Comparison of standard and stabilization free Virtual Elements on anisotropic elliptic problems

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Abstract

In this letter we compare the behaviour of standard Virtual Element Methods (VEM) and stabilization free Enlarged Enhancement Virtual Element Methods (E^2VEM) with the focus on some elliptic test problems whose solution and diffusivity tensor are characterized by anisotropies. Results show that the possibility to avoid an arbitrary stabilizing part, offered by E^2VEM methods, can reduce the magnitude of the error on general polygonal meshes and help convergence.

Keywords: Virtual Element Method, Poisson problem, Polygonal meshes, Anisotropy

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1. Introduction

In recent years, polytopal methods for the solution of PDEs have received a huge attention from the scientific community. VEM were introduced in [1, 2, 3] as a family of methods that deal with polygonal and polyhedral meshes without building an explicit basis of functions on each element, but rather defining the local discrete spaces and degrees of freedom in such a way that suitable polynomial projections of basis functions are computable. The problem is discretized

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with bilinear forms that consist of a polynomial part that mimics the operator and an arbitrary stabilizing bilinear form. In [4], error analysis focused on anisotropic elliptic problems shows that the stabilization term adds an isotropic component of the error, independently of the nature of the problem. In [5], a modified version of the method, E^2VEM, was proposed, designed to allow the definition of coercive bilinear forms that consist only of a polynomial approximation of the problem operator. In this letter, we apply the two methods to solve some test Poisson problems with anisotropic solutions and diffusivity tensors. For each test, we compare the relative energy errors done by each method.

Let $\Omega \subset \mathbb{R}^2$ be a bounded open set with Lipschitz boundary. We look for a solution of Poisson problem with homogeneous Dirichlet boundary conditions, that in variational form reads: find $u \in \mathrm{H}^1_0(\Omega)$ such that

$$\left(\mathcal{K}\nabla u, \nabla v\right)_\Omega = \left(f, v\right)_\Omega \quad \forall v \in \mathrm{H}^1_0(\Omega)\,, \tag{1}$$

where $(\cdot,\cdot)_\Omega$ denotes the $\mathrm{L}^2(\Omega)$ scalar product and we assume $f \in \mathrm{L}^2(\Omega)$ and $\mathcal{K} \in [\mathrm{L}^\infty(\Omega)]^{2\times 2}$ is a symmetric positive definite matrix.

2. Problem discretization

We consider a star-shaped polygonal tessellation $\mathcal{M}_h$ of Ω satisfying the standard VEM regularity assumptions (see [3, 5]). Let $k \in \mathbb{N}$ such that $k \geq 1$ and, $\forall E \in \mathcal{M}_h$, let $\Pi^\nabla_{k,E} \colon \mathrm{H}^1(E) \to \mathbb{P}_k(E)$ be such that, $\forall v \in \mathrm{H}^1(E)$,

$$\left(\nabla \Pi^\nabla_{k,E} v, \nabla p\right)_E = \left(\nabla v, \nabla p\right)_E \;\; \forall p \in \mathbb{P}_k(E) \;\text{ and }\; \begin{cases} \int_{\partial E} \Pi^\nabla_{k,E} v = \int_{\partial E} v & \text{if } k = 1\,, \\ \int_E \Pi^\nabla_{k,E} v = \int_E v & \text{if } k > 1\,. \end{cases}$$

2.1. Standard Virtual Element discretization

According to [3], we define the following virtual space on any $E \in \mathcal{M}_h$:

$$\begin{aligned} \mathcal{V}^E_{h,k} = \Big\{ v_h \in \mathrm{H}^1(E) \colon \Delta v_h \in \mathbb{P}_k(E)\,,\, v_{h|e} \in \mathbb{P}_k(e) \;\forall e \subset \partial E,\, v_{h|\partial E} \in C^0(\partial E)\,, \\ \left(v_h, p\right)_E = \left(\Pi^\nabla_{k,E} v, p\right)_E \;\forall p \in \mathbb{P}_k(E)\,/\mathbb{P}_{k-2}(E) \Big\}\,, \end{aligned} \tag{2}$$

and the relative global space $\mathcal{V}_{h,k} = \{v_h \in \mathrm{H}^1_0(\Omega) \colon v_{h|E} \in \mathcal{V}^E_{h,k}\ \forall E \in \mathcal{M}_h\}$. Then (1) can be discretized by defining, $\forall E \in \mathcal{M}_h$, the stabilizing bilinear $S^E \colon \mathcal{V}^E_{h,k} \times \mathcal{V}^E_{h,k} \to \mathbb{R}$ such that, denoting by $\chi^E(v_h)$ the vector of degrees of freedom of v_h on E (see [3]),

$$S^E(u_h, v_h) = \chi^E(u_h) \cdot \chi^E(v_h) \quad \forall u_h, v_h \in \mathcal{V}^E_{h,k}\,,$$

and looking for $u_h^{\mathcal{V}} \in \mathcal{V}_{h,k}$ that solves

$$\sum_{E\in\mathcal{M}_h} \left(\mathcal{K}\Pi^0_{k-1,E}\nabla u_h^{\mathcal{V}}, \Pi^0_{k-1,E}\nabla v_h\right)_E + \|\mathcal{K}\|_{\mathrm{L}^\infty(E)} S^E\left((I-\Pi^\nabla_{k,E})u_h^{\mathcal{V}}, (I-\Pi^\nabla_{k,E})v_h\right)$$
$$= \sum_{E\in\mathcal{M}_h} \left(f, \Pi^0_{k-1,E} v_h\right)_E \quad \forall v_h \in \mathcal{V}_{h,k}\,, \quad (3)$$

where $\Pi^0_{k-1,E}$ denotes the $\mathrm{L}^2(E)$-projection on $\mathbb{P}_{k-1}(E)$ or $[\mathbb{P}_{k-1}(E)]^2$, depending on the context.

2.2. Enlarged Enhancement Virtual Element discretization

In [5], the space defined in (2) has been modified in order to allow the discrete problem to be well-posed without the need of defining a stabilizing bilinear form. Let $\ell_E \in \mathbb{N}$ be given $\forall E$, such that, denoting by N_E the number of vertices of E,

$$(k+\ell_E)(k+\ell_E+1) \geq kN_E + k(k+1) - 3\,.$$

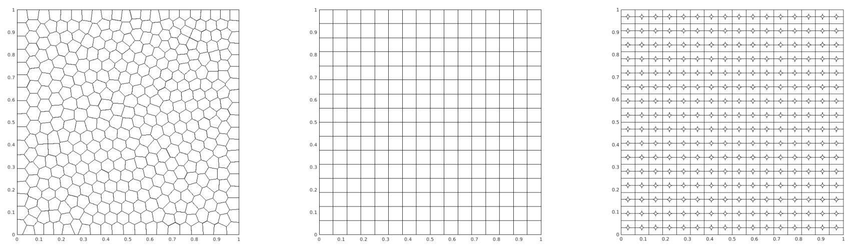
We define

$$\mathcal{W}^E_{h,k,\ell_E} = \big\{v_h \in \mathrm{H}^1(E) \colon \Delta v_h \in \mathbb{P}_{k+\ell_E}(E)\,,\ v_{h|e} \in \mathbb{P}_k(e)\ \forall e \subset \partial E,$$
$$v_{h|\partial E} \in C^0(\partial E)\,,\ (v_h, p)_E = \left(\Pi^\nabla_{k,E} v, p\right)_E\ \forall p \in \mathbb{P}_{k+\ell_E}(E)\,/\mathbb{P}_{k-2}(E)\Big\}\,, \quad (4)$$

that can be seen to have the same degrees of freedom of $\mathcal{V}^E_{h,k}$. Let $\mathcal{W}_{h,k,\ell} = \{v_h \in \mathrm{H}^1_0(\Omega) \colon v_{h|E} \in \mathcal{W}^E_{h,k,\ell_E}\ \forall E \in \mathcal{M}_h\}$. Then, we can discretize (1) by looking for $u_h^{\mathcal{W}} \in \mathcal{W}_{h,k,\ell}$ such that, $\forall v_h \in \mathcal{W}_{h,k,\ell}$,

$$\sum_{E\in\mathcal{M}_h} \left(\mathcal{K}\Pi^0_{k+\ell_E-1,E}\nabla u_h^{\mathcal{W}}, \Pi^0_{k+\ell_E-1,E}\nabla v_h\right)_E = \sum_{E\in\mathcal{M}_h} \left(f, \Pi^0_{k-1,E} v_h\right)_E\,. \quad (5)$$

The proof of well-posedness of (5) for $k = 1$ can be found in [5], while its extension to $k > 1$ will be the subject of an upcoming work.



(a) `Polymesher` (b) `Cartesian` (c) `Concave`

Figure 1: Meshes used in tests.

	`Polymesher`		`Cartesian`		`Concave`	
	order 1	order 3	order 1	order 3	order 1	order 3
$\mathrm{avg}\frac{\left\|\mathbb{A}^S\right\|_\infty}{\left\|\mathbb{A}^\Pi\right\|_\infty}$	1.05	0.27	1.00	0.23	0.93	0.27

Table 1: Test case 1. Average ratio through refinement between the infinity norms of the polynomial part $\mathbb{A}^\Pi$ and the stabilizing part $\mathbb{A}^S$ of the stiffness matrix in standard VEM.

3. Numerical results

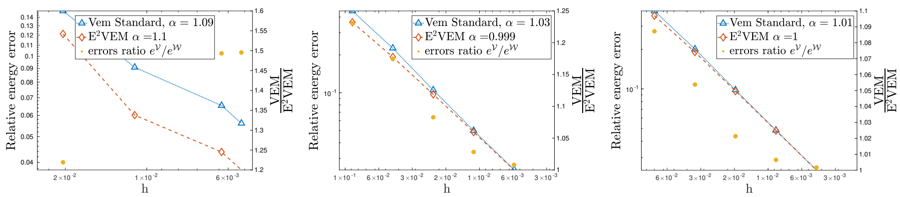
In all the test cases, we consider problem (1) on the unit square. We discretize the domain with the three families of polygonal meshes that are depicted in Figure 1, the first one being obtained using Polymesher [6], while the second one is a family of standard cartesian meshes and the third one is composed of concave polygons. We compare the two methods described in the previous section by observing the behaviour of the relative error computed in energy seminorm as

$$e^\star = \frac{\left(\sum_{E\in\mathcal{M}_h}\left\|\sqrt{\mathcal{K}}\nabla\left(u-\Pi^\nabla_{k,E}u_h^\star\right)\right\|^2_{\mathrm{L}^2(E)}\right)^{\frac{1}{2}}}{\left\|\sqrt{\mathcal{K}}\nabla u\right\|_{\mathrm{L}^2(\Omega)}} \quad \star=\mathcal{V},\mathcal{W}\,.$$

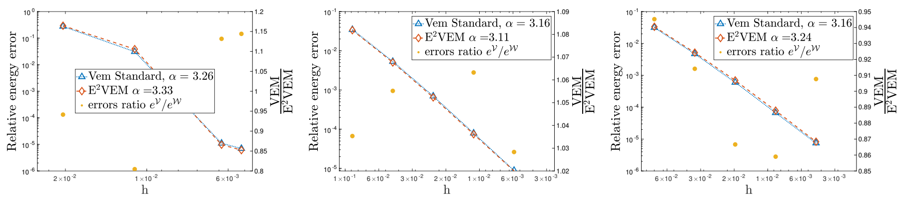
In the plots, we show the rate of convergence α computed using the last two computed errors.

3.1. Test case 1

In the first test, we define the forcing term f such that the exact solution is $u(x,y) = 10^{-2}xy(1-x)(1-y)(\mathrm{e}^{20x}-1)$, whereas $\mathcal{K} = 8\cdot10^{-3}(e_1e_1^\intercal)+e_2e_2^\intercal$,



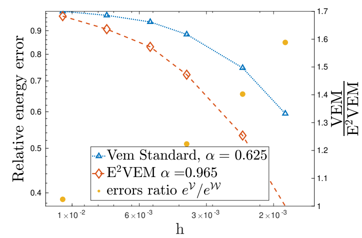
(a) Order 1, `Polymesher` mesh (b) Order 1, `Cartesian` mesh (c) Order 1, `Concave` mesh



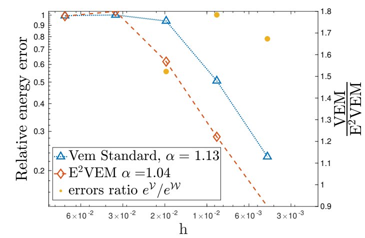
(d) Order 3, `Polymesher` mesh (e) Order 3, `Cartesian` mesh (f) Order 3, `Concave` mesh

Figure 2: Test case 1. Convergence plots.

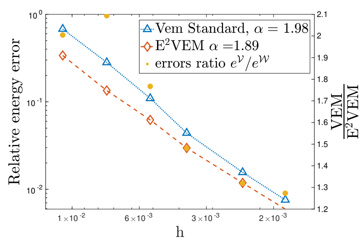
where e_1 and e_2 are the vectors of the canonical basis of $\mathbb{R}^2$. Figure 2 displays the behaviour of the errors obtained with the two methods and the ratio $e^{\mathcal{V}}/e^{\mathcal{W}}$, for orders 1 and 3, with respect to the maximum diameter of the discretization. The results show that the two methods behave equivalently on `Cartesian` and `Concave` meshes, whereas E^{2}VEM performs better on the `Polymesher` meshes with order 1, while the two methods tend to have the same behaviour with higher orders. This is due to the strong anisotropy both of the solution (that presents a strong boundary layer in the x-direction close to the boundary $x = 1$ of the domain) and of the diffusivity tensor $\mathcal{K}$. Indeed, as we can see from Table 1, for $k = 1$ the stabilizing part of the VEM bilinear form is of the same order of magnitude as the polynomial part, while for $k = 3$ we can see that the polynomial part is predominant. This induces larger errors (see Figure 2a) for the standard method on general polygonal meshes, such as the ones in the `Polymesher` family, since the stabilization is an isotropic operator. This effect is not felt by the E^{2}VEM method since its bilinear form consists only on a polynomial part that correctly takes into account the anisotropy of the tensor $\mathcal{K}$. The difference between the two methods is mitigated on `Cartesian` and `Concave` meshes since they are by construction aligned with the principal



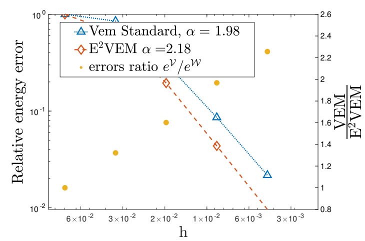
(a) Order 1, `Polymesher` mesh



(b) Order 1, `Concave` mesh



(c) Order 2, `Polymesher` mesh



(d) Order 2, `Concave` mesh

Figure 3: Test case 2. Convergence plots.

	`Polymesher`		`Cartesian`		`Concave`	
	order 1	order 2	order 1	order 2	order 1	order 2
avg $\frac{\left\Vert \mathbb{A}^S \right\Vert_\infty}{\left\Vert \mathbb{A}^\Pi \right\Vert_\infty}$	1.11	0.62	1.00	0.56	2.97	4.02

Table 2: Test case 2. Average ratio through refinement between the infinity norms of the polynomial part $\mathbb{A}^\Pi$ and the stabilizing part $\mathbb{A}^S$ of the stiffness matrix in standard VEM.

directions of the error (see the error analysis done in [4]). Notice that in this case the presence of non-convex polygons does not affect the results.

3.2. Test case 2

In the second test, the exact solution is $u(x, y) = \sin(2\pi x)\sin(80\pi y)$ and $\mathcal{K} = e_1 e_1^\intercal + 6.25 \cdot 10^{-4}(e_2 e_2^\intercal)$. In Figure 3 we display the error plots for orders 1 and 2 and Table 2 reports again the average ratio between the polynomial and stabilizing parts of the standard VEM bilinear form. The results for `Cartesian` meshes are consistent with the previous test, hence the convergence plots are not reported for brevity. However, we observe from Figure 3a that with `Polymesher` meshes the E^2VEM method reaches the asymptotic rate of convergence before the standard VEM method. This is due to the very strong anisotropy of the solution, due to its highly oscillating behaviour in the y-direction. Moreover, in this case the presence of non-convex polygons induces larger errors in the standard VEM, as we can see from Figures 3b and 3d. Notice that this fact can be related to the predominance of the stabilizing part as we can see in Table 2.

4. Conclusions

In this letter, we compare standard VEM and E^2VEM on some Poisson test problems. Numerical results show that in the presence of strong anisotropies of the solution and diffusivity tensor, with general convex polygonal meshes, lowest order E^2VEM perform better than VEM. In some strongly anisotropic cases, we observe better performance of E^2VEM also for higher orders in the presence of non-convex polygons. In all the other cases, the two methods behave equivalently.

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