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A group-theoretic approach to the disentanglement of generalized squeezing operators

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Abstract

The disentangled form of unitary operators is an indispensable tool for physical applications such as the study of squeezing properties or the time evolution of quantum systems. Here we derive a closed form disentanglement for the most general element of group ISp(2,$\mathbb{R}$), whose generating Lie algebra is obtained by joining the Heisenberg-Weyl algebra to su(1,1). We attain the disentanglement formula resorting to an extension of the Truax method and check our findings through an independent factorization approach, based on the use of displacement operators. As a result we obtain a new form of factorized squeezing operators, whose action on the light vacuum state is calculated.

Keywords: disentanglement, Baker-Campbell-Hausdorff formula, squeezing operators, squeezed states

1. Introduction

Unitary operators occurring in physics are defined in an algebraic framework in terms of the generators $\hat{g}_i$, $i = 1, ..., n$, of a Lie algebra $\mathfrak{a}$, where n is the dimension of the algebra. If $\hat{S} = \sum_{i=1}^{n} \alpha_i \hat{g}_i$ is an anti-hermitian element of $\mathfrak{a}$, with α_i complex coefficients such that $\hat{S}^\dagger = -\hat{S}$, the corresponding, generally entangled, unitary operator is $e^{\hat{S}}$. In most physical applications a disentangled or factorized expression of $e^{\hat{S}}$ is required and various forms of

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factorization can be defined, see, e.g., [1], [2]. In this work we propose the disentanglement in the form of the factorization of $e^{\hat{S}}$ into the product of the exponentials of the single generators $\hat{g}_i$. Therefore, the disentanglement problem consists in finding the complex unknown coefficients β_i such that the equation holds,

$$e^{\hat{S}} = \prod_{i=1}^{n} e^{\beta_i \, \hat{g}_i} \, . \tag{1}$$

The rhs of (1) depends on the order of the labelling i of the $\hat{g}_i$. Though the order of the factorization can be chosen without restrictions, we adopt the standard choice of normal-order currently used in quantum optics, according to which the exponentials of the lowering and raising operators of $\mathfrak{a}$ are in the right and left positions, respectively.

A well-known disentanglement problem is encountered in the definition of the conventional squeezing operator [3],

$$S(\tau) = e^{\tau K_+ - \bar{\tau} K_-} = e^{\tau \frac{\tanh|\tau|}{|\tau|} K_+} e^{-2\ln(\cosh|\tau|) K_0} e^{-\bar{\tau} \frac{\tanh|\tau|}{|\tau|} K_-} \, , \tag{2}$$

where $\tau \in \mathbb{C}$ and the bar denotes complex conjugation. In Eq. (2) the ladder operators $K_\pm$ and the Cartan operator K_0 are characterized by the commutation relations $[K_+, K_-] = -2K_0$, $[K_0, K_\pm] = \pm K_\pm$. The experimental realization of $S(\tau)$ is related to the process of degenerate parametric-down conversion [4],[5], thus in (2) the Schwinger one-mode two-boson realization of su(1,1) is adopted, whose generators are

$$K_- = \frac{1}{2} a^2 , \; K_+ = \frac{1}{2} a^{\dagger 2} , \; K_0 = \frac{1}{2} \left(\hat{n} + \frac{1}{2} \mathbb{I} \right) \, , \tag{3}$$

a and $a^\dagger$ being the harmonic oscillator annihilation and creation operators, while $\hat{n} = a^\dagger a$ is the occupation number operator and $\mathbb{I}$ denotes the identity operator.

In this work $\mathfrak{a}$ is the algebra isp(2,$\mathbb{R}$) obtained by joining the Heisenberg-Weyl algebra $\mathrm{w}_1(\mathbb{R})$ to the realization of su(1,1) given in Eq. (3). Note that we have adopted $\mathfrak{a} \equiv$ isp(2,$\mathbb{R}$) according to the notation given in [7], since su(1,1) $\approx$ sp(2,$\mathbb{R}$).

Many significant physical systems are characterized by the dynamical algebra isp(2,$\mathbb{R}$), namely, their Hamiltonians are linear combinations of the algebra generators. In units $\hbar = 1$, which are used throughout this paper, an important example in quantum optics is the Hamiltonian

$$H = \alpha K_0 + i(\bar{\tau} K_- - \tau K_+) + i(\bar{\lambda} a - \lambda a^\dagger) + \mu \mathbb{I}\,, \tag{4}$$

investigated in [7]-[10]. Specifically, H is a one-mode Hamiltonian describing two-photon processes, the linear terms being related to the squeezing of light coherent states. Another example is the well-known generalized Caldirola-Kanai model of the damped harmonic oscillator (see, e.g., [6]),

$$H = \frac{1}{2}\left[Ap^2 + Bq^2 + C(qp + pq)\right] + Dq + Ep + F\mathbb{I}\,, \tag{5}$$

where the coefficients from A to F are real functions of time and q, p are the canonically conjugate operators such that $[q, p] = i$. With $a = q + ip$, H is readily reduced to an element of isp(2,$\mathbb{R}$).

The physical motivation of our work lies in the prospective construction of the squeezing operator and the squeezed states naturally associated with isp(2,$\mathbb{R}$). Following Perelomov's definition of generalized coherent states [11]-[13] we use $U \doteq e^{\hat{S}} \in$ ISp(2,$\mathbb{R}$), where

$$\hat{S} = i\alpha K_0 + \tau K_+ - \bar{\tau} K_- + \lambda a^\dagger - \bar{\lambda} a + i\mu \mathbb{I}\,, \tag{6}$$

with $\alpha, \mu \in \mathbb{R}$ and $\tau, \lambda \in \mathbb{C}$, is the most general element of isp(2,$\mathbb{R}$) written for future convenience in anti-hermitian form. We let then U act on the highest weight vector $|\omega\rangle$ of isp(2,$\mathbb{R}$), which is defined as the most general state annihilated by the lowering operators a and K_-. In the Fock space $\mathfrak{F}$, state $|\omega\rangle$ is identified with the vacuum state, $|\omega\rangle \equiv |0\rangle$.

The key point in our work is the disentanglement of the group element U. To our knowledge, for $\hat{S}$ as in (6), the normal-order expression of U, disentangled as in Eq. (1), is not reported in literature. Here we derive the relevant expression analytically through an extension of the method originally devised by Truax [14] for the Lie algebras su(2) and su(1,1), which holds regardless of their specific realizations. Accordingly, the coefficients β_i in (1) are the solutions of the set of first-order differential equations associated with isp(2,$\mathbb{R}$). The resulting six-dimensional space of parameters is spanned by the coefficients $\alpha, \tau, \lambda, \mu$. Note that for $\lambda = \mu = 0$ in Eq. (6), disentangling is not an issue, the relevant results being available for example in [14], [15]. For completeness we reproduce the disentanglement formula in the Truax scheme by an independent separation method based on the use of the displacement operator ,

$$D(z) = e^{z a^\dagger - \bar{z} a}\,, \quad z \in \mathbb{C}\,. \tag{7}$$

The outline of the present paper is as follows. In section 2 we revisit the main features of the Baker-Campbell-Hausdorff (BCH) formula in order to highlight the differences between the latter and our approach. In section 3 the closed form expressions of the coefficients β_i in (1) are calculated. In section 4 we detail the $D(z)$-based method, while section 5 is devoted to the construction of both generalized squeezing operator and squeezed states. Concluding remarks are given in section 6.

2. Disentanglement and BCH formula

In the physicists' community the expression "Baker-Campbell-Hausdorff (BCH) formula" refers to the general result for the quantity $\hat{L} \doteq \ln\left(e^{\hat{X}}\, e^{\hat{Y}}\right)$, where $\hat{X}$ and $\hat{Y}$ are two noncommuting operators. A thorough exposition of the historical path leading to this result is reported in [16]. As for the mathematical features of the BCH formula, $\hat{L}$ contains nested commutators of $\hat{X}$ and $\hat{Y}$ and can be written in the integral form used, e.g., in [17],

$$\hat{L} = \hat{X} + \hat{Y} - \sum_{\ell=1}^{\infty} \frac{(-)^\ell}{\ell(\ell+1)} \int_0^1 \mathrm{d}t \left(e^{\mathrm{ad}_{\hat{X}}}\, e^{t\,\mathrm{ad}_{\hat{Y}}} - \mathbb{I}\right)^\ell \hat{Y}\,, \tag{8}$$

or in the equivalent form reported, e.g., in [18],

$$\hat{L} = \hat{X} + \hat{Y} + \sum_{\ell=1}^{\infty} \frac{(-)^\ell}{\ell+1} \int_0^1 \mathrm{d}t \left(e^{t\,\mathrm{ad}_{\hat{X}}}\, e^{\mathrm{ad}_{\hat{Y}}} - \mathbb{I}\right)^\ell \hat{X}\,. \tag{9}$$

In Eqs. (8), (9) one uses the standard notation $\mathrm{ad}_{\hat{X}}\, \hat{Y} \doteq [\hat{X}, \hat{Y}]$ and

$$e^{t\,\mathrm{ad}_{\hat{X}}}\, \hat{Y} = e^{t\hat{X}}\, \hat{Y}\, e^{-t\hat{X}} = \hat{Y} + \sum_{k=1}^{\infty} \frac{t^k}{k!}\, \hat{C}_k\,, \quad \hat{C}_k = (\mathrm{ad}_{\hat{X}})^k\, \hat{Y}\,,$$

where $(\mathrm{ad}_{\hat{X}})^k = \mathrm{ad}_{\hat{X}}(\mathrm{ad}_{\hat{X}})^{k-1}$, $k \geq 1$, with $\hat{C}_1 = [\hat{X}, \hat{Y}]$, $\hat{C}_2 = [\hat{X}, \hat{C}_1]$,..., $\hat{C}_{k+1} = [\hat{X}, \hat{C}_k]$, cf., e.g., [2]. Note that if $\hat{X}$ and $\hat{Y}$ are elements of a finite algebra, all the nested commutators $\hat{C}_k$ belong to the same algebra, i.e., they are cyclic linear combinations of the algebra generators.

Due to the importance of multiplying exponentials of non-commuting operators in quantum physics, many efforts have been devoted to the evaluation of the series expansions in Eqs. (8), (9), [17]-[20]. In [17], for example, Eq. (8) is applied to operators $\hat{X}$ and $\hat{Y}$, which obey the commutation relation

$$[\hat{X}, \hat{Y}] = u\hat{X} + v\hat{Y} + c\mathbb{I}\,, \tag{10}$$

with u, v, $c \in \mathbb{C}$ and $[\hat{Y}, \hat{X}] = -[\hat{X}, \hat{Y}]$. The ensuing closed form of $\hat{L}$ is $\hat{L} = \hat{X} + \hat{Y} + f(u,v)(u\hat{X} + v\hat{Y} + c\mathbb{I})$, where $f(u,v)$ is the scalar function

$$f(u,v) = \frac{(u-v)e^{u+v} - (ue^u - ve^v)}{uv(e^u - e^v)}\,.$$

A recent example of physical application of this method is related to the scattering function describing long-range coherent quantum tunneling [21].

Task (1) we pursue in this work is related to but different from the above BCH approach, because the disentangled form (1) depends strongly on the properties of the algebra to which $\hat{S}$ belongs. Specifically, disentangling $e^{\hat{S}}$ for $\hat{S}$ given by Eq. (6), i.e., writing $e^{\hat{S}}$ as the product of six exponentials of the generators of isp(2,$\mathbb{R}$), proves cumbersome if approached through the straightforward application of Eqs. (8) or (9). Choosing for instance $\hat{X} \in \mathrm{w}_1(\mathbb{R})$ and $\hat{Y}$ in the one-mode two-boson realization of su(1,1) and calculating the series expansions in Eqs. (8) or (9) entails the iteration of the BCH scheme in order to solve problem (1). In particular, further disentanglements and rearrangements of $e^{\hat{X}}$ and $e^{\hat{Y}}$ are required, depending on the order adopted for the factorized exponentials. In the following section we show that the Truax method can be efficiently extended to the unitary operator defined as the exponential of the element (6) of isp(2,$\mathbb{R}$).

3. Disentanglement through the Truax method

With $\hat{S}$ from Eq. (6), let $G_1(x)$ denote the unitary operator $e^{x\hat{S}}$, the real parameter x being instrumental in the following analysis,

$$G_1(x) = e^{x\hat{S}} = e^{x(i\alpha K_0 + \tau K_+ - \bar{\tau} K_- + \lambda a^\dagger - \bar{\lambda} a + i\mu\mathbb{I})}\,, \quad G_1(0) = \mathbb{I}\,. \tag{11}$$

The disentangled normal-order expression $G_2(x)$ of operator (11) can be written in terms of the set of the unknown complex coefficients $r_0(x)$, $p_0(x)$, $p_\pm(x)$, $q_\pm(x)$,

$$G_2(x) = e^{p_0(x)\mathbb{I}} e^{p_+(x)K_+} e^{q_+(x)a^\dagger} e^{r_0(x)\hat{n}} e^{q_-(x)a} e^{p_-(x)K_-}\,, \tag{12}$$

with $G_2(0) = \mathbb{I}$ so that the boundary conditions $r_0(0) = p_0(0) = p_\pm(0) = q_\pm(0) = 0$ must be imposed. It is worth noting that the disentanglement problem for operators living in the Lie group ISp(2,$\mathbb{C}$) has been investigated by Wünsche [7] resorting to a technique based on the three-dimensional fundamental representation of ISp(2,$\mathbb{C}$).

In the framework of the Truax method [14], the coefficients $r_0(x)$, $p_0(x)$, $p_\pm(x)$, $q_\pm(x)$ are evaluated requiring that $dG_2(x)/dx = dG_1(x)/dx$ and $G_2(x) = G_1(x)$, $\forall\ x \in \mathbb{R}$. Dispensing with the explicit dependence on x of the coefficients, the resulting system of first-order differential equations is, with the above boundary conditions,

$$p'_+ - 2p_+\, r'_0 + p_+^2\, p'_- e^{-2r_0} = \tau\,, \tag{13}$$

$$q'_+ - q_+\, r'_0 - p_+\, q'_- e^{-r_0} + p_+\, q_+\, p'_- e^{-2r_0} = \lambda\,, \tag{14}$$

$$r'_0 - p_+\, p'_- e^{-2r_0} = i\frac{\alpha}{2}\,, \tag{15}$$

$$q'_- e^{-r_0} - q_+\, p'_- e^{-2r_0} = -\bar{\lambda}\,, \tag{16}$$

$$p'_- e^{-2r_0} = -\bar{\tau}\,, \tag{17}$$

$$p'_0 - q_+ q'_- e^{-r_0} + \frac{1}{2}\left(q_+^2 - p_+\right)\, p'_- e^{-2r_0} = i\left(\frac{\alpha}{4} + \mu\right)\,, \tag{18}$$

where primes indicate differentiation with respect to x. The actual disentanglement coefficients are obtained setting $x = 1$ in the solutions of Eqs. (13)-(18). Since Eqs. (13)-(17) do not contain p_0, we solved them in the first place. For α and τ in the range $|\tau|^2 > \alpha^2/4$ or $\beta^2 \doteq |\tau|^2 - \alpha^2/4 > 0$, we calculated

$$p_+ = \frac{\tau \sinh\beta}{\beta\cosh\beta - i\frac{\alpha}{2}\sinh\beta}\,, \quad p_- = -\frac{\bar{\tau}}{\tau}\, p_+\,, \tag{19}$$

$$r_0 = -\ln\frac{\beta\cosh\beta - i\frac{\alpha}{2}\sinh\beta}{\beta}\,, \tag{20}$$

$$q_+ = \frac{\frac{1}{\beta}\left(\bar{\lambda}\tau + i\frac{\alpha}{2}\lambda\right)(1-\cosh\beta) + \lambda\sinh\beta}{\beta\cosh\beta - i\frac{\alpha}{2}\sinh\beta}\,, \tag{21}$$

$$q_- = \frac{1}{\tau}\,\frac{\left[\beta\lambda + \frac{1}{2}\frac{\alpha}{\beta}\left(\frac{1}{2}\alpha\lambda - i\bar{\lambda}\tau\right)\right](1-\cosh\beta) - \bar{\lambda}\tau\sinh\beta}{\beta\cosh\beta - i\frac{\alpha}{2}\sinh\beta}\,. \tag{22}$$

Solving Eq. (18) we obtained the coefficient p_0,

$$p_0 = P_1 + P_2 \ln \frac{\beta\cosh\beta - i\frac{\alpha}{2}\sinh\beta}{\beta} + \frac{P_3(1-\cosh\beta) + P_4\sinh\beta}{\beta\cosh\beta - i\frac{\alpha}{2}\sinh\beta}\,, \quad (23)$$

where

$$P_1 = i\mu + \frac{1}{\beta^2}\,\Xi\,,\; P_2 = -\frac{1}{2}\,,\; P_3 = \frac{1}{\beta^3}(-i\alpha\,\Xi + \beta^2|\lambda|^2)\,,\; P_4 = -\frac{1}{\beta^2}\,\Xi\,,$$

with $\Xi = \frac{1}{2}(\tau\bar{\lambda}^2 - \bar{\tau}\lambda^2 + i\alpha|\lambda|^2) \in i\mathbb{R}$. The results for $|\tau|^2 < \alpha^2/4$ are readily obtained by replacing β with $\pm i\beta$ throughout in Eqs. (19)-(23) regardless of the sign.

3.1. The disentanglement coefficients for $|\tau|^2 = \alpha^2/4$ or $\beta^2 = 0$

In this limit, from Eqs. (19)-(23) we found

$$\begin{aligned}
\lim_{\beta\to 0} p_+ &= \frac{2\tau}{2-i\alpha}\,, \quad \lim_{\beta\to 0} p_- = -\frac{\bar{\tau}}{\tau}\lim_{\beta\to 0} p_+\,, \quad \lim_{\beta\to 0} r_0 = -\ln\left(1 - i\frac{\alpha}{2}\right)\,,\\
\lim_{\beta\to 0} q_+ &= \frac{2\lambda - \left(\bar{\lambda}\tau + i\frac{\alpha}{2}\lambda\right)}{2-i\alpha}\,, \quad \lim_{\beta\to 0} q_- = \frac{1}{\tau}\,\frac{\bar{\lambda}\tau\left(i\frac{\alpha}{2} - 2\right) - \frac{\alpha^2}{4}\lambda}{2-i\alpha}\,,\\
\lim_{\beta\to 0} p_0 &= i\mu - \frac{1}{2}\ln\left(1 - i\frac{\alpha}{2}\right) + \frac{12|\lambda|^2 - (8-i\alpha)\Xi}{24}\,.
\end{aligned}$$

3.2. The special case of the algebra (3)

For $\lambda = \mu = 0$ the algebra isp(2,$\mathbb{R}$) reduces to the Schwinger realization of su(1,1) given in Eq. (3). From Eq. (6) $\hat{S} = i\alpha K_0 + \tau K_+ - \bar{\tau}K_-$, thus in this case the entangled unitary operator $e^{\hat{S}}$ is the element $e^{i\alpha K_0 + \tau K_+ - \bar{\tau}K_-}$ of the group SU(1,1) $\backsim$ Sp(2,$\mathbb{R}$). Eqs. (19)-(22) show that p_+, p_-, r_0 are unchanged and $q_+ = q_- = 0$, whereas Eq. (23) gives

$$p_0 = -\frac{1}{2}\ln\frac{\beta\cosh\beta - i\frac{\alpha}{2}\sinh\beta}{\beta} = \frac{1}{2}r_0\,. \quad (24)$$

As a check, we note that, in the disentangled expression $e^{p_0\mathbb{I}}e^{p_+K_+}e^{r_0\hat{n}}e^{p_-K_-}$ of $e^{\hat{S}}$, Eqs. (20) and (24) give $r_0\hat{n} + p_0\mathbb{I} = r_0\left(\hat{n} + \frac{1}{2}\mathbb{I}\right)$. This relation is useful for the comparison with the disentangled expression of $e^{\hat{S}}$ derived in [14],

$$e^{\hat{S}} = e^{t_+K_+}e^{t_0K_0}e^{t_-K_-}\,, \quad (25)$$

where

$$t_+ = p_+,\, t_0 = 2r_0,\, t_- = p_-\,. \tag{26}$$

Since $t_0 K_0 = 2r_0 K_0 = r_0\left(\hat{n} + \frac{1}{2}\mathbb{I}\right)$, the disentanglement formula for the algebra isp(2,$\mathbb{R}$) proves to contain the su(1,1) realization (3) as a special case, as it should.

3.3. The special case of the harmonic oscillator algebra

For $\tau = 0$ Eq. (6) shows that $\hat{S} = i\alpha K_0 + \lambda a^\dagger - \bar{\lambda} a + i\mu\mathbb{I}$ is the most general anti-hermitian element of the harmonic oscillator algebra generated by $\{\mathbb{I}, a, a^\dagger, \hat{n}\}$. The corresponding entangled unitary operator is $e^{\hat{S}} = e^{i\alpha K_0 + \lambda a^\dagger - \bar{\lambda} a + i\mu\mathbb{I}}$. From Eqs. (19)-(23) we found $p_+ = p_- = 0$ as well as

$$\begin{aligned} r_0 &= i\,\tfrac{\alpha}{2}\,,\; q_+ = i\,\tfrac{2\lambda}{\alpha}\left(1 - e^{i\alpha/2}\right)\,,\; q_- = -i\tfrac{2\bar{\lambda}}{\alpha}\left(1 - e^{i\alpha/2}\right)\,,\\ p_0 &= i\left[\left(\tfrac{\alpha}{4} + \mu\right) - \tfrac{2}{\alpha}|\lambda|^2\right] - \left(\tfrac{2}{\alpha}\right)^2 |\lambda|^2 \left(1 - e^{i\alpha/2}\right)\,, \end{aligned}$$

leading to the normal order disentangled expression of $e^{\hat{S}} = e^{p_0\mathbb{I}} e^{q_+ a^\dagger} e^{r_0\hat{n}} e^{q_- a}$.

4. The $D(z)$ approach

It is interesting to validate the result of the Truax approach by implementing a conceptually different approach based on the use of the unitary displacement $D(z)$ given in Eq. (7). According to this method, we write the operator (11) with $x = 1$ in the form

$$G_1(1,z) = D(z)\, e^{i\alpha K_0 + \tau K_+ - \bar{\tau} K_-}\, D^\dagger(z)\,, \tag{27}$$

Replacing in (27) $e^{i\alpha K_0 + \tau K_+ - \bar{\tau} K_-}$ with its disentangled expression (25) we obtain

$$\begin{aligned} G_1(1,z) &= D(z)\left(e^{t_+ K_+} e^{t_0 K_0} e^{t_- K_-}\right) D^\dagger(z)\\ &= e^{t_+ D(z) K_+ D^\dagger(z)}\, e^{t_0 D(z) K_0 D^\dagger(z)}\, e^{t_- D(z) K_- D^\dagger(z)}\,. \end{aligned}$$

Using $D(z) a D^\dagger(z) = a - z$ and its extensions to all relevant operators, we calculate

$$G_1(1,z) = e^{\Lambda\mathbb{I}}\, e^{t_+ K_+}\, e^{-t_+ \bar{z} a^\dagger}\, \mathcal{H}\, e^{-t_- z a}\, e^{t_- K_-}\,, \tag{28}$$

where $\Lambda = \frac{1}{2} t_+ \bar{z}^2 + \frac{1}{2} t_0\left(|z|^2 + \frac{1}{2}\right) + \frac{1}{2} t_- z^2 \in \mathbb{C}$, while $\mathcal{H} = e^{\frac{1}{2} t_0\left(\hat{n} - z a^\dagger - \bar{z} a\right)}$ is a non-unitary operator as in general $t_0 \in \mathbb{C}$. Its disentangled normal order expression,

$$\mathcal{H} = e^{g_1\mathbb{I}} e^{g_+\, a^\dagger} e^{g_0\, \hat{n}} e^{g_-\, a}\,, \tag{29}$$

is a special case of the exponential of the most general element of the harmonic oscillator algebra derived in Appendix A. Specifically, we obtain the coefficients $g_1, g_\pm, g_0$ in Eq. (29) setting $\gamma = \frac{1}{2}t_0$, $\delta = -\frac{1}{2}t_0 z$, $\nu = -\frac{1}{2}t_0\bar{z}$, $\varepsilon = 0$ in Eq. (A.1). With $\Theta \doteq e^{g_0} - 1$,

$$g_0 = \frac{1}{2}t_0,\; g_1 = |z|^2\left(\Theta - g_0\right),\; g_+ = -z\,\Theta,\; g_- = -\bar{z}\,\Theta\,. \tag{30}$$

The factorized normal order expression of operator (27) then follows,

$$G_2(1,z) = e^{(\Lambda+g_1)\mathbb{I}}\, e^{t_+K_+}\, e^{(g_+-t_+\bar{z})a^\dagger}\, e^{g_0\hat{n}}\, e^{(g_--t_-z)a}\, e^{t_-K_-}\,. \tag{31}$$

Clearly, there must be some compatibility relations between the group parameters α, τ, λ, μ in (11) and the additional parameter z in Eq. (27): indeed, the latter cannot be chosen arbitrarily as, for consistency, Eqs. (11), with $x = 1$, and (27) describe the same operator,

$$e^{i\alpha K_0+\tau K_+-\bar{\tau}K_-+\lambda a^\dagger-\bar{\lambda}a+i\mu\mathbb{I}} = e^{D(z)(i\alpha K_0+\tau K_+-\bar{\tau}K_-)D^\dagger(z)}\,. \tag{32}$$

Since

$$\begin{aligned} D(z)(i\alpha K_0+\tau K_+-\bar{\tau}K_-)D^\dagger(z) = i\alpha K_0+\tau K_+-\bar{\tau}K_- \\ -\left(i\tfrac{\alpha}{2}z+\tau\bar{z}\right)a^\dagger-\left(i\tfrac{\alpha}{2}\bar{z}-\bar{\tau}z\right)a+\tfrac{1}{2}\left(i\alpha|z|^2+\tau\bar{z}^2-\bar{\tau}z^2\right)\mathbb{I}\,, \end{aligned}$$

the coefficients of a and $a^\dagger$ in Eq. (32) are required satisfy the relations

$$i\frac{1}{2}\alpha\bar{z}-\bar{\tau}z = \bar{\lambda}\,, \quad i\frac{1}{2}\alpha z+\tau\bar{z} = -\lambda\,,$$

which give z in terms of the group parameters,

$$z = \frac{4\tau\bar{\lambda}+i2\alpha\lambda}{\alpha^2-4|\tau|^2}\,. \tag{33}$$

Note that different phase terms, $i\mu$ and

$$\varphi \doteq \frac{1}{2}\left(i\alpha|z|^2+\tau\bar{z}^2-\bar{\tau}z^2\right) = \frac{1}{2\beta^2}\left(\bar{\tau}\lambda^2-\tau\bar{\lambda}^2-i\alpha|\lambda|^2\right)\,,$$

are associated with the identity operators in Eq. (32). We can account for both of them redefining operator (27),

$$G_1(1,z) \doteq e^{(i\mu-\varphi)\mathbb{I}}\, D(z)e^{i\alpha K_0+\tau K_+-\bar{\tau}K_-}D^\dagger(z)\,, \tag{34}$$

so that, with z from (33), equality (32) is verified.
Besides, Eqs. (12), with $x = 1$, and (31) represent the same disentangled operator. This means that, provided that the phase factor $(i\mu - \varphi)\mathbb{I}$ is included in operator (31), in view of the result (34), the following condition must hold,

$$\begin{aligned} & e^{p_0\mathbb{I}}e^{p_+K_+}e^{q_+a^\dagger}e^{r_0\hat{n}}e^{q_-a}e^{p_-K_-} \\ & = e^{(\Lambda+g_1)\mathbb{I}}e^{(i\mu-\varphi)\mathbb{I}}\, e^{t_+K_+}\, e^{(g_+-t_+\bar{z})a^\dagger}\, e^{g_0\,\hat{n}}\, e^{(g_--t_-z)a}\, e^{t_-K_-}\,. \end{aligned} \tag{35}$$

Since, from Eqs. (26) and (30), $p_\pm = t_\pm$, $g_0 = \frac{1}{2}t_0 = r_0$, we need considering only the coefficients of operators a, $a^\dagger$ and $\mathbb{I}$ in Eq. (35). The three corresponding relations are

$$g_- - z\,p_- = q_-\,, \tag{36}$$

$$g_+ - \bar{z}\,p_+ = q_+\,, \tag{37}$$

$$\Lambda + g_1 + (i\mu - \varphi) = p_0\,. \tag{38}$$

For example, using Eqs. (19), (22) and (30) for p_-, q_- and g_-, condition (36) becomes

$$\begin{aligned} & -\bar{z}\left[\beta\,(1-\cosh\beta) + i\frac{\alpha}{2}\sinh\beta\right] + z\bar{\tau}\sinh\beta \\ & = \frac{1}{\tau}\left[\beta\lambda + \frac{1}{2}\frac{\alpha}{\beta}\left(\frac{1}{2}\alpha\lambda - i\bar{\lambda}\tau\right)\right](1-\cosh\beta) - \bar{\lambda}\sinh\beta\,, \end{aligned}$$

which, with z from (33), proves to be an identity. Conditions (37) and (38) are proved similarly.

5. Generalized squeezing operator and quadrature variances

The calculations in section 3 lead to the equality of the entangled unitary operator $e^{\hat{S}}$ written in terms of the group parameters, $U(\alpha, \tau, \lambda, \mu)$, and its normal-order factorized expression in terms of the disentanglement coefficients $p_\pm$, r_0, $q_\pm$, p_0, which in turn depend on the group parameters according to Eqs. (19)-(23). Therefore, for $\hat{S}$ as in Eq. (6), the following identity holds in the group ISp(2,$\mathbb{R}$),

$$\begin{aligned} U(\alpha,\tau,\lambda,\mu) = e^{\hat{S}} &= e^{i\alpha K_0+\tau K_+-\bar{\tau}K_-+\lambda a^\dagger-\bar{\lambda}a+i\mu\mathbb{I}} \\ &= e^{p_0\mathbb{I}}e^{p_+K_+}e^{q_+a^\dagger}e^{r_0\hat{n}}e^{q_-a}e^{p_-K_-}\,. \end{aligned} \tag{39}$$

We note that the normal-order disentangled expression of $U(\alpha,\tau,\lambda,\mu)$ given in Eq. (39) is quite a general result: indeed, the condition $|\tau|^2 > \alpha^2/4$ on the group parameters τ, α, which we used throughout our calculations, includes the cases of physical interest within the current quantum squeezing formalism, as we show in the subsequent discussion.

The effectiveness of result (39) emerges naturally in physical applications. As an example, if $\hat{S} = -iHt$, with $H^\dagger = H$, $U(\alpha,\tau,\lambda,\mu)$ plays the role of the evolution operator of a physical system, whose Hamiltonian H lives in the dynamical algebra isp(2,$\mathbb{R}$), as in the examples (4) and (5) discussed in section 1. The factorized expression of e^{-iHt} would then allow the analytical evaluation of the system dynamics [20], [24].

On the other side, and closer to the spirit of this work, we highlight that in quantum optics $U(\alpha,\tau,\lambda,\mu)$ includes in the group ISp(2,$\mathbb{R}$) both the conventional squeezing and displacement operators, (2) and (7), respectively: indeed, we retrieve such operators as particular cases of Eq. (39), $S(\tau) = U(0,\tau,0,0)$ and $D(z) = U(0,0,z,0)$. Acting with operator $U(\alpha,\tau,\lambda,\mu)$ on the vacuum state $|0\rangle \in \mathfrak{F}$ we obtain the generalized squeezed state $|\alpha,\tau,\lambda,\mu\rangle = U(\alpha,\tau,\lambda,\mu)|0\rangle$. Precisely, using the disentangled form of $U(\alpha,\tau,\lambda,\mu)$ in Eq. (39), we found

$$|\alpha,\tau,\lambda,\mu\rangle = \sum_{\ell,m=0}^{\infty} c_{\ell,m}\,|\ell+2m\rangle\,, \quad c_{\ell,m} = e^{p_0\mathbb{I}}\,\frac{p_+^m}{2^m\,m!}\,\frac{q_+^\ell}{\ell!}\sqrt{(\ell+2m)!}\,. \qquad (40)$$

Note that the coefficients $c_{\ell,m}$ are independent of r_0, q_-, p_-. Moreover, the state (40) is not a definite parity state, a feature which follows from the presence of q_+ in $c_{\ell,m}$, that testifies the role of the $\mathrm{w}_1(\mathbb{R})$ component of isp(2,$\mathbb{R}$).

Recalling that the quadrature operators are isomorphic with the position and momentum operators q and p, we calculated their variances with respect to state (40) using the definition $\Delta^2(\bullet) = \langle\bullet^2\rangle - \langle\bullet\rangle^2$, where $\bullet$ denotes the relevant operators and $\langle\bullet\rangle \doteq \langle\alpha,\tau,\lambda,\mu|\bullet|\alpha,\tau,\lambda,\mu\rangle = \langle 0|U^\dagger \bullet U|0\rangle$, with $U \equiv U(\alpha,\tau,\lambda,\mu)$. Since

$$U^\dagger(\alpha,\tau,\lambda,\mu)\,a^\dagger\,U(\alpha,\tau,\lambda,\mu) = e^{-r_0}\left(a^\dagger - p_- a - q_-\,\mathbb{I}\right)\,,$$

we could readily evaluate all the relevant transformations. With

$$\begin{matrix} q^2 \\ p^2 \end{matrix} = 2K_0 \pm (K_+ + K_-) = \hat{n} + \frac{1}{2} \pm \frac{1}{2}\left(a^{\dagger 2} + a^2\right)\,,$$

we calculated

$$\begin{matrix}\Delta^2(q)\\ \Delta^2(p)\end{matrix} = \frac{1}{2} + e^{-(r_0+\bar{r}_0)}|p_-|^2 \mp \frac{1}{2}\left(e^{-2r_0}p_- + e^{-2\bar{r}_0}\bar{p}_-\right) . \tag{41}$$

From our general formulation we derive and discuss a few particular cases of physical interest:

i) For $\lambda = \mu = 0$ we remove the $\mathrm{w}_1(\mathbb{R})$ component of algebra isp(2,$\mathbb{R}$), which consequently coincides with the su(1,1) realization (3), cf. section 3.2. Therefore, the three-dimensional space of parameters is defined by the coefficients α and τ. This particular case is investigated in [15], where the generalized squeezing operator $U(\alpha, \tau, 0, 0)$ is used to squeeze the vacuum of the system at one of the input ports of a Mach-Zehnder interferometer, leading to the emergence of new nonclassical regions of the interferometer.

ii) Adding the further constraint $\alpha = 0$ results in the two-dimensional space of parameters spanned by τ, thus re-establishing the standard experimental conditions associated with the squeezing operator $S(\tau) = U(0, \tau, 0, 0)$. Indeed, with $\tau = |\tau|e^{i\Phi_\tau}$, from Eqs. (19)-(23) we obtained $p_+ = e^{i\Phi_\tau}\tanh|\tau|$, $p_- = -e^{-i\Phi_\tau}\tanh|\tau|$, $r_0 = -\ln\cosh|\tau|$, $p_0 = \frac{1}{2}r_0$, $q_+ = q_- = 0$, so that state $|\alpha, \tau, \lambda, \mu\rangle$ of Eq. (40) reduces to the usual squeezed vacuum state $|\tau\rangle = S(\tau)|0\rangle$,

$$|\tau\rangle = (\cosh|\tau|)^{-\frac{1}{2}} \sum_{m=0}^{\infty} \left(\frac{e^{i\Phi_\tau}\tanh|\tau|}{2}\right)^m \sqrt{\binom{2m}{m}}|2m\rangle , \tag{42}$$

which is a definite parity state composed only of even Fock states. Analogously, the variances of the quadratures with respect to state $|\tau\rangle$, $\Delta^2_\tau(q)$ and $\Delta^2_\tau(p)$, are special cases of Eq. (41),

$$\begin{matrix}\Delta^2_\tau(q)\\ \Delta^2_\tau(p)\end{matrix} = \frac{1}{2} + \sinh^2|\tau| \pm \sinh|\tau|\cosh|\tau|\cos\Phi_\tau ,$$

from which, for $\Phi_\tau = 0$, we find the customary results [3],

$$\begin{matrix}\Delta^2_\tau(q)\\ \Delta^2_\tau(p)\end{matrix} = \frac{1}{2}\, e^{\pm 2|\tau|} .$$

5.1. Some remarks on the squeezed states of light

Currently, the squeezed states of light are generated from the vacuum state $|0\rangle$ in two different ways, cf. [4], [5]. We can first let the squeezing operator $S(\tau)$ act on the vacuum $|0\rangle$ obtaining the state (42), then the displacement operator (7) transforms $|\tau\rangle$ into the ideal squeezed states originally defined in [22]. With $\xi \doteq e^{i\Phi_\tau} \tanh|\tau|$ such states are

$$|z,\tau\rangle \doteq D(z)S(\tau)|0\rangle = D(z)|\tau\rangle = \frac{e^{\frac{1}{2}|z|^2}}{\sqrt{\cosh|\tau|}} \times$$
$$\sum_{j,\ell=0}^{\infty}\sum_{m=0}^{2\ell+j}\left(\frac{\xi}{2}\right)^{\ell}\sqrt{\binom{2\ell}{\ell}\binom{2\ell+j}{2\ell}\binom{2\ell+j}{m}}\,\frac{z^j}{\sqrt{j!}}\frac{(-\bar{z})^m}{\sqrt{m!}}|2\ell+j-m\rangle\,. \quad (43)$$

Reversing the order of the $D(z)$ and $S(\tau)$ operators leads to the 2-photon coherent states introduced in [23],

$$|\tau,z\rangle \doteq S(\tau)D(z)|0\rangle = S(\tau)|z\rangle\,, \quad (44)$$

where now the squeezing operator acts on the customary coherent state of light $|z\rangle = D(z)|0\rangle$, cf., e.g., [8], [9]. Naturally, states (43) and (44) are not the same physical state though a relation can be established between the two formalisms [4]. In fact, evaluating $S(\tau)D(z)$ gives $S(\tau)D(z) = D(v)S(\tau)$, where $D(v) = S(\tau)D(z)S(\tau)^\dagger$ with $v = (z + \xi\,\bar{z})\cosh|\tau|$, from which $|\tau, z\rangle = D(v)S(\tau)|0\rangle \doteq |v,\tau\rangle$.

The comparison of states (43) and (44) with our state (40) shows that the latter exhibits a more complex structure. This is due to the 6-dimensional structure of the space of parameters in which our formulation is framed, which displays two extra degrees of freedom with respect to the standard 4-dimensional space of parameters. As a consequence, the generalized unitary operator $U(\alpha,\tau,\lambda,\mu)$ in Eq. (39) embodies both the squeezing and displacement actions assigned to operators $S(\tau)$ and $D(z)$, respectively, then generalizing the usual notion of squeezing.

6. Conclusions

In this work we presented a generalization of the usual formulation of quantum squeezing, based on the use of the algebra isp(2,$\mathbb{R}$), which is defined as the merging of the single-mode two-boson Schwinger realization of su(1,1) and the Heisenberg-Weyl algebra $\mathrm{w}_1(\mathbb{R})$. The ensuing space of parameters is described by the six real coefficients corresponding to $\alpha, \tau, \lambda, \mu$.

The main motivation for our choice is related to the possibility to define, through the exponentiation of the most general anti-hermitian element of isp(2,$\mathbb{R}$), a single unitary operator $U(\alpha, \tau, \lambda, \mu)$, which combines the features of both the usual diplacement and squeezing operators, $D(z)$ and $S(\tau)$. In such theoretical framework we constructed the generalized squeezed states by letting $U(\alpha, \tau, \lambda, \mu)$ act on the vacuum state $|0\rangle$. To this aim the disentangled form of $U(\alpha, \tau, \lambda, \mu)$ is indispensable. We obtained it analytically in the normal-order form, resorting to a significant extension of the method first devised in [14]. The normal-order form of $U(\alpha, \tau, \lambda, \mu)$ is advantageous also in the analysis of the squeezing associated with more structured "vacuum states" as in [15], in the factorization of the unitary evolution operator, see, e.g., [20], and more generally for the construction of coherent states through a group-theoretic approach, cf. [9] and [11].

Our extension of the Truax method was confirmed by means of an independent approach, based on the use *ab initio* of the displacement operator $D(z)$.

In conclusion we observe that the normal-order disentangled expression of $U(\alpha, \tau, \lambda, \mu)$ given in Eq. (39) is a general result and remark that the condition $|\tau|^2 > \alpha^2/4$, characterizing our calculations, allows for the inclusion of the cases of physical interest in the current investigation into quantum squeezing by properly selecting the group parameters.

Appendix A. Disentanglement of a non-unitary operator in the group of the harmonic oscillator algebra

Following the Truax method as described in section 3, we write the exponential of the most general element of the harmonic oscillator algebra so as to obtain the non-unitary operator,

$$\hat{O}_1(x) = e^{x(\gamma \hat{n} + \delta a^\dagger + \nu a + \varepsilon \mathbb{I})}\,,$$

where $x \in \mathbb{R}$ with $\hat{O}_1(0) = \mathbb{I}$ and γ, δ, ν, $\varepsilon \in \mathbb{C}$. Then, in the disentangled normal order form of $\hat{O}_1(x)$,

$$\hat{O}_2(x) = e^{g_1(x)\mathbb{I}} e^{g_+(x)\, a^\dagger} e^{g_0(x)\, \hat{n}} e^{g_-(x)\, a}\,,$$

the coefficients $g_1(x)$, $g_\pm(x)$, $g_0(x)$ are the solutions of the system of first-order differential equations,

$$\begin{aligned} &g_1' - g_+ \, g_-' \, e^{-g_0} = \varepsilon \, , \\ &g_+' - g_0'(x) \, g_+ = \delta \, , \\ &g_-' \, e^{-g_0} = \nu \, , \\ &g_0' = \gamma \, , \end{aligned}$$

which must obey the boundary conditions $g_1(0) = g_\pm(0) = g_0(0) = 0$. For $x = 1$ and with $\Gamma \doteq e^\gamma - 1$, such solutions are

$$g_0 = \gamma, \; g_1 = \frac{\delta\nu}{\gamma^2} \, \Gamma + \left(\varepsilon - \frac{\delta\nu}{\gamma} \right), \; g_+ = \frac{\delta}{\gamma} \, \Gamma, \; g_- = \frac{\nu}{\gamma} \, \Gamma \, . \tag{A.1}$$

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