Zeckendorf representation of multiplicative inverses modulo a Fibonacci number

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Zeckendorf representation of multiplicative inverses modulo a Fibonacci number

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Abstract

Prempreesuk, Noppakaew, and Pongsriiam determined the Zeckendorf representation of the multiplicative inverse of 2 modulo F_n, for every positive integer n not divisible by 3, where F_n denotes the nth Fibonacci number. We determine the Zeckendorf representation of the multiplicative inverse of a modulo F_n, for every fixed integer $a \geq 3$ and for all positive integers n with $\gcd(a, F_n) = 1$. Our proof makes use of the so-called base-φ expansion of real numbers.

Keywords Base-φ expansion · Fibonacci number · Multiplicative inverse · Zeckendorf representation

Mathematics Subject Classification Primary 11B39 · Secondary 11A67, 11A99

1 Introduction

Let $(F_n)_{n \geq 1}$ be the sequence of Fibonacci numbers, which is defined by the initial conditions $F_1 = F_2 = 1$ and by the linear recurrence $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$. It is well known [22] that every positive integer n can be written as a sum of distinct non-

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consecutive Fibonacci numbers, that is, $n = \sum_{i=1}^{m} d_i F_i$, where $m \in \mathbb{N}$, $d_i \in \{0, 1\}$, and $d_i d_{i+1} = 0$ for all $i \in \{1, \ldots, m-1\}$. This is called the *Zeckendorf representation* of n and, apart from the equivalent use of F_1 instead of F_2 or vice versa, is unique.

The Zeckendorf representation of integer sequences has been studied in several works. For instance, Filipponi and Freitag [6, 7] studied the Zeckendorf representation of numbers of the form F_{kn}/F_n, F_n^2/d and L_n^2/d, where L_n are the Lucas numbers and d is a Lucas or Fibonacci number. Filipponi, Hart, and Sanchis [8, 13, 14] analyzed the Zeckendorf representation of numbers of the form mF_n. Filipponi [8] determined the Zeckendorf representation of mF_nF_{n+k} and mL_nL_{n+k} for $m \in \{1, 2, 3, 4\}$. Bugeaud [3] studied the Zeckendorf representation of smooth numbers. The study of Zeckendorf representations has been also approached from a combinatorial point of view [1, 9, 12, 21]. Moreover, generalizations of the Zeckendorf representation to linear recurrences other than the sequence of Fibonacci numbers have been considered [4, 5, 10, 11, 16].

For all integers a and $m \geq 1$ with $\gcd(a, m) = 1$, let $(a^{-1} \bmod m)$ denote the least positive multiplicative inverse of a modulo m, that is, the unique $b \in \{1, \ldots, m\}$ such that $ab \equiv 1 \pmod{m}$. Prempreesuk, Noppakaew, and Pongsriiam [17] determined the Zeckendorf representation of $(2^{-1} \bmod F_n)$, for every positive integer n that is not divisible by 3. (The condition $3 \nmid n$ is necessary and sufficient to have $\gcd(2, F_n) = 1$.) In particular, they showed [17, Theorem 3.2] that

$$(2^{-1} \bmod F_n) = \begin{cases} \sum_{k=0}^{(n-7)/2} F_{n-3k-2} + F_3 & \text{if } n \equiv 1 \mod 3; \\ \sum_{k=0}^{(n-8)/2} F_{n-3k-2} + F_4 & \text{if } n \equiv 2 \mod 3; \end{cases}$$

for every integer $n \geq 8$. We extend their result by determining the Zeckendorf representation of the multiplicative inverse of a modulo F_n, for every fixed integer $a \geq 3$ and every positive integer n with $\gcd(a, F_n) = 1$. Precisely, we prove the following result.

Theorem 1.1 *Let $a \geq 3$ be an integer. Then there exist integers $M, n_0, i_0 \geq 1$ and periodic sequences $\boldsymbol{z}^{(0)}, \ldots, \boldsymbol{z}^{(M-1)}$ and $\boldsymbol{w}^{(1)}, \ldots, \boldsymbol{w}^{(i_0)}$ with values in $\{0, 1\}$ such that, for all integers $n \geq n_0$ with $\gcd(a, F_n) = 1$, the Zeckendorf representation of $(a^{-1} \bmod F_n)$ is given by*

$$(a^{-1} \bmod F_n) = \sum_{i=i_0}^{n-1} z_{n-i}^{(n \bmod M)} F_i + \sum_{i=1}^{i_0-1} w_n^{(i)} F_i.$$

From the proof of Theorem 1.1 it follows that $M, n_0, i_0, \boldsymbol{z}^{(0)}, \ldots, \boldsymbol{z}^{(M-1)}$, and $\boldsymbol{w}^{(1)}, \ldots, \boldsymbol{w}^{(i_0)}$ can be computed from a (see also Remark 4.1 at the end of the paper).

2 Preliminaries on Fibonacci numbers

Let us recall that for every integer $n \geq 1$ it holds the *Binet formula*

$$F_n = \frac{\varphi^n - \overline{\varphi}^n}{\sqrt{5}},$$

where $\varphi := (1 + \sqrt{5})/2$ is the Golden ratio and $\overline{\varphi} := (1 - \sqrt{5})/2$ is its algebraic conjugate. Furthermore, it is well known that for every integer $m \geq 1$ the Fibonacci sequence $(F_n)_{n \geq 1}$ is (purely) periodic modulo m. Let $\pi(m)$ denote its period length, or the so-called *Pisano period*.

The next lemma gives a formula for the inverse of a modulo F_n.

Lemma 2.1 *For all integers $a \geq 1$ and $n \geq 3$ with $\gcd(a, F_n) = 1$, we have that*

$$(a^{-1} \bmod F_n) = \frac{bF_n + 1}{a},$$

where $b := (-F_r^{-1} \bmod a)$ and $r := (n \bmod \pi(a))$.

Proof Since $r \equiv n \pmod{\pi(a)}$, we have that $F_r \equiv F_n \pmod{a}$. In particular, it follows that $\gcd(a, F_r) = \gcd(a, F_n) = 1$. Hence, F_r is invertible modulo a, and consequently b is well defined. Moreover, we have that

$$bF_n + 1 \equiv -F_r^{-1} F_r + 1 \equiv 0 \pmod{a},$$

and thus $c := (bF_n + 1)/a$ is an integer. On the one hand, we have that

$$ac \equiv bF_n + 1 \equiv 1 \pmod{F_n}.$$

On the other hand, since $b \leq a - 1$ and $n \geq 3$, we have that

$$0 \leq c \leq \frac{(a-1)F_n + 1}{a} = F_n - \frac{F_n - 1}{a} < F_n.$$

Therefore, we get that $c = (a^{-1} \bmod F_n)$, as desired.

3 Preliminaries on base-φ expansion

We need some basic results regarding the so-called *base-φ expansion* of real numbers, which was introduced by Bergman [2] in 1957 (see also [19]), and which is a particular case of non-integer base expansion (see, e.g., [15, 18]). Let $\mathfrak{D}$ be the set of sequences in $\{0, 1\}$ that have no two consecutive terms equal to 1, and that are not ultimately equal to the periodic sequence $0, 1, 0, 1, \ldots$. Then for every $x \in [0, 1)$ there exists a unique sequence $\boldsymbol{\delta}(x) = (\delta_i(x))_{i \in \mathbb{N}}$ in $\mathfrak{D}$ such that $x = \sum_{i=1}^{\infty} \delta_i(x) \varphi^{-i}$. Precisely, $\delta_i(x) = \lfloor T^{(i)}(x) \rfloor$ for every $i \in \mathbb{N}$, where $T^{(i)}$ denotes the ith iterate of the map

$T : [0, 1) \to [0, 1)$ defined by $T(\hat{x}) := (\varphi\hat{x} \bmod 1)$ for every $\hat{x} \in [0, 1)$. Furthermore, letting $\mathcal{F} := \mathbb{Q}(\varphi) \cap [0, 1)$, if $x \in \mathcal{F}$ then $\boldsymbol{\delta}(x)$ is ultimately periodic. In particular, if $x \in \mathcal{F}$ is given as $x = x_1 + x_2\varphi$, where $x_1, x_2 \in \mathbb{Q}$, then the preperiod and the period of $\boldsymbol{\delta}(x)$ can be effectively computed by finding the smallest $i \in \mathbb{N}$ such that $T^{(i)}(x) = T^{(j)}(x)$ for some $j \in \mathbb{N}$ with $j < i$. Conversely, for every ultimately periodic sequence $\boldsymbol{d} = (d_i)_{i\in\mathbb{N}}$ in $\mathfrak{D}$ we have that the number $x = \sum_{i=1}^{\infty} d_i\varphi^{-i}$ belongs to $\mathcal{F}$, and $x_1, x_2 \in \mathbb{Q}$ such that $x = x_1 + x_2\varphi$ can be effectively computed in terms of the preperiod and period of $\boldsymbol{d}$ by using the formula for the sum of the geometric series. Moreover, in the case that x is a rational number in $[0, 1)$ then $\boldsymbol{\delta}(x)$ is purely periodic [20].

The next lemma collects two easy inequalities for sums involving sequences in $\mathfrak{D}$.

Lemma 3.1 *For every sequence $(d_i)_{i\in\mathbb{N}}$ in $\mathfrak{D}$ and for every $m \in \mathbb{N} \cup \{\infty\}$, we have:*

(1) $\sum_{i=1}^{m} d_i\varphi^{-i} \in [0, 1)$ *and*
(2) $\sum_{i=1}^{m} d_i(-\varphi)^{-i} \in (-1, \varphi^{-1})$.

Proof Since $(d_i)_{i\in\mathbb{N}}$ belongs to $\mathfrak{D}$, there exists $k \in \mathbb{N}$ such that $d_k = d_{k+1} = 0$. Let k be the minimum integer with such property. Then

$$\begin{aligned}\sum_{i=1}^{\infty} d_i\varphi^{-i} &= \sum_{i=1}^{k-1} d_i\varphi^{-i} + \sum_{i=k+2}^{\infty} d_i\varphi^{-i} < \sum_{j=1}^{\lfloor k/2\rfloor} \varphi^{-(2j-1)} + \sum_{i=k+2}^{\infty} \varphi^{-i}\\ &= \left(1 - \varphi^{-2\lfloor k/2\rfloor}\right) + \varphi^{-k} \le 1,\end{aligned}$$

and (1) is proved. Let us prove (2). On the one hand, we have

$$\sum_{i=1}^{m} d_i(-\varphi)^{-i} \le \sum_{j=1}^{m} d_{2j}\varphi^{-2j} < \sum_{j=1}^{\infty} \varphi^{-2j} = \varphi^{-1},$$

where the second inequality is strict because $\mathfrak{D}$ does not contain sequences that are ultimately equal to $(0, 1, 0, 1, \dots)$. On the other hand, similarly, we have

$$\sum_{i=1}^{m} d_i(-\varphi)^{-i} \ge -\sum_{j=1}^{m} d_{2j-1}\varphi^{-(2j-1)} > -\sum_{j=1}^{\infty} \varphi^{-(2j-1)} = -1.$$

Thus (2) is proved.

The following lemma relates base-φ expansion and Zeckendorf representation.

Lemma 3.2 *Let N be a positive integer and write $N = x\varphi^m/\sqrt{5}$ for some $x \in \mathcal{F}$ and some integer $m \ge 2$. Then the Zeckendorf representation of N is given by*

$$N = \sum_{i=1}^{m-1} \delta_{m-i}(x)F_i.$$

Moreover, we have $\delta_m(x) = 0$.

Proof Let $R := N - \sum_{i=1}^{m-1} \delta_{m-i}(x) F_i$. We have to prove that $R = 0$. Since R is an integer, it suffices to show that $|R| < 1$. We have

$$\begin{aligned}
\sqrt{5}N = x\varphi^m &= \sum_{i=1}^{\infty} \delta_i(x)\varphi^{m-i} = \sum_{i=1}^{m} \delta_i(x)\varphi^{m-i} + \sum_{i=m+1}^{\infty} \delta_i(x)\varphi^{m-i} \\
&= \sum_{i=0}^{m-1} \delta_{m-i}(x)\varphi^i + \sum_{i=1}^{\infty} \delta_{i+m}(x)\varphi^{-i} \\
&= \sum_{i=0}^{m-1} \delta_{m-i}(x)(\varphi^i - \overline{\varphi}^i) + \sum_{i=0}^{m-1} \delta_{m-i}(x)\overline{\varphi}^i + \sum_{i=1}^{\infty} \delta_{i+m}(x)\varphi^{-i} \\
&= \sqrt{5}\sum_{i=1}^{m-1} \delta_{m-i}(x)F_i + \sum_{i=0}^{m-1} \delta_{m-i}(x)(-\varphi)^{-i} + \sum_{i=1}^{\infty} \delta_{i+m}(x)\varphi^{-i}.
\end{aligned}$$

Hence, we get that

$$\sqrt{5}R = \sum_{i=0}^{m-1} \delta_{m-i}(x)(-\varphi)^{-i} + \sum_{i=1}^{\infty} \delta_{i+m}(x)\varphi^{-i}.$$

For the sake of contradiction, suppose that $\delta_m(x) = 1$. Then $\delta_{m+1}(x) = 0$ and, by Lemma 3.1, it follows that

$$\sqrt{5}R = 1 + \sum_{i=1}^{m-1} \delta_{m-i}(x)(-\varphi)^{-i} + \sum_{i=2}^{\infty} \delta_{i+m}(x)\varphi^{-i} \in (1-1+0,\, 1+\varphi^{-1}+\varphi^{-1}) = (0, \sqrt{5}),$$

which is a contradiction, since R is an integer.

Therefore, $\delta_m(x) = 0$ and, again by Lemma 3.1, we have

$$\sqrt{5}R = \sum_{i=1}^{m-1} \delta_{m-i}(x)(-\varphi)^{-i} + \sum_{i=1}^{\infty} \delta_{i+m}(x)\varphi^{-i} \in (-1+0,\, \varphi^{-1}+1) \subseteq (-\sqrt{5}, \sqrt{5}),$$

so that $|R| < 1$, as desired.

The next lemma regards the base-φ expansions of the sum of two numbers.

Lemma 3.3 *Let $x, y \in [0, 1)$, $m \in \mathbb{N}$, and put $v := x + y\varphi^{-m}$. Suppose that there exists $\lambda \in \mathbb{N}$ such that $\lambda + 2 \le m$ and $\delta_\lambda(x) = \delta_{\lambda+1}(x) = 0$. Then, putting*

$$w := \sum_{i=\lambda+2}^{\infty} \delta_i(x)\varphi^{-i} + \sum_{i=m+1}^{\infty} \delta_{i-m}(y)\varphi^{-i},$$

we have that $v, w \in [0, 1)$ *and*

$$\delta_i(v) = \begin{cases} \delta_i(x) & \text{if } i \le \lambda, \\ \delta_i(w) & \text{if } i > \lambda, \end{cases} \tag{1}$$

for every $i \in \mathbb{N}$.

Proof From Lemma 3.1(1), we have that

$$0 \le w < \varphi^{-(\lambda+1)} + \varphi^{-m} < \varphi^{-(\lambda+1)} + \varphi^{-(\lambda+2)} = \varphi^{-\lambda}.$$

Hence, $w \in [0, \varphi^{-\lambda}) \subseteq [0, 1)$ and so $w = \sum_{i=\lambda+1}^{\infty} \delta_i(w)\varphi^{-i}$. Therefore, recalling that $\delta_{\lambda+1}(x) = 0$, we get that

$$\begin{aligned} v = x + y\varphi^{-m} &= \sum_{i=1}^{\infty} \delta_i(x)\varphi^{-i} + \sum_{i=1}^{\infty} \delta_i(y)\varphi^{-i-m} = \sum_{i=1}^{\infty} \delta_i(x)\varphi^{-i} + \sum_{i=m+1}^{\infty} \delta_{i-m}(y)\varphi^{-i} \\ &= \sum_{i=1}^{\lambda} \delta_i(x)\varphi^{-i} + w = \sum_{i=1}^{\lambda} \delta_i(x)\varphi^{-i} + \sum_{i=\lambda+1}^{\infty} \delta_i(w)\varphi^{-i}, \end{aligned}$$

which is the base-φ expansion of v. (Note that $\delta_\lambda(x) = 0$.) In particular, by Lemma 3.1(1), we have that $v \in [0, 1)$. Thus (1) follows.

4 Proof of Theorem 1.1

Fix an integer $a \ge 3$. Let us begin by defining M, n_0, i_0, and $z^{(0)}, \dots, z^{(M-1)}$. Put $M := \pi(a)$. For each $r \in \{0, \dots, M-1\}$ with $\gcd(a, F_r) = 1$, let $b_r := (-F_r^{-1} \bmod a)$, $x_r := b_r/a$, and $z^{(r)} := \boldsymbol{\delta}(x_r)$. Note that $x_r \in (0, 1)$. Since x_r is a positive rational number, we have that $z^{(r)}$ is a (purely) periodic sequence belonging to $\mathfrak{D}$. Let ℓ be the least common multiple of the period lengths of $z^{(0)}, \dots, z^{(M-1)}$, and put $i_0 := \ell + 3$. Finally, let $n_0 := \max\{i_0 + 1, \lceil \log(2a)/\log\varphi \rceil\}$.

Pick an integer $n \ge n_0$ with $\gcd(a, F_n) = 1$ and, for the sake of brevity, put $r := (n \bmod M)$. From Lemma 2.1 and Binet's formula (2), we get that

$$(a^{-1} \bmod F_n) = \frac{b_r F_n + 1}{a} = \frac{b_r(\varphi^n - \overline{\varphi}^n)}{\sqrt{5}a} + \frac{1}{a} = (x_r + y_n\varphi^{-n})\frac{\varphi^n}{\sqrt{5}}, \tag{2}$$

where

$$y_n := \frac{\sqrt{5}}{a} - x_r(-\varphi)^{-n}.$$

Since $n \geq n_0$, it follows that $y_n \in (0, 1)$ and $x_r + y_n \varphi^{-n} \in (0, 1)$. Therefore, from (2) and Lemma 3.2, we get that

$$(a^{-1} \bmod F_n) = \sum_{i=1}^{n-1} \delta_{n-i}(x_r + y_n \varphi^{-n}) F_i.$$

Since $\boldsymbol{\delta}(x_r)$ is (purely) periodic and belongs to $\mathfrak{D}$, we have that $\boldsymbol{\delta}(x_r)$ contains infinitely many pairs of consecutive zeros. Furthermore, since the period length of $\boldsymbol{\delta}(x_r)$ is at most ℓ, we have that among every $\ell + 1$ consecutive terms of $\boldsymbol{\delta}(x_r)$ there are two consecutive zero. In particular, there exists $\lambda = \lambda(r)$ such that $n - \ell - 3 \leq \lambda \leq n - 2$ and $\delta_\lambda(x_r) = \delta_{\lambda+1}(x_r) = 0$. Consequently, by Lemma 3.3, we get that $\delta_i(x_r + y_n \varphi^{-n}) = \delta_i(x_r)$ for each positive integer $i \leq \lambda$ and, a fortiori, for each positive integer $i \leq n - i_0$. Therefore, we have that

$$\begin{aligned}(a^{-1} \bmod F_n) &= \sum_{i=i_0}^{n-1} \delta_{n-i}(x_r) F_i + \sum_{i=1}^{i_0-1} \delta_{n-i}(x_r + y_n \varphi^{-n}) F_i \\ &= \sum_{i=i_0}^{n-1} z_{n-i}^{(r)} F_i + \sum_{i=1}^{i_0-1} w_n^{(i)} F_i,\end{aligned} \tag{3}$$

where $\boldsymbol{w}^{(1)}, \cdots, \boldsymbol{w}^{(i_0)}$ are the sequences defined by $w_n^{(i)} := \delta_{n-i}(x_r + y_n \varphi^{-n})$. Note that, by construction,

$$z_1^{(r)}, z_2^{(r)}, \ldots, z_{n-i_0}^{(r)}, w_n^{(i_0-1)}, w_n^{(i_0-2)}, \ldots, w_n^{(1)}$$

is a string in $\{0, 1\}$ with no consecutive zeros. Hence, (3) is the Zeckendorf representation of $(a^{-1} \bmod F_n)$.

It remains only to prove that $\boldsymbol{w}^{(1)}, \cdots, \boldsymbol{w}^{(i_0)}$ are periodic. By (3) and the uniqueness of the Zeckendorf representation, it suffices to prove that

$$R(n) := (a^{-1} \bmod F_n) - \sum_{i=i_0}^{n-1} z_{n-i}^{(r)} F_i = \sum_{i=1}^{i_0-1} w_n^{(i)} F_i \tag{4}$$

is a periodic function of n. From the last equality in (4), we have that $0 \leq R(n) < \sum_{i=1}^{i_0-1} F_i$. (Actually, one can prove that $0 \leq R(n) < F_{i_0}$, but this is not necessary for our proof.) Fix a prime number $p > \max\{a, \sum_{i=1}^{i_0-1} F_i\}$. It suffices to prove that $R(n)$ is periodic modulo p. Recalling that $(a^{-1} \bmod F_n) = (b_r F_n + 1)/a$ and that the sequence of Fibonacci numbers is periodic modulo p, it follows that $(a^{-1} \bmod F_n)$ is periodic modulo p. Hence, it suffices to prove that $R'(n) := \sum_{i=i_0}^{n-1} z_{n-i}^{(r)} F_i$ is periodic modulo p. Using that $\boldsymbol{z}^{(r)}$ has period length dividing ℓ, we get that

$$\begin{aligned} R'(n+\ell M) - R'(n) &= \sum_{i=i_0}^{n+\ell M-1} z_{n+\ell M-i}^{((n+\ell M) \bmod M)} F_i - \sum_{i=i_0}^{n-1} z_{n-i}^{(r)} F_i \\ &= \sum_{i=i_0}^{n+\ell M-1} z_{n+\ell M-i}^{(r)} F_i - \sum_{i=i_0}^{n-1} z_{n-i}^{(r)} F_i \\ &= \sum_{i=n}^{n+\ell M-1} z_{n+\ell M-i}^{(r)} F_i + \sum_{i=i_0}^{n-1} (z_{n+\ell M-i}^{(r)} - z_{n-i}^{(r)}) F_i \\ &= \sum_{j=1}^{\ell M} z_j^{(r)} F_{n+\ell M-j}, \end{aligned}$$

which is a linear combination of sequences that are periodic modulo p. Hence $R'(n)$ is periodic modulo p. The proof is complete.

Remark 4.1 The proof of Theorem 1.1 provides a way to compute the positive integers M, i_0, n_0 and the periods of the periodic sequences $\boldsymbol{z}^{(0)}, \ldots, \boldsymbol{z}^{(M-1)}$ and $\boldsymbol{w}^{(1)}, \ldots, \boldsymbol{w}^{(i_0)}$. Indeed, going through the proof, we have that: $M = \pi(a)$ is the Pisano period of a, which can be computed in an obvious way; $\boldsymbol{z}^{(r)} = \boldsymbol{\delta}\big((-F_r^{-1} \bmod a)/a\big)$ and so the period of $\boldsymbol{z}^{(r)}$ can be computed as explained at the beginning of Section 3; i_0 and n_0 have simple formulas in terms of ℓ, which is the least common multiple of the period lengths of $\boldsymbol{z}^{(0)}, \ldots, \boldsymbol{z}^{(M-1)}$. Finally, the periods of $\boldsymbol{w}^{(1)}, \ldots, \boldsymbol{w}^{(i_0)}$ can be computed from (4) and the fact that $R(n)$ is periodic with period length at most $\pi(p)^2 \ell M$, which follows from the arguments after (4). However, note that proceeding in this way might be impractical, since ℓ might be exponential in M, and thus p might be double exponential in M; making the search for the periods of $\boldsymbol{w}^{(1)}, \ldots, \boldsymbol{w}^{(i_0)}$ extremely long.

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