Semigroups for quadratic evolution equations acting on Shubin-Sobolev and Gelfand-Shilov spaces

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Semigroups for quadratic evolution equations acting on Shubin–Sobolev and Gelfand–Shilov spaces

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Abstract. We consider the initial value Cauchy problem for a class of evolution equations whose Hamiltonian is the Weyl quantization of a homogeneous quadratic form with non-negative definite real part. The solution semigroup is shown to be strongly continuous on several spaces: the Shubin–Sobolev spaces, the Schwartz space, the tempered distributions, the equal index Beurling type Gelfand–Shilov spaces and their dual ultradistribution spaces.

Neliöllisten evoluutioyhtälöiden ratkaisupuoliryhmät Shubinin–Sobolevin ja Gelfandin–Shilovin avaruuksissa

Tiivistelmä. Tarkastelemme eräiden evoluutioyhtälöiden Cauchyn alkuarvo-ongelmaa tilanteessa, jossa yhtälön Hamiltonin operaattori on reaaliosaltaan positiivisesti semidefiniitin homogeenisen neliömuodon Weylin kvantisointi. Ratkaisupuoliryhmä osoitetaan vahvasti jatkuvaksi useissa avaruuksissa: Shubinin–Sobolevin avaruuksissa, Schwartzin avaruudessa, vaimennettujen distribuutioiden joukossa, Beurlingin-tyyppisissä Gelfandin–Shilovin avaruuksissa, joiden indeksit ovat keskenään yhtä suuret, sekä näiden ultradistribuutioista koostuvissa duaaliavaruuksissa.

1. Introduction

Consider the Cauchy problem for the evolution equation

$$\begin{cases} \partial_t u(t,x) + q^w(x,D)u(t,x) = 0, \quad t > 0, \ \ x \in \mathbf{R}^d, \\ u(0,\cdot) = u_0 \in L^2(\mathbf{R}^d), \end{cases}$$

where $q^w(x,D)$ is the Weyl quantization of a symbol q which is a homogeneous quadratic form on the phase space $T^*\mathbf{R}^d$, defined by a symmetric matrix $Q \in \mathbf{C}^{2d\times 2d}$ such that $\operatorname{Re} Q \geqslant 0$. Particular cases include the heat equation, the free Schrödinger equation and the harmonic oscillator Schrödinger equation.

Hörmander [19] showed that the solution operator $e^{-tq^w(x,D)}$ is a strongly continuous contraction semigroup on $L^2(\mathbf{R}^d)$ with respect to the parameter $t \geqslant 0$. Semigroup theory then guarantees that $u(t,x) = e^{-tq^w(x,D)}u_0$ is the unique solution to the Cauchy problem when $u_0 \in D(q^w(x,D)) \subseteq L^2(\mathbf{R}^d)$ where $D(q^w(x,D))$ denotes the domain of the closure of $q^w(x,D)$ considered as an unbounded operator on L^2. In this paper we show that the semigroup $e^{-tq^w(x,D)}$ is strongly continuous in several other functional frameworks.

First we show strong continuity on the Shubin–Sobolev spaces, or Hilbert modulation spaces $M^2_s(\mathbf{R}^d)$, with polynomial weights indexed by $s \in \mathbf{R}$. Since the $M^2_s(\mathbf{R}^d)$ norms for $s \geqslant 0$ is a system of seminorms for the Schwartz space $\mathscr{S}(\mathbf{R}^d)$ we obtain as

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byproduct the following results. The propagator $e^{-tq^w(x,D)}$ is a locally equicontinuous strongly continuous semigroup on $\mathscr{S}(\mathbf{R}^d)$. By duality it is also strongly continuous on the tempered distributions $\mathscr{S}'(\mathbf{R}^d)$, equipped with either the weak* or the strong topology. In the latter case the semigroup is moreover locally equicontinuous.

Then we consider the equal index Beurling type Gelfand–Shilov spaces $\Sigma_s(\mathbf{R}^d)$ for $s > \frac{1}{2}$. Again we prove that the propagator is a locally equicontinuous strongly continuous semigroup on $\Sigma_s(\mathbf{R}^d)$, that extends by duality to a strongly continuous semigroup on the Gelfand–Shilov ultradistribution space $\Sigma_s'(\mathbf{R}^d)$, equipped with either the weak* or the strong topology. In the latter case we show again local equicontinuity. In the process we show that the Gelfand–Shilov space $\Sigma_s(\mathbf{R}^d)$ is reflexive, which apparently has not been stated in the literature.

The proofs rely heavily on Hörmander's results [19]. We use both his formula for the Weyl symbol of the propagator $e^{-tq^w(x,D)}$, and his expression of the propagator as a Fourier integral operator with respect to a quadratic phase function. The latter is a particular case of an extension of the metaplectic group, called the metaplectic semigroup in [19], indexed by the semigroup of complex symplectic matrices that are positive in a certain sense.

The results presented here provide a link that is missing in our papers [4, 27, 33]. In fact a discussion on the action of the solution semigroup on tempered distributions and on Gelfand–Shilov ultradistributions is lacking in them.

The class of evolution equations under study in this paper is currently an active field of research [13, 24, 27]. In particular it has been studied with respect to Gelfand–Shilov smoothing effects [14, 15], where it turns out that the singular space [13] plays a crucial role. The singular space is a linear subspace of the phase space $T^*\mathbf{R}^d$ determined by the quadratic form q.

The paper is organized as follows. Section 2 treats the functional analytical background concerning the spaces of functions and (ultra-)distributions we study. In Section 3 we specify the investigated class of evolution equations, and we give a brief overview of Hörmander's results [19] on the propagator acting on L^2 expressed with Fourier integral operators. In Section 4 we prepare for the main results in Sections 5 and 6, in particular by using results from [19] to study the action of differential and monomial multiplication operators to the left of the propagator. Section 5 treats strong continuity of the semigroup on Shubin–Sobolev spaces and its consequences, and finally Section 6 concerns strong continuity on Gelfand–Shilov spaces and their duals.

2. Preliminaries

An open ball in a Banach space X with center $x_0 \in X$ and radius $r > 0$ is denoted $B_r(x_0) = \{x \in X : \|x - x_0\| < r\}$, and $B_r = B_r(0)$. We use $\langle x \rangle = (1 + |x|^2)^{\frac{1}{2}}$ for $x \in \mathbf{R}^d$, and the partial derivative $D_j = -i\partial_j$, $1 \leqslant j \leqslant d$, acting on functions and distributions on $\mathbf{R}^d$, with extension to multi-indices. The standard basis vector in $\mathbf{R}^d$ with index $1 \leqslant j \leqslant d$ is denoted $e_j \in \mathbf{R}^d$. The transpose of a matrix $A \in \mathbf{C}^{d\times d}$ is denoted A^T. The real (complex) quadratic matrices of dimension d is $\mathbf{R}^{d\times d}$ ($\mathbf{C}^{d\times d}$), the group of invertible real (complex) matrices is denoted $\mathrm{GL}(d, \mathbf{R}) \subseteq \mathbf{R}^{d\times d}$ ($\mathrm{GL}(d, \mathbf{C}) \subseteq \mathbf{C}^{d\times d}$), and the subgroup of real orthogonal matrices is denoted $\mathrm{O}(d) \subseteq \mathrm{GL}(d, \mathbf{R})$.

We write $f(x) \lesssim g(x)$ provided there exists $C > 0$ such that $f(x) \leqslant C\, g(x)$ for all x in the domain of f and of g. The symbol $f(x) \asymp g(x)$ means that $f(x) \lesssim g(x)$

and $g(x) \lesssim f(x)$. The normalization of the Fourier transform is

$$\mathscr{F} f(\xi) = \widehat{f}(\xi) = (2\pi)^{-\frac{d}{2}} \int_{\mathbf{R}^d} f(x) e^{-i\langle x,\xi \rangle} \, \mathrm{d}x, \quad \xi \in \mathbf{R}^d,$$

for $f \in \mathscr{S}(\mathbf{R}^d)$ (the Schwartz space), where $\langle \cdot \, , \, \cdot \rangle$ denotes the scalar product on $\mathbf{R}^d$. The conjugate linear action of a (ultra-)distribution u on a test function ϕ is written (u, ϕ), consistent with the L^2 inner product $(\cdot \, , \, \cdot) = (\cdot \, , \, \cdot)_{L^2}$ which is conjugate linear in the second argument.

Denote translation by $T_x f(y) = f(y-x)$ and modulation by $M_\xi f(y) = e^{i\langle y,\xi \rangle} f(y)$ for $x, y, \xi \in \mathbf{R}^d$ where f is a function or distribution defined on $\mathbf{R}^d$. The composition is denoted $\Pi(x, \xi) = M_\xi T_x$. Let $\varphi \in \mathscr{S}(\mathbf{R}^d) \setminus \{0\}$. The short-time Fourier transform of a tempered distribution $u \in \mathscr{S}'(\mathbf{R}^d)$ is defined by

$$V_\varphi u(x, \xi) = (2\pi)^{-\frac{d}{2}} (u, M_\xi T_x \varphi), \quad x, \xi \in \mathbf{R}^d.$$

Then $V_\varphi u$ is smooth and polynomially bounded [12, Theorem 11.2.3], and we have

$$(u, f) = (V_\varphi u, V_\varphi f)_{L^2(\mathbf{R}^{2d})} \tag{2.1}$$

for $u \in \mathscr{S}'(\mathbf{R}^d)$ and $f \in \mathscr{S}(\mathbf{R}^d)$, provided $\|\varphi\|_{L^2} = 1$, cf. [12, Theorem 11.2.5].

The Hilbert modulation space, also known as the Shubin–Sobolev space, $M^2_s(\mathbf{R}^d) \subseteq \mathscr{S}'(\mathbf{R}^d)$ of order $s \in \mathbf{R}$ [9, 12, 23, 31] has norm

$$\|u\|_{M^2_s} := \|\langle \cdot \rangle^s V_\varphi u\|_{L^2(\mathbf{R}^{2d})} = \left(\iint_{\mathbf{R}^{2d}} \langle (x, \xi) \rangle^{2s} |V_\varphi u(x, \xi)|^2 \, \mathrm{d}x \, \mathrm{d}\xi \right)^{1/2}. \tag{2.2}$$

Different functions $\varphi \in \mathscr{S}(\mathbf{R}^d) \setminus \{0\}$ give equivalent norms. We have $M^2_0(\mathbf{R}^d) = L^2(\mathbf{R}^d)$, and for any $s, t \in \mathbf{R}$ with $t \leqslant s$ the embeddings

$$\mathscr{S}(\mathbf{R}^d) \subseteq M^2_s(\mathbf{R}^d) \subseteq M^2_t(\mathbf{R}^d) \subseteq \mathscr{S}'(\mathbf{R}^d) \tag{2.3}$$

where $\mathscr{S}'$ is equipped with its weak* topology, and

$$\mathscr{S}(\mathbf{R}^d) = \bigcap_{s \in \mathbf{R}} M^2_s(\mathbf{R}^d), \quad \mathscr{S}'(\mathbf{R}^d) = \bigcup_{s \in \mathbf{R}} M^2_s(\mathbf{R}^d). \tag{2.4}$$

(Inclusions of function and distribution spaces understand embeddings.)

We need some elements from the calculus of pseudodifferential operators [10, 17, 23, 31]. Let $a \in C^\infty(\mathbf{R}^{2d})$ and $m \in \mathbf{R}$. Then a is a *Shubin symbol* of order m, denoted $a \in \Gamma^m$, if for all $\alpha, \beta \in \mathbf{N}^d$ there exists a constant $C_{\alpha,\beta} > 0$ such that

$$|\partial_x^\alpha \partial_\xi^\beta a(x, \xi)| \leqslant C_{\alpha,\beta} \langle (x, \xi) \rangle^{m-|\alpha+\beta|}, \quad x, \xi \in \mathbf{R}^d. \tag{2.5}$$

The Shubin symbols Γ^m form a Fréchet space where the seminorms are given by the smallest possible constants in (2.5).

For $a \in \Gamma^m$ a pseudodifferential operator in the Weyl quantization is defined by

$$a^w(x, D) f(x) = (2\pi)^{-d} \int_{\mathbf{R}^{2d}} e^{i\langle x-y,\xi \rangle} a\left(\frac{x+y}{2}, \xi \right) f(y) \, \mathrm{d}y \, \mathrm{d}\xi, \quad f \in \mathscr{S}(\mathbf{R}^d), \tag{2.6}$$

when $m < -d$. The definition extends to general $m \in \mathbf{R}$ if the integral is viewed as an oscillatory integral. The operator $a^w(x, D)$ then acts continuously on $\mathscr{S}(\mathbf{R}^d)$ and extends uniquely by duality to a continuous operator on $\mathscr{S}'(\mathbf{R}^d)$. By Schwartz's kernel theorem the Weyl quantization procedure may be extended to a weak formulation which yields operators $a^w(x, D) \colon \mathscr{S}(\mathbf{R}^d) \to \mathscr{S}'(\mathbf{R}^d)$, even if a is only an element of $\mathscr{S}'(\mathbf{R}^{2d})$.

For $a \in \mathscr{S}'(\mathbf{R}^{2d})$ and $f, g \in \mathscr{S}(\mathbf{R}^d)$ we have

$$(a^w(x,D)f, g) = (2\pi)^{-\frac{d}{2}}(a, W(g,f)) \tag{2.7}$$

where

$$W(g,f)(x,\xi) = (2\pi)^{-\frac{d}{2}} \int_{\mathbf{R}^d} g(x+y/2)\overline{f(x-y/2)}\, e^{-i\langle y,\xi\rangle}\, \mathrm{d}y \in \mathscr{S}(\mathbf{R}^{2d}) \tag{2.8}$$

is the Wigner distribution [10, 12].

According to [23, Theorem 1.7.16 and Corollary 1.7.17], [31, Theorem 25.2], the Weyl operators with $a \in \Gamma^m$ act continuously on the Hilbert modulation spaces as

$$a^w(x,D)\colon M^2_s(\mathbf{R}^d) \to M^2_{s-m}(\mathbf{R}^d), \quad s \in \mathbf{R}. \tag{2.9}$$

The real phase space $T^*\mathbf{R}^d \simeq \mathbf{R}^d \oplus \mathbf{R}^d$ is a real symplectic vector space equipped with the canonical symplectic form

$$\sigma((x,\xi),(x',\xi')) = \langle x', \xi\rangle - \langle x, \xi'\rangle, \quad (x,\xi),(x',\xi') \in T^*\mathbf{R}^d.$$

This form can be expressed with the inner product as $\sigma(X,Y) = \langle \mathcal{J}X, Y\rangle$ for $X, Y \in T^*\mathbf{R}^d$ where

$$\mathcal{J} = \begin{pmatrix} 0 & I_d \\ -I_d & 0 \end{pmatrix} \in \mathbf{R}^{2d\times 2d}. \tag{2.10}$$

The complex phase space $T^*\mathbf{C}^d \simeq \mathbf{C}^d \oplus \mathbf{C}^d$ is likewise a complex symplectic vector space with respect to the same symplectic form. (Note that $\langle\,\cdot\,,\,\cdot\,\rangle$ is not conjugate linear in one argument, but bilinear for arguments in $\mathbf{C}^d \times \mathbf{C}^d$.) The real (complex) symplectic group $\mathrm{Sp}(d,\mathbf{R})$ ($\mathrm{Sp}(d,\mathbf{C})$) is the set of matrices in $\mathrm{GL}(2d,\mathbf{R})$ ($\mathrm{GL}(2d,\mathbf{C})$) that leaves σ invariant. Hence $\mathcal{J} \in \mathrm{Sp}(d,\mathbf{R})$. A Lagrangian subspace $\lambda \subseteq T^*\mathbf{R}^d$ ($\lambda \subseteq T^*\mathbf{C}^d$) is a real (complex) linear space of dimension d such that $\sigma|_{\lambda\times\lambda} = 0$. A Lagrangian $\lambda \subseteq T^*\mathbf{C}^d$ is called positive [18, 19] if

$$i\sigma(\overline{X}, X) \geqslant 0, \quad X \in \lambda.$$

To each symplectic matrix $\chi \in \mathrm{Sp}(d,\mathbf{R})$ is associated an operator $\mu(\chi)$ that is unitary on $L^2(\mathbf{R}^d)$, and determined up to a complex factor of modulus one, such that

$$\mu(\chi)^{-1} a^w(x,D)\,\mu(\chi) = (a\circ\chi)^w(x,D), \quad a \in \mathscr{S}'(\mathbf{R}^{2d}) \tag{2.11}$$

(cf. [10, 17]). The operator $\mu(\chi)$ is a homeomorphism on $\mathscr{S}$ and on $\mathscr{S}'$.

The mapping $\mathrm{Sp}(d,\mathbf{R}) \ni \chi \to \mu(\chi)$ is called the metaplectic representation [10]. It is in fact a representation of the so called 2-fold covering group of $\mathrm{Sp}(d,\mathbf{R})$, which is called the metaplectic group. The metaplectic representation satisfies the homomorphism relation modulo a change of sign:

$$\mu(\chi\chi') = \pm\mu(\chi)\mu(\chi'), \quad \chi, \chi' \in \mathrm{Sp}(d,\mathbf{R}).$$

We will use two systems of seminorms on $\mathscr{S}(\mathbf{R}^d)$. The first is

$$\mathscr{S} \ni \varphi \mapsto \|\varphi\|_n := \max_{|\alpha+\beta|\leqslant n} \sup_{x\in\mathbf{R}^d} \left| x^\alpha D^\beta \varphi(x)\right|, \quad n \in \mathbf{N}, \tag{2.12}$$

and the second is

$$\mathscr{S} \ni \varphi \mapsto \|\varphi\|_{M^2_s}, \quad s \geqslant 0. \tag{2.13}$$

The fact that the seminorms (2.13) are equivalent to (2.12) follows from [12, Corollary 11.2.6 and Lemma 11.3.3].

Let $h, s > 0$ be fixed. The space denoted $\mathcal{S}_{s,h}(\mathbf{R}^d)$ is the set of all $f \in C^\infty(\mathbf{R}^d)$ such that

$$\|f\|_{\mathcal{S}_{s,h}} \equiv \sup \frac{|x^\alpha D^\beta f(x)|}{h^{|\alpha+\beta|}(\alpha!\,\beta!)^s} \tag{2.14}$$

is finite, where the supremum is taken over all $\alpha, \beta \in \mathbf{N}^d$ and $x \in \mathbf{R}^d$. The function space $\mathcal{S}_{s,h}$ is a Banach space which increases with h and s, and $\mathcal{S}_{s,h} \subseteq \mathscr{S}$. The topological dual $\mathcal{S}'_{s,h}(\mathbf{R}^d)$ is a Banach space and $\mathscr{S}'(\mathbf{R}^d) \subseteq \mathcal{S}'_{s,h}(\mathbf{R}^d)$. If $s > 1/2$, then $\mathcal{S}_{s,h}$ and $\bigcup_{h>0} \mathcal{S}_{1/2,h}$ contain all finite linear combinations of Hermite functions.

The Beurling type *Gelfand–Shilov space* $\Sigma_s(\mathbf{R}^d)$ is the projective limit of $\mathcal{S}_{s,h}(\mathbf{R}^d)$ with respect to h [11]. This means

$$\Sigma_s(\mathbf{R}^d) = \bigcap_{h>0} \mathcal{S}_{s,h}(\mathbf{R}^d) \tag{2.15}$$

and the Fréchet space topology of $\Sigma_s(\mathbf{R}^d)$ is defined by the seminorms $\|\cdot\|_{\mathcal{S}_{s,h}}$ for $h > 0$. Then $\Sigma_s(\mathbf{R}^d) \neq \{0\}$ if and only if $s > 1/2$ [26]. The topological dual of $\Sigma_s(\mathbf{R}^d)$ is the space of (Beurling type) *Gelfand–Shilov ultradistributions* [11, Section I.4.3]

$$\Sigma'_s(\mathbf{R}^d) = \bigcup_{h>0} \mathcal{S}'_{s,h}(\mathbf{R}^d). \tag{2.15$'$}$$

The dual space $\Sigma'_s(\mathbf{R}^d)$ may be equipped with several topologies: the weak* topology, the strong topology, the Mackey topology, and the topology defined by the union (2.15)$'$ as an inductive limit topology [30]. The latter topology is the strongest topology such that the inclusion $\mathcal{S}'_{s,h}(\mathbf{R}^d) \subseteq \Sigma'_s(\mathbf{R}^d)$ is continuous for all $h > 0$.

As we shall see shortly, the space $\Sigma_s(\mathbf{R}^d)$ may be equipped with Hilbert space seminorms, and thus it may be considered a countably-Hilbert space [1]. According to [1, Theorem 4.16] the strong, the Mackey and the inductive limit topologies on $\Sigma'_s(\mathbf{R}^d)$ coincide.

We will study $\Sigma'_s(\mathbf{R}^d)$ equipped with the weak* topology, denoted $\Sigma'_{s,\mathrm{w}}(\mathbf{R}^d)$, or with the strong topology, denoted $\Sigma'_{s,\mathrm{str}}(\mathbf{R}^d)$. The latter topology is defined by seminorms

$$\Sigma'_s(\mathbf{R}^d) \ni u \mapsto \sup_{\varphi \in B} |(u, \varphi)|$$

for each subset $B \subseteq \Sigma_s(\mathbf{R}^d)$ which is bounded, that is uniformly bounded with respect to each seminorm. Both spaces $\Sigma'_{s,\mathrm{w}}(\mathbf{R}^d)$ and $\Sigma'_{s,\mathrm{str}}(\mathbf{R}^d)$ are sequentially complete [11, Theorems I.5.1 and I.5.6]. From the latter result we also have: A sequence is convergent in $\Sigma'_{s,\mathrm{w}}(\mathbf{R}^d)$ exactly when it converges in the weak* topology of $\mathcal{S}'_{s,h}(\mathbf{R}^d)$ for some $h > 0$.

By the proof of Proposition 6.17 (see Section 6) it will follow that the space $\Sigma_s(\mathbf{R}^d)$ is a *perfect* space in the terminology of [11]: It is a space in which any bounded set is relatively compact. By [11, Theorem I.6.4] sequential convergence in $\Sigma'_{s,\mathrm{w}}$ and $\Sigma'_{s,\mathrm{str}}$ hence coincide.

The Roumieu type Gelfand–Shilov space is the union

$$\mathcal{S}_s(\mathbf{R}^d) = \bigcup_{h>0} \mathcal{S}_{s,h}(\mathbf{R}^d)$$

equipped with the inductive limit topology [30], that is the strongest topology such that each inclusion $\mathcal{S}_{s,h}(\mathbf{R}^d) \subseteq \mathcal{S}_s(\mathbf{R}^d)$ is continuous. Then $\mathcal{S}_s(\mathbf{R}^d) \neq \{0\}$ if and only if $s \geqslant 1/2$. The corresponding (Roumieu type) Gelfand–Shilov ultradistribution space is

$$\mathcal{S}_s'(\mathbf{R}^d) = \bigcap_{h>0} \mathcal{S}_{s,h}'(\mathbf{R}^d).$$

For every $s > 0$ and $\varepsilon > 0$

$$\Sigma_s(\mathbf{R}^d) \subseteq \mathcal{S}_s(\mathbf{R}^d) \subseteq \Sigma_{s+\varepsilon}(\mathbf{R}^d).$$

We will not use the Roumieu type spaces in this article but mention them as a service to a reader interested in a wider context. On a similar note we notice that $(\alpha!\beta!)^s$ in (2.14) may be replaced by $\alpha!^{s_1}\beta!^{s_2}$ for different parameters $s_1, s_2 > 0$ which leads to a more flexible family of spaces. In this paper we restrict to the equal index case.

The Gelfand–Shilov (ultradistribution) spaces enjoy invariance properties, with respect to translation, dilation, tensorization, coordinate transformation and (partial) Fourier transformation. The Fourier transform extends uniquely to homeomorphisms on $\mathscr{S}'(\mathbf{R}^d)$, $\mathcal{S}_s'(\mathbf{R}^d)$ and $\Sigma_s'(\mathbf{R}^d)$, and restricts to homeomorphisms on $\mathscr{S}(\mathbf{R}^d)$, $\mathcal{S}_s(\mathbf{R}^d)$ and $\Sigma_s(\mathbf{R}^d)$, and to a unitary operator on $L^2(\mathbf{R}^d)$. In particular the Wigner distribution (2.8) satisfies $W(g,f) \in \Sigma_s(\mathbf{R}^{2d})$ if $f, g \in \Sigma_s(\mathbf{R}^d)$, and the Weyl quantization formula (2.7) holds for $a \in \Sigma_s'(\mathbf{R}^{2d})$ and $f, g \in \Sigma_s(\mathbf{R}^d)$. Likewise (2.1) holds when $u \in \Sigma_s'(\mathbf{R}^d)$, $f \in \Sigma_s(\mathbf{R}^d)$, $\varphi \in \Sigma_s(\mathbf{R}^d)$ and $\|\varphi\|_{L^2} = 1$.

We will use the Hermite functions

$$h_\alpha(x) = \pi^{-\frac{d}{4}}(-1)^{|\alpha|}(2^{|\alpha|}\alpha!)^{-\frac{1}{2}} e^{\frac{|x|^2}{2}} \partial^\alpha e^{-|x|^2}, \quad x \in \mathbf{R}^d, \ \alpha \in \mathbf{N}^d,$$

and formal series expansions with respect to Hermite functions:

$$f = \sum_{\alpha \in \mathbf{N}^d} c_\alpha h_\alpha$$

where $\{c_\alpha\}$ is a sequence of complex coefficients defined by $c_\alpha = c_\alpha(f) = (f, h_\alpha)$.

Gelfand–Shilov spaces and their ultradistribution duals, as well as the Schwartz space $\mathscr{S}$ and the tempered distributions $\mathscr{S}'$, and L^2, can be identified by means of such series expansions, with characterizations in terms of the corresponding sequence spaces (see [7, 6, 22, 28]). Let

$$f = \sum_{\alpha \in \mathbf{N}^d} c_\alpha h_\alpha \quad \text{and} \quad \phi = \sum_{\alpha \in \mathbf{N}^d} d_\alpha h_\alpha$$

with sequences $\{c_\alpha\}$ and $\{d_\alpha\}$ of finite support. Then the sesquilinear form

$$(f, \phi) = \sum_{\alpha \in \mathbf{N}^d} c_\alpha \overline{d_\alpha} \tag{2.16}$$

agrees with the inner product on $L^2(\mathbf{R}^d)$ since $\{h_\alpha\}_{\alpha \in \mathbf{N}^d} \subseteq L^2(\mathbf{R}^d)$ is an orthonormal basis.

The form (2.16) extends uniquely to the duality on $\mathscr{S}'(\mathbf{R}^d) \times \mathscr{S}(\mathbf{R}^d)$, to the duality on $\mathcal{S}_s'(\mathbf{R}^d) \times \mathcal{S}_s(\mathbf{R}^d)$ for $s \geqslant 1/2$, as well as to the duality on $\Sigma_s'(\mathbf{R}^d) \times \Sigma_s(\mathbf{R}^d)$ for $s > 1/2$.

To wit Simon [28, Theorem V.13] showed that the family of Hilbert sequence spaces

$$\ell_r^2 = \ell_r^2(\mathbf{N}^d) = \left\{ \{c_\alpha\} \colon \|c_\alpha\|_{\ell_r^2} = \left(\sum_{\alpha \in \mathbf{N}^d} |c_\alpha|^2 \langle \alpha \rangle^{2r} \right)^{\frac{1}{2}} < \infty \right\}$$

for $r > 0$ provides a family of seminorms for $\mathscr{S}$ that is equivalent to (2.12), via the homeomorphism $\mathscr{S} \ni f \mapsto \{(f, h_\alpha)\}_{\alpha \in \mathbf{N}^d}$. Thus the Schwartz space $\mathscr{S}(\mathbf{R}^d)$ is identified topologically as the projective limit

$$\mathscr{S}(\mathbf{R}^d) = \bigcap_{r>0} \left\{ \sum_{\alpha \in \mathbf{N}^d} c_\alpha h_\alpha \colon \{c_\alpha\} \in \ell_r^2 \right\} \tag{2.17}$$

and $\mathscr{S}'(\mathbf{R}^d)$ is identified [28, Theorem V.14] as the union

$$\mathscr{S}'(\mathbf{R}^d) = \bigcup_{r>0} \left\{ \sum_{\alpha \in \mathbf{N}^d} c_\alpha h_\alpha \colon \{c_\alpha\} \in \ell_{-r}^2 \right\}$$

with weak* convergence of the sum for each element in $\mathscr{S}'$.

Likewise Langenbruch [22, Theorem 3.4] has shown that the family of Hilbert sequence spaces

$$\ell_{s,r}^2 = \ell_{s,r}^2(\mathbf{N}^d) = \left\{ \{c_\alpha\} \colon \|c_\alpha\|_{\ell_{s,r}^2} = \left(\sum_{\alpha \in \mathbf{N}^d} |c_\alpha|^2 e^{2r|\alpha|^{\frac{1}{2s}}} \right)^{\frac{1}{2}} < \infty \right\}$$

for $r > 0$ yields a family of seminorms that is equivalent to the family (2.14) for all $h > 0$, when $s \geqslant \frac{1}{2}$. For $s > 1/2$ this means that the space $\Sigma_s(\mathbf{R}^d)$ can be identified topologically as the projective limit

$$\Sigma_s(\mathbf{R}^d) = \bigcap_{r>0} \left\{ \sum_{\alpha \in \mathbf{N}^d} c_\alpha h_\alpha \colon \{c_\alpha\} \in \ell_{s,r}^2 \right\} \tag{2.18}$$

and for $s \geqslant 1/2$ the space $\mathcal{S}_s(\mathbf{R}^d)$ can be identified topologically as the inductive limit

$$\mathcal{S}_s(\mathbf{R}^d) = \bigcup_{r>0} \left\{ \sum_{\alpha \in \mathbf{N}^d} c_\alpha h_\alpha \colon \{c_\alpha\} \in \ell_{s,r}^2 \right\}.$$

Moreover [22, Corollary 3.5] shows, in particular, that $\Sigma_s'(\mathbf{R}^d)$ may be identified as the union

$$\Sigma_s'(\mathbf{R}^d) = \bigcup_{r>0} \left\{ \sum_{\alpha \in \mathbf{N}^d} c_\alpha h_\alpha \colon \{c_\alpha\} \in \ell_{s,-r}^2 \right\},$$

and $\mathcal{S}_s'(\mathbf{R}^d)$ may be identified as the intersection

$$\mathcal{S}_s'(\mathbf{R}^d) = \bigcap_{r>0} \left\{ \sum_{\alpha \in \mathbf{N}^d} c_\alpha h_\alpha \colon \{c_\alpha\} \in \ell_{s,-r}^2 \right\},$$

in both cases with weak* convergence of the sum for each ultradistribution.

Working with Gelfand–Shilov spaces we will occasionally need the inequality (cf. [4])

$$|x+y|^{1/s} \leqslant 2(|x|^{1/s} + |y|^{1/s}), \quad x, y \in \mathbf{R}^d,$$

which holds when $s \geqslant \frac{1}{2}$ and which implies

$$\begin{aligned} e^{A|x+y|^{1/s}} &\leqslant e^{2A|x|^{1/s}} e^{2A|y|^{1/s}}, \quad A > 0,\ x, y \in \mathbf{R}^d, \\ e^{-2A|x+y|^{1/s}} &\leqslant e^{-A|x|^{1/s}} e^{2A|y|^{1/s}}, \quad A > 0,\ x, y \in \mathbf{R}^d. \end{aligned} \tag{2.19}$$

Finally we state the basic definitions of a one-parameter semigroup of operators. Often semigroups of operators are considered on a Banach space [8, 25] but we need also the case of a locally convex space [21, 34]. Thus let X be a locally convex topological vector space, and let $\{T_t,\ t \geqslant 0\}$ be a one-parameter family of continuous linear operators on X. The family $\{T_t,\ t \geqslant 0\}$ is called a strongly continuous semigroup provided

$$T_0 = I, \quad T_t T_s = T_{t+s}, \quad t, s \geqslant 0, \quad \text{and} \quad \lim_{t \to 0^+} T_t x = x \quad \forall x \in X.$$

The infinitesimal generator A of the semigroup T_t is the linear, in general unbounded, operator

$$Ax = \lim_{t \to 0^+} t^{-1}(T_t - I)x$$

equipped with the domain $D(A) \subseteq X$ of all $x \in X$ such that the right-hand side limit is well defined in X.

A *locally equicontinuous* strongly continuous semigroup [21] is a strongly continuous semigroup $\{T_t\}_{t \geqslant 0}$ on X such that for all $t_0 > 0$ and each seminorm p on X there exists a seminorm q on X such that

$$p(T_t x) \leqslant q(x), \quad x \in X, \ \ 0 \leqslant t \leqslant t_0.$$

3. A class of evolution equations and the propagator on L^2

Let q be a homogeneous quadratic form on $T^*\mathbf{R}^d$, that is

$$q(x,\xi) = \langle (x,\xi), Q(x,\xi) \rangle, \quad (x,\xi) \in T^*\mathbf{R}^d, \tag{3.1}$$

where $Q \in \mathbf{C}^{2d \times 2d}$ is symmetric, and suppose its real part is non-negative definite, denoted $\operatorname{Re} Q \geqslant 0$. We study the initial value Cauchy problem for the following class of evolution equations.

$$\begin{cases} \partial_t u(t,x) + q^w(x,D)u(t,x) = 0, \quad t > 0, \ \ x \in \mathbf{R}^d, \\ u(0,\cdot) = u_0 \in L^2(\mathbf{R}^d). \end{cases} \tag{CP}$$

Here $q^w(x,D)$ acts on functions of the variable $x \in \mathbf{R}^d$. The *Hamilton map* F corresponding to q is

$$F = \mathcal{J} Q \in \mathbf{C}^{2d \times 2d}$$

with $\mathcal{J} \in \operatorname{Sp}(d, \mathbf{R})$ defined by (2.10). This framework of evolution equations has been studied in many papers, e.g. [13, 19, 24].

The symbol q is a Shubin symbol of order two, $q \in \Gamma^2$, which implies that $q^w(x,D)\colon M^2_{s+2}(\mathbf{R}^d) \to M^2_s(\mathbf{R}^d)$ is continuous for all $s \in \mathbf{R}$ by (2.9). There is a loss of regularity of order two.

The operator $q^w(x,D)$ can be considered as an unbounded operator in $L^2(\mathbf{R}^d)$. In [19, pp. 425–26] it is shown that its maximal realization equals its closure as an operator initially defined on $\mathscr{S}$, and the closure of $-q^w(x,D)$ generates a strongly continuous contraction semigroup on L^2 for $t \geqslant 0$ denoted by $e^{-tq^w(x,D)}$. The contraction property means that the L^2 operator norm satisfies $\|e^{-tq^w(x,D)}\| \leqslant 1$ for all $t \geqslant 0$.

By semigroup theory (see e.g. [25, Theorem I.2.4] and [20, pp. 483–84]) the unique solution in the space $C^1([0,\infty), L^2)$ to (CP) is

$$u(x,t) = e^{-tq^w(x,D)} u_0$$

where $u_0 \in D(q^w(x,D)) \subseteq L^2(\mathbf{R}^d)$ which denotes the domain of the closure of $q^w(x,D)$. The notation $C^1([0,\infty), L^2)$ understands that the derivative is right continuous at $t = 0$.

In the particular case when $\operatorname{Re} Q = 0$ the propagator is given by means of the metaplectic representation. In fact, then $e^{-tq^w(x,D)}$ is a group of unitary operators on $L^2(\mathbf{R}^d)$, and we have by [10, Theorem 4.45]

$$e^{-tq^w(x,D)} = \mu(e^{-2itF}), \quad t \in \mathbf{R}.$$

In this case F is purely imaginary and $iF \in \operatorname{sp}(d, \mathbf{R})$, the real symplectic Lie algebra, which implies that $e^{-2itF} \in \operatorname{Sp}(d, \mathbf{R})$ for any $t \in \mathbf{R}$ [10].

In the general case $\operatorname{Re} Q \geqslant 0$, Hörmander [19] has shown that the propagator $e^{-tq^w(x,D)}$ can be identified as a time-indexed family of Fourier integral operators, described briefly as follows. According to [19, Theorem 5.12] the Schwartz kernel of the propagator $e^{-tq^w(x,D)}$ for $t \geqslant 0$ is an oscillatory integral defined by a quadratic phase function. More precisely we have

$$e^{-tq^w(x,D)} = \mathscr{K}_{e^{-2itF}},$$

where $\mathscr{K}_{e^{-2itF}} : \mathscr{S}(\mathbf{R}^d) \to \mathscr{S}'(\mathbf{R}^d)$ is the linear continuous operator with kernel

$$K_{e^{-2itF}}(x,y) = (2\pi)^{-(d+N)/2} \sqrt{\det \begin{pmatrix} p''_{\theta\theta}/i & p''_{\theta y} \\ p''_{x\theta} & ip''_{xy} \end{pmatrix}} \int_{\mathbf{R}^N} e^{ip(x,y,\theta)} \mathrm{d}\theta \in \mathscr{S}'(\mathbf{R}^{2d}), \tag{3.2}$$

where the quadratic form p is specified below.

By [19, Proposition 5.8] $\mathscr{K}_{e^{-2itF}}$ is in fact continuous on $\mathscr{S}(\mathbf{R}^d)$. The kernel $K_{e^{-2itF}}$ is indexed by the matrix $e^{-2itF} \in \mathbf{C}^{2d \times 2d}$. By [27, Lemma 5.2] the matrix e^{-2itF} belongs to $\operatorname{Sp}(d, \mathbf{C})$, and its graph

$$\lambda' := \mathcal{G}(e^{-2itF}) = \{(e^{-2itF}X, X) \colon X \in T^*\mathbf{C}^d\} \subseteq T^*\mathbf{C}^d \times T^*\mathbf{C}^d, \tag{3.3}$$

is a positive Lagrangian with respect to the symplectic form σ_1 defined by [27, Eq. (5.1)]. As explained after [27, Lemma 5.1] the Lagrangian λ' can be twisted as in [27, Eq. (5.2)] to give a positive Lagrangian $\lambda \subseteq T^*\mathbf{C}^{2d}$. According to [19, Theorem 5.12 and p. 444] the oscillatory integral (3.2) is associated with the positive Lagrangian λ.

By [27, Proposition 4.4] there exists a quadratic form p on $\mathbf{R}^{2d+N}$ that defines λ, and this p defines (3.2). The factor in front of the integral (3.2) is designed to make the oscillatory integral independent of the quadratic form p on $\mathbf{R}^{2d+N}$, including possible changes of dimension N as discussed after [27, Proposition 4.2], as long as p defines λ by means of [27, Eq. (4.8)] with $x \in \mathbf{C}^d$ replaced by $(x,y) \in \mathbf{C}^{2d}$.

It is shown in [19, p. 444] that the kernel $K_{e^{-2itF}}$ is uniquely determined by the Lagrangian λ, apart from a sign ambiguity which is not essential for our purposes. For brevity we denote $\mathscr{K}_{e^{-2itF}} = \mathscr{K}_t$ for $t \geqslant 0$.

By [19, p. 446] the L^2 adjoint of $\mathscr{K}_t$, defined by

$$(\mathscr{K}_t f, g) = (f, \mathscr{K}_t^* g), \quad f, g \in L^2(\mathbf{R}^d), \tag{3.4}$$

is $\mathscr{K}_t^* = \mathscr{K}_T$ where

$$T = \overline{(e^{-2itF})^{-1}} = \overline{e^{2itF}} = e^{-2it\overline{F}}.$$

Thus the adjoint $\mathscr{K}_t^*$ is an operator of the same type as $\mathscr{K}_t$. It is obtained from the latter by conjugation of the matrix F, i.e. $\mathscr{K}_t^* = \mathscr{K}_{e^{-2itF}}^* = \mathscr{K}_{e^{-2it\overline{F}}}$.

4. The propagator, multiplication and differential operators

The following lemma is an important tool for our results. It can be seen as a commutator relation for the propagator $\mathscr{K}_t$ and $x^\alpha D^\beta$ operators, and particularly the limit behavior as $t \to 0^+$.

Lemma 4.1. *If $\alpha, \beta \in \mathbf{N}^d$, then*

$$x^\alpha D^\beta \mathscr{K}_t = \sum_{|\gamma+\kappa| \leqslant |\alpha+\beta|} C_{\gamma,\kappa}(t)\, \mathscr{K}_t\, x^\gamma D^\kappa \tag{4.1}$$

where $[0,\infty) \ni t \mapsto C_{\gamma,\kappa}(t)$ are continuous functions that satisfy

$$\begin{aligned} &\lim_{t \to 0^+} C_{\alpha,\beta}(t) = 1, \\ &\lim_{t \to 0^+} C_{\gamma,\kappa}(t) = 0, \quad (\gamma,\kappa) \neq (\alpha,\beta). \end{aligned} \tag{4.2}$$

Proof. Let $(x_0, \xi_0) \in T^*\mathbf{C}^d$ and set $(y_0(t), \eta_0(t)) = e^{2itF}(x_0, \xi_0) \in T^*\mathbf{C}^d$. By the proof of [19, Proposition 5.8] we have

$$(\langle D_x, x_0 \rangle - \langle x, \xi_0 \rangle)\, \mathscr{K}_t = \mathscr{K}_t\, (\langle D_x, y_0(t) \rangle - \langle x, \eta_0(t) \rangle)\,. \tag{4.3}$$

We first prove (4.1) and (4.2) when $\alpha = 0$ and $\beta \in \mathbf{N}^d$ using induction. Let $1 \leqslant j \leqslant d$ and set $x_0 = e_j$ and $\xi_0 = 0$. Then

$$\lim_{t \to 0^+} (y_0(t), \eta_0(t)) = (e_j, 0), \tag{4.4}$$

so (4.3) proves (4.1) and (4.2) when $|\beta| = 1$. Suppose (4.1) and (4.2) hold when $\alpha = 0$ and $|\beta| = n \geqslant 1$. Using (4.3) we have for $1 \leqslant j \leqslant d$, $x_0 = e_j$ and $\xi_0 = 0$

$$D^{e_j+\beta} \mathscr{K}_t = \sum_{|\gamma+\kappa| \leqslant n} C_{\gamma,\kappa}(t) \mathscr{K}_t\, (\langle D_x, y_0(t) \rangle - \langle x, \eta_0(t) \rangle)\, x^\gamma D^\kappa$$

where $\lim_{t \to 0^+} C_{0,\beta}(t) = 1$ and $\lim_{t \to 0^+} C_{\gamma,\kappa}(t) = 0$ when $(\gamma,\kappa) \neq (0,\beta)$. Again using (4.4) we obtain (4.1) and (4.2) for $\alpha = 0$ and $|\beta| = n+1$, which constitutes the induction step. Thus the claim (4.1) and (4.2) is true for $\alpha = 0$ and any $\beta \in \mathbf{N}^d$.

Next let $1 \leqslant j \leqslant d$ and set $x_0 = 0$ and $\xi_0 = -e_j$. Then

$$\lim_{t \to 0^+} (y_0(t), \eta_0(t)) = (0, -e_j). \tag{4.5}$$

By combining what we have shown with (4.3) we have for $\beta \in \mathbf{N}^d$

$$x_j D^\beta \mathscr{K}_t = \sum_{|\gamma+\kappa| \leqslant |\beta|} C_{\gamma,\kappa}(t) \mathscr{K}_t\, (\langle D_x, y_0(t) \rangle - \langle x, \eta_0(t) \rangle)\, x^\gamma D^\kappa$$

where $\lim_{t \to 0^+} C_{0,\beta}(t) = 1$ and $\lim_{t \to 0^+} C_{\gamma,\kappa}(t) = 0$ when $(\gamma,\kappa) \neq (0,\beta)$. Invoking (4.5) proves the claims (4.1) and (4.2) for $|\alpha| = 1$ and $\beta \in \mathbf{N}^d$. The generalization to $\alpha \in \mathbf{N}^d$ arbitrary follows again by induction. $\square$

In the next result we use the concept of a bounded set in $\mathscr{S}(\mathbf{R}^d)$. A subset $B \subseteq \mathscr{S}(\mathbf{R}^d)$ is bounded provided each seminorm is uniformly bounded. Using the system of seminorms (2.12) this can be expressed as

$$\sup_{\varphi \in B} \|\varphi\|_n = C_n < \infty \quad \forall n \in \mathbf{N}. \tag{4.6}$$

We prove a few preparatory results that are needed in Section 5, where we show that the propagator $\mathscr{K}_t$ is a strongly continuous semigroup on $M_s^2(\mathbf{R}^d)$ for all $s \in \mathbf{R}$.

Lemma 4.2. *If $B \subseteq \mathscr{S}(\mathbf{R}^d)$ is bounded and $\gamma, \kappa \in \mathbf{N}^d$, then $\{x^\gamma D^\kappa \varphi,\ \varphi \in B\} \subseteq \mathscr{S}(\mathbf{R}^d)$ is also bounded.*

Proof. We use the seminorms (2.12) so we assume that (4.6) is valid. For $\alpha,\beta \in \mathbf{N}^d$ we have

$$\begin{aligned}\left|x^\alpha D^\beta \left(x^\gamma D^\kappa \varphi(x)\right)\right| &= \left| \sum_{\sigma \leqslant \min(\beta,\gamma)} \binom{\beta}{\sigma} \frac{\gamma!\, i^{-|\sigma|}}{(\gamma-\sigma)!} x^{\alpha+\gamma-\sigma} D^{\kappa+\beta-\sigma}\varphi(x) \right| \\ &\leqslant |\gamma|! \sum_{\sigma \leqslant \min(\beta,\gamma)} \binom{\beta}{\sigma} \left| x^{\alpha+\gamma-\sigma} D^{\kappa+\beta-\sigma}\varphi(x)\right|\end{aligned}$$

which gives for any $n \in \mathbf{N}$

$$\begin{aligned}\|x^\gamma D^\kappa \varphi\|_n &= \max_{|\alpha+\beta|\leqslant n} \sup_{x\in\mathbf{R}^d} \left|x^\alpha D^\beta \left(x^\gamma D^\kappa \varphi(x)\right)\right| \\ &\leqslant |\gamma|! \max_{|\alpha+\beta|\leqslant n} \sum_{\sigma \leqslant \min(\beta,\gamma)} \binom{\beta}{\sigma} \sup_{x\in\mathbf{R}^d} \left| x^{\alpha+\gamma-\sigma} D^{\kappa+\beta-\sigma}\varphi(x)\right| \\ &\leqslant |\gamma|!\, \|\varphi\|_{n+|\gamma+\kappa|} \max_{|\alpha+\beta|\leqslant n} \sum_{\sigma \leqslant \min(\beta,\gamma)} \binom{\beta}{\sigma} \\ &\leqslant |\gamma|!\, C_{n+|\gamma+\kappa|} \max_{|\alpha+\beta|\leqslant n} 2^{|\beta|} \leqslant |\gamma|!\, 2^n\, C_{n+|\gamma+\kappa|}, \quad \varphi \in B. \qquad \square\end{aligned}$$

Lemma 4.3. *If $B \subseteq \mathscr{S}(\mathbf{R}^d)$ is bounded and $\varepsilon > 0$, then there exists $K \in \mathbf{N}$ and $\varphi_j \in \mathscr{S}(\mathbf{R}^d)$ for $1 \leqslant j \leqslant K$ such that*

$$B \subseteq \bigcup_{j=1}^K B_\varepsilon(\varphi_j)$$

where the open balls $B_\varepsilon(\varphi_j) \subseteq L^2(\mathbf{R}^d)$ refer to the L^2 norm.

Proof. We use the identification (2.17) of $\mathscr{S}(\mathbf{R}^d)$ as a projective limit of sequence spaces for Hermite series expansions. Then $L^2(\mathbf{R}^d)$ corresponds to $\ell^2(\mathbf{N}^d)$. We work on the side of the sequences $c = (c_\alpha)_{\alpha\in\mathbf{N}^d}$. Since $B \subseteq \mathscr{S}(\mathbf{R}^d)$ is bounded there exists for each $r > 0$ a bound $C_r > 0$ such that

$$\|c\|_{\ell^2_r}^2 = \sum_{\alpha\in\mathbf{N}^d} |c_\alpha|^2 \langle\alpha\rangle^{2r} \leqslant C_r^2, \quad c \in B.$$

For $r = 1$ and $N \in \mathbf{N}$ this gives

$$\begin{aligned}\sum_{\alpha\in\mathbf{N}^d,\, |\alpha|>N} |c_\alpha|^2 &= \sum_{\alpha\in\mathbf{N}^d,\, |\alpha|>N} |c_\alpha|^2 \langle\alpha\rangle^{2-2} \\ &\leqslant \langle N\rangle^{-2} \sum_{\alpha\in\mathbf{N}^d} |c_\alpha|^2\langle\alpha\rangle^2 \leqslant C_1^2 \langle N\rangle^{-2}, \quad c\in B.\end{aligned}$$

If we pick $N > 0$ sufficiently large we thus have

$$\sup_{c\in B} \sum_{\alpha\in\mathbf{N}^d,\, |\alpha|>N} |c_\alpha|^2 < \frac{\varepsilon^2}{2}. \tag{4.7}$$

On the other hand we have

$$B_{(N)} := \left\{ \{c_\alpha\}_{|\alpha|\leqslant N} \colon \{c_\alpha\}_{\alpha\in\mathbf{N}^d} \in B \right\} \subseteq \mathbf{C}^M$$

for some $M \in \mathbf{N}$, and

$$\sum_{|\alpha|\leqslant N} |c_\alpha|^2 \leqslant \sum_{\alpha\in\mathbf{N}^d} |c_\alpha|^2\langle\alpha\rangle^2 \leqslant C_1^2, \quad c \in B,$$

so $B_{(N)} \subseteq \overline{B}_{C_1} \subseteq \mathbf{C}^M$ where B_{C_1} denotes the open ball in $\mathbf{C}^M$, considered as a Hilbert space, with radius $C_1 > 0$. By the compactness of its closure $\overline{B}_{C_1} \subseteq \mathbf{C}^M$ there exist $\{c_j\}_{j=1}^K \subseteq \mathbf{C}^M$ such that

$$\min_{1 \leqslant j \leqslant K} \|c - c_j\|^2_{\ell^2_M} < \frac{\varepsilon^2}{2}, \quad c \in B_{(N)}. \tag{4.8}$$

We extend c_j to elements in $\ell^2(\mathbf{N}^d)$ by zero-padding:

$$c_{j,\alpha} = 0, \quad |\alpha| > N, \quad 1 \leqslant j \leqslant K.$$

Combining (4.7) and (4.8) gives

$$\min_{1 \leqslant j \leqslant K} \|c - c_j\|^2_{\ell^2(\mathbf{N}^d)} = \min_{1 \leqslant j \leqslant K} \sum_{|\alpha| \leqslant N} |c_\alpha - c_{j,\alpha}|^2 + \sum_{|\alpha| > N} |c_\alpha|^2 < \varepsilon^2, \quad c \in B.$$

Thus

$$B \subseteq \bigcup_{j=1}^K B_\varepsilon(c_j). \qquad \square$$

Lemma 4.4. *If $B \subseteq \mathscr{S}(\mathbf{R}^d)$ is bounded and $\alpha, \beta \in \mathbf{N}^d$, then*

$$\lim_{t \to 0^+} \sup_{\varphi \in B} \|x^\alpha D^\beta (\mathscr{K}_t - I)\varphi\|_{L^2} = 0.$$

Proof. From Lemma 4.1 we obtain for $\varphi \in \mathscr{S}$

$$\begin{aligned} x^\alpha D^\beta (\mathscr{K}_t - I)\varphi &= C_{\alpha,\beta}(t)(\mathscr{K}_t - I) x^\alpha D^\beta \varphi + (C_{\alpha,\beta}(t) - 1) x^\alpha D^\beta \varphi \\ &\quad + \sum_{\substack{|\gamma+\kappa| \leqslant |\alpha+\beta| \\ (\gamma,\kappa) \neq (\alpha,\beta)}} C_{\gamma,\kappa}(t) \mathscr{K}_t \, x^\gamma D^\kappa \varphi \end{aligned}$$

where (4.2) holds. The contraction property of $\mathscr{K}_t$ acting on L^2 yields for $0 < t \leqslant 1$

$$\begin{aligned} \|x^\alpha D^\beta (\mathscr{K}_t - I)\varphi\|_{L^2} &\leqslant C \|(\mathscr{K}_t - I) x^\alpha D^\beta \varphi\|_{L^2} + |C_{\alpha,\beta}(t) - 1| \, \|x^\alpha D^\beta \varphi\|_{L^2} \\ &\quad + \sum_{\substack{|\gamma+\kappa| \leqslant |\alpha+\beta| \\ (\gamma,\kappa) \neq (\alpha,\beta)}} |C_{\gamma,\kappa}(t)| \, \|x^\gamma D^\kappa \varphi\|_{L^2} \end{aligned} \tag{4.9}$$

where $C > 0$.

Let $\varepsilon > 0$. By Lemmas 4.2 and 4.3 there exists $K \in \mathbf{N}$ and $\varphi_j \in \mathscr{S}(\mathbf{R}^d)$, $1 \leqslant j \leqslant K$, such that

$$\min_{1 \leqslant j \leqslant K} \|x^\alpha D^\beta \varphi - \varphi_j\|_{L^2} < \frac{\varepsilon}{8C}, \quad \varphi \in B.$$

Next we use two properties of $\mathscr{K}_t$ acting on L^2: the contraction property and the strong continuity. This gives for $0 < t \leqslant \delta$

$$\begin{aligned} \|(\mathscr{K}_t - I) x^\alpha D^\beta \varphi\|_{L^2} &= \min_{1 \leqslant j \leqslant K} \|(\mathscr{K}_t - I)(x^\alpha D^\beta \varphi - \varphi_j + \varphi_j)\|_{L^2} \\ &\leqslant \min_{1 \leqslant j \leqslant K} \big(2 \|x^\alpha D^\beta \varphi - \varphi_j\|_{L^2} + \|(\mathscr{K}_t - I)\varphi_j\|_{L^2} \big) \\ &\leqslant \frac{\varepsilon}{4C} + \frac{\varepsilon}{4C} = \frac{\varepsilon}{2C}, \quad \varphi \in B, \end{aligned} \tag{4.10}$$

provided $\delta > 0$ is sufficiently small.

In the next step we use the seminorms (2.12) for $\mathscr{S}$ and (4.6). We also use

$$\langle x \rangle^{2d} = (1 + x_1^2 + \cdots + x_d^2)^d = \sum_{|\sigma| \leqslant d} C_\sigma x^{2\sigma} \tag{4.11}$$

where $C_\sigma > 0$ are constants. Thus we obtain for $|\gamma+\kappa| \leqslant |\alpha+\beta|$

$$\|x^\gamma D^\kappa \varphi\|_{L^2}^2 = \sum_{|\sigma|\leqslant d} C_\sigma \int_{\mathbf{R}^d} \langle x\rangle^{-2d} |x^{\sigma+\gamma} D^\kappa \varphi(x)|^2 \,\mathrm{d}x \leqslant D_1^2\, C_{|\alpha+\beta|+d}^2, \quad \varphi \in B, \tag{4.12}$$

for some $D_1 > 0$.

Finally we insert (4.10) and (4.12) into (4.9). We obtain then for $0 < t \leqslant \delta$, again after possibly decreasing $\delta > 0$,

$$\begin{aligned}
&\|x^\alpha D^\beta (\mathscr{K}_t - I)\varphi\|_{L^2}\\
&\leqslant C\|(\mathscr{K}_t - I)x^\alpha D^\beta \varphi\|_{L^2} + D_1 C_{|\alpha+\beta|+d} \left(|C_{\alpha,\beta}(t) - 1| + \sum_{\substack{|\gamma+\kappa|\leqslant|\alpha+\beta| \\ (\gamma,\kappa)\neq(\alpha,\beta)}} |C_{\gamma,\kappa}(t)| \right)\\
&\leqslant \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \quad \varphi \in B.
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary this proves the claim. □

Lemma 4.1 is useful in order to understand the behavior of the propagator $\mathscr{K}_t$ as $t \to 0^+$, witness Lemma 4.4. We will prove more results in this direction further on, see Theorems 5.2 and 6.12.

5. Strong continuity on Hilbert modulation spaces and tempered distributions

In this section we prove that $\mathscr{K}_t$ is a strongly continuous semigroup in several subspaces of the tempered distributions: $M_s^2(\mathbf{R}^d)$ for any $s \in \mathbf{R}$, the Schwartz space $\mathscr{S}(\mathbf{R}^d)$, and $\mathscr{S}'(\mathbf{R}^d)$ equipped with either the weak* or the strong topology. In the case of $\mathscr{S}'(\mathbf{R}^d)$ equipped with the strong topology, we show that the semigroup is locally equicontinuous.

We need the following tool in the proof of Theorem 5.2.

Lemma 5.1. *Let $s \in \mathbf{R}$ and $T > 0$. The propagator $\mathscr{K}_t$ is bounded on $M_s^2(\mathbf{R}^d)$ uniformly over $0 \leqslant t \leqslant T$.*

Proof. By [9, Theorem 4.5] (cf. [16, Proposition 1.2]) the modulation spaces are closed under complex interpolation of Banach spaces [2]. We may thus assume that $s = k \in \mathbf{Z}$. Suppose $k \geqslant 0$. By [23, Theorem 2.1.12]

$$\|u\| = \sum_{|\alpha+\beta|\leqslant k} \|x^\alpha D^\beta u\|_{L^2} \tag{5.1}$$

is a norm on $M_k^2(\mathbf{R}^d)$ that is equivalent to (2.2).

From Lemma 4.1 and the contraction property of $\mathscr{K}_t$ we obtain

$$\begin{aligned}
\|\mathscr{K}_t u\|_{M_k^2} \asymp \sum_{|\alpha+\beta|\leqslant k} \|x^\alpha D^\beta \mathscr{K}_t u\|_{L^2} &\leqslant \sum_{|\alpha+\beta|\leqslant k} \ \sum_{|\gamma+\kappa|\leqslant|\alpha+\beta|} |C_{\gamma,\kappa}(t)|\, \|x^\gamma D^\kappa u\|_{L^2}\\
&\lesssim \sum_{|\alpha+\beta|\leqslant k} \|x^\alpha D^\beta u\|_{L^2} \asymp \|u\|_{M_k^2},
\end{aligned}$$

in the last inequality using the consequence of Lemma 4.1 that the functions $C_{\gamma,\kappa}$ are continuous and therefore uniformly bounded with respect to $t \in [0,T]$. This proves the lemma when $k \geqslant 0$.

If $k<0$ we use duality. In fact the dual of M_k^2 can be identified with M_{-k}^2 with respect to an extension of the L^2 inner product [9], [12, Theorem 11.3.6]. We also use the expression of the adjoint of $\mathscr{K}_t$ as $\mathscr{K}_t^* = \mathscr{K}_{e^{-2it\overline{F}}}$, cf. (3.4). By the result above we have

$$\|\mathscr{K}_{e^{-2it\overline{F}}} u\|_{M^2_{-k}} \lesssim \|u\|_{M^2_{-k}}, \quad 0 \leqslant t \leqslant T,$$

which gives

$$\begin{aligned}\|\mathscr{K}_t u\|_{M_k^2} &= \sup_{\|g\|_{M^2_{-k}}\leqslant 1} |(\mathscr{K}_t u, g)| = \sup_{\|g\|_{M^2_{-k}}\leqslant 1} \left|\left(u, \mathscr{K}_{e^{-2it\overline{F}}} g\right)\right| \\ &\leqslant \|u\|_{M_k^2} \sup_{\|g\|_{M^2_{-k}}\leqslant 1} \|\mathscr{K}_{e^{-2it\overline{F}}} g\|_{M^2_{-k}} \lesssim \|u\|_{M_k^2}, \quad 0 \leqslant t \leqslant T. \qquad \square\end{aligned}$$

Theorem 5.2. *Let $s \in \mathbf{R}$. The propagator $\mathscr{K}_t = e^{-tq^w(x,D)}$ is for $t \geqslant 0$ a strongly continuous semigroup on $M_s^2(\mathbf{R}^d)$.*

Proof. By Lemma 5.1 the operators $\mathscr{K}_t$ are bounded on M_s^2, uniformly over $t \in [0,T]$ for any $T>0$. Pick $k \in \mathbf{N}$ such that $k \geqslant s$. For any $\varphi \in \mathscr{S}(\mathbf{R}^d)$ we obtain from (2.3), using the norm (5.1) on M_k^2, and Lemma 4.4

$$\|(\mathscr{K}_t - I)\varphi\|_{M_s^2} \lesssim \|(\mathscr{K}_t - I)\varphi\|_{M_k^2} \asymp \sum_{|\alpha+\beta|\leqslant k} \|x^\alpha D^\beta (\mathscr{K}_t - I)\varphi\|_{L^2} \longrightarrow 0, \quad t \to 0^+.$$

Since $\mathscr{S} \subseteq M_s^2$ is dense [12, Proposition 11.3.4], we may combine this find, Lemma 5.1 and [8, Proposition I.5.3]. The conclusion of the latter result is then the strong continuity of $\mathscr{K}_t$ on $M_s^2(\mathbf{R}^d)$.

Finally we consider the semigroup property. If $s \geqslant 0$, then $M_s^2 \subseteq L^2$. Thus $\mathscr{K}_0 = I$ and $\mathscr{K}_{t_1+t_2} = \mathscr{K}_{t_1}\mathscr{K}_{t_2}$ hold on $M_s^2(\mathbf{R}^d)$ due to the corresponding properties on L^2. If $s<0$, then let $u \in M_s^2(\mathbf{R}^d)$ and let $t_1, t_2 \geqslant 0$. From the extension of (3.4) to the duality on $M_{-s}^2 \times M_s^2$ we have for $\varphi \in \mathscr{S}(\mathbf{R}^d)$

$$((\mathscr{K}_{t_1+t_2} - \mathscr{K}_{t_1}\mathscr{K}_{t_2})u, \varphi) = (u, (\mathscr{K}_{t_1+t_2}^* - \mathscr{K}_{t_2}^*\mathscr{K}_{t_1}^*)\varphi) = 0$$

due to the semigroup property $\mathscr{K}_{t_1+t_2}^* = \mathscr{K}_{t_2}^*\mathscr{K}_{t_1}^*$ when the action refers to L^2. This proves the semigroup property $\mathscr{K}_{t_1+t_2} = \mathscr{K}_{t_1}\mathscr{K}_{t_2}$ for action on $M_s^2(\mathbf{R}^d)$, and likewise $\mathscr{K}_0 = I$ on $M_s^2(\mathbf{R}^d)$. $\square$

Corollary 5.3. *The propagator $\mathscr{K}_t$ is for $t \geqslant 0$ a locally equicontinuous strongly continuous semigroup on $\mathscr{S}(\mathbf{R}^d)$.*

Proof. We use the seminorms (2.13) on $\mathscr{S}(\mathbf{R}^d)$. The continuity of $\mathscr{K}_t$ on $\mathscr{S}$ follows from Lemma 5.1, as well as the local equicontinuity. The strong continuity is a consequence of the proof of Theorem 5.2. Finally the semigroup property $\mathscr{K}_{t_1+t_2} = \mathscr{K}_{t_1}\mathscr{K}_{t_2}$ for $t_1, t_2 \geqslant 0$, and $\mathscr{K}_0 = I$, are immediate consequences of the corresponding properties for the semigroup acting on L^2. $\square$

The generator of the semigroup $\mathscr{K}_t$ acting on $M_s^2(\mathbf{R}^d)$ according to Theorem 5.2 is

$$A_s f = \lim_{h\to 0^+} h^{-1}\left(\mathscr{K}_h - I\right) f \tag{5.2}$$

for all $f \in M_s^2(\mathbf{R}^d)$ such that the right-hand side limit exists in $M_s^2(\mathbf{R}^d)$ [25]. The linear space of all such $f \in M_s^2(\mathbf{R}^d)$ is the domain of A_s denoted $D(A_s) \subseteq M_s^2(\mathbf{R}^d)$. For each $s \in \mathbf{R}$ the operator A_s equipped with the domain $D(A_s)$ is an unbounded linear operator in $M_s^2(\mathbf{R}^d)$. The domain $D(A_s)$ is dense in $M_s^2(\mathbf{R}^d)$ and the operator A_s is closed [25, Corollary I.2.5].

It follows from (2.3) that $D(A_{s_2}) \subseteq D(A_{s_1})$ if $s_1 \leqslant s_2$ and $A_{s_2} f = A_{s_1} f$ if $f \in D(A_{s_2})$. Thus we have for $0 \leqslant s_1 \leqslant s_2$

$$A_{s_2} \subseteq A_{s_1} \subseteq -q^w(x,D) \subseteq A_{-s_1} \subseteq A_{-s_2} \tag{5.3}$$

where $-q^w(x,D) = A_0$ denotes the generator of the semigroup $\mathscr{K}_t$ on L^2.

According to Corollary 5.3 the propagator $\mathscr{K}_t$ is also a locally equicontinuous strongly continuous semigroup on $\mathscr{S}$. The generator of the semigroup $\mathscr{K}_t$ acting on $\mathscr{S}$ is

$$Af = \lim_{h\to 0^+} h^{-1} (\mathscr{K}_h - I) f \tag{5.4}$$

for all $f \in \mathscr{S}$ such that the limit is well defined in $\mathscr{S}$. The space of such f is the domain denoted $D(A) \subseteq \mathscr{S}$. According to [21, Propositions 1.3 and 1.4] A is a closed linear operator and $D(A) \subseteq \mathscr{S}$ is dense (cf. Remark 5.10).

Let $f \in D(A)$ and let $s \geqslant 0$. Then (5.4) converges in $\mathscr{S}$ and therefore also in M^2_s, to the same element in M^2_s. Thus $f \in D(A_s)$, so this means that $D(A) \subseteq D(A_s)$ and $A \subseteq A_s$. In particular for $s = 0$ we have $Af = -q^w(x,D)f$ if $f \in D(A) \subseteq \mathscr{S}$. By [31, p. 178] $q^w(x,D)$ is continuous on $\mathscr{S}$. Since A is closed and $D(A) \subseteq \mathscr{S}$ is dense we must have $D(A) = \mathscr{S}$. Combined with (5.3) his yields

$$\mathscr{S} = D(A) \subseteq \bigcap_{s\in\mathbf{R}} D(A_s).$$

If $f \in \bigcap_{s\in\mathbf{R}} D(A_s)$, then (5.4) converges in $\mathscr{S}$ so $f \in D(A) = \mathscr{S}$ and we can strengthen the inclusion into

$$\mathscr{S} = \bigcap_{s\in\mathbf{R}} D(A_s). \tag{5.5}$$

It follows from above that A is continuous on $\mathscr{S}$. We can thus extend A uniquely to $\mathscr{S}'$, using its formal L^2 adjoint $A^* = -\overline{q}^w(x,D)$ acting on $\mathscr{S}$, by

$$(Au,\varphi) = (u, A^*\varphi), \quad u \in \mathscr{S}'(\mathbf{R}^d), \ \varphi \in \mathscr{S}(\mathbf{R}^d). \tag{5.6}$$

The extension is continuous on $\mathscr{S}'$ equipped with its weak* topology.

Lemma 5.4. *For each $s \in \mathbf{R}$ we have $M^2_{s+2}(\mathbf{R}^d) \subseteq D(A_s)$.*

Proof. Since $q \in \Gamma^2$ we have by (2.9) for any $s \in \mathbf{R}$

$$\|q^w(x,D)f\|_{M^2_s} \lesssim \|f\|_{M^2_{s+2}}, \quad f \in \mathscr{S}. \tag{5.7}$$

Let $f \in M^2_{s+2}(\mathbf{R}^d)$. Since $\mathscr{S} \subseteq M^2_{s+2}$ is a dense subspace [12, Proposition 11.3.4] there exists a sequence $(f_n)_{n\geqslant 1} \subseteq \mathscr{S}$ such that $f_n \to f$ in M^2_{s+2} as $n \to \infty$. By (2.3) this implies that

$$f_n \to f \quad \text{in} \quad M^2_s \quad \text{as} \quad n \to \infty. \tag{5.8}$$

From (5.5) we know that $\mathscr{S} \subseteq D(A_s) \bigcap D(q^w(x,D))$ and hence using (5.7) we obtain

$$\|A_s(f_n - f_m)\|_{M^2_s} = \|q^w(x,D)(f_n - f_m)\|_{M^2_s} \lesssim \|f_n - f_m\|_{M^2_{s+2}}$$

for $n, m \geqslant 1$. Thus $(A_s f_n)_{n\geqslant 1}$ is a Cauchy sequence in M^2_s, which converges to an element $g \in M^2_s$. If we combine $A_s f_n \to g$ in M^2_s as $n \to \infty$ with (5.8) and the fact that A_s is closed, we may conclude that $f \in D(A_s)$ and $A_s f = g$. Hence $M^2_{s+2}(\mathbf{R}^d) \subseteq D(A_s)$. □

When we consider the equation (CP) in $M^2_s(\mathbf{R}^d)$, we identify $A_s = -q^w(x,D)$.

Corollary 5.5. *Let $s \in \mathbf{R}$ and consider the Cauchy problem* (CP) *in $M^2_s(\mathbf{R}^d)$. If $u_0 \in M^2_{s+2}(\mathbf{R}^d)$, then $\mathscr{K}_t u_0$ is the unique solution in $C^1([0,\infty), M^2_s)$.*

Proof. The claim is a consequence of Lemma 5.4, [25, Theorem I.2.4] and [20, pp. 483–84]. □

Finally we obtain from (2.4) the following consequence.

Corollary 5.6. *The Cauchy problem* (CP) *has the solution $\mathscr{K}_t u_0$ for any $u_0 \in \mathscr{S}'(\mathbf{R}^d)$. It is unique in the sense of Corollary 5.5.*

A version of Corollary 5.6 with additional information can be obtained in another fashion, as follows.

By Corollary 5.3 we may for fixed $t \geqslant 0$ extend $\mathscr{K}_t$ from domain $\mathscr{S}(\mathbf{R}^d)$ to $\mathscr{S}'(\mathbf{R}^d)$ uniquely by defining

$$(\mathscr{K}_t u, \varphi) = (u, \mathscr{K}_t{}^* \varphi) = (u, \mathscr{K}_{e^{-2it\overline{F}}} \varphi), \quad u \in \mathscr{S}'(\mathbf{R}^d),\ \varphi \in \mathscr{S}(\mathbf{R}^d), \tag{5.9}$$

since $\mathscr{K}_t{}^* \varphi \in \mathscr{S}$, cf. (3.4). Then $\mathscr{K}_{t_1+t_2} = \mathscr{K}_{t_1} \mathscr{K}_{t_2}$ for $t_1, t_2 \geqslant 0$ and $\mathscr{K}_0 = I$ for the action on $\mathscr{S}'$ follows as in the proof of Theorem 5.2.

Denote by $\mathscr{S}'_{\mathrm{w}}$ the space $\mathscr{S}'$ equipped with its weak* topology, with seminorms $\mathscr{S}' \ni u \mapsto |(u,\varphi)|$ for all $\varphi \in \mathscr{S}$. From Corollary 5.3 it follows that $\mathscr{K}_t \colon \mathscr{S}'_{\mathrm{w}} \to \mathscr{S}'_{\mathrm{w}}$ is continuous for each $t \geqslant 0$. Let $u \in \mathscr{S}'(\mathbf{R}^d)$. For some $s \geqslant 0$ we have for $\varphi \in \mathscr{S}$

$$|((\mathscr{K}_t - I)u, \varphi)| = |(u, (\mathscr{K}_t^* - I)\varphi)| \lesssim \|(\mathscr{K}_t^* - I)\varphi\|_{M^2_s}.$$

The right-hand side approaches zero as $t \to 0^+$ according to Theorem 5.2. We may conclude that $\mathscr{K}_t$ is a strongly continuous semigroup on $\mathscr{S}'_{\mathrm{w}}$.

The modulus of the right-hand side of (5.9) equals $|(u, \mathscr{K}_t{}^*\varphi)|$. For t in the interval $0 \leqslant t \leqslant T < \infty$ with $T > 0$ given, this is an indexed family of seminorms of $u \in \mathscr{S}'_{\mathrm{w}}$, but we cannot estimate $\{|(u, \mathscr{K}_t{}^*\varphi)|\}_{0 \leqslant t \leqslant T}$ by a *single* seminorm. Thus we cannot show that the semigroup $\mathscr{K}_t$ is locally equicontinuous on $\mathscr{S}'_{\mathrm{w}}$. For that purpose we need to equip $\mathscr{S}'$ with another topology.

The space $\mathscr{S}'_{\mathrm{str}}$ denotes $\mathscr{S}'$ equipped with its strong topology [28], with seminorms

$$\mathscr{S}' \ni u \mapsto \sup_{\varphi \in B} |(u,\varphi)|$$

for each bounded set $B \subseteq \mathscr{S}$. Expressed with the seminorms (2.13) a bounded set satisfies

$$\sup_{\varphi \in B} \|\varphi\|_{M^2_s} = C_s < \infty, \quad \forall s \geqslant 0.$$

If $B \subseteq \mathscr{S}$ is bounded and $0 \leqslant t \leqslant T$, then

$$\sup_{\varphi \in B} |(\mathscr{K}_t u, \varphi)| = \sup_{\varphi \in B} |(u, \mathscr{K}_t{}^*\varphi)| \leqslant \sup_{\varphi \in B,\ 0 \leqslant t \leqslant T} |(u, \mathscr{K}_t{}^*\varphi)|, \quad u \in \mathscr{S}'.$$

By Lemma 5.1 $\{\mathscr{K}_t{}^* B,\ 0 \leqslant t \leqslant T\} \subseteq \mathscr{S}$ is a bounded set. This shows that $\mathscr{K}_t$ is continuous on $\mathscr{S}'_{\mathrm{str}}$ for each $t \geqslant 0$, and $\{\mathscr{K}_t\}_{t \geqslant 0}$ is a locally equicontinuous semigroup on $\mathscr{S}'_{\mathrm{str}}$. It is also a strongly continuous semigroup on $\mathscr{S}'_{\mathrm{str}}$. In fact let $u \in \mathscr{S}'(\mathbf{R}^d)$ and let $B \subseteq \mathscr{S}$ be bounded. We have for some $k \in \mathbf{N}$ using (5.1) and Lemma 4.4

$$\begin{aligned}
\sup_{\varphi \in B} |((\mathscr{K}_t - I)u, \varphi)| &= \sup_{\varphi \in B} |(u, (\mathscr{K}_t^* - I)\varphi)| \lesssim \sup_{\varphi \in B} \|(\mathscr{K}_t^* - I)\varphi)\|_{M^2_k} \\
&\leqslant \sum_{|\alpha+\beta| \leqslant k} \sup_{\varphi \in B} \|x^\alpha D^\beta (\mathscr{K}_t^* - I)\varphi\|_{L^2} \longrightarrow 0, \quad t \to 0^+.
\end{aligned}$$

We have proved:

Theorem 5.7. *The semigroup $\mathscr{K}_t$ is:*

(i) *strongly continuous on $\mathscr{S}'_{\mathrm{w}}$, and*

(ii) *locally equicontinuous strongly continuous on $\mathscr{S}'_{\mathrm{str}}$.*

The generator of the semigroup $\mathscr{K}_t$ on $\mathscr{S}'_{\mathrm{w}}$ is denoted

$$A'_{\mathrm{w}} u = \lim_{h \to 0^+} h^{-1} \left(\mathscr{K}_h - I \right) u$$

for all $u \in \mathscr{S}'_{\mathrm{w}}$, denoted $D(A'_{\mathrm{w}}) \subseteq \mathscr{S}'_{\mathrm{w}}$, such that the limit is well defined in $\mathscr{S}'_{\mathrm{w}}$. The generator of the semigroup $\mathscr{K}_t$ on $\mathscr{S}'_{\mathrm{str}}$ is denoted A'_{str}. Note that $A'_{\mathrm{str}} \subseteq A'_{\mathrm{w}}$. By [21, Proposition 2.1] $A'_{\mathrm{w}} = A$ defined by (5.6) and hence $D(A'_{\mathrm{w}}) = \mathscr{S}'$.

The local equicontinuity of $\mathscr{K}_t$ acting on $\mathscr{S}'_{\mathrm{str}}$ guarantees by [21, Proposition 1.4] that the operator A'_{str} is closed. By [21, Proposition 1.3], the inclusion $D(A'_{\mathrm{str}}) \subseteq \mathscr{S}'$ is dense. Combining the latter two facts gives $D(A'_{\mathrm{str}}) = \mathscr{S}'$ and $A'_{\mathrm{str}} = A'_{\mathrm{w}} = A$. The generators of the two semigroups are identical.

We denote $A' = A'_{\mathrm{str}} = A'_{\mathrm{w}}$, and A is defined by (5.4). Extending (5.3) we thus have for $s_1 \leqslant s_2$

$$A \subseteq A_{s_2} \subseteq A_{s_1} \subseteq A'.$$

Remark 5.8. There is also a more abstract motivation for some of the conclusions above, based on the fact that the space $\mathscr{S}$ is reflexive [28, Theorem V.24]. Theorem 5.7 (i) is an immediate consequence of the definition (5.9), cf. [21, p. 262]. The reflexivity of $\mathscr{S}$ entails the following consequence by [21, Theorem 1 and its Corollary]. The semigroup $\mathscr{K}_t$, considered as a strongly continuous semigroup on $\mathscr{S}'_{\mathrm{w}}$, is automatically a strongly continuous semigroup on $\mathscr{S}'_{\mathrm{str}}$, and the two semigroups have identical infinitesimal generators.

An appeal to [21, Proposition 1.2] and [20, pp. 483–84] gives a version of Corollary 5.6 with a continuity statement. Note that the uniqueness space is larger than the solution space: $C^1([0,\infty), \mathscr{S}'_{\mathrm{str}}) \subseteq C^1([0,\infty), \mathscr{S}'_{\mathrm{w}})$.

Corollary 5.9. *For any $u_0 \in \mathscr{S}'(\mathbf{R}^d)$ the Cauchy problem* (CP) *has the solution $\mathscr{K}_t u_0$ in the space $C^1([0,\infty), \mathscr{S}'_{\mathrm{str}})$. The solution is unique in the space $C^1([0,\infty), \mathscr{S}'_{\mathrm{w}})$.*

Remark 5.10. A strongly continuous semigroup T_t in a locally convex space X has the following interesting property. The map $[0,\infty) \ni t \mapsto T_t u_0$ is a solution to (CP) (with $q^w(x,D)$ replaced by $-A$) in $C^1([0,\infty), X)$ when $u_0 \in D(A)$ where A denotes the generator of the semigroup [21, Proposition 1.2]. The proof in [21] uses integrals of $T_t u_0$ with respect to t over finite intervals in $[0,\infty)$. Thanks to the strong continuity such integrals are well defined as Riemann integrals. Local equicontinuity is not needed to define integrals, as is done e.g. in the proof of [34, Theorem IX.3.1]. The solution $T_t u_0$ is unique in $C^1([0,\infty), X)$ by the argument in [20, pp. 483–84].

If the space X is sequentially complete, then the domain $D(A) \subseteq X$ is dense [21, Proposition 1.3]. If the semigroup T_t is locally equicontinuous, then the generator A is a closed operator [21, Proposition 1.4].

6. Strong continuity on Gelfand–Shilov (ultradistribution) spaces

In this section we study the semigroup $\mathscr{K}_t$ acting on the Gelfand–Shilov space $\Sigma_s(\mathbf{R}^d)$ for $s > \frac{1}{2}$ and its dual space of ultradistributions $\Sigma'_s(\mathbf{R}^d)$.

We need the following lemma which is similar to [23, Theorem 6.1.6]. It is basically a special case of [22, Remark 2.1], but we provide an elementary proof in order to give a selfcontained account as a service to the reader.

Lemma 6.1. *If $s > \frac{1}{2}$, then the family of seminorms*

$$\|f\|_h \equiv \sup_{\alpha,\beta\in\mathbf{N}^d} \frac{\|x^\alpha D^\beta f\|_{L^2}}{h^{|\alpha+\beta|}(\alpha!\,\beta!)^s} \tag{6.1}$$

for $h>0$, is equivalent to the family $\{\|\cdot\|_{\mathcal{S}_{s,h}}\}_{h>0}$ as seminorms on $\Sigma_s(\mathbf{R}^d)$.

Proof. Using $(\alpha+\gamma)! \leqslant 2^{|\alpha+\gamma|}\alpha!\gamma!$ (cf. [23, Eq. (0.3.6)]) we have for $\alpha,\beta\in\mathbf{N}^d$ and $0<h\leqslant 1$, cf. (4.11),

$$\begin{aligned}\|x^\alpha D^\beta f\|_{L^2} &= \|\langle x\rangle^{-d}\langle x\rangle^d x^\alpha D^\beta f\|_{L^2} \lesssim \sum_{|\gamma|\leqslant d} \|x^{\alpha+\gamma} D^\beta f\|_{L^\infty}\\ &\leqslant \|f\|_{\mathcal{S}_{s,h}} \sum_{|\gamma|\leqslant d} h^{|\alpha+\gamma+\beta|}((\alpha+\gamma)!\,\beta!)^s \lesssim \|f\|_{\mathcal{S}_{s,h}} (2^s h)^{|\alpha+\beta|}(\alpha!\,\beta!)^s.\end{aligned}$$

This gives $\|f\|_{2^s h} \lesssim \|f\|_{\mathcal{S}_{s,h}}$, or equivalently $\|f\|_h \lesssim \|f\|_{\mathcal{S}_{s,2^{-s}h}}$ for $0<h\leqslant 2^s$. Since $\|\cdot\|_{h_1} \leqslant \|\cdot\|_{h_2}$ when $h_1 \geqslant h_2 > 0$ this shows that any seminorm $\|\cdot\|_h$ with $h>0$ can be estimated by a seminorm from $\{\|\cdot\|_{\mathcal{S}_{s,h}}\}_{h>0}$.

For an opposite estimate, again for $\alpha,\beta\in\mathbf{N}^d$ and $h>0$ we have using Fourier's inversion formula and Plancherel's identity for $x\in\mathbf{R}^d$

$$\begin{aligned}|x^\alpha D^\beta f(x)| &= \left|(2\pi)^{-\frac{d}{2}} \int_{\mathbf{R}^d} \langle\xi\rangle^{-2d}\langle\xi\rangle^{2d}\widehat{x^\alpha D^\beta f}(\xi) e^{i\langle x,\xi\rangle}\,\mathrm{d}\xi\right|\\ &\lesssim \left\|\sum_{|\gamma|\leqslant 2d} C_\gamma\,\xi^\gamma \widehat{x^\alpha D^\beta f}(\xi)\right\|_{L^2} \lesssim \sum_{|\gamma|\leqslant 2d} \left\|\mathscr{F}\left(D^\gamma\left(x^\alpha D^\beta f\right)\right)\right\|_{L^2}\\ &= \sum_{|\gamma|\leqslant 2d} \left\|D^\gamma\left(x^\alpha D^\beta f\right)\right\|_{L^2} \leqslant \sum_{|\gamma|\leqslant 2d}\ \sum_{\kappa\leqslant\min(\gamma,\alpha)} \binom{\gamma}{\kappa}\frac{\alpha!}{(\alpha-\kappa)!}\left\|x^{\alpha-\kappa}D^{\beta+\gamma-\kappa}f\right\|_{L^2}\\ &\leqslant 2^{|\alpha|}\sum_{|\gamma|\leqslant 2d}\ \sum_{\kappa\leqslant\min(\gamma,\alpha)} \binom{\gamma}{\kappa}\kappa!\left\|x^{\alpha-\kappa}D^{\beta+\gamma-\kappa}f\right\|_{L^2}\end{aligned}$$

in the last step using $\alpha! = (\alpha-\kappa+\kappa)! \leqslant (\alpha-\kappa)!\,\kappa!\,2^{|\alpha|}$.

Next we use $1 = 2s-\delta$ where $\delta>0$, and $\kappa! \geqslant |\kappa|!\,d^{-|\kappa|}$ for $\kappa\in\mathbf{N}^d$ [23, Eq. (0.3.3)] which gives

$$\kappa!^{-\delta}h^{-2|\kappa|} = \left(\frac{h^{-\frac{2|\kappa|}{\delta}}}{\kappa!}\right)^\delta \leqslant \left(\frac{\left(dh^{-\frac{2}{\delta}}\right)^{|\kappa|}}{|\kappa|!}\right)^\delta \leqslant \exp\left(\delta d h^{-\frac{2}{\delta}}\right). \tag{6.2}$$

Thus for $0<h\leqslant 1$ and $x\in\mathbf{R}^d$

$$\begin{aligned}|x^\alpha D^\beta f(x)| &\leqslant \|f\|_h\,2^{|\alpha|}\sum_{|\gamma|\leqslant 2d}\ \sum_{\kappa\leqslant\min(\gamma,\alpha)}\binom{\gamma}{\kappa}\kappa!^{2s-\delta}h^{|\alpha+\beta+\gamma-2\kappa|}\left((\alpha-\kappa)!(\beta+\gamma-\kappa)!\right)^s\\ &\leqslant \|f\|_h\,2^{|\alpha|}h^{|\alpha+\beta|}\sum_{|\gamma|\leqslant 2d}\ \sum_{\kappa\leqslant\min(\gamma,\alpha)}\binom{\gamma}{\kappa}\kappa!^{-\delta}h^{-2|\kappa|}\left(\alpha!(\beta+\gamma)!\right)^s\\ &\lesssim \|f\|_h\,(2h)^{|\alpha+\beta|}\,(\alpha!\beta!)^s\sum_{|\gamma|\leqslant 2d}\ \sum_{\kappa\leqslant\min(\gamma,\alpha)}\binom{\gamma}{\kappa}2^{s|\beta|}\\ &\lesssim \|f\|_h\,(2^{1+s}h)^{|\alpha+\beta|}\,(\alpha!\beta!)^s\end{aligned}$$

which gives $\|f\|_{\mathcal{S}_{s,2^{1+s}h}} \lesssim \|f\|_h$, or equivalently $\|f\|_{\mathcal{S}_{s,h}} \lesssim \|f\|_{2^{-1-s}h}$ for any $0 < h \leqslant 2^{1+s}$. Since $\|\cdot\|_{\mathcal{S}_{s,h_1}} \leqslant \|\cdot\|_{\mathcal{S}_{s,h_2}}$ when $h_1 \geqslant h_2 > 0$ this shows that any seminorm $\|\cdot\|_{\mathcal{S}_{s,h}}$ can be estimated by a seminorm from $\{\|\cdot\|_h\}_{h>0}$. $\square$

The next project is to prove the fundamental Theorem 6.7 which shows that $\mathscr{K}_t$ is uniformly continuous on $\Sigma_s(\mathbf{R}^d)$ for $0 \leqslant t \leqslant T$, for any $T > 0$. In order to prove it we need several auxiliary results. First we study the derivatives of a Gaussian type function $g_\lambda(x) = e^{\lambda x^2/2}$ for $x \in \mathbf{R}$ and $\lambda \in \mathbf{C}$. It is clear that

$$\partial^k g_\lambda(x) = p_{\lambda,k}(x)\, g_\lambda(x) \tag{6.3}$$

where $p_{\lambda,k}$ is a polynomial of order $k \in \mathbf{N}$. This polynomial is essentially a rescaled Hermite polynomial with complex argument [32].

Lemma 6.2. *Suppose $g_\lambda(x) = e^{\lambda x^2/2}$ for $x \in \mathbf{R}$ and $\lambda \in \mathbf{C}$, let $p_{\lambda,k}$ be the polynomial defined in* (6.3) *for $k \in \mathbf{N}$, and let $s > \frac{1}{2}$. For each $\mu > 0$ there exists $0 < \delta \leqslant 1$ such that $p_{\lambda,k}$ satisfy the following estimates provided $|\lambda| \leqslant \delta$: For any $h > 0$*

$$|p_{\lambda,k}(x)| \lesssim h^k k!^s e^{\mu\, h^{-\frac{1}{s}} |x|^{\frac{1}{s}}}, \quad x \in \mathbf{R}, \ \ k \in \mathbf{N}.$$

Proof. By a straightforward induction argument one may confirm the formula (cf. [32, Eq. (5.5.4)])

$$p_{\lambda,k}(x) = k! \sum_{m=0}^{\lfloor k/2 \rfloor} \frac{x^{k-2m} \lambda^{k-m}}{m!(k-2m)!2^m}.$$

Since $k! \leqslant 2^k (k-2m)!(2m)!$ we can estimate $|p_{\lambda,k}(x)|$ as

$$|p_{\lambda,k}(x)| \leqslant \sum_{m=0}^{\lfloor k/2 \rfloor} \frac{(|\lambda|^{\frac{1}{2}}\,|x|)^{k-2m}(2m)!}{m!2^{m-k}} \leqslant \sum_{m=0}^{\lfloor k/2 \rfloor} (\delta^{\frac{1}{2}}|x|)^{k-2m} m! 2^{m+k}.$$

Combining with $m! = m!^{2s-\varepsilon}$ where $\varepsilon = 2s - 1 > 0$, this gives for any $h > 0$ and $b > 0$

$$\begin{aligned}
|p_{\lambda,k}(x)| h^{-k} k!^{-s} &\leqslant \sum_{m=0}^{\lfloor k/2 \rfloor} (\delta^{\frac{1}{2}}\,|x|)^{k-2m} m!^{2s-\varepsilon} 2^{m+k} h^{-k} k!^{-s} \\
&= \sum_{m=0}^{\lfloor k/2 \rfloor} \left(\frac{\left(\frac{b}{s} (\delta^{\frac{1}{2}}\,|x|)^{\frac{1}{s}} \right)^{k-2m}}{(k-2m)!} \right)^s \left(\frac{b}{s} \right)^{s(2m-k)} \left(\frac{(k-2m)! m!^2}{k!} \right)^s \frac{2^{m+k} h^{-k}}{m!^{\varepsilon}} \\
&\leqslant e^{b\, \delta^{\frac{1}{2s}} |x|^{\frac{1}{s}}} \left(2 \left(\frac{s}{b} \right)^s h^{-1} \right)^k \sum_{m=0}^{\lfloor k/2 \rfloor} \left(\frac{\left(2 \left(\frac{b}{s} \right)^{2s} \right)^{\frac{m}{\varepsilon}}}{m!} \right)^{\varepsilon} \\
&\leqslant e^{\varepsilon \left(2 \left(\frac{b}{s} \right)^{2s} \right)^{\frac{1}{\varepsilon}}} e^{b\, \delta^{\frac{1}{2s}} |x|^{\frac{1}{s}}} \left(4 \left(\frac{s}{b} \right)^s h^{-1} \right)^k = C_{s,b}\, e^{b\, \delta^{\frac{1}{2s}} |x|^{\frac{1}{s}}},
\end{aligned}$$

where $C_{s,b} > 0$, provided $b = s\, 4^{\frac{1}{s}} h^{-\frac{1}{s}}$. Thus if $\delta \leqslant 4^{-2} \left(\frac{\mu}{s} \right)^{2s}$, then

$$b\, \delta^{\frac{1}{2s}} = s\, 4^{\frac{1}{s}} \delta^{\frac{1}{2s}} h^{-\frac{1}{s}} \leqslant \mu h^{-\frac{1}{s}}$$

and therefore

$$|p_{\lambda,k}(x)| \lesssim h^k k!^s e^{\mu\, h^{-\frac{1}{s}} |x|^{\frac{1}{s}}}. \qquad \square$$

Corollary 6.3. *Let $\lambda > 0$ and $s > \frac{1}{2}$. Suppose $\Lambda \in \mathbf{C}^{2d\times 2d}$ is a diagonal matrix with entries λ_j that are bounded as $|\lambda_j| \leqslant \lambda$ for all $1 \leqslant j \leqslant 2d$. If $g(z) = e^{\frac{1}{2}\langle \Lambda z, z\rangle}$, $z \in \mathbf{R}^{2d}$, then*

$$\partial^\alpha g(z) = p_{\Lambda,\alpha}(z) g(z), \quad \alpha \in \mathbf{N}^{2d}, \tag{6.4}$$

where $p_{\Lambda,\alpha}$ are polynomials of order $|\alpha|$. For each $\mu > 0$ there exists $0 < \delta \leqslant 1$ such that the polynomials $p_{\Lambda,\alpha}$ satisfy the following estimates provided $\lambda \leqslant \delta$: For any $h > 0$

$$|p_{\Lambda,\alpha}(z)| \lesssim h^{|\alpha|} \alpha!^s e^{\mu\, h^{-\frac{1}{s}} |z|^{\frac{1}{s}}}, \quad z \in \mathbf{R}^{2d}, \ \alpha \in \mathbf{N}^{2d}. \tag{6.5}$$

Proposition 6.4. *Let $\lambda > 0$ and $\varepsilon > 0$. Suppose $T_t \in \mathbf{C}^{2d\times 2d}$, $0 \leqslant t \leqslant \varepsilon$, is a parametrized family of symmetric matrices such that for all $t \in [0,\varepsilon]$ we have $\operatorname{Re} T_t \leqslant 0$, and $\operatorname{Re} T_t$ and $\operatorname{Im} T_t$ both have eigenvalues in the interval $[-\lambda, \lambda]$. Let $a_t(z) = e^{\frac{1}{2}\langle T_t z, z\rangle}$, $z \in \mathbf{R}^{2d}$ and let $s > \frac{1}{2}$. For each $\mu > 0$ there exists $\delta > 0$ such that if $\lambda \leqslant \delta$, then for any $h > 0$*

$$|\partial^\alpha a_t(z)| \lesssim h^{|\alpha|} \alpha!^s e^{\mu\, h^{-\frac{1}{s}} |z|^{\frac{1}{s}}}, \quad z \in \mathbf{R}^{2d}, \ \alpha \in \mathbf{N}^{2d}, \ 0 \leqslant t \leqslant \varepsilon. \tag{6.6}$$

Proof. We may factorize $\operatorname{Re} T_t = U_t^T \Lambda_t U_t$ where $U_t \in \mathrm{O}(2d)$ and $\Lambda_t \in \mathbf{R}^{2d\times 2d}$ is diagonal, with the non-positive eigenvalues of $\operatorname{Re} T_t$ on the diagonal. The coefficients of U_t satisfy the bound

$$|(U_t)_{j,k}| \leqslant \|U_t\| = 1, \quad 1 \leqslant j,k \leqslant 2d, \tag{6.7}$$

where $\|U_t\|$ denotes the operator matrix norm.

Thus $a_{t,1}(z) = e^{\frac{1}{2}\langle \operatorname{Re} T_t z, z\rangle} = g_{t,1}(U_t z)$ where $g_{t,1}$ satisfies the assumptions of Corollary 6.3. We pick $\delta > 0$ so that the polynomials $p_{\Lambda,\alpha,t,1}$, that correspond to $g_{t,1}$ as in (6.4), satisfy (6.5) with μ replaced by $\mu_1 = \mu\, 2^{-2-\frac{2}{s}} d^{-1-\frac{1}{s}}$. We have

$$\partial_j a_{t,1}(z) = \sum_{k=1}^{2d} (U_t)_{k,j} \partial_k g_{t,1}(U_t x), \quad 1 \leqslant j \leqslant 2d.$$

Taking into account (6.7), it follows that we may express $\partial^\alpha a_{t,1}(z)$ for $\alpha \in \mathbf{N}^{2d}$ as a sum of $(2d)^{|\alpha|}$ terms, consisting of coefficients the modulus of which are upper bounded by one, times $\partial^\beta g_{t,1}(U_t x)$ where $\beta \in \mathbf{N}^{2d}$ satisfies $|\beta| = |\alpha|$.

Let $h > 0$. We obtain using Corollary 6.3, [23, Eq. (0.3.3)] and the assumption $\operatorname{Re} T_t \leqslant 0$

$$\begin{aligned}
|\partial^\alpha a_{t,1}(z)| &\leqslant (2d)^{|\alpha|} \max_{|\beta|=|\alpha|} |\partial^\beta g_{t,1}(U_t z)| \\
&\lesssim (2dh)^{|\alpha|} |\alpha|!^s e^{\mu_1\, h^{-\frac{1}{s}} |U_t z|^{\frac{1}{s}}} |g_{t,1}(U_t z)| \\
&\leqslant ((2d)^{1+s} h)^{|\alpha|} \alpha!^s e^{\mu_1\, h^{-\frac{1}{s}} |z|^{\frac{1}{s}}} e^{\frac{1}{2}\langle \operatorname{Re} T_t z, z\rangle} \\
&\leqslant ((2d)^{1+s} h)^{|\alpha|} \alpha!^s e^{\mu_1\, h^{-\frac{1}{s}} |z|^{\frac{1}{s}}}, \quad 0 \leqslant t \leqslant \varepsilon.
\end{aligned}$$

We apply the same argument to $a_{t,2}(z) = e^{\frac{i}{2}\langle \operatorname{Im} T_t z, z\rangle}$. This gives new matrices $U_t \in \mathrm{O}(2d)$ and $a_{t,2}(z) = g_{t,2}(U_t z)$ where $g_{t,2}$ again satisfies the assumptions of Corollary 6.3. We obtain

$$\begin{aligned}
|\partial^\alpha a_{t,2}(z)| &\lesssim ((2d)^{1+s} h)^{|\alpha|} \alpha!^s e^{\mu_1\, h^{-\frac{1}{s}} |z|^{\frac{1}{s}}} |a_{t,2}(z)| \\
&= ((2d)^{1+s} h)^{|\alpha|} \alpha!^s e^{\mu_1\, h^{-\frac{1}{s}} |z|^{\frac{1}{s}}}, \quad 0 \leqslant t \leqslant \varepsilon.
\end{aligned}$$

Finally Leibniz' rule gives

$$\begin{aligned}|\partial^\alpha a_t(z)| = |\partial^\alpha (a_{t,1}(z)\, a_{t,2}(z))| &\leqslant \sum_{\beta\leqslant\alpha} \binom{\alpha}{\beta} |\partial^{\alpha-\beta} a_{t,1}(z)|\, |\partial^\beta a_{t,2}(z)| \\ &\lesssim \sum_{\beta\leqslant\alpha} \binom{\alpha}{\beta} ((2d)^{1+s}h)^{|\alpha-\beta|+|\beta|} (\alpha-\beta)!^s \beta!^s e^{2\mu_1 h^{-\frac{1}{s}}|z|^{\frac{1}{s}}} \\ &\leqslant (2^{2+s} d^{1+s} h)^{|\alpha|} \alpha!^s e^{2\mu_1 h^{-\frac{1}{s}}|z|^{\frac{1}{s}}}, \quad 0 \leqslant t \leqslant \varepsilon.\end{aligned}$$

The result now follows by replacing $2^{2+s}d^{1+s}h$ by h. □

Lemma 6.5. *Let $\varepsilon > 0$ and $s > \frac{1}{2}$. Suppose that $a_t \in C^\infty(\mathbf{R}^{2d})$ is a family of functions parametrized by $t \in [0,\varepsilon]$ that for any $h > 0$ satisfy the estimates*

$$|\partial^\alpha a_t(z)| \lesssim h^{|\alpha|} \alpha!^s e^{\mu h^{-\frac{1}{s}}|z|^{\frac{1}{s}}}, \quad z \in \mathbf{R}^{2d}, \ \alpha \in \mathbf{N}^{2d}, \ 0 \leqslant t \leqslant \varepsilon,$$

where $\mu = s\, 2^{-4-\frac{3}{2s}} d^{-\frac{1}{2s}}$. Let $\Phi \in \Sigma_s(\mathbf{R}^{2d}) \setminus 0$. Then for any $b > 0$ there exists $C_b > 0$ such that

$$|V_\Phi a_t(z,\zeta)| \leqslant C_b\, e^{\frac{b}{4}|z|^{\frac{1}{s}} - b|\zeta|^{\frac{1}{s}}}, \quad z,\zeta \in \mathbf{R}^{2d}, \ 0 \leqslant t \leqslant \varepsilon.$$

Proof. We will use the fact that

$$f \mapsto \sup_{x\in\mathbf{R}^d, \ \beta\in\mathbf{N}^d} \beta!^{-s} A^{|\beta|} e^{A|x|^{\frac{1}{s}}} |\partial^\beta f(x)|$$

for all $A > 0$ is a family of seminorms for $\Sigma_s(\mathbf{R}^d)$, equivalent to (2.14) for all $h > 0$ (cf. [4, Proposition 3.1]).

Integration by parts and (2.19) gives for any $h_1, h_2 > 0$

$$\begin{aligned}|\zeta^\alpha V_\Phi a_t(z,\zeta)| &= (2\pi)^{-d} \left| \int_{\mathbf{R}^{2d}} a_t(w) \partial_w^\alpha \left(e^{-i\langle\zeta,w\rangle}\right) \overline{\Phi(w-z)}\, \mathrm{d}w \right| \\ &\leqslant (2\pi)^{-d} \sum_{\beta\leqslant\alpha} \binom{\alpha}{\beta} \int_{\mathbf{R}^{2d}} \left|\partial^\beta a_t(w)\right| \left|\partial^{\alpha-\beta}\Phi(w-z)\right| \mathrm{d}w \\ &\lesssim \sum_{\beta\leqslant\alpha} \binom{\alpha}{\beta} h_1^{|\beta|} h_2^{|\alpha-\beta|} \beta!^s (\alpha-\beta)!^s \int_{\mathbf{R}^{2d}} e^{\mu h_1^{-\frac{1}{s}}|w|^{\frac{1}{s}}} e^{-h_2^{-1}|w-z|^{\frac{1}{s}}}\, \mathrm{d}w \\ &\leqslant \alpha!^s e^{2\mu h_1^{-\frac{1}{s}}|z|^{\frac{1}{s}}} \sum_{\beta\leqslant\alpha} \binom{\alpha}{\beta} h_1^{|\beta|} h_2^{|\alpha-\beta|} \int_{\mathbf{R}^{2d}} e^{(2\mu h_1^{-\frac{1}{s}} - h_2^{-1})|w-z|^{\frac{1}{s}}}\, \mathrm{d}w \\ &\lesssim \alpha!^s (h_1+h_2)^{|\alpha|} e^{2\mu h_1^{-\frac{1}{s}}|z|^{\frac{1}{s}}}, \quad z,\zeta \in \mathbf{R}^{2d}, \quad \alpha \in \mathbf{N}^{2d}, \quad 0 \leqslant t \leqslant \varepsilon,\end{aligned}$$

provided $h_2^{-1} > 2\mu\, h_1^{-\frac{1}{s}}$.

Let $b > 0$. Using

$$|\zeta|^n \leqslant (2d)^{\frac{n}{2}} \max_{|\alpha|=n} |\zeta^\alpha|$$

we obtain

$$\begin{aligned} e^{\frac{b}{s}|\zeta|^{\frac{1}{s}}} |V_\Phi a_t(z,\zeta)|^{\frac{1}{s}} &= \sum_{n=0}^\infty 2^{-n} n!^{-1} \left(\frac{2b}{s} |\zeta|^{\frac{1}{s}} \right)^n |V_\Phi a_t(z,\zeta)|^{\frac{1}{s}} \\ &\leqslant 2 \left(\sup_{n \geqslant 0} n!^{-s} \left(\left(\frac{2b}{s} \right)^s |\zeta| \right)^n |V_\Phi a_t(z,\zeta)| \right)^{\frac{1}{s}} \\ &\lesssim \left(\sup_{n \geqslant 0} \left(\left(\frac{2b}{s} \right)^s (2d)^{\frac{1}{2}} \right)^n \max_{|\alpha|=n} \frac{|\zeta^\alpha V_\Phi a_t(z,\zeta)|}{n!^s} \right)^{\frac{1}{s}} \\ &\lesssim e^{\frac{1}{s} 2\mu h_1^{-\frac{1}{s}} |z|^{\frac{1}{s}}} \left(\sup_{n \geqslant 0} \left(\left(\frac{2b}{s} \right)^s (2d)^{\frac{1}{2}} (h_1 + h_2) \right)^n \right)^{\frac{1}{s}}. \end{aligned}$$

The result now follows provided the following three conditions are true:

$$2\mu\, h_1^{-\frac{1}{s}} = \frac{b}{4}, \tag{6.8}$$

$$h_2^{-1} > 2\mu\, h_1^{-\frac{1}{s}}, \tag{6.9}$$

$$\left(\frac{2b}{s} \right)^s (2d)^{\frac{1}{2}} (h_1 + h_2) \leqslant 1. \tag{6.10}$$

We first pick

$$h_1 = \frac{1}{2} \left(\frac{s}{2b} \right)^s (2d)^{-\frac{1}{2}} = s^s 2^{-\frac{3}{2}-s} d^{-\frac{1}{2}} b^{-s}$$

which means that (6.8) is satisfied. Since

$$\left(\frac{2b}{s} \right)^s (2d)^{\frac{1}{2}} h_1 = \frac{1}{2},$$

we may pick $h_2 > 0$ sufficiently small so that (6.9) and (6.10) are satisfied. □

Finally we are in a position to prove that estimates for a family of symbols as required in Lemma 6.5 give rise to operators that are uniformly bounded on $\Sigma_s(\mathbf{R}^d)$. It is interesting to compare this result with [3, Theorem 4.10]. The conditions that are sufficient for continuity given in [3, Theorem 4.10] and here are quite similar, but neither condition implies the other.

Proposition 6.6. *Suppose $s > \frac{1}{2}$ and $\varepsilon > 0$. Let $a_t \in C^\infty(\mathbf{R}^{2d})$ be a family of functions parametrized by $t \in [0,\varepsilon]$, that for any $h > 0$ satisfy the estimates*

$$|\partial^\alpha a_t(z)| \lesssim h^{|\alpha|} \alpha!^s e^{\mu\, h^{-\frac{1}{s}} |z|^{\frac{1}{s}}}, \quad z \in \mathbf{R}^{2d},\ \alpha \in \mathbf{N}^{2d},\ 0 \leqslant t \leqslant \varepsilon,$$

where $\mu = s\, 2^{-4-\frac{3}{2s}} d^{-\frac{1}{2s}}$. Then for any $h > 0$ there exists $h_1 = h_1(h) > 0$ and $C = C_h > 0$ such that

$$\|a_t^w(x,D) f\|_h \leqslant C \|f\|_{h_1}, \quad 0 \leqslant t \leqslant \varepsilon,\ \ f \in \Sigma_s(\mathbf{R}^d).$$

Proof. Let $\varphi \in \Sigma_s(\mathbf{R}^d)$ be such that $\Phi = W(\varphi,\varphi) \in \Sigma_s(\mathbf{R}^{2d})$ satisfies $\|\Phi\|_{L^2} = 1$. We use the Weyl quantization formula (2.7), involving the Wigner distribution (2.8), and (2.1). This gives for $f, g \in \Sigma_s(\mathbf{R}^d)$ and $w \in \mathbf{R}^{2d}$

$$\begin{aligned} (a_t^w(x,D) f, \Pi(w) g) &= (2\pi)^{-\frac{d}{2}} (a_t, W(\Pi(w) g, f)) \\ &= (2\pi)^{-\frac{d}{2}} (V_\Phi a_t, V_\Phi W(\Pi(w) g, f)). \end{aligned} \tag{6.11}$$

Since $\Phi = W(\varphi,\varphi)$ we obtain from [12, Lemma 14.5.1 and Lemma 3.1.3]

$$\begin{aligned}
|V_\Phi W(\Pi(w)\varphi, f)(z,\zeta)| &= \left| V_\varphi f\left(z+\frac{1}{2}\mathcal{J}\zeta\right)\right| \left| V_\varphi\left(\Pi(w)\varphi\right)\left(z-\frac{1}{2}\mathcal{J}\zeta\right)\right| \\
&= \left| V_\varphi f\left(z+\frac{1}{2}\mathcal{J}\zeta\right)\right| \left| V_\varphi \varphi\left(z-w-\frac{1}{2}\mathcal{J}\zeta\right)\right|.
\end{aligned}$$

Inserting this into (6.11), using Lemma 6.5 and (2.19) we obtain for any $b>0$

$$\begin{aligned}
&|V_\varphi(a_t^w(x,D)f)(w)| = (2\pi)^{-\frac{d}{2}}|(a_t^w(x,D)f, \Pi(w)\varphi)| \\
&\leqslant (2\pi)^{-d}\int_{\mathbf{R}^{4d}} |V_\Phi a_t(z,\zeta)|\,|V_\Phi W(\Pi(w)\varphi,f)(z,\zeta)|\,\mathrm{d}z\,\mathrm{d}\,\zeta \\
&\leqslant C_b \int_{\mathbf{R}^{4d}} e^{\frac{b}{4}|z|^{\frac{1}{s}}-b|\zeta|^{\frac{1}{s}}} \left| V_\varphi f\left(z+\frac{1}{2}\mathcal{J}\zeta\right)\right| \left| V_\varphi\varphi\left(z-w-\frac{1}{2}\mathcal{J}\zeta\right)\right| \mathrm{d}z\,\mathrm{d}\zeta \qquad (6.12)\\
&= C_b \int_{\mathbf{R}^{4d}} e^{\frac{b}{4}\left|z-\frac{1}{2}\mathcal{J}\zeta\right|^{\frac{1}{s}}-b|\zeta|^{\frac{1}{s}}} \,|V_\varphi f(z)|\,|V_\varphi\varphi\left(z-w-\mathcal{J}\zeta\right)|\,\mathrm{d}z\,\mathrm{d}\zeta \\
&\leqslant C_b \int_{\mathbf{R}^{4d}} e^{\frac{b}{2}|z|^{\frac{1}{s}}-b\left(1-2^{-1-\frac{1}{s}}\right)|\zeta|^{\frac{1}{s}}} \,|V_\varphi f(z)|\,|V_\varphi\varphi\left(z-w-\mathcal{J}\zeta\right)|\,\mathrm{d}z\,\mathrm{d}\zeta.
\end{aligned}$$

The estimate is uniform with respect to $t\in[0,\varepsilon]$.

Next we use the seminorms on $\Sigma_s(\mathbf{R}^d)$ defined by

$$\Sigma_s(\mathbf{R}^d) \ni f \mapsto \|f\|''_A = \sup_{z\in\mathbf{R}^{2d}} e^{A|z|^{\frac{1}{s}}}|V_\varphi f(z)|, \quad A>0, \qquad (6.13)$$

where $\varphi \in \Sigma_s(\mathbf{R}^d)\setminus\{0\}$ is fixed but arbitrary (cf. [4, Propositon 3.1]). Using $2^{-1-\frac{1}{s}} < 2^{-1}$ and again (2.19) we obtain for any $a>0$

$$\begin{aligned}
&|V_\varphi(a_t^w(x,D)f)(w)| \\
&\leqslant C_b\|f\|''_{b+2a}\|\varphi\|''_{4a}\int_{\mathbf{R}^{4d}} e^{-\left(\frac{b}{2}+2a\right)|z|^{\frac{1}{s}}-b\left(1-2^{-1-\frac{1}{s}}\right)|\zeta|^{\frac{1}{s}}-4a|z-w-\mathcal{J}\zeta|^{\frac{1}{s}}}\,\mathrm{d}z\,\mathrm{d}\zeta \\
&\leqslant C_b\|f\|''_{b+2a}\|\varphi\|''_{4a}\int_{\mathbf{R}^{4d}} e^{-\left(\frac{b}{2}+2a\right)|z|^{\frac{1}{s}}-\frac{b}{2}|\zeta|^{\frac{1}{s}}-4a|z-w-\mathcal{J}\zeta|^{\frac{1}{s}}}\,\mathrm{d}z\,\mathrm{d}\zeta \qquad (6.14)\\
&\leqslant C_b\|f\|''_{b+2a}\|\varphi\|''_{4a}\int_{\mathbf{R}^{4d}} e^{-\left(\frac{b}{2}+2a\right)|z|^{\frac{1}{s}}-\left(\frac{b}{2}-4a\right)|\zeta|^{\frac{1}{s}}-2a|z-w|^{\frac{1}{s}}}\,\mathrm{d}z\,\mathrm{d}\zeta \\
&\leqslant C_b\|f\|''_{b+2a}\|\varphi\|''_{4a}\, e^{-a|w|^{\frac{1}{s}}}\int_{\mathbf{R}^{4d}} e^{-\frac{b}{2}|z|^{\frac{1}{s}}-\left(\frac{b}{2}-4a\right)|\zeta|^{\frac{1}{s}}}\,\mathrm{d}z\,\mathrm{d}\zeta, \quad w\in\mathbf{R}^{2d}.
\end{aligned}$$

Let $B>0$ be arbitrary. If we first pick $a \geqslant B$ and then $b>8a$ we obtain

$$\|a_t^w(x,D)f\|''_B = \sup_{w\in\mathbf{R}^{2d}} e^{B|w|^{\frac{1}{s}}}|V_\varphi(a_t^w(x,D)f)(w)| \leqslant C\|f\|''_{b+2a} \qquad (6.15)$$

for a constant $C>0$ and for all $t\in[0,\varepsilon]$.

Finally we combine Lemma 6.1 and [4, Proposition 3.1], which admits the conclusion that the seminorms (6.13) are equivalent to the seminorms $\|\cdot\|_h$ for $h>0$, defined in (6.1). This implies the claim. □

We have reached a point at which we may prove the theorem for which Lemma 6.2, Corollary 6.3, Proposition 6.4, Lemma 6.5, and Proposition 6.6 are preparations.

Theorem 6.7. *Let* $\operatorname{Re} Q \geqslant 0$, $s>\frac{1}{2}$ *and* $T>0$. *For every* $h>0$ *there exists* $h_1 = h_1(h) > 0$ *and* $C = C_{T,h}>0$ *such that*

$$\|\mathscr{K}_t f\|_h \leqslant C\|f\|_{h_1}, \quad 0\leqslant t\leqslant T, \;\; f\in\Sigma_s(\mathbf{R}^d).$$

Proof. It suffices to show the following statement. There exists $\varepsilon > 0$ such that for any $h > 0$ there exists $h_1 = h_1(h) > 0$ and $C = C(h) > 0$ such that

$$\|\mathscr{K}_t f\|_h \leqslant C\|f\|_{h_1}, \quad 0 \leqslant t \leqslant \varepsilon. \tag{6.16}$$

In fact, suppose that (6.16) holds, for given $\varepsilon > 0$, all $h > 0$ and some $C, h_1 > 0$. Take $n \in \mathbf{N}$ such that $n \geqslant T\varepsilon^{-1}$, which implies $t/n \leqslant \varepsilon$ for $0 \leqslant t \leqslant T$. We use the semigroup property $\mathscr{K}_{t_1+t_2} = \mathscr{K}_{t_1}\mathscr{K}_{t_2}$ for $t_1, t_2 \geqslant 0$. Thus we obtain from (6.16) the existence of $C_1, C_2, \cdots, C_n > 0$ and $h_1, h_2, \cdots, h_n > 0$

$$\begin{aligned}\|\mathscr{K}_t f\|_h = \|(\mathscr{K}_{t/n})^n f\|_h &\leqslant C_1\|(\mathscr{K}_{t/n})^{n-1} f\|_{h_1} \leqslant C_1C_2\|(\mathscr{K}_{t/n})^{n-2} f\|_{h_2} \\ &\leqslant C_1C_2\cdots C_n\|f\|_{h_n}, \quad 0 \leqslant t \leqslant T,\end{aligned}$$

which implies the claim of the theorem.

Thus we may concentrate on the proof of (6.16) for some $\varepsilon > 0$, and for all $h > 0$, some $h_1 = h_1(h) > 0$ and some $C = C(h) > 0$. We express $\mathscr{K}_t$ as a Weyl operator (2.6) as $\mathscr{K}_t = a_t^w(x, D)$. Then we can benefit from Hörmander's [19, Theorem 4.3] explicit formula for the Weyl symbol

$$a_t(z) = (\det(\cos(tF)))^{-\frac{1}{2}} \exp\left(\sigma(\tan(tF)z, z)\right), \quad z \in \mathbf{R}^{2d},$$

where $F = \mathcal{J}Q$ and $\tan(tF) = \sin(tF)(\cos(tF))^{-1}$, which is valid for all $t \geqslant 0$ such that $\det(\cos(tF)) \neq 0$. According to [19, Theorem 4.1], $\det(\cos(tF)) \neq 0$ unless $t\lambda \in \pi\left(\frac{1}{2} + \mathbf{Z}\right)$ where $\lambda \in \mathbf{C}$ is an eigenvalue of F. Clearly it is possible to pick $\varepsilon > 0$ such that $\det(\cos(tF)) \neq 0$ for $0 \leqslant t \leqslant \varepsilon$.

The exponent of a_t is

$$\sigma(\tan(tF)z, z) = \langle \mathcal{J}\tan(tF)z, z\rangle = \frac{1}{2}\langle T_t z, z\rangle$$

where the symmetric matrix $T_t \in \mathbf{C}^{2d\times 2d}$ is

$$T_t = \mathcal{J}\tan(tF) - (\tan(tF))^T\mathcal{J}$$

due to $\mathcal{J}^T = -\mathcal{J}$.

Since $\cos(tF) \to I$ as $t \to 0^+$ we may assume that the factor $(\det(\cos(tF)))^{-\frac{1}{2}}$ satisfies

$$(\det(\cos(tF)))^{-\frac{1}{2}} \leqslant 2, \quad 0 \leqslant t \leqslant \varepsilon,$$

after possibly decreasing $\varepsilon > 0$.

According to [19, Theorem 4.6] we have $\operatorname{Re} T_t \leqslant 0$ for $t \in [0, \varepsilon]$. Since $T_t \to 0$ as $t \to 0^+$, we may assume that $\operatorname{Re} T_t$ and $\operatorname{Im} T_t$ both have small eigenvalues, uniformly over $t \in [0, \varepsilon]$, again after possibly decreasing $\varepsilon > 0$. Specifically we assume that the eigenvalues belong to $[-\delta, \delta]$ for $t \in [0, \varepsilon]$, where $\delta > 0$ is chosen small enough to guarantee by Proposition 6.4 that the estimates (6.6) hold for all $h > 0$ with $\mu = s\,2^{-4-\frac{3}{2s}} d^{-\frac{1}{2s}}$. The claim is now a consequence of Proposition 6.6. □

By Theorem 6.7 we may extend $\mathscr{K}_t$ uniquely from the domain $\Sigma_s(\mathbf{R}^d)$ to $\Sigma_s'(\mathbf{R}^d)$ by the assignment

$$(\mathscr{K}_t u, \varphi) = (u, \mathscr{K}_t^* \varphi) = (u, \mathscr{K}_{e^{-2it\overline{F}}}\varphi), \quad u \in \Sigma_s'(\mathbf{R}^d), \ \varphi \in \Sigma_s(\mathbf{R}^d). \tag{6.17}$$

Corollary 6.8. *If $s > \frac{1}{2}$ and $t \geqslant 0$, then $\mathscr{K}_t$ is a continuous linear operator on $\Sigma_s(\mathbf{R}^d)$, that extends uniquely to a continuous linear operator on $\Sigma_s'(\mathbf{R}^d)$ equipped with its weak* topology.*

Theorem 6.7 implies in particular that $\mathscr{K}_t = \mathscr{K}_{e^{-2itF}} \colon \Sigma_s(\mathbf{R}^d) \to \Sigma_s(\mathbf{R}^d)$ is continuous for each fixed $t \geqslant 0$.

Remark 6.9. The continuity of $\mathscr{K}_t\colon \Sigma_s(\mathbf{R}^d) \to \Sigma_s(\mathbf{R}^d)$ can be generalized as follows. The operator $\mathscr{K}_T\colon \Sigma_s(\mathbf{R}^d) \to \Sigma_s(\mathbf{R}^d)$ is continuous for any matrix $T \in \mathrm{Sp}(d,\mathbf{C})$ which is positive in the sense of

$$i(\sigma(\overline{TX}, TX) - \sigma(\overline{X}, X)) \geqslant 0, \quad X \in T^*\mathbf{C}^d, \tag{6.18}$$

(cf. [19]), where $\mathscr{K}_T$ is the operator with kernel K_T defined as in (3.2) with e^{-2itF} replaced by T. Condition (6.18) means that the graph of T is a positive Lagrangian in $T^*\mathbf{C}^d \times T^*\mathbf{C}^d$. The operator $\mathscr{K}_t$ with kernel $K_{e^{-2itF}}$ defined by the oscillatory integral kernel (3.2) is a particular case with $T = e^{-2itF}$. The matrix e^{-2itF} is a positive matrix in $\mathrm{Sp}(d,\mathbf{C})$ according to [27, Lemma 5.2].

This generalization of Theorem 6.7 has been stated in [4, Proposition 8.1]. The proof there is unfortunately wrong but it has been corrected [5].

The next result is a Gelfand–Shilov version of Lemma 4.2.

Lemma 6.10. *If $B \subseteq \Sigma_s(\mathbf{R}^d)$ is bounded and $N > 0$ is an integer, then*

$$\{x^\gamma D^\kappa f,\ f \in B,\ |\gamma+\kappa| < N\} \subseteq \Sigma_s(\mathbf{R}^d)$$

is also bounded.

Proof. Using the seminorms (6.1) the assumption means that

$$\sup_{f\in B} \|f\|_h = C_h < \infty \quad \forall h > 0. \tag{6.19}$$

We have for $f \in B$ and $\alpha,\beta,\gamma,\kappa \in \mathbf{N}^d$

$$\begin{aligned}\left\|x^\alpha D^\beta\left(x^\gamma D^\kappa f\right)\right\|_{L^2} &= \left\|\sum_{\sigma\leqslant\min(\beta,\gamma)} \binom{\beta}{\sigma}\frac{\gamma!\, i^{-|\sigma|}}{(\gamma-\sigma)!} x^{\alpha+\gamma-\sigma} D^{\kappa+\beta-\sigma} f\right\|_{L^2} \\ &\leqslant \sum_{\sigma\leqslant\min(\beta,\gamma)} \binom{\beta}{\sigma} \sigma!\, 2^{|\gamma|} \left\|x^{\alpha+\gamma-\sigma} D^{\kappa+\beta-\sigma} f\right\|_{L^2}.\end{aligned}$$

As in the proof of Lemma 6.1 we next use $1 = 2s - \delta$ where $\delta > 0$. Let $h > 0$. Since $\|\cdot\|_{h_1} \leqslant \|\cdot\|_{h_2}$ when $h_1 \geqslant h_2 > 0$ we may assume that $h \leqslant 1$. Provided $|\gamma+\kappa| < N$ we obtain using (6.2) and (6.19)

$$\begin{aligned}\left\|x^\alpha D^\beta\left(x^\gamma D^\kappa f\right)\right\|_{L^2} &\leqslant 2^N \sum_{\sigma\leqslant\min(\beta,\gamma)} \binom{\beta}{\sigma} \sigma!^{2s-\delta} \left\|x^{\alpha+\gamma-\sigma} D^{\kappa+\beta-\sigma} f\right\|_{L^2} \\ &\leqslant 2^N C_h \sum_{\sigma\leqslant\min(\beta,\gamma)} \binom{\beta}{\sigma} \sigma!^{2s-\delta} h^{|\alpha+\beta+\gamma+\kappa-2\sigma|}((\alpha+\gamma-\sigma)!(\kappa+\beta-\sigma)!)^s \\ &\leqslant 2^N C_h\, h^{|\alpha+\beta|} \sum_{\sigma\leqslant\min(\beta,\gamma)} \binom{\beta}{\sigma} \sigma!^{-\delta} h^{-2|\sigma|}((\alpha+\gamma)!(\kappa+\beta)!)^s \\ &\leqslant 2^N C_{\delta,d,h}\, C_h\, h^{|\alpha+\beta|}(\alpha!\beta!)^s \sum_{\sigma\leqslant\min(\beta,\gamma)} \binom{\beta}{\sigma} 2^{s|\alpha+\beta+\gamma+\kappa|}(\gamma!\kappa!)^s \\ &\leqslant C_N\, C_{\delta,d,h}\, C_h\, (2^{s+1}h)^{|\alpha+\beta|}(\alpha!\beta!)^s.\end{aligned}$$

This gives for some $C'_{\delta,d,h,N} > 0$

$$\|x^\gamma D^\kappa f\|_{2^{s+1}h} \leqslant C'_{\delta,d,h,N}, \quad |\gamma+\kappa| < N, \ \ f \in B.$$

Since $0 < h \leqslant 1$ is arbitrary we have proved the claim. $\square$

The proof of the next result is omitted since it is conceptually identical to the proof of Lemma 4.3.

Lemma 6.11. *If $B \subseteq \Sigma_s(\mathbf{R}^d)$ is bounded and $\varepsilon > 0$, then there exists $K \in \mathbf{N}$ and $\varphi_j \in \Sigma_s(\mathbf{R}^d)$ for $1 \leqslant j \leqslant K$ such that*

$$B \subseteq \bigcup_{j=1}^{K} B_\varepsilon(\varphi_j)$$

where the open balls $B_\varepsilon(\varphi_j) \subseteq L^2(\mathbf{R}^d)$ refer to the L^2 norm.

We have now reached a point where we may prove that $\mathscr{K}_t$ is a strongly continuous semigroup on $\Sigma_s(\mathbf{R}^d)$. It is a consequence of the following result.

Theorem 6.12. *The map $[0,\infty) \ni t \mapsto \mathscr{K}_t$ is a semigroup on $\Sigma_s(\mathbf{R}^d)$, which satisfies for each bounded set $B \subseteq \Sigma_s(\mathbf{R}^d)$ and all $h > 0$*

$$\lim_{t \to 0^+} \sup_{\varphi \in B} \|(\mathscr{K}_t - I)\varphi\|_h = 0. \tag{6.20}$$

Proof. The semigroup property $\mathscr{K}_{t_1+t_2} = \mathscr{K}_{t_1}\mathscr{K}_{t_2}$ for $t_1, t_2 \geqslant 0$, as well as $\mathscr{K}_0 = I$, are immediate since they hold on L^2 and $\Sigma_s(\mathbf{R}^d) \subseteq L^2$, and Corollary 6.8 shows that $\mathscr{K}_t \colon \Sigma_s(\mathbf{R}^d) \to \Sigma_s(\mathbf{R}^d)$ is continuous for each $t \geqslant 0$.

It remains to show (6.20) where $h > 0$ and $B \subseteq \Sigma_s(\mathbf{R}^d)$ is bounded as in (6.19). We may assume that $h \leqslant 1$.

Let $\varepsilon > 0$ and $N \in \mathbf{N}$. If $|\alpha + \beta| \geqslant N$ and $0 < t \leqslant 1$, then we obtain from Theorem 6.7

$$\begin{aligned} \frac{\|x^\alpha D^\beta (\mathscr{K}_t - I)\varphi\|_{L^2}}{h^{|\alpha+\beta|}(\alpha!\beta!)^s} &\leqslant 2^{-|\alpha+\beta|} \frac{\|x^\alpha D^\beta \mathscr{K}_t \varphi\|_{L^2} + \|x^\alpha D^\beta \varphi\|_{L^2}}{\left(\frac{h}{2}\right)^{|\alpha+\beta|}(\alpha!\beta!)^s} \\ &\leqslant 2^{-N}\left(\|\mathscr{K}_t\varphi\|_{\frac{h}{2}} + \|\varphi\|_{\frac{h}{2}}\right) \\ &\lesssim 2^{-N}\left(\|\varphi\|_{h_1} + \|\varphi\|_{\frac{h}{2}}\right) \leqslant \varepsilon, \quad \varphi \in B, \end{aligned} \tag{6.21}$$

for some $h_1 > 0$, provided $N \in \mathbf{N}$ is sufficiently large, taking into account (6.19).

We also have to consider $\alpha, \beta \in \mathbf{N}^d$ such that $|\alpha + \beta| < N$. From Lemma 4.1 and the contraction property of $\mathscr{K}_t$ acting on L^2 we obtain for $0 < t \leqslant 1$

$$\begin{aligned} \|x^\alpha D^\beta(\mathscr{K}_t - I)\varphi\|_{L^2} &\leqslant |C_{\alpha,\beta}(t)|\, \|(\mathscr{K}_t - I)x^\alpha D^\beta \varphi\|_{L^2} + |C_{\alpha,\beta}(t) - 1|\, \|x^\alpha D^\beta\varphi\|_{L^2} \\ &\quad + \sum_{\substack{|\gamma+\kappa| \leqslant |\alpha+\beta| \\ (\gamma,\kappa) \neq (\alpha,\beta)}} |C_{\gamma,\kappa}(t)|\, \|x^\gamma D^\kappa \varphi\|_{L^2} \\ &\leqslant C \|(\mathscr{K}_t - I)x^\alpha D^\beta\varphi\|_{L^2} + |C_{\alpha,\beta}(t) - 1|\, \|x^\alpha D^\beta\varphi\|_{L^2} \\ &\quad + \sum_{\substack{|\gamma+\kappa| \leqslant |\alpha+\beta| \\ (\gamma,\kappa) \neq (\alpha,\beta)}} |C_{\gamma,\kappa}(t)|\, \|x^\gamma D^\kappa \varphi\|_{L^2} \end{aligned} \tag{6.22}$$

where $C > 0$ and (4.2) hold.

By Lemma 6.10, $\{x^\alpha D^\beta \varphi \colon \varphi \in B,\ |\alpha+\beta| < N\} \subseteq \Sigma_s(\mathbf{R}^d)$ is bounded. Thus by Lemma 6.11 there exists $K \in \mathbf{N}$ and $\varphi_j \in \Sigma_s(\mathbf{R}^d)$, $1 \leqslant j \leqslant K$, such that

$$\min_{1 \leqslant j \leqslant K} \|x^\alpha D^\beta \varphi - \varphi_j\|_{L^2} < \frac{\varepsilon h^N}{8C}, \quad |\alpha+\beta| < N, \ \varphi \in B.$$

The strong continuity and the contraction property of $\mathscr{K}_t$ acting on L^2 gives for $0 < t \leqslant \delta$

$$\begin{aligned} \|(\mathscr{K}_t - I)x^\alpha D^\beta \varphi\|_{L^2} &= \min_{1\leqslant j\leqslant K} \|(\mathscr{K}_t - I)(x^\alpha D^\beta \varphi - \varphi_j + \varphi_j)\|_{L^2} \\ &\leqslant \min_{1\leqslant j\leqslant K} \big(2\|x^\alpha D^\beta \varphi - \varphi_j\|_{L^2} + \|(\mathscr{K}_t - I)\varphi_j\|_{L^2}\big) \\ &\leqslant \frac{\varepsilon h^N}{4C} + \frac{\varepsilon h^N}{4C} = \frac{\varepsilon h^N}{2C}, \quad |\alpha+\beta| < N, \ \varphi \in B, \end{aligned} \tag{6.23}$$

provided $\delta > 0$ is sufficiently small.

We have

$$\|x^\alpha D^\beta \varphi\|_{L^2} \leqslant C_h (\alpha!\beta!)^s h^{|\alpha+\beta|}, \quad \alpha,\beta \in \mathbf{N}^d, \ \varphi \in B, \tag{6.24}$$

and for $|\gamma+\kappa| \leqslant |\alpha+\beta| < N$ we have

$$\|x^\gamma D^\kappa \varphi\|_{L^2} \leqslant C_h h^{|\gamma+\kappa|} (\gamma!\kappa!)^s \leqslant C_h \max_{|\gamma+\kappa|<N} (\gamma!\kappa!)^s := C_{h,N}, \quad \varphi \in B.$$

Due to (4.2) the latter gives for $0 < t \leqslant \delta$

$$\sum_{\substack{|\gamma+\kappa|\leqslant|\alpha+\beta| \\ (\gamma,\kappa)\neq(\alpha,\beta)}} |C_{\gamma,\kappa}(t)| \, \|x^\gamma D^\kappa \varphi\|_{L^2} < \frac{\varepsilon h^N}{4}, \quad |\alpha+\beta| < N, \ \varphi \in B, \tag{6.25}$$

after possibly decreasing $\delta > 0$.

Finally we insert (6.23), (6.24) and (6.25) into (6.22). Using (4.2) we obtain then for $0 < t \leqslant \delta$, again after possibly decreasing $\delta > 0$,

$$\begin{aligned} &\frac{\|x^\alpha D^\beta (\mathscr{K}_t - I)\varphi\|_{L^2}}{h^{|\alpha+\beta|}(\alpha!\beta!)^s} \\ &\leqslant \frac{C\|(\mathscr{K}_t - I)x^\alpha D^\beta \varphi\|_{L^2}}{h^N} + |C_{\alpha,\beta}(t) - 1| \, C_h + \sum_{\substack{|\gamma+\kappa|\leqslant|\alpha+\beta| \\ (\gamma,\kappa)\neq(\alpha,\beta)}} |C_{\gamma,\kappa}(t)| \, \frac{\|x^\gamma D^\kappa \varphi\|_{L^2}}{h^N} \\ &\leqslant \frac{\varepsilon}{2} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \varepsilon, \quad |\alpha+\beta| < N, \ \varphi \in B. \end{aligned}$$

If we combine this estimate with (6.21) we obtain for $0 < t \leqslant \delta$

$$\frac{\|x^\alpha D^\beta (\mathscr{K}_t - I)\varphi\|_{L^2}}{h^{|\alpha+\beta|}(\alpha!\beta!)^s} \leqslant \varepsilon, \quad \alpha,\beta \in \mathbf{N}^d, \ \varphi \in B.$$

Since $\varepsilon > 0$ is arbitrary this proves (6.20). □

As a consequence, picking the bounded set B as a single element in Σ_s, we obtain the following result. The local equicontinuity is a consequence of Theorem 6.7.

Corollary 6.13. *For $t \geqslant 0$, $\mathscr{K}_t$ is a locally equicontinuous strongly continuous semigroup on $\Sigma_s(\mathbf{R}^d)$.*

We denote the generator of the semigroup $\mathscr{K}_t$ acting on $\Sigma_s(\mathbf{R}^d)$ by L_s, to distinguish from the generator A_s defined in (5.2). Because of $\Sigma_s \subseteq \mathscr{S}$ we have $L_s \subseteq A$, cf. (5.4).

By [21, Proposition 1.3] the domain $D(L_s) \subseteq \Sigma_s(\mathbf{R}^d)$ is dense, and by [21, Proposition 1.4] L_s is a closed operator in $\Sigma_s(\mathbf{R}^d)$.

Proposition 6.14. *The generator L_s is a continuous operator on $\Sigma_s(\mathbf{R}^d)$ and thus $D(L_s) = \Sigma_s(\mathbf{R}^d)$.*

Proof. First we consider the Weyl symbol $q \in \Gamma^2$ defined in (3.1). It satisfies

$$\begin{aligned} |\partial^\alpha q(z)| &\lesssim \langle z \rangle^{2-|\alpha|}, \quad z \in \mathbf{R}^{2d}, \ \alpha \in \mathbf{N}^{2d}, && |\alpha| \leqslant 2, \\ &\equiv 0, && |\alpha| > 2. \end{aligned}$$

Combining with (cf. (6.2))

$$\alpha!^{-s} h^{-|\alpha|} = \left(\frac{h^{-\frac{|\alpha|}{s}}}{\alpha!} \right)^s \leqslant \left(\frac{\left(2dh^{-\frac{1}{s}}\right)^{|\alpha|}}{|\alpha|!} \right)^s \leqslant \exp\left(2sdh^{-\frac{1}{s}}\right), \quad \alpha \in \mathbf{N}^{2d}, \ h > 0,$$

this gives

$$\alpha!^{-s} h^{-|\alpha|} e^{-|z|^{\frac{1}{s}}} |\partial^\alpha q(z)| \lesssim \exp\left(2sdh^{-\frac{1}{s}}\right) \langle z \rangle^2 e^{-|z|^{\frac{1}{s}}} \leqslant C, \ \ z \in \mathbf{R}^{2d}, \ \alpha \in \mathbf{N}^{2d}, \ h > 0,$$

where $C = C_{s,d,h} > 0$.

We have proved the estimates

$$|\partial^\alpha q(z)| \lesssim h^{|\alpha|} \alpha!^s e^{|z|^{\frac{1}{s}}}, \quad z \in \mathbf{R}^{2d}, \ \alpha \in \mathbf{N}^{2d}, \ \forall h > 0,$$

which by [3, Definition 2.4] implies that q belongs to a space there denoted $\Gamma^\infty_{0,s}(\mathbf{R}^{2d})$. According to [3, Theorem 4.10] the operator $q^w(x,D)\colon \Sigma_s(\mathbf{R}^d) \to \Sigma_s(\mathbf{R}^d)$ is thereby continuous. Hence, referring to the seminorms (6.1), for any $h_1 > 0$ there exists $h_2 > 0$ such that

$$\|q^w(x,D)\varphi\|_{h_1} \lesssim \|\varphi\|_{h_2}, \quad \varphi \in \Sigma_s(\mathbf{R}^d). \tag{6.26}$$

We have $D(L_s) \subseteq \Sigma_s(\mathbf{R}^d) \subseteq D(q^w(x,D))$. If $f \in D(L_s)$, then the limit

$$L_s f = \lim_{h \to 0^+} h^{-1} \left(\mathscr{K}_h - I \right) f$$

exists in Σ_s. Since $\| \cdot \|_{L^2} \leqslant \| \cdot \|_h$ for any $h > 0$, the limit also exists in L^2. It follows that $L_s f = -q^w(x,D) f$ for $f \in D(L_s)$, that is $L_s \subseteq -q^w(x,D)$.

Let $f \in \Sigma_s(\mathbf{R}^d)$. By the density $D(L_s) \subseteq \Sigma_s(\mathbf{R}^d)$ there exists a sequence $(f_n)_{n \geqslant 1} \subseteq D(L_s)$ such that $f_n \to f$ in Σ_s. The estimate (6.26) gives for any $h_1 > 0$

$$\|L_s(f_n - f_m)\|_{h_1} = \|q^w(x,D)(f_n - f_m)\|_{h_1} \lesssim \|f_n - f_m\|_{h_2}$$

for some $h_2 > 0$. Thus $(L_s f_n)_{n \geqslant 1}$ is a Cauchy sequence in $\Sigma_s(\mathbf{R}^d)$ which converges to an element $g \in \Sigma_s(\mathbf{R}^d)$. From the closedness of L_s it follows that $f \in D(L_s)$ and $L_s f = g$. Hence $D(L_s) = \Sigma_s(\mathbf{R}^d)$ and L_s is continuous on $\Sigma_s(\mathbf{R}^d)$. $\square$

As in (5.6) we may extend L_s uniquely to a continuous operator on $\Sigma_s'(\mathbf{R}^d)$ equipped with its weak* topology, denoted $\Sigma_{s,\mathrm{w}}'(\mathbf{R}^d)$. In fact we set, using the formal L^2 adjoint $L_s^* = -\overline{q}^w(x,D)$ acting on Σ_s,

$$(L_s u, \varphi) = (u, L_s^* \varphi), \quad u \in \Sigma_s'(\mathbf{R}^d), \ \varphi \in \Sigma_s(\mathbf{R}^d). \tag{6.27}$$

The space $\Sigma_s'(\mathbf{R}^d)$ equipped with its strong topology is denoted $\Sigma_{s,\mathrm{str}}'(\mathbf{R}^d)$, and the topology is defined by the seminorms

$$\Sigma_s'(\mathbf{R}^d) \ni u \mapsto \sup_{\varphi \in B} |(u, \varphi)|$$

for all bounded subsets $B \subseteq \Sigma_s(\mathbf{R}^d)$. Then L_s defined by (6.27) is continuous on $\Sigma_{s,\mathrm{str}}'(\mathbf{R}^d)$.

We can now formulate and prove a Gelfand–Shilov distribution version of Theorem 5.7.

Theorem 6.15. *The semigroup $\mathscr{K}_t$ is:*

(i) *strongly continuous on* $\Sigma'_{s,\mathrm{w}}$, *and*
(ii) *locally equicontinuous strongly continuous on* $\Sigma'_{s,\mathrm{str}}$.

Proof. The semigroup property $\mathscr{K}_{t_1+t_2} = \mathscr{K}_{t_1}\mathscr{K}_{t_2}$ for $t_1, t_2 \geqslant 0$, as well as $\mathscr{K}_0 = I$, on $\Sigma'_s(\mathbf{R}^d)$ defined by (6.17) follow from the corresponding properties on $\Sigma_s(\mathbf{R}^d)$, as in the proof of Theorem 5.2.

Let $u \in \Sigma'_s(\mathbf{R}^d)$ and let $T > 0$. For $0 \leqslant t \leqslant T$ fixed, a seminorm of $\mathscr{K}_t u$ considered as an element in $\Sigma'_{s,\mathrm{str}}$ is defined by a bounded set $B \subseteq \Sigma_s(\mathbf{R}^d)$ as

$$\sup_{\varphi \in B} |(\mathscr{K}_t u, \varphi)| = \sup_{\varphi \in B} |(u, \mathscr{K}_t^* \varphi)|$$

and the right-hand side is a seminorm of u, since $\mathscr{K}_t^* B \subseteq \Sigma_s(\mathbf{R}^d)$ is a bounded set according to Theorem 6.7. This shows the continuity $\mathscr{K}_t \colon \Sigma'_{s,\mathrm{str}}(\mathbf{R}^d) \to \Sigma'_{s,\mathrm{str}}(\mathbf{R}^d)$ as well as the continuity $\mathscr{K}_t \colon \Sigma'_{s,\mathrm{w}}(\mathbf{R}^d) \to \Sigma'_{s,\mathrm{w}}(\mathbf{R}^d)$ for each fixed t such that $0 \leqslant t \leqslant T$. Theorem 6.7 also shows that $\{\mathscr{K}_t^* B,\ 0 \leqslant t \leqslant T\} \subseteq \Sigma_s(\mathbf{R}^d)$ is a bounded set so $\mathscr{K}_t$ is locally equicontinuous on $\Sigma'_{s,\mathrm{str}}(\mathbf{R}^d)$.

Finally let $B \subseteq \Sigma_s(\mathbf{R}^d)$ be a bounded set and let $u \in \Sigma'_s(\mathbf{R}^d)$. For some $h > 0$ we obtain using Theorem 6.12

$$\sup_{\varphi \in B} |((\mathscr{K}_t - I)u, \varphi)| = \sup_{\varphi \in B} |(u, (\mathscr{K}_t^* - I)\varphi)| \lesssim \sup_{\varphi \in B} \|(\mathscr{K}_t^* - I)\varphi\|_h \longrightarrow 0, \quad t \longrightarrow 0^+,$$

which shows that $\mathscr{K}_t$ is strongly continuous on $\Sigma'_{s,\mathrm{str}}(\mathbf{R}^d)$ as well as on $\Sigma'_{s,\mathrm{w}}(\mathbf{R}^d)$. $\square$

The generator of the semigroup $\mathscr{K}_t$ acting on $\Sigma'_{s,\mathrm{w}}(\mathbf{R}^d)$ is denoted L'_{w}, and the generator of the semigroup $\mathscr{K}_t$ acting on $\Sigma'_{s,\mathrm{str}}(\mathbf{R}^d)$ is denoted L'_{str}. By [21, Proposition 2.1] we have $L'_{\mathrm{w}} = L_s$ defined by (6.27), and hence $D(L'_{\mathrm{w}}) = \Sigma'_s$.

The argument that proves $A'_{\mathrm{str}} = A'_{\mathrm{w}}$ after Theorem 5.7 again shows that $L'_{\mathrm{str}} = L'_{\mathrm{w}}$. Again we may thus conclude that the two semigroups have identical generators. Denoting $L' = L'_{\mathrm{str}} = L'_{\mathrm{w}}$ we have $D(L') = \Sigma'_s$. We may again invoke [21, Proposition 1.2] and [20, pp. 483–84] to yield the following result which is conceptually similar to Corollary 5.9. Note that the uniqueness space is again larger than the solution space: $C^1([0,\infty), \Sigma'_{s,\mathrm{str}}) \subseteq C^1([0,\infty), \Sigma'_{s,\mathrm{w}})$.

Corollary 6.16. *For any* $u_0 \in \Sigma'_s(\mathbf{R}^d)$ *the Cauchy problem* (CP) *has the solution* $\mathscr{K}_t u_0$ *in the space* $C^1([0,\infty), \Sigma'_{s,\mathrm{str}})$. *The solution is unique in the space* $C^1([0,\infty), \Sigma'_{s,\mathrm{w}})$.

There is also an alternative way to show $D(L'_{\mathrm{str}}) = \Sigma'_s$, cf. Remark 5.8. In fact, if we can show that $\Sigma_s(\mathbf{R}^d)$ is a reflexive space, then [21, Theorem 1 and its Corollary] show that $\mathscr{K}_t$, considered as a strongly continuous semigroup on $\Sigma'_{s,\mathrm{w}}$, is necessarily also strongly continuous on $\Sigma'_{s,\mathrm{str}}$, and the two semigroups have identical generators.

Thus it remains to show that $\Sigma_s(\mathbf{R}^d)$ is a reflexive space (cf. [11, Theorem I.6.2]), which may be of independent interest. A locally convex space X is called reflexive provided $X \mapsto (X'_\beta)'_\beta$ is a topological isomorphism [30, p. 144]. Here X'_β denotes the dual of X, equipped with its strong topology.

Proposition 6.17. *If* $s > \frac{1}{2}$, *then the space* $\Sigma_s(\mathbf{R}^d)$ *is reflexive.*

Proof. By [28, Exercise V.52] the Fréchet space $\Sigma_s(\mathbf{R}^d)$ carries the Mackey topology. By [28, Exercise V.56 (a) and Lemma on p. 166] it remains to prove the following statement: Every weakly closed and weakly bounded subset $B \subseteq \Sigma_s(\mathbf{R}^d)$ is weakly compact.

By [28, Theorem V.23] $B \subseteq \Sigma_s(\mathbf{R}^d)$ is bounded in the Fréchet space topology of $\Sigma_s(\mathbf{R}^d)$. The Fréchet space topology on $\Sigma_s(\mathbf{R}^d)$ is stronger than the weak topology. This fact implies that $B \subseteq \Sigma_s(\mathbf{R}^d)$ is closed in its Fréchet space topology, and if B is shown to be compact in the Fréchet space topology, then it is also weakly compact.

Thus it remains to show that $B \subseteq \Sigma_s(\mathbf{R}^d)$ is compact in its Fréchet space topology. Since the Fréchet space topology of $\Sigma_s(\mathbf{R}^d)$ is metric we may prove compactness of B by showing that any sequence $(f_n)_{n\geqslant 1} \subseteq B$ has a convergent subsequence. The space $\Sigma_s(\mathbf{R}^d)$ is complete and B is closed so it is suffices to show the existence of a Cauchy subsequence of $(f_n)_{n\geqslant 1} \subseteq \Sigma_s(\mathbf{R}^d)$. We accomplish this by constructing a subsequence which is Cauchy in the seminorm (2.14) for the space $\mathcal{S}_{s,h}$ for each $0 < h \leqslant 1$.

We have since $B \subseteq \Sigma_s(\mathbf{R}^d)$ is bounded

$$|x^\alpha D^\beta f_n(x)| \leqslant C_h h^{|\alpha+\beta|}(\alpha!\beta!)^s, \quad x \in \mathbf{R}^d, \ \ \alpha, \beta \in \mathbf{N}^d, \ \ n \geqslant 1, \ \ h > 0, \tag{6.28}$$

for some constants $C_h > 0$.

Let $0 < h \leqslant 1$ and let $\varepsilon > 0$. The bound (6.28) gives

$$\begin{aligned}
|x^\alpha D^\beta f_n(x)| &= |x|^{-2} \left| \sum_{j=1}^d x_j^2 x^\alpha D^\beta f_n(x) \right| \\
&\leqslant |x|^{-2} C_{\frac{h}{2}} \sum_{j=1}^d \left(\frac{h}{2}\right)^{|\alpha+\beta|+2} (\alpha!(\alpha_j+1)(\alpha_j+2)\beta!)^s \\
&\leqslant |x|^{-2} C_{\frac{h}{2}} \left(\frac{h}{2}\right)^{|\alpha+\beta|} (\alpha!\beta!)^s d(|\alpha|+2)^{2s} \\
&\leqslant |x|^{-2} C \, C_{\frac{h}{2}} \, h^{|\alpha+\beta|}(\alpha!\beta!)^s, \quad x \in \mathbf{R}^d \setminus 0, \ \ \alpha, \beta \in \mathbf{N}^d, \ \ n \geqslant 1,
\end{aligned}$$

for some $C > 0$. This gives

$$\sup_{\alpha,\beta \in \mathbf{N}^d, \ |x| \geqslant L} \frac{|x^\alpha D^\beta (f_n(x) - f_m(x))|}{h^{|\alpha+\beta|}(\alpha!\beta!)^s} < \varepsilon, \quad n, m \geqslant 1, \tag{6.29}$$

provided $L > 0$ is sufficiently large.

Next we consider the sequences $(x^\alpha D^\beta f_n(x))_{n\geqslant 1}$ for $|\alpha+\beta| > N$ where $N \in \mathbf{N}$ is to be chosen. Again (6.28) yields

$$\begin{aligned}
|x^\alpha D^\beta f_n(x)| &\leqslant C_{\frac{h}{2}} \left(\frac{h}{2}\right)^{|\alpha+\beta|} (\alpha!\beta!)^s \\
&\leqslant C_{\frac{h}{2}} 2^{-N} h^{|\alpha+\beta|}(\alpha!\beta!)^s, \quad x \in \mathbf{R}^d, \ \ |\alpha+\beta| > N, \ \ n \geqslant 1,
\end{aligned}$$

which proves the estimate

$$\sup_{|\alpha+\beta| > N, \ x \in \mathbf{R}^d} \frac{|x^\alpha D^\beta (f_n(x) - f_m(x))|}{h^{|\alpha+\beta|}(\alpha!\beta!)^s} < \varepsilon, \quad n, m \geqslant 1, \tag{6.30}$$

provided $N \in \mathbf{N}$ is sufficiently large.

Finally we study the sequences of functions $(x^\alpha D^\beta f_n(x))_{n\geqslant 1}$ restricted to the compact ball $\overline{B}_L = \{x \in \mathbf{R}^d\colon \ |x| \leqslant L\}$, where $\alpha, \beta \in \mathbf{N}^d$ satisfy $|\alpha+\beta| \leqslant N$. If $1 \leqslant j \leqslant d$ we obtain from (6.28) if $\alpha_j = 0$

$$\begin{aligned}
|D_j(x^\alpha D^\beta f_n)(x)| = |x^\alpha D^{\beta+e_j} f_n(x)| &\leqslant C_h h^{|\alpha+\beta|+1}(\alpha!\beta!)^s(|\beta|+1)^s \\
&\leqslant C_h (\alpha!\beta!)^s(|\beta|+1)^s, \quad x \in \mathbf{R}^d,
\end{aligned}$$

and if $\alpha_j > 0$

$$
\begin{aligned}
|D_j(x^\alpha D^\beta f_n)(x)| &= |i^{-1}\alpha_j x^{\alpha-e_j} D^\beta f_n(x) + x^\alpha D^{\beta+e_j} f_n(x)| \\
&\leqslant C_h(\alpha!\beta!)^s(|\alpha| h^{|\alpha+\beta|-1} + h^{|\alpha+\beta|+1}(|\beta|+1)^s) \\
&\leqslant C_h(\alpha!\beta!)^s(|\alpha| + (|\beta|+1)^s), \quad x \in \mathbf{R}^d.
\end{aligned}
$$

The gradient is thus uniformly bounded with respect to $x \in \mathbf{R}^d$:

$$
\sup_{x\in\mathbf{R}^d} |\nabla(x^\alpha D^\beta f_n)(x)| \leqslant C_{h,\alpha,\beta} < \infty.
$$

The mean value theorem gives

$$
|(x^\alpha D^\beta f_n)(x) - (x^\alpha D^\beta f_n)(y)| \leqslant C_{h,\alpha,\beta}|x-y|, \quad x, y \in \mathbf{R}^d,
$$

which shows that $\{x^\alpha D^\beta f_n,\ n \geqslant 1\}$ is an equicontinuous set of functions on $\mathbf{R}^d$ for all $\alpha, \beta \in \mathbf{N}^d$, particularly if $|\alpha+\beta| \leqslant N$.

Combining with the bound (6.28) which is uniform with respect to $x \in \mathbf{R}^d$ and $n \geqslant 1$ we find that the assumptions for the Arzelà–Ascoli theorem [29, Theorem 11.28] are satisfied for $\{x^\alpha D^\beta f_n,\ n \geqslant 1\}$, for each $\alpha, \beta \in \mathbf{N}^d$.

Thus we start by extracting a subsequence of $(f_n)_{n\geqslant 1}$ that converges uniformly on $\overline{B}_L$. We apply $x^\alpha D^\beta$ to the subsequence and extract a new subsequence that converges uniformly on $\overline{B}_L$, consecutively, first for all $\alpha, \beta \in \mathbf{N}^d$ such that $|\alpha + \beta| = 1$, and after that for all multi-indices $\alpha, \beta \in \mathbf{N}^d$ of increasing orders $|\alpha + \beta| = 2, \dots, N$. After a finite number of such subsequence extractions we obtain a subsequence $(f_{n_k})_{k\geqslant 1}$ such that

$$
\sup_{|\alpha+\beta|\leqslant N,\ |x|\leqslant L} \frac{|x^\alpha D^\beta(f_{n_k}(x) - f_{n_m}(x))|}{h^{|\alpha+\beta|}(\alpha!\beta!)^s} < \varepsilon, \quad k, m \geqslant K, \tag{6.31}
$$

provided $K \in \mathbf{N}$ is sufficiently large. When we combine (6.29), (6.30) and (6.31) it follows that $(f_{n_k})_{k\geqslant 1}$ is a Cauchy sequence in $\mathcal{S}_{s,h}$. □

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