by construction,

$$z_1^{(r)}, z_2^{(r)}, \dots, z_{n-i_0}^{(r)}, w_n^{(i_0-1)}, w_n^{(i_0-2)}, \dots, w_n^{(1)}$$

is a string in $\{0, 1\}$ with no consecutive zeros. Hence, (3) is the Zeckendorf representation of $(a^{-1} \bmod F_n)$.

It remains only to prove that $w^{(1)}, \dots, w^{(i_0)}$ are periodic. By (3) and the uniqueness of the Zeckendorf representation, it suffices to prove that

$$R(n) := (a^{-1} \bmod F_n) - \sum_{i=i_0}^{n-1} z_{n-i}^{(r)}F_i = \sum_{i=1}^{i_0-1} w_n^{(i)}F_i \tag{4}$$

is a periodic function of n . From the last equality in (4), we have that $0 \leq R(n) < \sum_{i=1}^{i_0-1} F_i$. (Actually, one can prove that $0 \leq R(n) < F_{i_0}$, but this is not necessary for our proof.) Fix a prime number $p > \max\{a, \sum_{i=1}^{i_0-1} F_i\}$. It suffices to prove that $R(n)$ is periodic modulo p . Recalling that $(a^{-1} \bmod F_n) = (b_r F_n + 1)/a$ and that the sequence of Fibonacci numbers is periodic modulo p , it follows that $(a^{-1} \bmod F_n)$ is periodic modulo p . Hence, it suffices to prove that $R'(n) := \sum_{i=i_0}^{n-1} z_{n-i}^{(r)}F_i$ is periodic modulo p . Using that $z^{(r)}$ has period length dividing ℓ , we get that

$$\begin{aligned}
R'(n + \ell M) - R'(n) &= \sum_{i=i_0}^{n+\ell M-1} z_{n+\ell M-i}^{((n+\ell M) \bmod M)} F_i - \sum_{i=i_0}^{n-1} z_{n-i}^{(r)} F_i \\
&= \sum_{i=i_0}^{n+\ell M-1} z_{n+\ell M-i}^{(r)} F_i - \sum_{i=i_0}^{n-1} z_{n-i}^{(r)} F_i \\
&= \sum_{i=n}^{n+\ell M-1} z_{n+\ell M-i}^{(r)} F_i + \sum_{i=i_0}^{n-1} (z_{n+\ell M-i}^{(r)} - z_{n-i}^{(r)}) F_i \\
&= \sum_{j=1}^{\ell M} z_j^{(r)} F_{n+\ell M-j},
\end{aligned}$$

which is a linear combination of sequences that are periodic modulo p . Hence $R'(n)$ is periodic modulo p . The proof is complete.

Remark 4.1 The proof of Theorem 1.1 provides a way to compute the positive integers M, i_0, n_0 and the periods of the periodic sequences $z^{(0)}, \dots, z^{(M-1)}$ and $w^{(1)}, \dots, w^{(i_0)}$. Indeed, going through the proof, we have that: $M = \pi(a)$ is the Pisano period of a , which can be computed in an obvious way; $z^{(r)} = \delta((-F_r^{-1} \bmod a)/a)$ and so the period of $z^{(r)}$ can be computed as explained at the beginning of Section 3; i_0 and n_0 have simple formulas in terms of ℓ , which is the least common multiple of the period lengths of $z^{(0)}, \dots, z^{(M-1)}$. Finally, the periods of $w^{(1)}, \dots, w^{(i_0)}$ can be computed from (4) and the fact that $R(n)$ is periodic with period length at most $\pi(p)^2 \ell M$, which follows from the arguments after (4). However, note that proceeding in this way might be impractical, since ℓ might be exponential in M , and thus p might be double exponential in M ; making the search for the periods of $w^{(1)}, \dots, w^{(i_0)}$ extremely long.

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References

1. Artz, J., Rowell, M.: A tiling approach to Fibonacci product identities. *Involve* **2**(5), 581–587 (2009)
2. Bergman, G.: A number system with an irrational base. *Math. Mag.* **31**, 98–110 (1957/58)
3. Bugeaud, Y.: On the Zeckendorf representation of smooth numbers. *Moscow Math. J.* **21**(1), 31–42 (2021)
4. Daykin, D.E.: Representation of natural numbers as sums of generalized Fibonacci numbers. *J. London Mathematical Society* **35**, 143–160 (1960)
5. Demontigny, P., Do, T., Kulkarni, A., Miller, S.J., Moon, D., Varma, U.: Generalizing Zeckendorf's Theorem to f -decompositions. *J. Number Theory* **141**, 136–158 (2014)
6. Filippini, P., Freitag, H.T.: On the F -Representation of Integral Sequences $\{F_n^2/d\}$ and $\{L_n^2/d\}$ where d is Either a Fibonacci or a Lucas Number. *Fibonacci Quart.* **27**(3), 276–282 (1989)
7. Filippini, P., Freitag, H.T.: The Zeckendorf Representation of $\{F_{kn}/F_n\}$. *Applications of Fibonacci Numbers* **5**, 217–19 (1993)
8. Filippini, P., Hart, E.L.: The Zeckendorf decomposition of certain Fibonacci-Lucas products. *Fibonacci Quart.* **36**(3), 240–247 (1998)
9. Gerdemann, D.: Combinatorial proofs of Zeckendorf family identities. *Fibonacci Quart.* **46**(47), 249–261 (2009)
10. Grabner, P.J., Tichy, R.F.: Contributions to digit expansions with respect to linear recurrences. *J. Number Theory* **35**, 160–169 (1990)
11. Grabner, P.J., Tichy, R.F.: Generalized Zeckendorf expansions. *Appl. Math. Lett.* **7**(2), 25–28 (1994)
12. McGregor, D., Rowell, M.J.: Combinatorial proofs of Zeckendorf representations of Fibonacci and Lucas products. *Involve* **4**(1), 75–89 (2011)
13. Hart, E.L.: On Using Patterns in Beta-Expansions To Study Fibonacci-Lucas Products. *Fibonacci Quart.* **36**, 396–406 (1998)
14. Hart, E., Sanchis, L.: On the occurrence of F_n in the Zeckendorf decomposition of nF_n . *Fibonacci Quart.* **37**, 21–33 (1999)
15. Parry, W.: On the β -expansions of real numbers. *Acta Math. Acad. Sci. Hungar.* **11**, 401–416 (1960)
16. Pethő, A., Tichy, R.F.: On digit expansions with respect to linear recurrences. *J. Number Th.* **33**, 243–256 (1989)
17. Premreesuk, B., Noppakaew, P., Pongsriam, P.: Zeckendorf representation and multiplicative inverse of $F_m \bmod F_n$. *Int. J. Math. Comput. Sci.* **15**(1), 17–25 (2020)
18. Rényi, A.: Representations for real numbers and their ergodic properties. *Acta Math. Acad. Sci. Hungar.* **8**, 477–493 (1957)
19. Rousseau, C.: The phi number system revisited. *Math. Mag.* **68**(4), 283–284 (1995)
20. Schmidt, K.: On periodic expansions of Pisot numbers and Salem numbers. *Bull. London Math. Soc.* **12**, 269–278 (1980)
21. Wood, P.M.: Bijective proofs for Fibonacci identities related to Zeckendorf's theorem. *Fibonacci Quart.* **45**(2), 138–145 (2007)
22. Zeckendorf, E.: Représentation des nombres naturels par une somme de nombres de Fibonacci ou de nombres de Lucas. *Bull. Soc. Roy. Sci. Liege* **41**, 179–82 (1972)

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