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ANNALI DELL'UNIVERSITA' DI FERRARA (2020) 66:231–254
https://doi.org/10.1007/s11565-020-00343-5

Injective linear series of algebraic curves on quadrics

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Received: 5 May 2020 / Accepted: 11 August 2020 / Published online: 16 August 2020

Abstract
We study linear series on curves inducing injective morphisms to projective space, using zero-dimensional schemes and cohomological vanishings. Albeit projections of curves and their singularities are of central importance in algebraic geometry, basic problems still remain unsolved. In this note, we study cuspidal projections of space curves lying on irreducible quadrics (in arbitrary characteristic).

Keywords Linear series · Zero-dimensional schemes · Cuspidal projections

Mathematics Subject Classification (Primary)14H45 · 14H50; (Secondary)14H20 · 14H51

1 Introduction

Projections and singularities of curves are of central importance in algebraic geometry. The projective geometry of singular curves is a delightful chapter of classical algebraic geometry that remains active even up to this date: many questions await to be settled, and in turn they inspire the introduction of tools entailing deformation theory, zero-dimensional schemes, and combinatorics, among other techniques.

A natural direction of research is the classification of singularities that may arise on a curve X, in some specific ranges of the numerical invariants attached to X. An approach to this classification issue, relying on osculating spaces and combinatorics of semigroups of valuations, has been recently employed in [4]. Other classification results, leveraging the structure of free resolutions of certain ideals, have been achieved in [7].

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In this note, we employ to some extent the standpoint of Greuel, Lossen, and Shustin [11], using the geometry of zero-dimensional schemes and the cohomology of their ideal sheaves, to study *cuspidal* (or unibranch) singularities. These types of singular points are usually related to some *tangency* conditions and so carry interesting geometric information about the curve.

Let k be an algebraically closed field. Let X be a complete smooth curve of genus g over k, i.e., an integral scheme of dimension one, smooth and proper over k. Every such X is projective and can be embedded in projective 3-space, independently of the characteristic of k. A natural question to wonder about is whether every X admits a projection to $\mathbb{P}^2$ with only cuspidal singularities, i.e. X admits a cuspidal projection. Ferrand [9] showed that, when $\mathrm{char}(k) > 0$, if X admits a cuspidal projection then X is a set-theoretic complete intersection.

Thereafter, back to characteristic zero, Piene [24] proved that every $X \subset \mathbb{P}^3$ of degree d and genus g, when $\binom{d-1}{2} - g \leq 3$, admits a cuspidal projection. However, a general canonical curve $X \subset \mathbb{P}^3$ of genus 4 does not admit a cuspidal projection [24, Theorem 2]. (Note that the latter curve is a complete intersection, so the converse to Ferrand's result does not hold in $\mathrm{char}(k) = 0$.) Sacchiero [26] showed that, in the range $\binom{d-1}{2} - g \geq 4$, a generic projection of an $X \subset \mathbb{P}^n$ $(n \geq 4)$, without inflection points and without hyperosculating planes, is not cuspidal.

For cuspidal curves, a classical open problem is to determine the maximum number of cusps realizable on a plane curve of degree d; recent asymptotic results were proven by Calabri, Paccagnan, and Stagnaro [6]. Interestingly, Koras and Palka [19] showed that complex plane rational cuspidal curves possess at most four singular points.

Cuspidal projections play an important role in the theory of X-ranks. Let $X \subset \mathbb{P}^3$ be a smooth curve and $p \in \mathbb{P}^3 \setminus X$. Then the X-rank of p satisfies $\mathrm{rk}_X(p) > 2$ if and only if the projection of X away from p is cuspidal. See [2] for more results in this direction.

What originally triggered this work has a topological source. Motivated by the study of regular topological maps [5] in the case of smooth curves, Michałek posed the problem [21]:

Question 1.1 *For any X over a field k as above, does there exist an injective morphism $\varphi : X \to \mathbb{P}^2$?*

This is also studied for other projective varieties by Görlach [10]. The curve $\varphi(X)$ is then an integral plane curve possibly with only cuspidal singularities. The map $\varphi : X \to \varphi(X)$ is a closed bijection and so a homeomorphism in Zariski topology; cuspidal projections of X to $\mathbb{P}^2$ are instances of injective morphisms.

The aim of this note is to study base-point free (not necessarily complete) two-dimensional linear series g^2_d on some smooth algebraic curves X inducing separable and injective morphisms to $\mathbb{P}^2$; we call these linear series *injective*. In this article, these are usually constructed by cuspidal projections of X to $\mathbb{P}^2$ with the help of zero-dimensional schemes and their cohomology, which will let us give positive instances to Question 1.1.

In this context, we propose the following conjecture:

Conjecture 1.2 *For large g, a very general smooth curve of genus g has no injective linear series g^2_d.*

We spell out the meaning of "very general" in the statement of Conjecture 1.2. Let Y be an integral quasi-projective variety. Fix a property $\wp$ that a point $p \in Y$ may satisfy. We say that $\wp$ is true for a very general point of Y if the set of all $p \in Y$ for which p fails $\wp$ is contained in a countable family of proper subvarieties of Y. In Conjecture 1.2, the generality is applied to the moduli scheme $\mathcal{M}_g$ ($g \geq 2$) of all smooth curves of genus g (over some fixed algebraically closed field). We guess that more should be true: for large g and for every positive integer d, the set of all $X \in \mathcal{M}_g$ with an injective g^2_d sits inside a proper subvariety of $\mathcal{M}_g$.

We now clarify the meaning of "large g" in the statement of Conjecture 1.2: this refers to the existence of an integer $g_0(k)$ (depending on the fixed algebraically closed ground field k) such that, for all $g \geq g_0(k)$, a very general curve of genus g has no injective linear series g^2_d.

We believe it would be interesting to have partial results on Conjecture 1.2, for non-complete g^2_d, i.e., for g^2_d inducing a non-degenerate injective map $j : X \to \mathbb{P}^n$, $n > 2$, composed with a linear projection. This is the setting of Piene [24] and Sacchiero [26], except that they require j to be an embedding. Furthermore, we ask the following

Question 1.3 *Let X be a smooth curve of genus g. Are there infinitely many integers d such that X has injective g^2_d?*

Even if Conjecture 1.2 fails, we ask whether, for all sufficiently large g, there exists an $X \in \mathcal{M}_g$ with no injective g^2_d. In such a case, one may still wonder whether, for infinitely many genera g, a very general curve of genus g has no injective g^2_d.

Most of our results arise from looking at the quadric $Q = \mathbb{P}^1 \times \mathbb{P}^1$. Let X be a smooth projective curve of genus g. By the universal property of the fibered product of schemes, giving a morphism $f : X \to \mathbb{P}^1 \times \mathbb{P}^1$ is equivalent to prescribing two morphisms $u_i : X \to \mathbb{P}^1$, $i = 1, 2$, i.e., two base-point free linear series $g^1_{d_1}$ and $g^1_{d_2}$, where $d_1 = \deg u_1^*(\mathcal{O}_{\mathbb{P}^1}(1))$ and $d_2 = \deg u_2^*(\mathcal{O}_{\mathbb{P}^1}(1))$. (Here we assume $d_1, d_2 \neq 0$, as otherwise $f(X)$ is contained in a line of $\mathbb{P}^3$.) The morphism f is birational onto its image if and only if there is no 4-tuple (D, h, v_1, v_2), where D is a smooth projective curve, $h : X \to D$ is a finite morphism with $\deg(h) > 1$, $v_i : D \to \mathbb{P}^1$, $i = 1, 2$, are morphisms and $u_i = v_i \circ h$, $i = 1, 2$. In classical terminology, f is birational onto its image if and only if $g^1_{d_1}$ and $g^1_{d_2}$ are not composed with the same involution. The pair (d_1, d_2) is defined to be the bidegree of f.

Question 1.4 *For which (X, d_1, d_2), is there an f of bidegree (d_1, d_2) that is injective and separable? For which X, are there infinitely many (d_1, d_2) such that there is an injective and separable f? For which pair (X, d_1), with $d_1 > 1$, are there infinitely many integers d_2, such that there exists an injective and separable $f : X \to \mathbb{P}^1 \times \mathbb{P}^1$ of bidegree (d_1, d_2)?*

Main result. Let X be a complete smooth curve of genus g over k. An *injective linear series* on X is a (not necessarily complete) series g^2_d inducing a separable and injective morphism $\varphi : X \to \mathbb{P}^2$. We organize injective linear series into natural *types*:

Definition 1.5 Let $L = \varphi^*(\mathcal{O}_{\mathbb{P}^2}(1))$ be the line bundle on X associated to g^2_d, i.e. for a divisor $D \in g^2_d$ one has $L = \mathcal{O}_X(D)$. An injective g^2_d on X has one of the following *types*:

(I) a complete g^2_d, i.e. $h^0(L) = 3$;
(II) an incomplete g^2_d, i.e. $h^0(L) \geq 4$, with L very ample line bundle;
(III) an incomplete g^2_d with L not very ample.

(See Proposition 2.1 for some geometric remarks about them.) This is our main result:
Main Theorem (**Theorems** 4.8 **and** 4.16) *Let k be an algebraically closed field of arbitrary characteristic and let $d_2 \geq d_1 \geq 1$. Then there exists a smooth genus g curve with an injective $g^2_{d_1+d_2}$ of type* **II** *with $g = d_1d_2 - d_1 - d_2 + 1$.*

Let k be an algebraically closed field of characteristic zero. Let $d_2 \geq d_1 \geq 16$ and $h > 0$ such that $3h + 2 \leq \binom{d_1-1}{2}$. Fix an integer κ such that $0 < \kappa \leq 2h$ and set $g = d_1d_2 - d_1 - d_2 + 1 - \kappa$. Then there exists a smooth genus g curve with an injective $g^2_{d_1+d_2}$ of type **III**.

More contributions and structure of the paper. In §2, we work in $\mathrm{char}(k) = 0$. We first record some observations about the geometry behind the types of linear series in Proposition 2.1. In Theorem 2.5, we use an existence result of Barkats [3] to show that in every genus there exist curves equipped with *type* **I** injective linear series. We recall an important (existence and smoothness) result of Greuel, Lossen, and Shustin about the variety $V(d, \kappa)$, parametrizing plane curves of given degree d and with κ ordinary cusps as their only singularities; see Theorem 2.6. In Remark 2.7, we point out that the curves from this result are essentially different from those arising in our Theorems 4.8 and 4.16.

In §3, we offer a study of injective linear series on hyperelliptic curves. Surprisingly, this material seems new and it is interesting on its own right. We explicitly describe $2g+2$ families of ∞^1-many injective linear series g^2_{g+3} of degree $g+3$, see Theorem 3.1. These families are in correspondence to the Weierstrass points of a given hyperelliptic curve along with a double cover of $\mathbb{P}^1$. Although the statement of Theorem 3.1 is in characteristic zero, we point out that the result holds in any characteristic with suitable modifications of the arguments; this is observed in Remark 3.2. Proposition 3.4 and Proposition 3.5 give a characterization of injective (non-special) linear series of degree $g+3$. Proposition 3.6 provides a description of injective linear series of degree $g+2$.

In §4, we study more closely space curves lying on irreducible quadrics in $\mathbb{P}^3$. With the help of zero-dimensional schemes in arbitrary $\mathrm{char}(k)$, we prove Theorem 4.1 and Theorem 4.6, showing the existence of smooth curves on smooth quadrics and cones admitting cuspidal projections; these two results yield Theorem 4.8.

In order to establish Theorem 4.16, we employ results of Roé and of Greuel, Lossen, and Shustin, extending them to smooth quadrics $Q \subset \mathbb{P}^3$; this is achieved in Lemmas 4.12, 4.13, 4.14, and 4.15.

In §5, we introduce two sets $\mathcal{A}$ and $\mathcal{B}$, naturally attached to a (inner smooth) cuspidal projection of a curve from a point. Theorem 5.3 and Theorem 5.4 provide a characterization of curves in $\mathbb{P}^3$, lying on smooth quadrics and quadric cones, with only cuspidal singularities in terms of $\mathcal{A}$ and $\mathcal{B}$.

2 Types and Castelnuovo's bound

In this section, we work in $\mathrm{char}(k) = 0$. We start formulating a proposition recording some geometric remarks behind the types of injective linear series:

Proposition 2.1 *Keep the notation from Definition 1.5. Then the following hold:*

(i) If g^2_d has type **II**, *then there exists $g^3_d \subseteq |L|$ with $g^3_d \supset g^2_d$; the latter g^3_d induces an embedding $j : X \to \mathbb{P}^3$ such that the morphism φ is the composition of j with a linear projection of $\mathbb{P}^3$ from a point of $\mathbb{P}^3 \setminus j(X)$* [24, p. 110, parts 4) and 5)].

(ii) If g^2_d is of type **III**, *then for any $3 \leq s \leq h^0(L) - 1$, any $g^s_d \subseteq |L|$ containing g^2_d is base-point free and it induces a morphism $u : X \to \mathbb{P}^s$ birational onto its image, but not an embedding; the curve $\varphi(X)$ is obtained from $u(X)$ by a linear projection from an $(s-3)$-dimensional linear subspace of $\mathbb{P}^s$ not intersecting $u(X)$.*

Remark 2.2 Since a cuspidal g^2_d has no base points and it induces an injective morphism, a necessary condition for the existence of a g^2_d of *type* **II** or **III** on X is that X has a degree d and genus $g = p_a(X)$ non-degenerate birational model in $\mathbb{P}^3$.

Fix integers d, g such that $g \geq 0$ and $d \geq 3$. Define:

$$\pi(d, 3) = m(m-1) + m\varepsilon, \text{ where } \varepsilon \in \{0, 1\} \text{ and } d = 2m + 1 + \varepsilon.$$

Halphen and later Castelnuovo proved that if there is a non-degenerate space curve $X \subset \mathbb{P}^3$ of degree d and arithmetic genus g, then $g \leq \pi(d, 3)$ [15, Theorem 3.7]. Not all the possible integers $g \leq \pi(d, 3)$ arise as arithmetic genera, even allowing singular curves.

Thus the question of existence of the possible pairs (g, d) is natural:

Question 2.3 *For which g and d, does there exist a smooth curve of genus g with an injective g^2_d of types* **I** *or* **II**?

Remark 2.4 Question 2.3 was partially answered by Ephraim and Kulkarni, who proved that for each genus $g \geq 0$ and each integer $d > 2g$ there exists a curve of genus g with a type **II** injective g^2_d [8, Corollary 3.9].

Theorem 2.5 *For each integer $g \geq 0$ there exists a smooth curve of genus g with a type* **I** *injective linear series.*

Proof Let d be the minimal integer ≥ 2 such that $g \leq (d-1)(d-2)/2$. If $g = (d-1)(d-2)/2$ it is sufficient to take as X a smooth degree d plane curve. Thus we may assume $d \geq 4$ and $(d-2)(d-3)/2 < g < (d-1)(d-2)/2$. Define:

$$\kappa = (d-1)(d-2)/2 - g.$$

Note that $1 \leq \kappa \leq d-3$ and hence $5\kappa \leq 5(d-3) \leq (d+2)(d+1)/2 - d - 1$. Since $5\kappa \leq (d+2)(d+1)/2 - d - 1$, there exists an integral plane curve $Y \subset \mathbb{P}^2$ of degree d with κ ordinary cusps as its only singularities by the work of Barkats [3] for

$d \geq 5$ (for $d = 4$ there exists a plane curve with a unique cusp, i.e. the projection of a curve of genus $g = 2$).

Thus there exists an injective g^2_d on a smooth genus g curve. Now we discuss why we may find some complete g^2_d. If $g = 0$ (resp. $g = 1$, resp. $g = 3$) we take a smooth plane curve of degree 2 (resp. 3, resp. 4). If $g = 2$, we take a degree 4 plane curve with an ordinary cusp as its unique singular point; this linear series g^2_4 is complete, because no genus 2 curve has a g^3_4 (see e.g. [16, Corollary IV.6.2]).

Assume $d \geq 5$ and $1 \leq \kappa \leq d - 3$. Let $Y \subset \mathbb{P}^2$ be any integral plane curve with exactly κ ordinary cusps as singularities. Let $u : X \to Y$ be the normalization map and let $\varphi : X \to \mathbb{P}^2$ denote the composition of u with the inclusion $Y \subset \mathbb{P}^2$. Define $L = \varphi^*(\mathcal{O}_{\mathbb{P}^2}(1))$.

We need to prove that $h^0(L) = 3$. Letting $g = p_a(X)$, by Riemann-Roch, it is enough to show that $3 - h^1(L) = d + 1 - g$, i.e. $h^1(L) = g + 2 - d$. Let $S = \mathrm{Sing}(Y)$. Since each singular point of Y is an ordinary cusp, the classical Plücker formulas affirm that $g = (d-1)(d-2)/2 - \kappa$ [14, page 280]. The equality is in fact derived from the cohomological equality $H^0(X, \omega_X) = \varphi^*(H^0(\mathbb{P}^2, \mathcal{I}_S(d-3)))$. By Serre duality, $h^1(L) = h^0(\omega_X \otimes L^*)$. Since $H^0(X, \omega_X) = \varphi^*(H^0(\mathbb{P}^2, \mathcal{I}_S(d-3))$, we have $h^1(L) = g + 2 - d$ if and only if $h^1(\mathbb{P}^2, \mathcal{I}_S(d-4)) = 0$. Note that for a general S we have $h^1(\mathbb{P}^2, \mathcal{I}_S(d-4)) = 0$, because $\#S = \kappa \leq (d-3)(d-2)/2 = h^0(\mathcal{O}_{\mathbb{P}^2}(d-4))$. $\square$

Besides Barkats' results [3] on the existence of curves with prescribed singularities, another more recent result is due to Greuel, Lossen, and Shustin [12, Corollary 2.4]. (Here we explicitly state a special case of the latter for cuspidal curves.)

Theorem 2.6 *(***Greuel, Lossen, Shustin***) Let $V(d, \kappa)$ denote the set parametrizing all irreducible plane curves $X \subset \mathbb{P}^2$ with $\deg(X) = d$ and κ ordinary cusps as its only singularities. Assume $9\kappa < d^2 + 6d + 8$. If $V(d, \kappa) \neq \emptyset$, then $V(d, \kappa)$ is smooth of pure dimension $(d^2 + 3d)/2 - 2\kappa$.*

Remark 2.7 Irreducibility of $V(d, \kappa) \neq \emptyset$ requires $18\kappa < d^2$, see [12, Corollary 3.2]. The non-emptyness $V(d, \kappa) \neq \emptyset$ was proven over $\mathbb{R}$ by Shustin [27, Theorem 3.3]. For all positive integers $d > 0$, let $\kappa(d)$ denote the maximal integer such that for all $0 \leq \kappa \leq \kappa(d)$ there exists $Y \in V(d, \kappa)$ defined over $\mathbb{R}$ with $\mathrm{Sing}(Y) \subset \mathbb{P}^2(\mathbb{R})$. One sees $\kappa(1) = \kappa(2) = 0$, $\kappa(4) = 3$, $\kappa(5) = 5$ and $\kappa(6) = 7$ [27, page 851]. More generally: for all $d \geq 7$, one has $\kappa(d) \geq (d^2 - 3d + 4)/4$ if $d \equiv 0, 3 \pmod 4$ and $\kappa(d) \geq (d^2 - 3d + 2)/4$ if $d \equiv 1, 2 \pmod 4$ [27, Theorem 3]. Concerning the smoothness of varieties parametrizing plane curves (not necessarily with only unibranch singularities) see the results in [11, §4.3]. The beautiful book [11] contains an extensive bibliography about this venerable subject; in particular, more information about singular curves on more surfaces other than the projective plane are discussed.

Remark 2.8 The plane curves X realized in Theorem 2.6 do not come from injective linear series g^2_d of *type* **II** or **III** whenever

$$p_a(X) = (d-1)(d-2)/2 - \kappa > \pi(d, 3).$$

Hence such injective linear series g^2_d must be complete and so of *type* **I**. Therefore, the curves from Theorem 2.6 are different from those arising in our Theorems 4.8 and 4.16.

3 Injective linear series on hyperelliptic curves

From the classical analytic definition of complex hyperelliptic curves, i.e., as the Riemann surface of the algebraic function $y = \sqrt{(x-a_1)\cdots(x-a_{2g+2})}$ [1, §2], it is clear that they admit a cuspidal model in $\mathbb{P}^2$ (the cuspidal point being at infinity) and therefore they carry an injective linear series from their very definition. However, we describe other natural injective linear series on any hyperelliptic curve in any characteristic. Their construction is highlighted in the course of the proof of the next Theorem 3.1; to our knowledge this is new and interesting on its own right. The result is stated and proven first in characteristic zero, and in the subsequent Remark 3.2 the general case is treated.

Theorem 3.1 *Let X be a smooth hyperelliptic curve of genus $g \geq 2$ over a field k with* $\mathrm{char}(k) = 0$. *Then there is a base-point free g^2_{g+3} inducing an injective morphism $\varphi : X \to \mathbb{P}^2$. The image $\varphi(X)$ has exactly two singular points: one ordinary cusp and one unibranch singularity. Moreover, X has ∞^1-many such g^2_{g+3}, each of them being a sublinear series of a different complete and very ample g^3_{g+3}; X has $2g+2$ such one-dimensional families of g^2_{g+3} and g^3_{g+3}.*

Proof Let $\mathcal{W}$ be the set of Weierstrass points of X, i.e., the support of the ramification divisor of the $2:1$ cover $u_2 : X \to \mathbb{P}^1$, induced by the linear series g^1_2 on X. (This exists on X, as it is hyperelliptic.)

Fix a point $o \in \mathcal{W}$. Then $o \in \mathcal{W}$ if and only if $2o \in g^1_2$ by definition of ramification divisor. Thus $2 = h^0(g^1_2) = h^0(\mathcal{O}_X(2o))$. For each $p \in X \setminus \mathcal{W}$, set $N_p := \mathcal{O}_X(2o + (g+1)p)$. We now split the proof into four claims.

Claim 1: For a general $p \in X \setminus \mathcal{W}$, we have $h^0(\mathcal{O}_X(gp)) = 1$, $h^0(\mathcal{O}_X((g+1)p)) = 2$ and $h^0(N_p) = 4$.
Proof of Claim 1: By Riemann-Roch, we have $h^0(\mathcal{O}_X(gp)) = 1 + h^1(\mathcal{O}_X(gp))$, $h^0(\mathcal{O}_X((g+1)p)) = 2 + h^1(\mathcal{O}_X((g+1)p))$ and $h^0(N_p) = 4 + h^1(N_p)$. Notice that $h^1(\mathcal{O}_X(2o + (g+1)p)) \leq h^1(\mathcal{O}_X((g+1)p)) \leq h^1(\mathcal{O}_X(gp))$. Hence, in order to finish the proof of *Claim 1*, it is sufficient to prove that $h^1(\mathcal{O}_X(gp)) = 0$. To see this, as $\mathrm{char}(k) = 0$, note that for any invertible sheaf $\mathcal{N}$ on X and for a general $p \in X$, one has $h^0(\mathcal{N}(-tp)) = \max\{0, h^0(\mathcal{N}) - t\}$, for any positive integer t. Letting $\mathcal{N} = \omega_X$ and $t = g$, we obtain $h^0(\omega_X(-gp)) = 0$. Finally, Serre duality gives $h^1(\mathcal{O}_X(gp)) = 0$.

Claim 2: For a general $p \in X$, N_p is very ample.
Proof of Claim 2: The base locus of $|\mathcal{O}_X((g+1)p)|$ is contained in $\{p\}$. Since $h^0(\mathcal{O}_X(gp)) < h^0(\mathcal{O}_X((g+1)p))$ by *Claim 1*, it follows that p is not a base point of $|\mathcal{O}_X((g+1)p)|$, as subtracting a base point from a divisor does not decrease dimension of global sections. Thus $\mathcal{O}_X((g+1)p)$ is base-point free and so its

linear series $|\mathcal{O}_X((g+1)p)|$ defines a degree $g+1$ morphism $u_1 : X \to \mathbb{P}^1$. As above, denote $u_2 : X \to \mathbb{P}^1$ the degree 2 cover of $\mathbb{P}^1$ induced by g^1_2 on X. The pair (u_1, u_2) induces a morphism $w : X \to \mathbb{P}^1 \times \mathbb{P}^1$. Now since u_2 is a degree 2 cover, either w is birational onto its image or it factors through u_2. The latter case is not possible, because u_1 cannot factor through u_2. Indeed, for the sake of contradiction, assume that u_1 factors through u_2. Then $\mathcal{O}_X((g+1)p)$ would be isomorphic to the invertible sheaf $(g^1_2)^{\otimes(g+1)/2}$. Since the dimension of the linear series of the latter is $(g+1)$ and $g \geq 2$, this isomorphism implies $h^0(\mathcal{O}_X((g+1)p)) > 2$, which contradicts *Claim 1* above. Hence w is birational onto its image.

Recall that the canonical sheaf of $\mathbb{P}^1 \times \mathbb{P}^1$ is $\omega_{\mathbb{P}^1\times\mathbb{P}^1} \cong \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(-2,-2)$. For $D \in |\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(2, g+1)|$, by adjunction, $\omega_D \cong \omega_{\mathbb{P}^1\times\mathbb{P}^1} \otimes \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(2, g+1)$. (For singular D, replace ω_D with the dualizing sheaf ω°_D and every later statement holds as well.) This implies that the arithmetic genus of each $D \in |\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(2, g+1)|$ is $p_a(D) = g$.

Since w is birational onto its image, $w(X)$ has bidegree $(2, g+1)$ and so $w(X) \in |\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(2, g+1)|$. Since $w(X)$ has arithmetic genus g, the morphism w is an embedding.

The linear series $|\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,1)|$ embeds $\mathbb{P}^1 \times \mathbb{P}^1$ as a quadric surface in $\mathbb{P}^3$. Call f the composition of w and the inclusion $\mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^3$. By construction, f is the map induced by $|N_p|$; hence N_p is very ample.

Take a very ample divisor of the form N_p with associated embedding $f : X \to \mathbb{P}^3$ and, as in the proof of *Claim 2*, regard $f(X) \subset \mathbb{P}^1 \times \mathbb{P}^1$ as a divisor of bidegree $(2, g+1)$ on the quadric surface $\mathbb{P}^1 \times \mathbb{P}^1$. Let $q \in \mathbb{P}^1 \times \mathbb{P}^1$ be the point $(u_1(p), u_2(o))$. For a general p, we may assume $u_1(p) \neq u_2(o)$. With this assumption, we show the next

Claim 3: We have $q \notin f(X)$.

Proof of Claim 3: Assume $q \in f(X)$. The line $L_1 := \mathbb{P}^1 \times \{u_2(o)\}$ is tangent to $f(X)$ at $f(o)$ because it intersects $f(X)$ at $f(o)$ with multiplicity two. The line $L_2 := \{u_1(p)\} \times \mathbb{P}^1$ is tangent to $f(X)$ at $u_2(p)$, because it intersects $f(X)$ at $f(p)$ with multiplicity $g+1$. Since L_1, L_2 are lines in a different ruling of the quadric surface $\mathbb{P}^1 \times \mathbb{P}^1$, $L_1 \neq L_2$ and $L_1 \cap L_2$ is a single point. Note that $\{q\} = L_1 \cap L_2$. Since $f(X)$ is smooth at q, it has a unique tangent line at q. Thus $L_1 = L_2$, which is a contradiction.

Let $\pi_q : \mathbb{P}^3 \setminus \{q\} \to \mathbb{P}^2$ denote the linear projection from q. Since $q \notin f(X)$, $\pi_{q|f(X)}$ induces a morphism $\varphi : X \to \mathbb{P}^2$. Since $\deg(u_2) = 2$ and u_1 does not factor through u_2, φ is birational onto its image. To conclude the proof of the theorem, it is sufficient to prove the following claim.

Claim 4: The morphism φ is injective, o and p are the only ramification points of φ, and $\varphi(o)$ is an ordinary cusp of $\varphi(X)$.

Proof of Claim 4: Since f is an embedding and φ is induced by the linear projection from $q \in \mathbb{P}^1 \times \mathbb{P}^1 \setminus f(X)$, it is sufficient to prove that $|L \cap f(X)| \leq 1$ for each line $L \subset \mathbb{P}^3$ containing q. Fix a line $L \subset \mathbb{P}^3$ such that $q \in L$ and $\deg(f(X) \cap L) \geq 2$. Since $q \notin f(X)$ and $f(X) \subset \mathbb{P}^1 \times \mathbb{P}^1$, Bézout's theorem gives $L \subset \mathbb{P}^1 \times \mathbb{P}^1$. Thus L is one of the two lines of the smooth quadric $\mathbb{P}^1 \times \mathbb{P}^1$ passing through q. One of these lines meet $f(X)$ only at $f(o)$ (with multiplicity two), whereas the other one meets $f(X)$ only at $f(p)$ (with multiplicity $g+1$). In both cases, the set-theoretic intersection $L \cap f(X)$ consists only of one point. Thus φ is injective. Moreover, by the discussion above, o and p are the only ramification points. Since $\varphi^{-1}(o)$ is a curvilinear double point, $f(o)$ is a double point with one branch and so an ordinary cusp.

In conclusion, X has ∞^1-many g^2_{g+3} base-point free linear series, corresponding to the morphism $\varphi : X \to \mathbb{P}^2$, as p varies in $X \setminus \mathcal{W}$. (The degree of the linear series is indeed $g+3$, as φ is an injective morphism.) They sit inside a very ample g^3_{g+3}, given by N_p. Again, there are ∞^1-many of such, as p varies in $X \setminus \mathcal{W}$. Moreover, X has $2g+2$ such one-dimensional families of g^2_{g+3} and g^3_{g+3}, given by the choice $o \in \mathcal{W}$. □

Let $\sigma : X \to X$ be the *hyperelliptic involution* and let $R \in \mathrm{Pic}^2(X)$ be the hyperelliptic divisor of degree two, i.e., $g^1_2 = |R| = \{a + \sigma(a)\}_{a \in X}$.

Remark 3.2 (**Arbitrary characteristic**) We summarize the ingredients providing a similar proof of Theorem 3.1 in arbitrary characteristic. Unless otherwise stated, the statements used in the proof are valid for any algebraically closed field k and any $\mathrm{char}(k)$.

(In $\mathrm{char}(k) \neq 2$, the scheme $\mathcal{W}$ of ramification points satisfies $\deg(\mathcal{W}) = 2g+2$. In $\mathrm{char}(k) = 2$, one has $1 \leq \deg(\mathcal{W}) \leq g+1$, where each integer in this interval may occur for some hyperelliptic curve of genus g; see [28, p. 226], [20, §7]. In particular, in any characteristic, there is at least one ramification point.)

Independently of $\mathrm{char}(k)$, we show that if $p \in X \setminus \mathcal{W}$, then $h^1(\mathcal{O}_X(gp)) = 0$. Since X is hyperelliptic, the canonical map $\eta : X \to \mathbb{P}^{g-1}$ has as its image the degree $g-1$ rational normal curve and its fibers are the elements of $|R|$. Thus we have the following recipe to see if an effective divisor D on X is special. Let $D' \supset D$ be the following effective divisor: for each $o \in \mathcal{W}$, let m_o denote the multiplicity of o in $\mathcal{W}$; the multiplicity of $o \in D'$ is the minimal even integer $\geq m_o$. If $a \in X \setminus \mathcal{W}$ and $m_1, m_2 \in \mathbb{N}$ are the multiplicities of a and $\sigma(a)$ in D, then both a and $\sigma(a)$ appears in D' with multiplicity $\max\{m_1, m_2\}$. By construction, D' has even degree and it is the minimal divisor containing $\eta^{-1}(\eta(D))$, where, given $D = \sum m_i p_i$, we set $\eta(D) := \sum m_i \eta(p_1)$. Let $k := \deg(D')/2$. Note that $D' \in |R^{\otimes k}|$. By Serre duality $H^1(D) \cong H^0(K_X \otimes D^\vee)^\vee$ and the latter is isomorphic to $H^0(R^{\otimes(g-1)} \otimes D^\vee)$. Hence $h^1(D) > 0$ if and only if $k \leq g-1$. In particular, let $p \in X \setminus \mathcal{W}$ and $D = \mathcal{O}_X(gp)$. Hence $D' = \mathcal{O}_X(gp + g\sigma(p))$ and so $\deg(D') = 2g$. Thus $h^1(\mathcal{O}_X(gp)) = 0$.

Remark 3.3 Take $f(X)$ as in the proof of Theorem 3.1. Call $\mathcal{S}$ the set of all $q \in \mathbb{P}^3 \setminus f(X)$ such that the linear projection $\pi_q : \mathbb{P}^3 \setminus \{q\} \to \mathbb{P}^2$ induces an injective map $\varphi_q : f(X) \to \mathbb{P}^2$. Call Q the quadric surface containing $f(X)$ as an element of $|\mathcal{O}_Q(2, g+1)|$.

(i) We describe the set $\mathcal{S} \cap Q$. Fix $q = (q_1, q_2) \in Q \setminus f(X)$ and set $L_1 := \mathbb{P}^1 \times \{q_2\} \in |\mathcal{O}_Q(0,1)|$ and $L_2 := \{q_1\} \times \mathbb{P}^1 \in |\mathcal{O}_Q(1,0)|$. Let $L \subset \mathbb{P}^3$ be a line such that $q \in L$ and $\deg(L \cap f(X)) \geq 2$. Since $q \notin f(X)$, Bézout's theorem gives

$L \subset Q$. Hence $L \in \{L_1, L_2\}$. Thus $q \in \mathcal{S}$ if and only if both L_1 and L_2 contain a unique point of $f(X)$. By definition of f, given in the proof of Theorem 3.1, L_1 meets $f(X)$ at a unique point, a_1, if and only if $a_1 = f(p_1)$ for some $p_1 \in \mathcal{W}$. Recall that $h^0(\mathcal{O}_X((g+1)p)) = 2$, $w = (u_1, u_2)$ where u_1 is induced by the linear series $|\mathcal{O}_X((g+1)p))|$. By definition of f given in the proof of Theorem 3.1, L_2 meets $f(X)$ at a unique point, a_2, if and only if $a_2 = f(p_2)$ for some $p_2 \in X$ such that $\mathcal{O}_X((g+1)p_2) \cong \mathcal{O}_X((g+1)p)$. The number of these points may depend on g, X and p. However, there is at least one such pair of points $(p_1, p_2) \in X \times X$, i.e., the pair (o, p) ($\mathcal{W} \neq \emptyset$ in any characteristic).

(ii) Fix $q \notin Q$ and take a line L such that $q \in L$ and $\deg(L \cap f(X)) \geq 2$. Since $q \notin Q$, we have $L \nsubseteq Q$. Thus $\deg(L \cap f(X)) = 2$, by Bézout's theorem. In particular, each line L through q which is tangent to $f(X)$, say at a point q', has order of vanishing two with $f(X)$ at q', and $L \cap (f(X) \setminus \{q'\}) = \emptyset$. Thus any unibranch point of $\pi_q(f(X))$ is an ordinary cusp. If the degree $g+3$ curve $\pi_q(f(X))$ is unibranch, then it has $(g+2)(g+1)/2 - g$ cusps. Tono [29, Theorem 1.1] showed that a cuspidal plane curve has at most $(21g+17)/2$ cusps. Thus, since $(g+2)(g+1)/2 - g > (21g+17)/2$ for $g \gg 0$, one has $\mathcal{S} \subset Q$ for $g \gg 0$. A generalization of Tono's result to Hirzebruch surfaces was found by Moe [22].

Proposition 3.4 *Let X be a hyperelliptic curve of genus $g \geq 3$. Take any non-special and base-point free $N \in \mathrm{Pic}^{g+3}(X)$ inducing an injective map $\varphi : X \to \mathbb{P}^3$. Either φ is an embedding and its image $\varphi(X)$ is contained in a smooth quadric Q as a divisor of bidegree $(2, g+1)$ or $(g+1, 2)$, or $\varphi(X)$ is contained in a quadric cone.*

Proof Since N is non-special, $h^1(N) = 0$. Thus Riemann-Roch gives $h^0(N) = 4$.

Set $M = N \otimes R^\vee \in \mathrm{Pic}^{g+1}(X)$. Fix $a \in X$. Since $h^0(N) = 4$, we have $h^0(N(-a-\sigma(a))) \geq 2$ and so $h^0(M) \geq 2$. Since φ is injective, we have $\varphi(a) \neq \varphi(\sigma(a))$ for $a \in X \setminus \mathcal{W}$. Thus $h^0(M) = h^0(N) - 2 = 2$.

Since $\deg(M) = g+1$, and $h^0(M) = 2$, Riemann-Roch implies that M is non-special.

Assume that M has a base point, say $b \in X$. Since $h^0(M(-b)) = 2$ and $\deg(M(-b)) = g$, $M(-b)$ is special, with $h^0(M(-b)) = 2$. Thus $|M(-b)| = R \otimes E$ for a fixed effective divisor E with $\deg(E) = g - 2 > 0$. Note that $M \cong R(E+b)$. Since by definition $M = N \otimes R^\vee$, tensoring by R both sides yields $N \cong R^{\otimes 2}(E+b)$.

Note that the divisor E is a fixed component for the linear series associated to $N(-b)$. Indeed, this holds if and only if $h^0(N(-b-E)) = h^0(N(-b))$. Moreover, $N(-b-E) \cong R^{\otimes 2}$ and so $h^0(N(-b-E)) = h^0(R^{\otimes 2}) = 3$. Furthermore, $h^0(N(-b)) = 3 = h^0(N) - 1$, as N is base-point free. Since $N(-b-E)$ is base-point free and $h^0(N(-b-E)) = 3$, this induces a map from X to $\mathbb{P}^2$, which factor through φ (the morphism induced by N). More precisely, $N(-b-E)$ induces $\pi_{\varphi(b)} \circ \varphi : X \to \mathbb{P}^2$, where $\pi_{\varphi(b)}$ is the the linear projection with center the point $\varphi(b)$. On the other hand, $R^{\otimes 2}$ has $\deg(R^{\otimes 2}) = 4$ and induces a $2:1$ cover $X \to \mathbb{P}^1 \subset \mathbb{P}^2$, where $\mathbb{P}^1 \subset \mathbb{P}^2$ is a smooth conic. As φ is injective, $\pi_{\varphi(b)}$ is a $2:1$ cover of a smooth conic. This is possible only if $\varphi(X)$ is contained in a quadric cone such that $\varphi(b)$ is a vertex.

The map $\pi_{\varphi(b)} \circ \varphi$ sends all the points in the support $\mathrm{Supp}(E)$ of E to $\varphi(b)$ (because E is a fixed component of $N(-b)$) and since φ is injective, $\mathrm{Supp}(E) \subseteq \{b\}$. Since $\deg(E) = g-2$, then $E = (g-2)b$. Hence $N = R^{\otimes 2} \otimes \mathcal{O}_X((g-1)b)$.

Conversely for a general $b \in X$, the linear series $|R^{\otimes 2}((g-1)b)|$ is non-special by Remark 3.2 and gives an injective map with image contained in a quadric cone.

Suppose M is base-point free and call $\psi : X \to \mathbb{P}^1$ the morphism induced by $|M|$. Since M is base-point free, $h^0(M(-b)) = h^0(M) - 1 = 1$ for every $b \in X$. Then Riemann-Roch gives $h^1(M(-b)) = 0$ for every $b \in X$. As in the proof of Theorem 3.1, we see that N induces an embedding with image contained in a smooth quadric Q as a divisor of bidegree $(2, g+1)$ or $(g+1, 2)$. □

Proposition 3.5 *Keep the notation from Proposition 3.4. Assume $\varphi(X)$ is contained in a quadric cone. Then φ is an embedding if and only if $g = 2$.*

Proof Recall that in this case $M = N \otimes R^\vee$ has b as base point.

Suppose $g = 2$, then $\deg(N) = g + 3 = 2g + 1$ and so N is very ample, and hence an embedding. Suppose $g > 2$, then $E = (g-2)b$ is non-zero. Recall that E is the fixed component of $N(-b)$. Hence $h^0(N(-b)) = h^0(N(-2b))$. Thus φ is not an embedding, as N does not divide tangent directions, i.e., the differential of φ is not injective at b. (Note that for $g = 2$, $\varphi(X)$ has degree 5 [16, Example V.2.9].)

Assume $g \geq 3$ and that $\varphi(X)$ is contained in a quadric cone $\mathcal{C}$ with vertex $v = \varphi(b)$ and take $q \in \mathcal{C} \setminus \varphi(X)$. Here we check that the linear projection from q does not induce an injective map $X \to \mathbb{P}^2$. Call R_q the unique line on $\mathcal{C}$ containing q. By Bézout's theorem, for each line L containing q and with $\deg(L \cap \varphi(X)) \geq 2$, we have $L \subset \mathcal{C}$. Recall that $v = \varphi(b)$ is the vertex of $\mathcal{C}$. Since the vertex $\varphi(b) \in \varphi(X)$, the projection π_q of $\varphi(X)$ from q is a cuspidal projection if and only if $R_q \cap \varphi(X) = \{\varphi(b)\}$ set-theoretically.

We show this is not the case. The linear projection π_v of $\varphi(X)$ from v is a $2:1$ morphism (away from $v \in \varphi(X)$), whose image is a smooth conic, i.e., the base of the cone $\mathcal{C}$. Thus the ramification points of π_v are the images of the Weierstrass points $\varphi(\mathcal{W})$, the image of the ramification points of the covering map induced by R. However, since $b \notin \mathcal{W}$, the point $\varphi(b)$ is not a ramification point. Thus R_q cannot intersect $\varphi(X)$ only at $\varphi(b)$, i.e., π_q is not a cuspidal projection. □

Recall that $\sigma : X \to X$ denotes the hyperelliptic involution and $R \in \mathrm{Pic}^2(X)$ is the hyperelliptic divisor of degree two, i.e., $g^1_2 = |R| = \{a + \sigma(a)\}_{a \in X}$. With this notation, we are ready to prove the next result.

Proposition 3.6 *Let X be a hyperelliptic curve of genus $g \geq 3$. There is an injective morphism $f : X \to \mathbb{P}^2$ with $\deg(f(X)) = g + 2$ and each such map f is induced by a complete linear series $|N|$ with $h^1(N) = 0$ and $N \cong R(gp)$ with $p \in X$ such that $h^1(\mathcal{O}_X(gp)) = 0$ (e.g., with p general in X).*

Proof Since every special base-point free linear series on X is composed with the g^1_2, each injective morphism $X \to \mathbb{P}^2$ must be induced by a non-special base-point free linear series g^2_d. By Riemann-Roch this linear series is complete if and only if $d = g + 2$, whereas if $d > g + 2$ this g^2_d is a linear subspace of a base-point free and non-special complete g^{d-g}_d.

Claim 1: For every $N \in \mathrm{Pic}^{g+2}(X)$, $g \geq 3$, with $h^1(N) = 0$ and N base-point free there is a degree g effective divisor B with $N \cong R(B)$ and $h^1(B) = 0$.

Proof of Claim 1: Fix $a \in X$. Since every degree two effective divisor of X is contained in a prescribed g^2_k, there is a degree g effective divisor B such that $a + \sigma(a) + B \in |N|$. Note that $a + \sigma(a) \in g^1_2$. Hence $h^0(N \otimes R^\vee) > 0$. Take $B \in |N \otimes R^\vee|$. If $h^1(B) > 0$, then B is special and so $B = R \otimes N'$, where N' is some effective divisor of degree $g-2$. Thus $N \cong R^{\otimes 2} \otimes N'$. Since $\deg(N') = g - 2 > 0$, and $h^0(R^{\otimes 2}) = 3 = h^0(N)$, every point in the support of a divisor of N' is a base point of N, a contradiction.

Thus, so far we have shown that $N \cong R(B)$ for some effective divisor B with $\deg(B) = g$ and $h^1(B) = 0$. Since by assumption $R \otimes B$ induces a map to $\mathbb{P}^2$, it is base-point free. Note that $R(B)$ is base-point free if and only if $h^0(R(B-p)) = 2$ for each p in the support of B; indeed, since R is base-point free, the base locus of $R(B)$ has to be contained in the support of B. Moreover, since by assumption N induces an injective morphism and $h^0(R(B-p)) = 2 = h^0(R)$, $|R(B)|$ maps all the points in the support of B to the same point of $\mathbb{P}^2$. Therefore $B = gp$ for some $p \in X$ (and such that $h^1(\mathcal{O}_X(gp)) = 0$).

Conversely, assume $h^1(\mathcal{O}_X(gp)) = 0$ and set $N := R(gp)$. Call $\varphi : X \to \mathbb{P}^2$ the morphism induced by the non-special and base-point free linear series $|N|$. We claim that φ is injective. Fix $a, b \in X$ with $a \neq b$. First assume $\varphi(a) = \varphi(p)$. Thus $|R(gp-a)| = |R((g-1)p)|$, with $h^0(R(g-1)p) = 2 = h^0(R)$. Hence $(g-1)p$ is the base locus of $R((g-1)p)$ and so of $R(gp-a)$. This implies $a = p$.

Now assume $a \neq p$ and $b \neq p$. Since $\varphi(a) = \varphi(b)$, by definition b is a base point of $|R(gp-a)|$. Thus $|R(gp-a)| = \{b+E\}_{E \in |R(gp-a-b)|}$. Since $\deg(R(gp-a-b)) = g$ and $h^0(R(gp-a-b)) = 2$, by Riemann-Roch we obtain $h^1(R(gp-a-b)) > 0$ and hence $R(gp-a-b)$ is a special divisor. Thus $R(gp-a-b) = R \otimes F$, for some effective degree $g-2$ divisor F on X. So $gp - a - b$ is an effective divisor. This is possible if and only if $a = b = p$, which is a contradiction. □

4 Quadrics and cuspidal projections

In this section, we study more closely cuspidal projections of curves lying on irreducible quadrics in $\mathbb{P}^3$. In the first results the characteristic of our ground field k is arbitrary. Only later, we will switch to characteristic zero. We start off considering curves on smooth quadrics.

Proposition 4.1 *Let $Q \subset \mathbb{P}^3$ be a smooth quadric surface. Fix integers $1 \leq d_1 \leq d_2$ such that $d_2 \geq 2$. Let Y be an integral element of $|\mathcal{O}_Q(d_1, d_2)|$ with only unibranch singularities and let $\varphi : X \to Y$ be its normalization. Take $q \in Q \setminus Y$ and let $L \in |\mathcal{O}_Q(1,0)|$ and $L' \in |\mathcal{O}_Q(0,1)|$ be the unique lines of Q through q. The linear projection from q induces an injective map $\pi_q : Y \to \mathbb{P}^2$ (and hence an injective map $\eta = \pi_q \circ \varphi : X \to \mathbb{P}^2$) if and only if each L and L' contains a unique point of Y. Moreover, the morphisms π_q and η are separable.*

Proof Notice that π_q is a morphism, because $q \notin Y$. Since φ is bijective and separable, and an isomorphism outside finitely many points of X, π_q is injective (resp., separable) if and only if η has the same property. If η is injective then $|L \cap Y| = |L' \cap Y| = 1$. To show the converse, it is sufficient to remark that $|L'' \cap Y| = 1$ for each line $L'' \subset \mathbb{P}^3$ such that $q \in L''$ and $L'' \notin \{L, L'\}$.

Now we explain why π_q and η are separable morphisms. As mentioned above, η is separable if and only if π_q is separable. Separability must be checked only if $\mathrm{char}(k) > 0$, as it is immediate in characteristic zero. Fix $p \in Y_{\mathrm{reg}}$ such that $p \notin L \cup L'$; call $L_p \subset \mathbb{P}^3$ the line spanned by $\{q, p\}$. Since $p \notin L \cup L'$, we have $L_p \notin \{L, L'\}$ and hence $\deg(L_p \cap Q) = 2$. Thus the differential of η at p is injective. This shows that the differential is generically injective, and so η is separable. □

Remark 4.2 The proof of Proposition 4.1 shows that if $d_1 > 1$ there are only finitely many cuspidal projections. (Note that if $(d_1, d_2) \neq (2, 2)$, having at least one cuspidal projection is a closed condition on the smooth curves of bidegree (d_1, d_2).) For curves of bidegree $(1, d_2)$, which are smooth and rational, if there is a cuspidal projection from a point o, then any point on $Q \setminus Y$ and on the line of bidegree $(1, 0)$ containing o induces a cuspidal projection.

Let $Q \subset \mathbb{P}^3$ be a smooth quadric. For any $p \in Q$, let $(2p, Q)$ denote the closed subscheme of Q with $(\mathcal{I}_{p,Q})^2$ as its ideal sheaf. The scheme $(2p, Q)$ has degree 3 and $\{p\}$ is its support.

For any zero-dimensional scheme $W \subset Q$ and any effective divisor $E \subset Q$, let $\mathrm{Res}_E(W)$ denote the closed subscheme of Q whose ideal sheaf is $\mathcal{I}_W : \mathcal{I}_E$. If $E \in |\mathcal{O}_Q(a, b)|$ the following sequence (usually called the *residual exact sequence* with respect to E)

$$0 \to \mathcal{I}_{\mathrm{Res}_E(W)}(k - a, h - b) \to \mathcal{I}_W(k, h) \to \mathcal{I}_{W \cap E, E}(k, h) \to 0$$

is exact. If $p \notin E$ (resp. p is a smooth point of E, resp. p is a singular point of E), then $E \cap (2p, Q) = \emptyset$ (resp. $\deg(E \cap (2p, Q)) = 2$, resp. $(2p, Q) \subset E$) and hence $\mathrm{Res}_E((2p, Q)) = (2p, Q)$ (resp. $\mathrm{Res}_E((2p, Q)) = \{p\}$, resp. $\mathrm{Res}_E((2p, Q)) = \emptyset$).

The next result shows how zero-dimensional schemes may naturally provide information on the *existence* of cuspidal projections and therefore injective linear series. (Note that, for $\mathrm{char}(k) = 0$, its proof may be simplified using the classical Bertini's theorem [16, Corollary III.10.9].)

Theorem 4.3 *Fix integers $d_2 \geq d_1 > 0$, a smooth quadric $Q \subset \mathbb{P}^3$, lines $L \in |\mathcal{O}_Q(1, 0)|$, $L' \in |\mathcal{O}_Q(0, 1)|$ and set $\{q\} := L \cap L'$. Fix $o \in L \setminus \{q\}$ and $o' \in L' \setminus \{q\}$. Let $Z \subset L$ (resp. $Z' \subset L'$) be the zero-dimensional subscheme of L (resp. L') of degree d_2 (resp. degree d_1) with support o (resp. o'). Then there is a smooth divisor $Y \in |\mathcal{O}_Q(d_1, d_2)|$ such that $Y \cap L = Z$ and $Y \cap L' = Z'$ (scheme-theoretically and hence set-theoretically they intersect at a unique point).*

Proof Since $h^1(\mathcal{O}_Q(d_1 - 1, d_2 - 1)) = 0$, from the residual exact sequence

$$0 \to \mathcal{O}_Q(d_1 - 1, d_2 - 1) \to \mathcal{I}_{Z \cup Z'}(d_1, d_2) \to \mathcal{I}_{Z \cup Z', L \cup L'}(d_1, d_2) \to 0,$$

it follows $h^0(\mathcal{I}_{Z \cup Z'}(d_1, d_2)) = d_1 d_2 + 1$. Similarly, one can directly check that $h^0(\mathcal{I}_{Z'}(d_1 - 1, d_2)) = d_1(d_2 + 1) - d_1$, and $h^0(\mathcal{I}_Z(d_1, d_2 - 1)) = (d_1 + 1)d_2 - d_2$. (This is the stabilization of the Hilbert function to the Hilbert polynomial.)

To prove the statement, it is sufficient to prove that a general $Y \in |\mathcal{I}_{Z\cup Z'}(d_1, d_2)|$ is smooth. Since $h^0(\mathcal{I}_{Z'}(d_1-1, d_2)) = d_1(d_2+1) - d_1$ and $h^0(\mathcal{I}_Z(d_1, d_2-1)) = (d_1+1)d_2 - d_2$, neither L nor L' is an irreducible component of Y, by dimensional count.

Assume $d_2 \ge d_1 \ge 2$. Since $|\mathcal{I}_{Z\cup Z'}(d_1, d_2)|$ contains all curves of the form $F \cup L \cup L'$, where $F \in |\mathcal{O}_Q(d_1-1, d_2-1)|$, and $\mathcal{O}_Q(d_1-1, d_2-1)$ is very ample, the linear series $|\mathcal{I}_{Z\cup Z'}(d_1, d_2)|$ separates points and tangent vectors of $Q \setminus (L \cup L')$, i.e., it induces an embedding $\psi : Q \setminus (L \cup L') \to \mathbb{P}^{d_1 d_2}$. Thus the general element in $|\mathcal{I}_{Z\cup Z'}(d_1, d_2)|$ is smooth outside the locus $L \cup L'$.

Claim: In order to prove the theorem it is sufficient to prove the following statements:

(i) $h^0(\mathcal{I}_{Z\cup Z'\cup(2o,Q)}(d_1, d_2)) = d_1 d_2$;
(ii) $h^0(\mathcal{I}_{Z\cup Z'\cup(2o',Q)}(d_1, d_2)) = d_1 d_2$;
(iii) $h^0(\mathcal{I}_{Z\cup Z'\cup(2q,Q)}(d_1, d_2)) = d_1 d_2$;
(iv) for each $m \in L \setminus \{q, o\}$ we have $h^0(\mathcal{I}_{Z\cup Z'\cup(2m,Q)}(d_1, d_2)) = d_1 d_2 - 1$;
(v) for each $m' \in L' \setminus \{q, o'\}$ we have $h^0(\mathcal{I}_{Z\cup Z'\cup(2m',Q)}(d_1, d_2)) = d_1 d_2 - 1$.

Proof of the Claim: Recall that it is sufficient to prove that a general $Y \in |\mathcal{I}_{Z\cup Z'}(d_1, d_2)|$ is smooth at each point of $(L \cup L') \cap Y$. Since $h^0(\mathcal{I}_{Z\cup Z'}(d_1, d_2)) = d_1 d_2 + 1$, (i) (resp. (ii), resp. (iii)) shows that $(2o, Q) \nsubseteq Y$ (resp. $(2o', Q) \nsubseteq Y$, resp. $(2q, Q) \nsubseteq Y$) for a general $Y \in |\mathcal{I}_{Z\cup Z'}(d_1, d_2)|$, i.e. (i), (ii) and (iii) give the smoothness of Y at o, o' and (if $q \in Y$) at q. Since $\dim L = \dim L' = 1$, (iv) and (v) imply that Y is smooth at all points of $L \cup L' \setminus \{o, o', q\}$.

Now we prove (i). Since $d_2 \ge d_1 \ge 2$, $(Z \cup (2o, Q)) \cap L = Z$. Since L is smooth, $\mathrm{Res}_L(Z \cup Z' \cup (2o, Q)) = Z' \cup \{o\}$. Consider the residual exact sequence with respect to L:

$$0 \to \mathcal{I}_{Z'\cup\{o\}}(d_1-1, d_2) \to \mathcal{I}_{Z\cup Z'\cup(2o,Q)}(d_1, d_2) \to \mathcal{I}_{Z,L}(d_1, d_2) \to 0. \tag{1}$$

Since $d_1 - 1 > 0$, $h^0(\mathcal{I}_{Z'\cup\{o\}}(d_1-1, d_2)) = d_1(d_2+1) - d_1 - 1$ and $h^1(\mathcal{I}_{Z'\cup\{o\}}(d_1-1, d_2)) = 0$. Note that $\mathcal{O}_L(d_1, d_2)$ is the degree d_2 line bundle on the smooth genus 0 curve L. Thus $h^0(L, \mathcal{I}_{Z,L}(d_1, d_2)) = 1$. Hence the cohomology exact sequence of (1) gives (i). The proof of (ii) is similar.

To show (iii), consider the residual exact sequence with respect to $L \cup L'$ (we use that $L \cup L'$ is singular at q):

$$0 \to \mathcal{O}_Q(d_1-1, d_2-1) \to \mathcal{I}_{Z\cup Z'\cup(2q,Q)}(d_1, d_2) \to \mathcal{I}_{Z\cup Z'\cup(2q,Q),L\cup L'}(d_1, d_2) \to 0 \tag{2}$$

We have $h^0(\mathcal{O}_Q(d_1-1, d_2-1)) = d_1 d_2$. Since $q \notin \{o, o'\}$, $\deg(L \cap (Z \cup (2q, Q))) = d_2 + 2$ and $\deg(L' \cap (Z' \cup (2q, Q))) = d_1 + 2$. Since $\mathcal{O}_L(d_1, d_2)$ is the degree d_2 line bundle on L and $\mathcal{O}_{L'}(d_1, d_2)$ is the degree d_1 line bundle on L', $h^0(L \cup L', \mathcal{I}_{Z\cup Z'\cup(2q,Q),L\cup L'}(d_1, d_2)) = 0$. Thus the cohomology exact sequence of (2) gives (iii).

For (iv), fix $m \in L \setminus \{q, o\}$. Since $L \cap Z' = \emptyset$, the residual exact sequence with respect to L gives the following exact sequence:

$$0 \to \mathcal{I}_{Z' \cup \{m\}}(d_1 - 1, d_2) \to \mathcal{I}_{Z \cup Z' \cup (2m,Q)}(d_1, d_2) \to \mathcal{I}_{(Z \cup (2m,Q)) \cap L, L}(d_1, d_2) \to 0 \quad (3)$$

We have $h^0(\mathcal{I}_{Z' \cup \{m\}}(d_1-1, d_2)) = d_1(d_2+1)-d_1-1$. Since $\deg(L \cap (Z \cup (2m, Q))) = d_2 + 2$, we have $h^0(L, \mathcal{I}_{(Z \cup (2m,Q)) \cap L, L}(d_1, d_2)) = 0$. Thus the cohomology exact sequence of (3) gives $h^0(\mathcal{I}_{Z \cup Z' \cup (2m,Q)}(d_1, d_2)) = h^0(\mathcal{I}_{Z' \cup \{m\}}(d_1-1, d_2)) = d_1 d_2 - 1$. Similarly, (v) is derived using the residual exact sequence with respect to L'.

Now assume $d_1 = 1$. Take any $Y \in |\mathcal{O}_Q(1, d_2)|$ and assume that Y has a singular point $z \in Q \setminus \{o, o'\}$. Let R_z be the element of $|\mathcal{O}_Q(0, 1)|$ passing through z. Since Y is singular at z, Bézout's theorem gives $Y = R_z \cup G$, for some $G \in |\mathcal{O}_Q(1, d_2 - 1)|$. If $Y \in |\mathcal{I}_{Z \cup Z'}(1, d_2)|$ and $R_z \cap \{o, o'\} = \emptyset$ (this condition holds for a general Y), we obtain $G \in |\mathcal{I}_{Z \cup Z'}(1, d_2 - 1)|$. Notice that $h^0(\mathcal{I}_{Z \cup Z'}(1, d_2 - 1)) \le d_2$. Thus a general $Y \in |\mathcal{I}_{Z \cup Z'}(1, d_2)|$ is smooth outside $\{o, o'\}$. One also verifies that the general Y is smooth at o and o'. □

Remark 4.4 Let $d_1 = 1$, $d_2 = d - 1 \ge 2$ and Y be a smooth rational curve. By Proposition 4.1 and Theorem 4.3, we obtain that for each integer $d \ge 3$ there is a smooth rational curve $Y \subset \mathbb{P}^3$ with $\deg(Y) = d$ and admitting ∞^1 cuspidal projections to $\mathbb{P}^2$. Compare this observation with the statement [24, (a) of Remark, p. 102] saying that, for $d \ge 5$, no smooth degree d rational curve has a cuspidal projection with as its image a plane curve with only ordinary cusps.

We now establish Proposition 4.1 and Theorem 4.3 for quadric cones in $\mathbb{P}^3$.

Proposition 4.5 *Let $\mathcal{C} \subset \mathbb{P}^3$ be a quadric cone with vertex v. Fix an integer $d \ge 2$. Let $Y \in |\mathcal{O}_{\mathcal{C}}(d)|$ be an integral curve with only unibranch singularities and let $\varphi : X \to Y$ be its normalization. Fix $q \in \mathcal{C} \setminus Y$ with $q \neq v$ and let R_q be the line of $\mathcal{C}$ containing q. The linear projection from q induces a cuspidal projection of Y (and hence of X taking the composition with the injective map φ) if and only if $|R_q \cap Y| = 1$.*

Proof Take a line $L \subset \mathbb{P}^3$ such that $q \in L$ with $\deg(Y \cap L) \ge 2$. Since $q \in \mathcal{C} \setminus Y$, we have $\deg(L \cap \mathcal{C}) \ge 3$. Thus $L \subset \mathcal{C}$, by Bézout's theorem. Since $q \in L$, $L = R_q$. □

Theorem 4.6 *Let $\mathcal{C} \subset \mathbb{P}^3$ be a quadric cone with vertex v. Fix an integer $d \ge 2$ and $q \in \mathcal{C} \setminus \{v\}$. Let $R_q \subset \mathcal{C}$ be the line spanned by $\{v, q\}$. Then there is a smooth divisor $Y \in |\mathcal{O}_{\mathcal{C}}(d)|$ such that $v \notin Y$, $q \notin Y$ and R_q meets Y at a unique point.*

Proof Fix $p \in R_q \setminus \{v\}$. Let $Z = d \cdot p$ be the effective divisor of degree d of R_q supported at $p \in \mathcal{C}$, regarded as a zero-dimensional subscheme of $\mathcal{C}$. It is sufficient to prove that a general element of $|\mathcal{I}_Z(d)|$ is smooth and it does not contain the vertex v.

Let $\eta : \mathbb{F}_2 \to \mathcal{C}$ be the minimal desingularization of $\mathcal{C}$; here $\mathbb{F}_2$ denotes the second *Hirzebruch surface*: this is the rational ruled surface $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-2))$. Thus we have a projection map $\pi : \mathbb{F}_2 \to \mathbb{P}^1$ and a section $H := \eta^{-1}(v)$ with self-intersection $H^2 = -2$. Its Picard group $\mathrm{Pic}(\mathbb{F}_2)$ is freely generated by the Cartier divisors H and a fiber F of π, with $F \cdot H = 1$ and $F^2 = 0$ [16, Proposition V.2.3].

The linear series $|\mathcal{O}_{\mathbb{F}_2}(H+2F)|$ is base-point free, induces η and indeed contracts H to a point. Hence $\eta^*(\mathcal{O}_{\mathcal{C}}(d)) = \mathcal{O}_{\mathbb{F}_2}(dH+2dF)$, for every $d \geq 1$. In fact, $\eta^*(H^0(\mathcal{O}_{\mathcal{C}}(d))) = H^0(\mathcal{O}_{\mathbb{F}_2}(dH+2dF))$. Indeed, since $\mathcal{C}$ is a quadric and it is projectively normal, one has $h^0(\mathcal{O}_{\mathcal{C}}(d)) = h^0(\mathcal{O}_{\mathbb{P}^3}(d)) - h^0(\mathcal{O}_{\mathbb{P}^3}(d-2)) = (d+1)^2$. On the other hand, note that $\pi_*\mathcal{O}_{\mathbb{F}_2}(dH) \cong \mathrm{Sym}^d(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-2)) \cong \bigoplus_{i=0}^d \mathcal{O}_{\mathbb{P}^1}(-2i)$ by [16, Proposition V.2.8.] and [16, Exercise III. 8.4]. As F is a fiber of π over $\mathbb{P}^1$, $\mathcal{O}_{\mathbb{F}_2}(F) \cong \pi^*(\mathcal{O}_{\mathbb{P}^1}(1))$. The projection formula [16, Exercise II. 5.1] gives the isomorphism $\pi_*(\mathcal{O}_{\mathbb{F}_2}(dH+2dF)) \cong \oplus_{i=0}^d \mathcal{O}_{\mathbb{P}^1}(2d-2i)$. Now, the dimension of the space of global sections of the latter coincides with $h^0(\mathcal{O}_{\mathcal{C}}(d))$. Thus $\eta^*(H^0(\mathcal{O}_{\mathcal{C}}(d))) = H^0(\mathcal{O}_{\mathbb{F}_2}(dH+2dF))$.

Since $p \neq v$, the scheme $A := \eta^{-1}(Z)$ is a degree d zero-dimensional scheme and $h^0(\mathcal{C}, \mathcal{I}_Z(d)) = h^0(\mathbb{F}_2, \mathcal{I}_A(dH+2dF))$. Therefore, to prove the statement it is sufficient to prove that a general $W \in |\mathcal{I}_A(dH+2dF)|$ is smooth and $H \cap W = \emptyset$.

Let $R_A \in |\mathcal{O}_{\mathbb{F}_2}(F)|$ denote the element containing A (i.e., it is the strict transform of R_q). The residual exact sequence of R_A in $\mathbb{F}_2$ gives the exact sequence

$$0 \to \mathcal{O}_{\mathbb{F}_2}(dH+(2d-1)F) \to \mathcal{I}_A(dH+2dF) \to \mathcal{I}_{A,R_A}(dH+2dF) \to 0. \quad (4)$$

With analogous computations as above, we have $\pi_*(\mathcal{O}_{\mathbb{F}_2}(dH+(2d-1)F)) \cong \oplus_{i=0}^d \mathcal{O}_{\mathbb{P}^1}(2d-1-2i)$ and so $h^1(\pi_*(\mathcal{O}_{\mathbb{F}_2}(dH+(2d-1)F)) = 0$. By [16, Lemma V.2.4], we have $H^1(\mathcal{O}_{\mathbb{F}_2}(dH+(2d-1)F))) \cong H^1(\mathbb{P}^1, \pi_*(\mathcal{O}_{\mathbb{F}_2}(dH+(2d-1)F)) = 0$ and hence $h^1(\mathcal{O}_{\mathbb{F}_2}(dH+(2d-1)F)) = 0$.

Since $R_A \cong \mathbb{P}^1$ and A is a zero-dimensional scheme of degree d, $\mathcal{O}_{R_A}(dH+2dF)$ has degree d and so $h^0(R_A, \mathcal{I}_{A,R_A}(dH+2dF)) = 1$. Thus $W \in |\mathcal{I}_A(dH+2dF)|$ containing R_A are of codimension one. Therefore, the general W does not contain R_A, which yields that $\eta(W)$ does not intersect R_q outside Z.

The divisor $\mathcal{O}_{\mathbb{F}_2}((d-1)H+(2d-1)F)$ is very ample. Since $H \cup R_A \cup G \in |\mathcal{I}_A(dH+2dF)|$ for all $G \in |\mathcal{O}_{\mathbb{F}_2}(d-1)H+(2d-1)F)|$, the linear system $|\mathcal{I}_A(dH+2dF)|$ induces an embedding outside $H \cup R_A$. By a characteristic free version of Bertini's theorem for embeddings of quasi-projective varieties [17, Th. 6.3, (3)], a general $W \in |\mathcal{I}_A(dH+2dF)|$ is smooth outside $H \cup R_A$.

We only need to check that a general W is smooth at $q = \eta^{-1}(p)$, the support of A. Since smoothness at q is an open condition, it is sufficient to exhibit a $W' \in |I_A(dH+2dF)|$ that is smooth at q: take $W' = G \cup H \cup R_A$ with $G \in |\mathcal{O}_{F_2}((d-1)H+(2d-1)F)|$ and $q \notin G$; it is possible to choose such a G as $\mathcal{O}_{\mathbb{F}_2}((d-1)H+(2d-1)F)$ is very ample and in particular base-point free.

Moreover, $H \cap W = \emptyset$ for a general $W \in |\mathcal{I}_A(dH+2dF)|$, as $h^0(\mathcal{O}_{\mathbb{F}_2}((d-1)H+2dF)) < h^0(\mathcal{O}_{\mathbb{F}_2}(dH+2dF))$ and one has zero intersection index between the two: $W \cdot H = (dH+2dF) \cdot H = dH^2 + 2dF \cdot H = -2d + 2d = 0$. □

Example 4.7 Let $X \subset \mathbb{P}^3$ be the canonical model of a smooth and non-hyperelliptic curve of genus 4. It is known that X is the complete intersection of an integral quadric $\mathcal{C}$ and a cubic surface; moreover, such $\mathcal{C}$ is smooth if and only if X has two different g_3^1's (in this case the g_3^1's are induced by the two rulings of $\mathcal{C}$), whereas if $\mathcal{C}$ is a quadric cone with vertex v, then $v \notin X$ and X has set-theoretically a unique g_3^1. This g_3^1 is induced by the linear projection from v; see [18, §4]. The case $\mathcal{C}$ smooth is the one described in Proposition 4.1 with $X = Y$ and $d_1 = d_2 = 3$. By Proposition 4.1 and Proposition 4.3, we obtain that X has a cuspidal projection if all the g^1_3 on X have a total ramification point, i.e., $3p \in g^1_3$ for some $p \in X$. The converse holds if we only consider projections from points of the quadric surface containing X. Furthermore, [24, Theorem 2] gives a different and stronger result: the canonical model of a general curve of genus 4 has no cuspidal projections.

Equipped with the terminology in Definition 1.5, we may state the following result, which provides positive examples to Question 1.1.

Theorem 4.8 *Let k be an algebraically closed field of arbitrary characteristic and let $d_2 \ge d_1 \ge 1$. Then there exists a smooth genus g curve with an injective $g^2_{d_1+d_2}$ of type* **II** *with $g = d_1 d_2 - d_1 - d_2 + 1$.*

Proof This is a consequence of Theorem 4.1 and Theorem 4.6 on smooth quadrics and cones, respectively. □

Henceforth, we assume $\mathrm{char}(k) = 0$.

Definition 4.9 For each germ (Y, z) of an isolated one-dimensional singularity, the *singularity degree* $\delta(Y, z)$ is the nonnegative integer such that, given the normalization $\nu : X \to Y$, the arithmetic genus of Y is $p_a(Y) = p_a(X) + \sum_{z \in \mathrm{Sing}(Y)} \delta(Y, z)$.

Definition 4.10 For each $h \ge 1$, a singularity A_{2h} is a plane curve singularity which is formally equivalent to the singularity $y^2 = x^{2h+1}$ at $(0, 0) \in \mathbb{A}^2$; see [13, Example 3 at p. 549], [11, Corollary 1.1.41, Lemma 1.1.78].

Its *singularity scheme* is any degree $3h + 2$ connected zero-dimensional $Z \subset \mathbb{A}^2$ isomorphic to the subscheme of $\mathbb{A}^2$ with ideal $I_Z = (y^2, yx^{h+1}, x^{2h+1})$. An A_{2h}-*singularity scheme* is any connected subscheme of a smooth surface isomorphic to the singularity scheme Z. The singularity degree of an A_{2h}-singularity is h. We use the convention that the A_0-singularity scheme is a smooth point.

We use the following theorem proved by Roé [25, Theorem 1.2]:

Theorem 4.11 *(**Roé**) Fix positive integers h and $t \ge 13$. If $3h + 2 \le \binom{t+2}{2}$, then there is an A_{2h}-singularity scheme $Z \subset \mathbb{P}^2$ such that $h^1(\mathbb{P}^2, \mathcal{I}_Z(t)) = 0$.*

Lemma 4.12 *Fix positive integers h and $t \ge 14$. If $3h + 2 \le \binom{t+1}{2}$, then there exist two A_{2h}-singularity schemes $A, B \subset Q$ such that $h^1(Q, \mathcal{I}_{A \cup B}(t, t)) = 0$.*

Proof Fix $o \in \mathbb{P}^3 \setminus Q$. Let $\pi_o : \mathbb{P}^3 \setminus \{o\} \to \mathbb{P}^2$ denote the linear projection from o. Since $o \notin Q$, $\pi_{o|Q}$ defines a degree 2 finite morphism $\pi : Q \to \mathbb{P}^2$. The ramification locus $R \subset Q$ of π is a smooth conic which is the intersection of Q with the plane polar to o with respect to Q. The branch locus $\pi(R) \subset \mathbb{P}^2$ is a smooth conic. By Roé's Theorem 4.11, there is an A_{2h}-singularity scheme $Z \subset \mathbb{P}^2$ such that $h^1(\mathbb{P}^2, \mathcal{I}_Z(t - 1)) = 0$. Applying an automorphism to $\mathbb{P}^2$, we may assume $Z \cap f(R) = \emptyset$. Thus $\pi^{-1}(Z)$ is the disjoint union of the A_{2h}-singularity schemes, say A and B. Since π is a degree 2 covering between smooth varieties and the branch locus of π is a conic, $\pi_*(\mathcal{O}_Q) \cong \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-1)$. (This can be checked on a local chart.) Since π is a finite morphism, $R^i\pi_*(\mathcal{F}) = 0$ for all $i > 0$ and any coherent sheaf $\mathcal{F}$ on Q. So π_* induces an isomorphism of all cohomology groups. Since $\pi^*(\mathcal{I}_Z) = \mathcal{I}_{A\cup B}$, the projection formula gives $\pi_*(\mathcal{I}_{A\cup B}(t,t)) \cong \mathcal{I}_Z(t) \oplus \mathcal{I}_Z(t-1)$. Thus $h^1(\mathcal{I}_{A\cup B}(t,t)) = h^1(\mathbb{P}^2, \mathcal{I}_Z(t)) + h^1(\mathbb{P}^2, \mathcal{I}_Z(t-1)) = 0$. □

Lemma 4.13 *Fix positive integers h and $d_2 \ge d_1 \ge 15$. If $3h+2 \le \binom{d_1}{2}$, then there are two A_{2h}-singularity schemes $A, B \subset Q$ such that $h^1(Q, \mathcal{I}_{A\cup B}(d_1-1, d_2-1)) = h^1(Q, \mathcal{I}_{A\cup B}(d_1, d_2)) = 0$ and a general $Y \in |\mathcal{I}_{A\cup B}(d_1, d_2)|$ is irreducible and with exactly two singular points, A_{red} and B_{red}, both of them A_{2h}-singularities.*

Proof Take A and B as in Lemma 4.12 for the integer $t = d_1 - 1$. Set $\{q_A\} = A_{\mathrm{red}}$ and $\{q_B\} := B_{\mathrm{red}}$. Lemma 4.12 shows that $h^1(Q, \mathcal{I}_{A\cup B}(d_1-1, d_1-1)) = 0$. The Castelnuovo-Mumford's lemma for zero-dimensional schemes gives $h^1(Q, \mathcal{I}_{A\cup B}(d_1, d_2)) = 0$ and that $\mathcal{I}_{A\cup B}(d_1, d_2)$ is globally generated [23, Lecture 14].

Fix a general $D \in |\mathcal{I}_{A\cup B}(d_1, d_2)|$. Since $\mathcal{I}_{A\cup B}(d_1, d_2)$ is globally generated and D is general, D has an A_{2h}-singularity at both q_A and q_B [11, Lemma 1.1.33]. Since $\mathcal{I}_{A\cup B}(d_1, d_2)$ is globally generated, D is smooth outside $\{q_A, q_B\}$ by Bertini's theorem in characteristic zero [16, Corollary III.10.9]. Thus, to conclude, it is sufficient to prove that D is irreducible. Since D has only finitely many singular points, D has no multiple components. Since $d_2 \ge d_1 \ge 2$, $\mathcal{O}_Q(d_1, d_2)$ is very ample and hence D is connected. Thus if D were reducible, there would be irreducible component D' and D'' of D, $D' \ne D''$ and passing both through q_A or q_B. This is a contradiction, because D has unibranch singularity A_{2h}, at q_A and q_B. □

Lemma 4.14 *Fix integers $t_2 \ge t_1 \ge 16$ and $h > 0$ such that $3h+2 \le \binom{t_1-1}{2}$. Fix $L \in |\mathcal{O}_Q(1,0)|$, $R \in |\mathcal{O}_Q(0,1)|$, $o_1 \in L \setminus L\cap R$ and $o_2 \in R \setminus R \cap L$. Let Z_1 be the degree t_2 divisor of L supported at o_1. Let Z_2 be the degree t_1 divisor of R supported at o_2. There are two A_{2h}-singularity schemes $A, B \subset Q \setminus (L \cup R)$ such that $h^1(Q, \mathcal{I}_{Z_1\cup Z_2\cup A\cup B}(t_1, t_2)) = 0$ and a general $Y \in |\mathcal{I}_{Z_1\cup Z_2\cup A\cup B}(t_1, t_2)|$ is irreducible with* $\mathrm{Sing}(Y) = \{q_A, q_B\}$ *and Y has A_{2h}-singularities at $\{q_A\} = A_{\mathrm{red}}$ and at $\{q_B\} = B_{\mathrm{red}}$.*

Proof We split the proof into two claims.

Claim 1: $h^1(Q, \mathcal{I}_{Z_1\cup Z_2\cup A\cup B}(t_1, t_2)) = 0$, $h^1(Q, \mathcal{I}_{Z_2\cup A\cup B}(t_1-1, t_2)) = 0$, and $h^1(Q, \mathcal{I}_{Z_1\cup A\cup B}(t_1, t_2-1)) = 0$.

Proof of Claim 1: Since $L \cap \{o_2, q_A, q_B\} = \emptyset$, the residual exact sequence with respect to L gives the exact sequence

$$0 \to \mathcal{I}_{Z_2\cup A\cup B}(t_1-1, t_2) \to \mathcal{I}_{Z_1\cup Z_2\cup A\cup B}(t_1, t_2) \to \mathcal{I}_{Z_1,L}(t_1, t_2) \to 0. \quad (5)$$

Since $\deg(Z_1) = t_2$ and $\mathcal{O}_L(t_1, t_2)$ is the degree t_2 line bundle on $L \cong \mathbb{P}^1$, we have $h^1(L, \mathcal{I}_{Z_1,L}(t_1, t_2)) = 0$. Thus, by (5), to prove the assertion, it is sufficient to show the vanishing $h^1(Q, \mathcal{I}_{Z_2\cup A\cup B}(t_1-1, t_2)) = 0$. Since $R \cap \{q_A, q_B\} = \emptyset$, the residual exact sequence with respect to R gives the exact sequence

$$0 \to \mathcal{I}_{A\cup B}(t_1-1, t_2-1) \to \mathcal{I}_{Z_2\cup A\cup B}(t_1-1, t_2) \to \mathcal{I}_{Z_2,R}(t_1-1, t_2) \to 0 \quad (6)$$

Now, Lemma 4.13, with the choice $d_1 = t_1 - 1$ and $d_2 = t_2 - 1$, gives $h^1(Q, \mathcal{I}_{A\cup B}(t_1 - 1, t_2 - 1)) = 0$. Since $\deg(Z_2) = t_1$ and $\mathcal{O}_R(t_1 - 1, t_2)$ is the degree $t_1 - 1$ line bundle on R, we have $h^1(R, \mathcal{I}_{Z_2,R}(t_1 - 1, t_2)) = 0$. The long cohomology exact sequence of (6) concludes the proof of the claim.

Claim 2: A general $Y \in |\mathcal{I}_{Z_1\cup Z_2\cup A\cup B}(t_1, t_2)|$ is smooth outside $\{q_A, q_B\}$, has A_{2h}-singularities at $\{q_A\} = A_{\mathrm{red}}$ and at $\{q_B\} = B_{\mathrm{red}}$, and it is irreducible.
Proof of Claim 2: By Lemma 4.13, the general $D \in |\mathcal{I}_{A\cup B}(t_1 - 1, t_2 - 1)|$ is smooth outside $\{q_A, q_B\}$. Using the action of $\mathrm{Aut}(\mathbb{P}^1)\times\mathrm{Aut}(\mathbb{P}^1)$, we may assume that L and R are transversal to D. With this assumption, the curve $Y' = D \cup L \cup R \in |\mathcal{I}_{Z_1\cup Z_2\cup A\cup B}(t_1, t_2)|$ is smooth at o_1 and o_2 and it has an A_{2h}-singularity at q_A and at q_B. Since each element of $|\mathcal{I}_{Z_1\cup Z_2\cup A\cup B}(t_1, t_2)|$ has at least an A_{2h}-singularity at q_A and at q_B, a general element of $|\mathcal{I}_{Z_1\cup Z_2\cup A\cup B}(t_1, t_2)|$ has A_{2h}-singularities at q_A and q_B. Since $\{o_1, o_2\}$ is a finite set and smoothness is an open condition, a general $Y \in |\mathcal{I}_{Z_1\cup Z_2\cup A\cup B}(t_1, t_2)|$ is smooth at o_1 and o_2. By *Claim 1*, we have $h^1(Q, \mathcal{I}_{Z_1\cup Z_2\cup A\cup B}(t_1, t_2)) = 0$. Thus

$$
\begin{aligned}
h^0(Q, \mathcal{I}_{Z_1\cup Z_2\cup A\cup B}(t_1, t_2)) &= (t_1 + 1)(t_2 + 1) - \deg(A) - \deg(B) - t_1 - t_2 \\
&= t_1 t_2 - \deg(A) - \deg(B) + 1.
\end{aligned}
$$

Since $h^1(Q, \mathcal{I}_{Z_2\cup A\cup B}(t_1 - 1, t_2)) = 0$, we have

$$
\begin{aligned}
h^0(Q, \mathcal{I}_{Z_2\cup A\cup B}(t_1 - 1, t_2)) &= t_1(t_2 + 1) - \deg(A) - \deg(B) - t_1 \\
&= t_1 t_2 - \deg(A) - \deg(B) < t_1 t_2 - \deg(A) - \deg(B) + 1 = h^0(Q, \mathcal{I}_{Z_1\cup Z_2\cup A\cup B}(t_1, t_2).
\end{aligned}
$$

Thus L is not in the base locus of $|\mathcal{I}_{Z_1\cup Z_2\cup A\cup B}(t_1, t_2)|$.
Since $h^1(Q, \mathcal{I}_{Z_1\cup A\cup B}(t_1, t_2 - 1)) = 0$, one has

$$
\begin{aligned}
h^0(Q, \mathcal{I}_{Z_1\cup A\cup B}(t_1, t_2 - 1)) &= (t_1 + 1)t_2 - \deg(A) - \deg(B) - t_2 \\
&= t_1 t_2 - \deg(A) - \deg(B) < t_1 t_2 - \deg(A) - \deg(B) + 1 = h^0(Q, \mathcal{I}_{Z_1\cup Z_2\cup A\cup B}(t_1, t_2)).
\end{aligned}
$$

Analogously R is not in the base locus of $|\mathcal{I}_{Z_1\cup Z_2\cup A\cup B}(t_1, t_2)|$.
The base locus of $|\mathcal{I}_{Z_1\cup Z_2\cup A\cup B}(t_1, t_2)|$ is then strictly contained in $L \cup R \cup \{q_A, q_B\}$. Take any $Y \in |\mathcal{I}_{Z_1\cup Z_2\cup A\cup B}(t_1, t_2)|$ smooth at o_1 and o_2. Since $Y \cdot \mathcal{O}_Q(1, 0) = t_2$, this curve meets L only at o_1. Similarly, Y meets R only at o_2. Thus Y is smooth along $L \cup R$. By Bertini's theorem in characteristic zero, Y is smooth outside $L \cup R \cup \{q_A, q_B\}$. From what we have just proven, a general Y is smooth outside $\{q_A, q_B\}$. We proved that Y has an A_{2h}-singularity at q_A and at q_B. Since Y is connected and it has only unibranch singularities, Y is irreducible, as we argued in the proof of Lemma 4.13.

□

Lemma 4.15 *Fix integers $t_2 \geq t_1 \geq 16$ and $h > 0$ such that $3h + 2 \leq \binom{t_1 - 1}{2}$, and $0 < \alpha, \beta \leq h$. Fix $L \in |\mathcal{O}_Q(1, 0)|$, $R \in |\mathcal{O}_Q(0, 1)|$, $o_1 \in L \setminus L\cap R$ and $o_2 \in R \setminus R\cap L$. Let Z_1 be the degree t_2 divisor of L supported at o_1. Let Z_2 be the degree t_1 divisor of*

R supported at o_2. There are an $A_{2\alpha}$-singularity scheme A' and an $A_{2\beta}$-singularity scheme B' contained in $Q \setminus (L \cup R)$ such that $h^1(Q, \mathcal{I}_{Z_1 \cup Z_2 \cup A' \cup B'}(t_1, t_2)) = 0$ and a general $Y \in |\mathcal{I}_{Z_1 \cup Z_2 \cup A' \cup B'}(t_1, t_2)|$ is irreducible with $\mathrm{Sing}(Y) = \{q_{A'}, q_{B'}\}$ and Y has $A_{2\alpha}$-singularity at $\{q_{A'}\} = A'_{\mathrm{red}}$ and and $A_{2\beta}$-singularity at $\{q_{B'}\} = B'_{\mathrm{red}}$.

Proof Take A and B as in Lemma 4.14 and fix an $A_{2\alpha}$-singularity scheme $A' \subseteq A$ and an $A_{2\beta}$ singularity scheme $B' \subseteq B$. Since $A' \subseteq A$ and $B' \subseteq B$, all vanishing occurring in the proof of Lemma 4.13 holds true as well. Indeed, for $A' \subseteq A$ we have the exact sequence:

$$0 \longrightarrow \mathcal{I}_A \longrightarrow \mathcal{I}_{A'} \longrightarrow \mathcal{I}_{A'}/\mathcal{I}_A \longrightarrow 0.$$

Taking the long exact sequence in cohomology, for each $i > 0$, we have

$$\cdots \longrightarrow H^i(\mathcal{I}_A) \longrightarrow H^i(\mathcal{I}_{A'}) \longrightarrow H^i(\mathcal{I}_{A'}/\mathcal{I}_A) = 0,$$

because the sheaf $\mathcal{I}_{A'}/\mathcal{I}_A$ is supported on a zero-dimensional scheme. So the vanishing of the left-most implies the vanishing of the middle cohomology group. Then, as in the proof of Lemma 4.14, we reach the same conclusion for any scheme $A' \cup B' \subset A \cup B$ as above. □

Theorem 4.16 *Let k be an algebraically closed field of characteristic zero. Fix integers $d_2 \geq d_1 \geq 16$ and $h > 0$ such that $3h + 2 \leq \binom{d_1 - 1}{2}$. Fix an integer κ such that $0 < \kappa \leq 2h$ and set $g = d_1 d_2 - d_1 - d_2 + 1 - \kappa$. Then there exists a smooth genus g curve with an injective $g^2_{d_1+d_2}$ of type* **III**.

Proof Fix $L \in |\mathcal{O}_Q(1,0)|$, $R \in |\mathcal{O}_Q(0,1)|$, $o_1 \in L \setminus L \cap R$ and $o_2 \in R \setminus R \cap L$. Let Z_1 be the degree d_2 divisor of L supported at o_1. Let Z_2 be the degree d_1 divisor of R supported at o_2. Any $Y \in |\mathcal{O}_Q(d_1, d_2)|$ has arithmetic genus $p_a(Y) = d_1 d_2 - d_1 - d_2 + 1$. By an application of Proposition 4.1, it is sufficient to prove the existence of $\{q_A, q_B\} \subset Q \setminus (L \cup R)$ and an integral $Y \in |\mathcal{O}_Q(d_1, d_2)|$ with only unibranch singularities $\mathrm{Sing}(Y) \subseteq \{q_A, q_B\}$ and with singularity degree $\delta(Y, q_A) + \delta(Y, q_B) = \kappa$. (Recall that, for any A_{2m}-singularity, its singularity degree is m.) So it is enough to fix two positive integers $0 < \beta \leq \alpha \leq h$ such that $\alpha + \beta = \kappa$ and find a Y that has an $A_{2\alpha}$-singularity at q_A and an $A_{2\beta}$-singularity at q_B. This is the content of Lemma 4.15. □

The case $\kappa = 0$ is covered by Theorem 4.8. Theorem 4.16 provides positive examples to Question 1.1.

Remark 4.17 Piene [24] considered only injective linear series g^2_d of *type* **II**. In $\mathrm{char}(k) = 0$, Proposition 4.1 and Proposition 4.5 provide curves X possessing an injective (and separable) non-complete g^2_d which cannot occur from [24], since in Piene's setting the smooth curve X is required to be embedded in $\mathbb{P}^3$. So the same observation applies to the curves arising from Theorem 4.16.

5 Inner cuspidal projections

In this section, we work in $\mathrm{char}(k) = 0$. We study inner cuspidal projections and introduce two natural sets $\mathcal{A}$ and $\mathcal{B}$ attached to them.

Let $\pi_o : \mathbb{P}^3 \setminus \{o\} \to \mathbb{P}^2$ denote the linear projection from o. We look at inner *smooth* projections, i.e. projection from smooth points of a curve. (We only allow projection from smooth points of the curve, because projecting from a singular point of the curve is more complicated and depends on the germ of the singularity.)

Let $X \subset \mathbb{P}^n$ be an integral and non-degenerate curve with only cuspidal singularities. For any $o \in X_{\mathrm{reg}}$ the restriction $\pi_{o|X\setminus\{o\}}$ extends to a unique morphism $\pi_o : X \to \mathbb{P}^2$. Define:

$$\mathcal{A} = \left\{o \in X_{\mathrm{reg}} \mid \pi_{o|X\setminus\{o\}} \text{ is injective}\right\}$$

and

$$\mathcal{B} = \left\{o \in X_{\mathrm{reg}} \mid \pi_o : X \to \mathbb{P}^2 \text{ is injective}\right\}.$$

It is clear that $\mathcal{B} \subseteq \mathcal{A}$ and sometimes $\mathcal{A} \neq \mathcal{B}$, see Proposition 5.1, Theorem 5.3, and Remark 5.2.

Proposition 5.1 *Le $Q \subset \mathbb{P}^3$ be a smooth quadric surface. For any $o \in Q$, let $L_1(o)$ (resp. $L_2(o)$) denote the element of $|\mathcal{O}_Q(1,0)|$ (resp. $|\mathcal{O}_Q(0,1)|$) containing o. Let $X \in |\mathcal{O}_Q(d_1,d_2)|$, $d_1 > 0, d_2 > 0$, $(d_1,d_2) \neq (1,1)$, be an integral curve with only cuspidal singularities. Fix $o \in X_{\mathrm{reg}}$. Then:*

(i) $o \in \mathcal{A}$ if and only if $\#(L_1(o) \cap X)_{\mathrm{red}} \leq 2$ and $\#(L_2(o) \cap X)_{\mathrm{red}} \leq 2$;
(ii) $o \in \mathcal{B}$ if and only if $\#(L_1(o) \cap X)_{\mathrm{red}} = 1$ and $\#(L_2(o) \cap X)_{\mathrm{red}} = 1$. This is equivalent to $L_1(o)$ (resp. $L_2(o)$) and X having intersection multiplicity d_2 (resp. d_1) at o.

Proof The assumptions on d_1, d_2 imply X is non-degenerate. Let $L \subset \mathbb{P}^3$ be a line containing o and different from $L_1(o)$ and $L_2(o)$. Since $o \in Q$ and $L \nsubseteq Q$, Bézout's theorem implies $\#((X \setminus \{o\}) \cap L)_{\mathrm{red}} \leq 1$. If $\#((X \setminus \{o\}) \cap L)_{\mathrm{red}} = 1$, then L is not tangent to X at o. Thus to check whether π_o is injective, it is sufficient to check the sets $\#(L_1(o) \cap X)_{\mathrm{red}}$ and $(L_2(o) \cap X)_{\mathrm{red}}$. We have $\#(L_1(o) \cap X)_{\mathrm{red}} = 1$ (resp. $\#(L_2(o) \cap X)_{\mathrm{red}} = 1$) if and only if $L_1(o)$ and X have intersection multiplicity d_2 at o (resp. $L_2(o)$ and X have intersection multiplicity d_1 at o). □

Remark 5.2 Note that if $L_1(o)$ and X have intersection multiplicity at least $d_2 - 1$ at o, and $L_2(o)$ and X have intersection multiplicity at least $d_1 - 1$ at o, then $\#(L_1(o) \cap X)_{\mathrm{red}} \leq 2$ and $\#(L_2(o) \cap X)_{\mathrm{red}} \leq 2$ and hence $o \in \mathcal{A}$ by Proposition 5.1(i).

Theorem 5.3 *Let $\mathcal{C} \subset \mathbb{P}^3$ be a quadric cone with vertex v. Let $X \subset \mathcal{C}$ be an integral and non-degenerate curve with only cuspidal singularities. Fix $o \in X_{\mathrm{reg}}$ such that $o \neq v$ and let L_o be the line contained in $\mathcal{C}$ and passing through o. The following statements hold:*

(i) $o \in \mathcal{B}$ *if and only if* $\#(L_o \cap X)_{\text{red}} = 1$.
(ii) $o \in \mathcal{A}$ *if and only if* $\#(L_o \cap X)_{\text{red}} \leq 2$.
(iii) Assume $v \in X_{\text{reg}}$. *Then:*

$$v \in \mathcal{A} \Leftrightarrow v \in \mathcal{B} \Leftrightarrow X \textit{ is a rational normal curve}.$$

Proof The first two statements are verified as the corresponding ones in Proposition 5.2. As in the proof of Theorem 4.6, let $\mathbb{F}_2$ denote the second Hirzebruch surface. Let $\eta : \mathbb{F}_2 \to \mathcal{C}$ be the minimal desingularization of the quadric cone $\mathcal{C}$, and let $Y \subset \mathbb{F}_2$ be the strict transform of X in $\mathbb{F}_2$. Then $Y \in |\mathcal{O}_{\mathbb{F}_2}(aH + bF)|$ for some $a > 0$ and $b \geq 2a$. The assumption that X is smooth at v is equivalent to $b = 2a + 1$.

We show $v \in \mathcal{B}$ if and only if X is a rational normal curve. If X is a rational normal curve, one has $\mathcal{B} = X$ and so $v \in \mathcal{B}$. Conversely, assume $v \in \mathcal{B}$. Then the map $\pi_{v|X}$ is birational onto its image, which happens if and only if $a = 1$. Indeed, since the strict transform of X is $Y \in |\mathcal{O}_{\mathbb{F}_2}(aH + bF)|$, the degree of $\pi_{v|X}$ coincides with the intersection number of Y with the ruling F, i.e. $\deg(\pi_{v|X}) = (aH + bF) \cdot F = a$. Since $a = 1$, one has $b = 3$ and hence Y is smooth of genus zero. So X is a rational normal curve.

Finally, by definition $v \in \mathcal{B}$ implies $v \in \mathcal{A}$. Suppose $v \in \mathcal{A}$, then again $\pi_{v|X}$ is birational onto its image, so X is a rational normal curve and hence $v \in X = \mathcal{B}$. This completes the proof of statement (iii). ◻

Theorem 5.4 *Let* $X \subset \mathbb{P}^3$ *be an integral and non-degenerate curve with only cuspidal singularities. The following conditions are equivalent:*

(i) $\mathcal{B}$ *is infinite;*
(ii) $\mathcal{B} = X_{\text{reg}}$*;*
(iii) $\deg(X) \in \{3, 4\}$ *and if* $\deg(X) = 4$, *then* $p_a(X) = 1$.

Proof Let $d = \deg(X)$. If $d \leq 4$, then X is contained in a quadric surface. Proposition 5.1 and Theorem 5.3 show that (iii) implies (ii). It is clear that (ii) implies (i).

We show that (i) implies (iii). For any $o \in \mathcal{A}$, the map π_o has degree one and hence $\deg(\pi_o(X)) = d - 1$. Thus $p_a(\pi_o(X)) = (d-2)(d-3)/2$. By assumption, $\mathcal{A}$ is infinite. Since $o \in X_{\text{reg}}$ and $\pi_{o|X}$ is birational onto its image, we have $\deg(\pi_o(X)) = d - 1$. Since $\text{Sing}(X)$ is finite and each singular point of X has Zariski tangent space of dimension two, for infinitely many $o \in \mathcal{B}$, the morphism $\pi_o : X \to \pi_o(X)$ is a local isomorphism at each singular point of X. Since π_o is injective, we have $\pi_o(X_{\text{reg}}) \cap \pi_o(\text{Sing}(X)) = \emptyset$. Since $\text{char}(k) = 0$, at all points of X_{reg}, except finitely many, the order of contact of $T_o X$ with X is two. Hence, for infinitely many $o \in \mathcal{B}$, $\pi_o(o)$ is a smooth point of $\pi_o(X)$.

By Zariski's Main Theorem, the morphism $\pi_o : X \to \pi_o(X)$ is then an isomorphism for some $o \in X_{\text{reg}}$. Thus $p_a(X) = p_a(\pi_o(X)) = (d-2)(d-3)/2$.

Recall from Remark 2.2 that Castelnuovo's bound gives $\pi(d, 3) = m(m-1) + m\varepsilon$, where $\varepsilon \in \{0, 1\}$ and $d = 2m + 1 + \varepsilon$ and $m > 0$. So

$$p_a(X) = \frac{(2m + \varepsilon - 1)(2m + \varepsilon - 2)}{2} = \begin{cases} (2m-1)(m-1), & \text{for } \varepsilon = 0, \\ m(2m-1), & \text{for } \varepsilon = 1. \end{cases}$$

The inequalities $(2m-1)(m-1) \leq \pi(d,3)$ and $m(2m-1) \leq \pi(d,3)$ both imply $m=1$ and so $d \leq 4$. Therefore, X is a rational normal curve if $d=3$, or $p_a(X)=1$ if $d=4$. □

Remark 5.5 In particular, X is smooth with $\mathcal{A}=X$ if and only if X is either a rational normal curve or a linearly normal elliptic curve.

Acknowledgements The first author is partially supported by MIUR and GNSAGA of INdAM (Italy). The second author is supported by Vici Grant 639.033.514 of Jan Draisma from the Netherlands Organisation for Scientific Research. We warmly thank the reviewers for their useful suggestions and corrections.

Funding Open access funding provided by University of Bern.

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