Relation between the spectral properties of wall turbulence and the scaling of the Darcy-Weisbach friction factor

Relation between the spectral properties of wall turbulence and the scaling of the Darcy-Weisbach friction factor

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Empirical formulas describing the Darcy-Weisbach friction factor remain indispensable for applications in sciences and engineering dealing with turbulent flows. Despite their practical significance, these formulas have remained without theoretical interpretation for many decades. To close this knowledge gap, much research has been devoted to the development of the so-called "spectral link" introduced in the early 2000s. Such a theory is entirely based on elegant phenomenological arguments that make no contact with equations describing turbulent wall flows. The spectral link spawned alternative approaches, now labeled "cospectral budget" (or CSB) models, that describe how turbulent eddies contribute to wall stresses. The CSB overcomes some of the shortcomings of the phenomenological approach and is here employed to provide a thorough clarification of the link between spectral properties of velocity fluctuations and the scaling of friction factors in turbulent pipe flows in the hydraulically smooth and fully rough regimes.

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I. INTRODUCTION

The seminal work by Nikuradse [1] identified that the Darcy-Weisbach friction factor f of a rough-walled pipe depends on the bulk Reynolds number $\mathrm{Re} = 2VR/\nu$ and the relative roughness of the wall d/R, where V is the time and cross-sectional averaged velocity (hereafter referred to as bulk velocity), R is the pipe radius, ν is the fluid kinematic viscosity, and d is the size of the roughness elements (uniform sand grains in the experiments carried out by Nikuradse). The f, as is well known, is a measure of the shear stress that the flow exerts on the pipe wall and according to the Darcy-Weisbach formula (see, e.g., [2]) is equal to $8f_d$, where $f_d = \tau_0/(\rho V^2)$, τ_0 being the wall shear stress and ρ the fluid density.

The data gathered by Nikuradse allowed identifying the so-called hydrodynamically smooth, transitionally and fully rough regimes, whose existence is commonly justified using a phenomenological argument, based on a competing mechanism between the roughness size d and the viscous length scale ν/u_*, where $u_* = \sqrt{\tau_0/\rho}$ is the shear velocity. If the roughness elements are immersed within the viscous sublayer, the flow functions as a hydraulically smooth regime and the friction

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factor f_d depends on Re only. Conversely, if the wall roughness is much larger than the viscous sublayer thickness, f_d becomes independent of Re and depends on the relative roughness d/R only. Intermediate conditions, whereby the viscous sublayer and the roughness have comparable size, pertain to the transitionally rough regime, in which f_d depends on both Re and d/R. From a physical point of view, the hydraulically smooth and the fully rough flow conditions can be considered as two limiting regimes, occurring for $d/R \to 0$ and $\mathrm{Re} \to \infty$, respectively [3].

Experimental evidence suggests that when $3 \times 10^3 < \mathrm{Re} < 10^5$, the f_d in the hydraulically smooth regime follows the so-called Blasius scaling, namely $f_d \sim \mathrm{Re}^{-1/4}$ [4], where the $\sim$ symbol means "scales as." For higher Reynolds numbers ($\mathrm{Re} > 10^5$), experiments suggest deviation from the Blasius scaling. For the fully-rough regime, the f_d vary, in good approximation, according to the so-called Strickler scaling, namely as $f_d \sim (d/R)^{1/3}$ [5].

While there is a good body of empirical evidence in support of the proposed scaling regimes for both limiting regimes, theoretical arguments remain lacking. A major advance on this problem emerged in the past 20 years from the work by Gioia, Chakraborty, and co-workers (hereafter referred to as GC) [3,6–10]. These authors developed a framework that justifies the existence of both the Blasius scaling and the Strickler scaling in f_d on the basis of near-wall momentum transport mechanisms driven by turbulent velocity fluctuations, following the Kolmogorov scaling of energy spectra. In other words, a so-called "spectral link" was established for the first time to connect the scaling of the friction factor with the (universal) scaling of turbulence in the so-called inertial subrange. This result created a stir in the turbulence-research community. In fact, the arguments proposed by GC have been used in a variety of studies to address problems related to turbulent friction in canopy and permeable beds [11], momentum and scalar transport in atmospheric flows [12–16], turbulent friction caused by non-Newtonian fluids [9,10], local scour [17–21], and sediment transport [22], to name just a few. Moreover, the allegedly proven theoretical origin of the Blasius and Strickler scaling has led to the development of further theoretical (and original) ideas about turbulent flows. A notable example is an analogy between critical phenomena (typically studied and observed within the remit of thermodynamics) and turbulence [3]. Goldenfeld [3] found out that, assuming the Blasius and Strickler scalings exist as limiting regimes, the Nikuradse data can collapse on one single curve exploiting the Widom arguments that were originally developed to infer thermodynamic properties of ferromagnets near the critical temperature. A further refinement of this work was developed by Mehrafarin and Pourtolami [7], who argued that intermittency corrections to the Kolmogorov energy spectrum could further improve the imperfect data collapse obtained by Goldenfeld [3].

Despite its appeal, elegance, and simplicity, though, the spectral link theory proposed by GC is entirely phenomenological. Moreover, a number of later studies (published by Katul and co-workers and hereafter referred to as KCW) for simplicity, identified various issues in the GC analysis, the most critical concerning the adopted range of eddy scales contributing to near-wall turbulent momentum transport and the adopted scaling for the near-wall dissipation rate of turbulent kinetic energy (TKE) [23–25]. Therefore, KCW proposed a different approach derived from the analysis of a cospectral budget equation [24–26], which overcomes the aforementioned issues and opens new avenues for the study of the spectral link (see Bonetti *et al.* [23] for an in-depth discussion on the comparison between the GC and cospectral budget approach). Such avenues have been only partially explored by KCW, mainly in two papers, namely Bonetti *et al.* [23] and Katul and Manes [24]. These works, in fact, focus mostly on finding a link between spectral properties of turbulence and the shape of mean velocity profiles in wall flows. This analysis requires the numerical integration of the cospectral budget equation over the wall-normal direction. While the friction factor is surely linked to the shape of mean velocity profiles, in these two works it is difficult to identify a clean spectral link, as this is hidden behind a thick curtain of calculations (this aspect is clarified in the following section).

The present paper intends to explore the full potential of the CSB equation to reveal and clarify the spectral link, while maintaining analytical tractability. The aim is to do so within the contexts of the hydraulically smooth and fully rough flow conditions. With the results from the proposed approach, we also intend to contribute to the ongoing discussion on whether (or to what extent) the Blasius and Strickler scalings, which allegedly characterize friction factors in the aforementioned limiting regimes, are founded on turbulence theories or must be considered merely as a concise summary of experiments.

The paper is organized as follows: Section II reviews the co-spectral budget (CSB) model as per steady and turbulent pipe flows; building upon the analysis presented in Sec. II, the spectral link in hydrodynamically smooth and fully rough conditions is explored in Secs. III and IV, respectively. Sections V and VI are devoted to discussion and conclusions, respectively.

II. COSPECTRAL BUDGET MODEL

The case of a steady turbulent flow within a uniform rough-walled pipe is considered for illustration. The momentum balance equation at any wall normal distance z can be written as

$$\frac{d(\tau_t(z)+\tau_v(z))}{dz}=\frac{d\overline{p_s}}{dx} \tag{1}$$

where τ_v is the viscous shear stress component, $\tau_t=-\rho\overline{u'w'}$ is the turbulent shear stress component, the overbar indicates averaging over coordinates of statistical homogeneity (here assumed to be time averaging), u' and w' are the longitudinal and wall-normal velocity components, respectively; p_s is the modified pressure [27].

The turbulent momentum flux in Eq. (1) can be expressed as

$$\rho\overline{u'w'}(z)=-\rho\int_0^\infty F_{uw}(k)dk, \tag{2}$$

where $F_{uw}(k)$ is referred to as the "cospectrum" and is defined as the real part of the Fourier transform of the cross-correlation function between u' and w'; k is the wave number reflecting the inverse of eddy sizes. The related CSB was given by [24,25]

$$\frac{\partial F_{uw}(k)}{\partial t}+2\nu k^2F_{uw}(k)=G(k), \tag{3}$$

where t is time and

$$G(k)=P_{uw}(k)+T_{uw}(k)+\Pi(k). \tag{4}$$

In Eq. (4) the term $P_{uw}(k)=\Gamma(z)E_{ww}(k)$ represents the covariance production term, Γ being the mean velocity gradient externally imposed on the cospectrum, $E_{ww}(k)$ the turbulent energy spectrum of the vertical velocity, $T_{uw}(k)$ the cospectral flux-transfer term (that differs from its turbulent kinetic energy transfer counterpart), and $\Pi(k)$ the velocity-pressure interaction term, which acts as to decorrelate u' and w' [27]. To maintain analytical tractability, it is necessary to introduce simplifying assumptions. As discussed by Katul *et al.* [25], it is convenient to locate the whole analysis (i.e., momentum balance and cospectral budget) at a distance z where the viscous component of the shear stress τ_v can be neglected in Eq. (1). This hypothesis is key for the correct application of the cospectral budget approach because the turbulent momentum flux can be considered, in good approximation, as an estimate of the wall shear stress, or equivalently, $\tau_t=-\rho\overline{u'w'}\approx\rho u_*^2=\tau_0$. As reported below, this is true in the fully rough regime, where $\mathrm{Re}\to\infty$, and under specific and afterward discussed hypotheses in the smooth regime (see Sec. IV). Assuming negligible viscous effects allows also neglecting the second term in the left-hand side of Eq. (3) relative to the pressure-decorrelation term. A good candidate elevation z for the proposed analysis is the top edge of the buffer layer [28]. Here, as pointed out by [25,26], the transfer term $T_{uw}(k)$, which does not alter the outcome of the cospectral approach [23], can be neglected and, upon further assuming steady conditions, the cospectral budget equation reduces to an interplay between production and decorrelation at z as

$$\Gamma E_{ww}(k)+\Pi(k)=0. \tag{5}$$

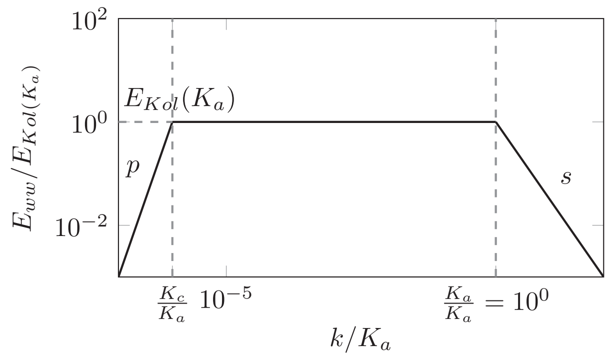

FIG. 1. Scheme of the idealized dimensionless vertical velocity spectrum $E_{ww}/E_{\mathrm{Kol}}(K_a)$.

If the Rotta model [29] is invoked for $\Pi(k)$ and upon neglecting the wall-blockage effect [24] by selecting z at the top of the buffer layer, then

$$\Pi(k) = -C_R \frac{F_{uw}(k)}{\tau_r(k)} - C_I P_{uw}(k), \tag{6}$$

where $\tau_r(k) = \epsilon^{-1/3} \min(k^{-2/3}, K_a^{-2/3})$ is a wave-number-dependent relaxation timescale [26,30], $C_R \approx 1.8$ is the known Rotta constant, $C_I = 3/5$ is a constant associated with the isotropization of the production term correcting the original Rotta model [27], and K_a is the wave-number scaling as the inverse of the characteristic inner length scale of flow, namely the roughness size and the viscous length scale, for the fully rough and hydraulically smooth regimes, respectively. As previously discussed [31], the relaxation timescale may be interpreted as the mean duration of an eddy of wave number k to be dissipated with a constant local dissipation rate of TKE, ϵ. Bonetti *et al.* [23] provide an in-depth discussion about the relaxation timescale. In particular, they have verified that it is more than reasonable to assume that $\tau_r(k)$ varies only with k and ϵ as $\epsilon^{-1/3}k^{-2/3}$. From this consideration alone, it can be shown that

$$F_{uw}(k) = \frac{1}{A_\pi} \Gamma \epsilon^{-1/3} E_{ww}(k) k^{-2/3}, \tag{7}$$

where $A_\pi = C_R/(1 - C_I) \approx 4.5$. Combining Eqs. (2) and (7) leads to

$$|\Gamma|\epsilon^{-1/3} = A_\pi \frac{-\overline{u'w'}}{I_k} = A_\pi \frac{\tau_0/\rho}{I_k}, \tag{8}$$

where

$$I_k = \int_0^\infty k^{-2/3} E_{ww}(k) dk. \tag{9}$$

To evaluate I_k [Eq. (9)], it is now necessary to make assumptions about the shape of the spectrum $E_{ww}(k)$ (instead of the TKE spectrum as proposed of GC). Towards this end, the approach used by Bonetti *et al.* [23] and Katul and Manes [24] is followed. The E_{ww}, at the top edge of the buffer layer, is approximated as shown in Fig. 1. Starting from low wave numbers, for $k < K_c$, E_{ww} displays a power law ($E_{ww} = k^p$), where p is an exponent dependent on the geometry of the flow domain [27], and K_c is a crossover wave number linked to large scale eddies and, in pipe flows, presumably scaling as $K_c \sim 1/R$. Moving towards higher wave numbers, at $K_c < k < K_a$, the spectrum typically displays a plateau (presumably attributed to wall effects resulting in "energy splashing" across scales), with K_a scaling as the inverse of the characteristic inner length scale of flow, namely the roughness size and the viscous length scale, for the fully rough and hydraulically smooth regimes, respectively. For $k > K_a$, E_{ww} follows the Kolmogorov spectrum, namely, $E_{ww} = E_{\mathrm{Kol}} = C_0 \epsilon^{2/3} k^s$

(with $s \approx -5/3$), where $C_0 = (24/55)C'_k$ is the universal Kolmogorov constant for the vertical velocity component and $C'_k = 1.5$ [27,32]. In the inertial subrange where $E_{ww}(k) = E_{\mathrm{Kol}}(k)$, the cospectrum reduces to $F_{uw}(k) = (C_0/A_\pi)\Gamma\epsilon^{1/3}k^{-7/3}$. Thus, the CSB model recovers the accepted cospectral shape and the correct similarity constant $(C_0/A_\pi) = 0.15$ [30,33]. Inclusion of the flux transfer term $T_{uw}(k)$ in the CSB necessarily yields deviations from a $F_{uw}(k) \sim k^{-7/3}$ scaling, which may be used as an indirect justification to ignoring $T_{uw}(k)$ altogether noting that $\int_0^\infty T_{uw}(k)dk = 0$ and including $T_{uw}(k)$ must lead to deviations from the $-7/3$ cospectral scaling. Analogous to Bonetti *et al.* [23], the viscous correction to the inertial subrange ($k > K_a$), usually modeled by an exponential cutoff [27,34], is herein neglected, so that the inertial range holds up to $k \to \infty$. We highlight that this approximation is not strictly needed to maintain analytical tractability. However, retaining this correction has very minor effects on the final results presented herein and, hence, it was neglected for clarity.

The chosen shape of the spectrum E_{ww} is, to some extent, arbitrary. However, in depth discussion and justifications for the proposed spectral model can be found in Bonetti *et al.* [23] and are only briefly summarized here. In particular, Bonetti *et al.* [23] noted that mean velocity profiles (and hence, implicitly, friction factors) obtained from the integration of the CSB equation were essentially insensitive to the exact value of the exponent p and the extent of the plateau in $E_{ww}(k)$.

A more conceptual criticism stems from the fact that the chosen spectral model for $E_{ww}(k)$ includes a well developed inertial range, which is unlikely to exist at the top of the buffer layer (see Zhao and Smits [35]). It is straightforward to demonstrate that the results presented herein are rather independent of this strong hypothesis (see Sec. V). However, before addressing this issue, an elaboration on the proposed spectral link is offered.

Equation (9) transforms to

$$I_k = \frac{E_{\mathrm{Kol}}(K_a)}{K_c^p}\int_0^{K_c} k^p k^{-2/3}\,dk + E_{\mathrm{Kol}}(K_a)\int_{K_c}^{K_a} k^{-2/3}\,dk + \int_{K_a}^{\infty} k^{-2/3}E_{\mathrm{Kol}}(k)\,dk. \tag{10}$$

From Eqs. (8) and (10), the friction factor f_d can be estimated as

$$f_d = \frac{\tau_0}{\rho V^2} = \left[\frac{C_o\epsilon^{2/3}K_a^s}{K_c^p}\int_0^{K_c} k^{p-2/3}\,dk + C_o\epsilon^{2/3}K_a^s\int_{K_c}^{K_a} k^{-2/3}\,dk \right.$$
$$\left. + C_o\epsilon^{2/3}\int_{K_a}^{\infty} k^{-2/3}k^s\,dk\right]\frac{|\Gamma|\epsilon^{-1/3}}{A_\pi V^2}. \tag{11}$$

This equation is further featured in the next two sections to assess the spectral link and the scaling of f_d for the hydraulically smooth and fully rough regimes. Towards this end, it is necessary to link ϵ and Γ, which are local properties of the mean flow and turbulence at the height where the CSB model is formulated, to bulk flow and geometry variables such as V, R, and d. As already noted in Bonetti *et al.* [23] (and later discussed here), this link between local and global variables remains one of the major deficiencies in the work of GC [6,36,37]. To avoid the *ad hoc* nature of this transformation, integrating Eq. (11) (twice) across the entire flow domain and across all wave numbers is necessary to enable the determination of how different spectral shapes affect the mean velocity profile and, for the fully rough case only, the scaling of f_d. As already introduced in Sec. I, this type of analysis does not allow for the finding of an explicit spectral link. For this reason, the local-to-global transformation is adopted here despite its shortcoming. However, its potential (and pitfalls) to clarify the link between friction factors and velocity spectra at limiting regime conditions is now explored.

III. THE SCALING OF THE FRICTION FACTOR IN THE FULLY ROUGH REGIME

Taking into account the fact that, in the fully rough regime, $K_c \sim 1/R$ and $K_a \sim 1/d$ [23], Eq. (11) becomes

$$f_d \approx \left[\frac{R^p}{d^s}\int_0^{1/R} k^{p-2/3}\,dk + d^{-s}\int_{1/R}^{1/d} k^{-2/3}\,dk + \int_{1/d}^{\infty} k^{-2/3}k^s dk\right]\frac{|\Gamma|C_o\epsilon^{1/3}}{A_\pi V^2}. \tag{12}$$

As proposed by Bonetti *et al.* [23] (and also, although in a different context, by Coscarella *et al.* [19]), the local-to-global transformation can be carried out as follows. The mean velocity gradient can be transformed as $\Gamma(z) = d\overline{u}(z)/dz = \alpha_1 V/d$ and the local TKE dissipation rate ϵ can be taken as $\alpha_2^3 V^3/R$, where α_1 and α_2 are *ad hoc* scaling functions, depending in general, as reported by Bonetti *et al.* [23] and below, on d/R and Re, that are not proportionality constants as implicitly assumed in GC.

In particular, in fully rough conditions, when Re $\to \infty$, α_1 and α_2 depend only on d/R [19,23]. Upon implementation of the local-to-global transformation and after some algebraic steps, the friction factor reads

$$f_d = \left[-\frac{3p}{p+1/3}\frac{d^{-1-s}}{R^{2/3}} + \frac{3s}{s+1/3}\frac{d^{-4/3-s}}{R^{1/3}}\right]\frac{C_0\alpha_1\alpha_2}{A_\pi}. \tag{13}$$

Taking into account that $s = -5/3$, Eq. (13) can be further reworked to obtain

$$f_d = \left[-\frac{3p}{p+1/3}\left(\frac{d}{R}\right)^{2/3} + \frac{15}{4}\left(\frac{d}{R}\right)^{1/3}\right]\frac{C_0\alpha_1\alpha_2}{A_\pi}, \tag{14}$$

where p is an unknown coefficient of the order of unity and, owing to the fully rough hypothesis, α_1 and α_2 depend on d/R only [24].

Equations (13) and (14) display interesting features. The power-law exponent p of the spectrum characterizing the low-wave-number range appears only in the first term on the right-hand side of the equation, which for small values of d/R, becomes negligible with respect to the second term, displaying the Strickler scaling, i.e., $(d/R)^{1/3}$. Therefore, Eq. (14) suggests that the Strickler scaling is retrieved for low relative roughness values and when the product $\alpha_1\alpha_2$ does not depend on d/R. This condition will be further explored using the Nikuradse experimental data [1]. Towards this end, $\alpha_1\alpha_2$ can be computed as follows:

$$\alpha_1\alpha_2 = \frac{A_\pi}{C_0}\frac{f_d}{-\frac{8}{3}\left(\frac{d}{R}\right)^{2/3} + \frac{15}{4}\left(\frac{d}{R}\right)^{1/3}}. \tag{15}$$

As an example, Bonetti *et al.* [23] considered $p = 8/3$ (i.e., analogously to the von Kármán spectrum, in which $p = 17/6$ [34]). Then, considering, for example, the empirical formula proposed by Nikuradse for the assessment of the friction factor in rough pipes [1],

$$f_d = \frac{1}{\left[1.74 + 2\log\left(\frac{d}{R}\right)^{-1}\right]^2}, \tag{16}$$

$\alpha_1\alpha_2$ can be calculated as a function of d/R, as shown in Fig. 2.

From Fig. 2 it is possible to see that the trend of $\alpha_1\alpha_2$ is approximately constant for values of d/R less than 0.02, since variations are contained within 5% of the mean value of $\alpha_1\alpha_2$ computed over this range. In contrast, for higher values of the relative roughness d/R, $\alpha_1\alpha_2$ varies by about 25% within the range of data explored by Nikuradse and as used in Fig. 2. Deviations of the friction factor from the Strickler scaling for high values of d/R have been repeatedly reported (see, e.g., [38], but it seems these results have been overlooked by the theoretical physics community) and it is interesting to see that the proposed model, although simplistic, captures this aspect, which is ascribed to a combined effect of the shape of the vertical spectrum in the low-wave-number range

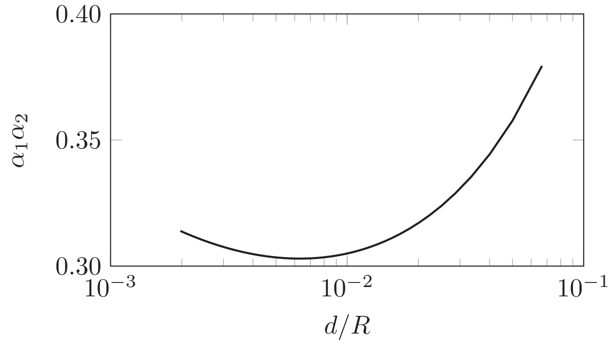

FIG. 2. $\alpha_1\alpha_2$ as a function of d/R in the fully rough turbulent flow case.

(i.e., in the range $k < K_a$) as well as the product of the scaling functions α_1 and α_2, which links global to local variables.

The above analysis suggests that the presence of the product $\alpha_1\alpha_2$ in Eq. (14), which depends also on d/R, prevents finding a clear spectral link, i.e., a well-defined relation between the spectral exponent $s = -5/3$ and the $1/3$ exponent in the Strickler formulation [i.e., $f_d \sim (d/R)^{1/3}$]. However, theoretical considerations presented herein (as well as experimental data) suggest that the Strickler scaling might be valid not only within the remit of the limiting condition identified by the fully rough regime (i.e., $\mathrm{Re} \to \infty$), but also for flow conditions involving large inner-outer length-scale separation, namely small values of d/R.

IV. THE SCALING OF THE FRICTION FACTOR IN THE SMOOTH REGIME

In the hydraulically smooth regime, the roughness size d is submerged within a viscous sublayer and no longer contributes to flow resistance, meaning that the viscous length scale ν/u_* becomes the dominant one in the inner region. The analysis here proceeds analogously to the fully rough case, but with the difference that the crossover wave number K_a can be expressed as $K_a \sim 1/(\nu/u_*)$, where ν/u_* can be written as η/η^+, η being the Kolmogorov length scale and η^+ the Kolmogorov length scale normalized with the viscous length scale. Consistent with the choice of carrying out the whole analysis at the top edge of the buffer layer [28], we set K_a as $1/(50\nu/u_*)$. The $50\nu/u_*$ in smooth flows is often considered as a good estimator of the wall normal distance where the mean velocity profile starts becoming independent of the viscous shear [27]; at this normal distance, η^+ can be taken equal to 2.71 [39]. In addition, the selection of z just above the buffer region of the mean velocity profiles justifies the choice of neglecting the wall-blockage effects in Eq. (6). In fact, the so-called "missing" curvature in the mean velocity profile owing to unconsidered wall blockage can be significant in the buffer layer [26], i.e., when $5\nu/u_* < z < 30\nu/u_*$ [27]. With this choice of K_a, Eq. (11) transforms as

$$f_d = \left[-\frac{3p}{p+1/3}\left(\frac{\eta^+}{50}\right)^{s}\eta^{-1-s}R^{-2/3} + \frac{3s}{s+1/3}\left(\frac{\eta^+}{50}\right)^{s+1/3}\eta^{-4/3-s}R^{-1/3}\right]\frac{C_0\alpha_1\alpha_2\eta^+}{50A_\pi}, \tag{17}$$

which provides the relation between the friction factor and spectral exponents, consistent with the chosen spectral model. Before further commenting on Eq. (17), it is important to define its limits of validity. As specified in Sec. II, a key requirement for the whole analysis is that the wall-normal elevation where the cospectrum and momentum balance are analyzed (i.e., for the case of smooth-wall flows, $z = 50\nu/u_*$) must be close enough to the wall so that the turbulent shear stress τ_t computed at this location can be considered in good approximation equal to $\tau_0 = \rho u_*^2$. An acceptable hypothesis could be to consider a mismatch not greater than 5%, which means that

$$\frac{\tau(50\nu/u_*)}{\tau_0} \geqslant 0.95, \tag{18}$$

or, given the linear dependence of the total shear stress on the radial direction,

$$\frac{Ru_*}{\nu} = R^+ \geqslant 10^3. \tag{19}$$

Now, since in smooth-wall pipe flows u_* can be up to two order of magnitude smaller than the bulk velocity V, a conservative approach would be to consider Eq. (17) valid for $\mathrm{Re} = 2RV/\nu \geqslant 5 \times 10^4$.

Taking $s = -5/3$ [27], Eq. (17) becomes

$$f_d = \left[-\frac{3p}{p+1/3}\left(\frac{50}{\eta^+}\right)^{5/3}\left(\frac{\eta}{R}\right)^{2/3} + \frac{15}{4}\left(\frac{50}{\eta^+}\right)^{4/3}\left(\frac{\eta}{R}\right)^{1/3}\right]\frac{C_0\alpha_1\alpha_2\eta^+}{50A_\pi}, \tag{20}$$

where

$$\begin{aligned}\frac{\eta}{R} &= \left(\frac{\nu^3}{\epsilon}\right)^{1/4}\frac{1}{R} = \left(\frac{\nu^3 R}{\alpha_2^3 V^3}\right)^{1/4}\frac{1}{R}\\ &= \frac{1}{\alpha_2^{3/4}}\frac{\nu^{3/4}}{V^{3/4}R^{3/4}} = \left(\frac{\alpha_2}{2}\mathrm{Re}\right)^{-3/4}\end{aligned} \tag{21}$$

and

$$f_d = \left[-\frac{3p}{p+1/3}\left(\frac{50}{\eta^+}\right)^{5/3}\left(\frac{\alpha_2}{2}\mathrm{Re}\right)^{-1/2} + \frac{15}{4}\left(\frac{50}{\eta^+}\right)^{4/3}\left(\frac{\alpha_2}{2}\mathrm{Re}\right)^{-1/4}\right]\frac{C_0\alpha_1\alpha_2\eta^+}{50A_\pi}. \tag{22}$$

The first term on the right-hand side of Eq. (22) scales as $\mathrm{Re}^{-1/2}$ and is the one containing the exponent p associated with the low wave number range of E_{ww} (large-scale eddies). For high Re [as required by Eq. (19)], this term is much smaller than the other term on the right hand side of the same equation, which scales as $\eta^{-4/3-s}R^{-1/3}$ or $\mathrm{Re}^{-1/4}$. This implies that, when the Reynolds number is high, the friction factor scaling is not significantly influenced by the low-wave-number range of spectra. In Eq. (22) the Blasius scaling is retrieved if the product $\alpha_1\alpha_2$ does not depend on Re. As per the fully-rough case, the scaling of $\alpha_1\alpha_2$ cannot be deduced from first principles and, hence, its dependence on Re must be assessed with the aid of experimental data. Towards this end and by taking $p = 8/3$ for illustration, $\alpha_1\alpha_2$ can be computed as

$$\alpha_1\alpha_2 = \frac{50A_\pi}{\eta^+ C_0}\frac{f_d}{-\frac{8}{3}\left(\frac{50}{\eta^+}\right)^{\frac{5}{3}}\left(\frac{\alpha_2}{2}\mathrm{Re}\right)^{-\frac{1}{2}} + \frac{15}{4}\left(\frac{50}{\eta^+}\right)^{\frac{4}{3}}\left(\frac{\alpha_2}{2}\mathrm{Re}\right)^{-\frac{1}{4}}} \tag{23}$$

and, neglecting the term scaling as $\mathrm{Re}^{-1/2}$, Eq. (23) becomes

$$\alpha_1\alpha_2^{3/4} = \frac{50A_\pi}{\eta^+ C_0}\frac{f_d}{\frac{15}{4}\left(\frac{50}{\eta^+}\right)^{4/3}\left(\frac{\mathrm{Re}}{2}\right)^{-1/4}}. \tag{24}$$

The dependence of $\alpha_1\alpha_2^{3/4}$ on Re is now explored using the original data by Nikuradse and the Princeton group as interpolated by Yang and Joseph [40] in Eq. (6) (not reported here for the sake of brevity) of their paper.

Figure 3 shows that $\alpha_1\alpha_2^{3/4}$ is approximately constant for values of Re less than 10^5, where, however, according to Eq. (19), the approach proposed herein cannot be applied with confidence. In contrast, at high Reynolds numbers, $\alpha_1\alpha_2^{3/4}$ varies by about 50% within the range of data explored experimentally by Nikuradse and the Princeton group. Deviations of the friction factors from the Blasius scaling for high values of Re have already been reported [1,4,40] and, according to the CSB

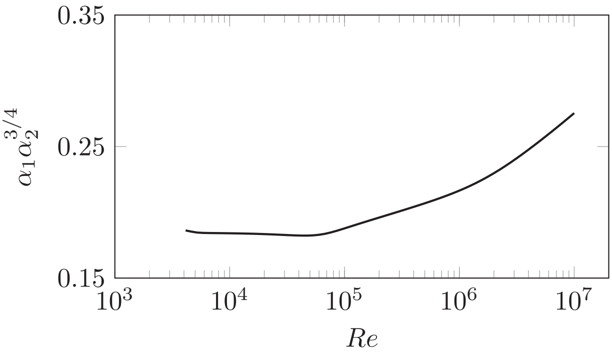

FIG. 3. $\alpha_1\alpha_2^{3/4}$ as a function of Re in the smooth turbulent flow case.

model here, they can be ascribed mostly to the product of the scaling functions α_1 and α_2, which link global to local variables.

As a final check on whether the superpipe experiments did maintain a smooth wall flow state at such high Reynolds number, it was verified by us that the internal microroughness of the superpipe (reported as $r = 1.5274 \times 10^{-7}$m) remained much smaller than the viscous sublayer thickness ($5\nu/u_*$) for all Re except the three highest Re values (about 3×10^7). At such high Re values, $r/(5\nu/u_*) \leqslant 0.5$ in all cases. Hence, the viscous sublayer thickness exceeded the micro-roughness of the walls to a leading approximation for the superpipe experiments.

V. DISCUSSION

As already mentioned in Sec. II, at a first glance the model spectrum used to represent $E_{ww}(k)$ might seem contradictory to common expectations in the buffer layer. We indeed assumed $E_{ww}(k)$ having an inertial range following the Kolmogorov scaling. It is known that this is not the case in the buffer layer and we seek to resolve this apparent contradiction and discuss the associated implications here.

Towards this end, the extreme case of an inertial layer that collapses to one point only at $k = K_a$ is presented followed by a classical exponential decay for $k > K_a$. Considering the smooth regime, and starting from Eqs. (8) and (10), the friction factor f_d can be estimated as follows

$$f_d = \left[\frac{K_a^s}{K_c^p}\int_0^{K_c} k^{p-2/3}\,dk + K_a^s\int_{K_c}^{K_a} k^{-2/3}\,dk + K_a^s\int_{K_a}^{\infty} k^{-2/3}\exp(-\beta k\eta)\,dk\right]\frac{C_0|\Gamma|\epsilon^{1/3}}{A_\pi V^2}, \tag{25}$$

where $\exp(-\beta k\eta)$ is the well-known exponential decay and β is a dimensionless constant equal to 2.1 [27]. Setting $x = k\eta$, the last integral of the previous equation becomes $\eta^{-1/3}\int_{\eta^+/50}^{\infty} x^{-2/3}\exp(-\beta x)\,dx$. After some algebraic steps and numerically solving the integral, we obtain

$$f_d = \left\{-\frac{3p}{p+1/3}\left(\frac{50}{\eta^+}\right)^{5/3}\left(\frac{\alpha_2}{2}\mathrm{Re}\right)^{-1/2} + \left[3 + 0.99\left(\frac{50}{\eta^+}\right)^{1/3}\right]\left(\frac{50}{\eta^+}\right)^{4/3}\left(\frac{\alpha_2}{2}\mathrm{Re}\right)^{-1/4}\right\}\frac{C_0\alpha_1\alpha_2\eta^+}{50A_\pi}. \tag{26}$$

This latter equation is equal to Eq. (22), except for the coefficient of the term $(50/\eta^+)^{4/3}\cdot(\alpha_2\mathrm{Re}/2)^{-1/4}$. This confirms that a well developed inertial range in E_{ww} is not required to recover the results presented so far. The only requirement is that K_a (i.e., wave-numbers scaling linearly with the elevation, which can be therefore associated with "attached eddies") follows the $-5/3$ scaling, which is a hypothesis in line with GC works [37]. There is no conceptual reason preventing similar conclusions from being drawn for hydraulically rough surfaces. However, this cannot be demonstrated with the proposed CSB approach, as the last integral of Eq. (25) depends on η, Re, and d/R and, hence, cannot be analytically integrated.

In further support of the robustness of the chosen spectrum shape, it can be shown that $\int_0^\infty E_{ww}(k)\,dk$ reduces to

$$\frac{\sigma_w}{u_*} = \frac{1}{u_*}\left[\int_0^\infty E_{ww}(k)\,dk\right]^{0.5} = \cdots = C_0^{0.5}\left(\frac{d}{R}\frac{-p}{p+1}+\frac{5}{2}\right)^{0.5} \tag{27}$$

in the fully rough regime and

$$\frac{\sigma_w}{u_*} = \frac{1}{u_*}\left[\int_0^\infty E_{ww}(k)\,dk\right]^{0.5} = \cdots = C_0^{0.5}\left[\frac{50}{\eta^+}\left(\frac{\alpha_2}{2}\mathrm{Re}\right)^{-3/4}\frac{-p}{p+1}+\frac{5}{2}\right]^{0.5} \tag{28}$$

in the hydraulically smooth regime, σ_w being the root mean square of the vertical velocity fluctuations.

In the limit of d/R and $\mathrm{Re}^{-3/4} \ll 1$ (i.e., in the fully rough and smooth regimes, respectively), the ratio $\sigma_w/u_* \approx \sqrt{C_0(5/2)} \approx 1.28$ is independent of p, d/R, or Re. This finding is consistent with numerous laboratory and field experiments [27].

VI. CONCLUSIONS

The present paper investigates the link between the scaling of the vertical velocity spectra and the well-known Blasius and Strickler scaling of f_d in hydraulically smooth and fully rough pipe flows, respectively. Such a link was initially established and investigated almost 20 years ago by GC using phenomenological arguments describing turbulent momentum transfer in proximity of the wall. Here, building upon the work by KCW, the existence of this link is refined through the integration of the CSB equation.

To maintain analytical tractability, the CSB equation must be complemented by relations linking the local variables ϵ and Γ to global quantities such as V and R. In doing so, it was necessary to introduce the functions α_1 and α_2, whose scaling behavior cannot be explicitly derived from first principles. The CSB equation also required an externally supplied shape of E_{ww}, which was idealized in Fig. 1.

Integration of the CSB equation for the fully rough and hydraulically smooth regimes leads to Eqs. (14) and (22), respectively. These equations provide the sought link between spectral properties of turbulence and friction factor, indicating that, for the fully rough and hydraulically smooth regimes, the characteristics of the spectra at low wave numbers (i.e., for $k < K_a$) are negligible, provided that d/R is small and $\mathrm{Re} \to \infty$, respectively.

If eddies at scale K_a follow the Kolmogorov scaling with $s = -5/3$ (see Sec. V), the Blasius and the Strickler scaling are recovered in Eqs. (14) and (22) provided that the product $\alpha_1\alpha_2$ is constant. Since the dependence of this product to either Re or d/R cannot be explored from first principles, we conclude that a link between the Kolmogorov and the Blasius/Strickler scaling cannot be theoretical. Caution must be exercised when using such a link (as established in previous studies) to investigate aspects related to wall turbulence and turbulent friction.

However, comparison between Eq. (15) and experimental data (Fig. 2) indicates that, for the fully rough regime, $\alpha_1\alpha_2$ is approximately constant for $d/R < 0.02$. For the hydraulically smooth case, $\alpha_1\alpha_2$ is constant only for $\mathrm{Re} < 10^5$ (Fig. 3), where, however, the proposed theoretical approach cannot be applied with confidence [see Eq. (24)]. Hence, it can be concluded that the existence of the Strickler scaling and its link with spectral properties of turbulence is supported by combining experimental evidence and theoretical arguments. The same cannot be claimed with certainty for the Blasius scaling that, instead, seems to have only empirical support.

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[1] J. Nikuradse, Laws of flow in rough pipes, National Advisory Committee for Aeronautics, Technical Memorandum 1292, 1950 (unpublished).

[2] I. H. Shames and I. H. Shames, *Mechanics of Fluids* (McGraw-Hill, New York, 1982), Vol. 2.

[3] N. Goldenfeld, Roughness-Induced Critical Phenomena in a Turbulent Flow, Phys. Rev. Lett. **96**, 044503 (2006).

[4] H. Blasius, Das aehnlichkeitsgesetz bei reibungsvorgängen in flüssigkeiten, in *Mitteilungen über Forschungsarbeiten auf dem Gebiete des Ingenieurwesens* (Springer, Berlin, 1913), pp. 1–41.

[5] A. Strickler, Contributions to the question of a velocity formula and roughness data for streams, channels and closed pipelines, Swiss Department of the Interior, Report of the Bureau of Water Affairs No. 16, 1923 (unpublshed).

[6] G. Gioia and P. Chakraborty, Turbulent Friction in Rough Pipes and the Energy Spectrum of the Phenomenological Theory, Phys. Rev. Lett. **96**, 044502 (2006).

[7] M. Mehrafarin and N. Pourtolami, Intermittency and rough-pipe turbulence, Phys. Rev. E **77**, 055304(R) (2008).

[8] T. Tran, P. Chakraborty, N. Guttenberg, A. Prescott, H. Kellay, W. Goldburg, N. Goldenfeld, and G. Gioia, Macroscopic effects of the spectral structure in turbulent flows, Nat. Phys. **6**, 438 (2010).

[9] H. R. Anbarlooei, D. O. A. Cruz, F. Ramos, and A. P. Silva Freire, Phenomenological Blasius-type friction equation for turbulent power-law fluid flows, Phys. Rev. E **92**, 063006 (2015).

[10] H. R. Anbarlooei, D. O. A. Cruz, F. Ramos, C. M. M. Santos, and A. P. Silva Freire, Phenomenological friction equation for turbulent flow of Bingham fluids, Phys. Rev. E **96**, 023107 (2017).

[11] C. Manes, L. Ridolfi, and G. Katul, A phenomenological model to describe turbulent friction in permeable-wall flows, Geophys. Res. Lett. **39**, L14403 (2012).

[12] G. G. Katul, A. G. Konings, and A. Porporato, Mean Velocity Profile in a Sheared and Thermally Stratified Atmospheric Boundary Layer, Phys. Rev. Lett. **107**, 268502 (2011).

[13] S. T. Salesky, G. G. Katul, and M. Chamecki, Buoyancy effects on the integral lengthscales and mean velocity profile in atmospheric surface layer flows, Phys. Fluids **25**, 105101 (2013).

[14] D. Li, S. T. Salesky, and T. Banerjee, Connections between the Ozmidov scale and mean velocity profile in stably stratified atmospheric surface layers, J. Fluid Mech. **797**, 29 (2016).

[15] G. Gioia and P. Chakraborty, Spectral derivation of the classic laws of wall-bounded turbulent flows, Proc. R. Soc. London A **473**, 20170354 (2017).

[16] D. Li, The O'KEYPS equation and 60 years beyond, Boundary-Layer Meteorol., **179**, 19 (2021).

[17] G. Gioia and F. A. Bombardelli, Localized Turbulent Flows on Scouring Granular Beds, Phys. Rev. Lett. **95**, 014501 (2005).

[18] F. A. Bombardelli and G. Gioia, Scouring of granular beds by jet-driven axisymmetric turbulent cauldrons, Phys. Fluids **18**, 088101 (2006).

[19] F. Coscarella, R. Gaudio, and C. Manes, Near-bed eddy scales and clear-water local scouring around vertical cylinders, J. Hydraul. Res. **58**, 968 (2020).

[20] C. Manes and M. Brocchini, Local scour around structures and the phenomenology of turbulence, J. Fluid Mech. **779**, 309 (2015).

[21] M. Musa, M. Heisel, and M. Guala, Predictive model for local scour downstream of hydrokinetic turbines in erodible channels, Phys. Rev. Fluids **3**, 024606 (2018).

[22] S. Z. Ali and S. Dey, Impact of phenomenological theory of turbulence on pragmatic approach to fluvial hydraulics, Phys. Fluids **30**, 045105 (2018).

[23] S. Bonetti, G. Manoli, C. Manes, A. Porporato, and G. Katul, Mannings formula and Stricklers scaling explained by a co-spectral budget model, J. Fluid Mech. **812**, 1189 (2017).
[24] G. G. Katul and C. Manes, Cospectral budget of turbulence explains the bulk properties of smooth pipe flow, Phys. Rev. E **90**, 063008 (2014).
[25] G. G. Katul, A. Porporato, C. Manes, and C. Meneveau, Co-spectrum and mean velocity in turbulent boundary layers, Phys. Fluids **25**, 091702 (2013).
[26] K. A. McColl, G. G. Katul, P. Gentine, and D. Entekhabi, Mean-velocity profile of smooth channel flow explained by a cospectral budget model with wall-blockage, Phys. Fluids **28**, 035107 (2016).
[27] S. B. Pope, *Turbulent Flows* (IOP, Bristol, 2001).
[28] B. McKeon, C. Swanson, M. Zagarola, R. Donnelly, and A. J. Smits, Friction factors for smooth pipe flow, J. Fluid Mech. **511**, 41 (2004).
[29] J. C. Rotta, Turbulent boundary layers in incompressible flow, Prog. Aerosp. Sci. **2**, 1 (1962).
[30] W. Bos, H. Touil, L. Shao, and J.-P. Bertoglio, On the behavior of the velocity-scalar cross correlation spectrum in the inertial range, Phys. Fluids **16**, 3818 (2004).
[31] D. Li and G. G. Katul, On the linkage between the $k^{-5/3}$ spectral and $k^{-7/3}$ cospectral scaling in high-Reynolds number turbulent boundary layers, Phys. Fluids **29**, 065108 (2017).
[32] A. N. Kolmogorov, The local structure of turbulence in incompressible viscous fluid for very large Reynolds numbers, Dokl. Akad. Nauk SSSR **30**, 301 (1941).
[33] J. Lumley, Similarity and the turbulent energy spectrum, Phys. Fluids **10**, 855 (1967).
[34] U. Frisch, *Turbulence: The Legacy of A.N. Kolmogorov* (Cambridge University Press, Cambridge, 1995).
[35] R. Zhao and A. J. Smits, Scaling of the wall-normal turbulence component in high-Reynolds-number pipe flow, J. Fluid Mech. **576**, 457 (2007).
[36] G. Gioia and F. A. Bombardelli, Scaling and Similarity in Rough Channel Flows, Phys. Rev. Lett. **88**, 014501 (2001).
[37] G. Gioia, N. Guttenberg, N. Goldenfeld, and P. Chakraborty, Spectral Theory of the Turbulent Mean-Velocity Profile, Phys. Rev. Lett. **105**, 184501 (2010).
[38] R. Ferguson, Time to abandon the Manning equation? Earth Surf. Processes Landforms **35**, 1873 (2010).
[39] C. Zuniga Zamalloa, Experiments on turbulent flows in rough pipes: spectral scaling laws and the spectral link, Ph.D. thesis, University of Illinois at Urbana-Champaign, 2012 (unpublished).
[40] B. H. Yang and D. D. Joseph, Virtual Nikuradse, J. Turbul. **10**, N11 (2009).