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ISS control for continuous-time systems with filtered time-varying parameter and saturating actuators

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Abstract

This paper presents new convex conditions to design robust and LPV gains for continuous time-varying systems subject to saturating actuator and energy bounded disturbances. The input-to-state stability conditions are used to design controllers ensuring the minimization of the $\mathscr{L}_2$-gain between the disturbance input and the controlled output. Furthermore, optimization procedures to maximize the estimate of the region of attraction, and the bound to the control signal as well, are formulated. The efficacy of the proposed methods is illustrated with numerical examples, including a reference tracking problem, where realistic simulations are performed.

KEYWORDS:
Linear parameter-varying systems; Saturation; Linear matrix inequalities; Stabilization.

1 | INTRODUCTION

Linear parameter varying (LPV) systems can be used to represent nonlinear models, simplifying the search for stability certificates, and the design of filters and controllers. Currently, studies of LPV processes are on the rise, and several works can be found in the literature [14]. A great part of the existing results has been developed by using the Lyapunov theory. These problems have been solved with the use of Linear Matrix Inequalities (LMIs).

One of the first methods employed on the study of LPV systems consists in finding a Lyapunov function with a constant matrix that can guarantee the stability of the entire domain of the time-varying parameters. These results shed light on the solution to a great number of problems, although they may provide conservative results. To diminish the conservatism associated with the use of a single Lyapunov matrix, new methods based on the use of parameter-dependent Lyapunov functions have emerged [6, 23, 10], including delayed Lyapunov functions used in the context of discrete-time systems [24]. The main challenge when considering parameter-dependent Lyapunov functions for LPV continuous-time systems is the computation of the time derivative of the Lyapunov matrix. The works in [11] proposed LMI conditions to certify the stability of LPV continuous-time systems. The conditions consider the variation rate of the time-varying parameters to have known bounds and belonging to a convex polytopic domain. This fact led to less conservative results when compared with quadratic stability conditions. In [22], computational issues associated with solving parameter-dependent Lyapunov functions problems are discussed, exhibiting the growth of LMI number when the Lyapunov matrix depends on continuous time-varying parameters.

In [1], LPV systems with piecewise constant parameters under constant and minimum dwell-time are investigated yielding conditions to certificate stability for such systems and to design stabilizing controller. In [3], the stability of LPV systems with differentiable parameters is investigated. LPV systems with piecewise constant parameters subject to spontaneous Poissonian jumps are studied in [2], where conditions for stability analysis and state-feedback control are developed. A recent approach consists of using a Lyapunov matrix depending on the filtered time-varying parameter [5], where the authors show that it allows instantaneous jumps in the parameter, overcoming the inconvenience of bounding the time-derivative of the Lyapunov function.

Working with continuous time-varying parameters may be a challenging task due to the size and complexity of the problem. Moreover, the design of controllers in the presence of saturating actuators makes necessary estimating the set of initial conditions, as well the maximum energy disturbance tolerable, such that the closed-loop system remains stable [7]. Even for unstable linear open-loop systems, the actuator saturation makes the closed-loop nonlinear, and, for any state-feedback control law, the trajectories to the origin depend on the initial states. See [26] for details.

There are many ways of treating the problem of actuator saturation and one known method is applying the modified sector condition, introduced by [12] and used in [10] for discrete-time LPV systems. In [13] the authors investigate precisely known and time-invariant systems with asymmetric saturating actuators using nonlinear sector condition and piecewise quadratic Lyapunov candidate functions. In [8], $\mathscr{L}_2$ control of continuous-time LPV systems with saturating actuators is studied, employing the modified sector condition to deal with the saturation problem, and the Pólya theorem to reduce the conservatism of the proposed method. Homogeneous polynomially parameter-dependent state-feedback controllers for LPV systems with saturating actuators are also studied in [21]. The authors developed conditions to improve the estimates of the region of attraction (R.A.) for the closed-loop systems. The results obtained with parameter-dependent control laws are compared, in terms of the size of R.A., with robust controllers. Recently, the fault estimation problem for aero-engine LPV system has been considered in [28] with actuator and sensor faults under disturbances. See also [30] for a model predictive based approach. Local stabilization conditions under saturation, control signal quantisation, and state nonlinearity are given in [25] where amplitude bounded disturbances are taken into account. However, the nonlinearity is required to fulfill (local) Lipschitz conditions and the state space matrices must be time-invariant. In [19], the input-to-state practical stability, and a common Lyapunov function were employed to design adaptive fault-tolerant tracking controllers. A switched nonlinear model was employed, and a constant Lyapunov matrix was used in the derivation of the conditions. In [31], the control design problem dealing with unavailable states was addressed. The unknown states were compensated via a new form of K-filters. In [15], the dynamic output-feedback control problem for continuous-time LPV systems was addressed, but still requiring the continuity of the parameter variation. Thus the main motivating issues for this work are *i)* relaxing the requirement of continuity of the time-varying parameter, *ii)* designing a controller depending on a filtered version of the time-varying parameter, reducing actuator stress due to noise parameter measurements, *iii)* handling saturating actuators and energy disturbance signals. In particular, all the above-mentioned works require bounded parameter variation rate, which may limit the applicability of conditions. Our proposal overcomes such an issue.

This work proposes conditions, written in the form of LMIs to design robust, and LPV state-feedback gains for time-varying continuous systems subject to saturating actuator and energy bounded disturbances. The state-feedback gains designed depend on the filtered time-varying parameter. Additionally, when designing time-varying controllers for time-varying systems in real situations, the noise on the parameter measures may induce abrupt changes in the gain values and, consequently, in the control signal. Therefore, one may expect early damages on actuator due to wear and fatigue. Hence, filtering the time-varying parameter and guaranteeing that the closed-loop filtered system is stable may produce state-feedback gains with smaller variance over time. Inspired by [20], the methods proposed in this article consider the filtered parameter on the stabilization problems and, as an advantage over the existing literature, there is no need to know the bounds of the time-derivative of the parameter and such function does not need to be continuous. Moreover, the proposed technique provides local stability certificates for the closed-loop system for a set of initial conditions and disturbance limits. In this sense, different optimization problems can be addressed: *i)* minimize the $\mathscr{L}_2$ gain between the disturbance and measured output, *ii)* maximize the set of initial conditions such that the closed-loop remains stable, *iii)* minimize the control signal limit such that the closed-loop system is stable, or *iv)* maximize the disturbance energy tolerance.

The main contributions of this paper are to provide new design conditions formulated as convex optimization procedures. Moreover, the presented approach relaxes the fundamental assumption of constrained parameter time-derivative of LPV systems. Therefore, the new local stabilization design conditions for polytopic LPV systems present the following features under less restrictive hypothesis than the aforementioned methods:

1. the parameter of the LPV system may be discontinuous, allowing unbounded time-derivative rates;
2. the designed controller employs a filtered version of the system's parameter, naturally leading to smoother control signals;
3. minimization of the $\mathscr{L}_2$ gain between the disturbance and measured output;
4. maximization of the estimate of the region of attraction.

This paper is structured as follows: in the next section, we formulate the main problem. In Section 3, some preliminaries and auxiliary results are given. In Section 4, we present the main contributions followed by some selected convex optimization procedures such as to maximize either the estimate of the region of attraction, the minimization of the $\mathscr{L}_2$ gain between the exogenous signal and the output of the system. Numerical examples are provided in Section 5, where the efficacy of our approach is illustrated. The conclusions of this work are summarized in Section 6.

Notation: The symbol $\star$ stands for symmetric blocks in a matrix. Identity and zero matrices are represented by $\mathbf{I}$ and $\mathbf{0}$, respectively. The set of real numbers is denoted by $\mathbb{R}$. $M \in \mathbb{R}^{n \times n_u}$ is a matrix of dimensions $n \times n_u$ with real entries and $x \in \mathbb{R}^n$ is a vector with n positions and real entries. The transpose of a matrix M is $M^\top$, and $M^\top = M > (\geq) 0$ denotes a symmetric positive definite (semi-definite) matrix M. $\text{He}(M) = M + M^\top$ and $\text{diag}(X, Y)$ represents a block diagonal matrix, and $\texttt{trace}(M)$ denotes the sum of the diagonal elements of a square matrix M.

2 | PROBLEM FORMULATION

Consider the following LPV system subject to saturating actuators (to simplify the notation, the dependence on t will be omitted for the time-varying parameters):

$$\begin{aligned} \dot{x} &= A(\alpha)x + B(\alpha)\texttt{sat}(u) + B_w(\alpha)w \\ y &= C(\alpha)x + D_w(\alpha)w \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$ is the state vector and $u \in \mathbb{R}^{n_u}$ is the control input. The symmetric and decentralized saturation function, $\texttt{sat}(u)$, can be written as

$$\texttt{sat}\left(u_{(r)}\right) = \texttt{sign}\left(u_{(r)}\right) \min(|u_{(r)}|, \rho_{(r)}), \quad r = 1, \dots, n_u, \tag{2}$$

where $\rho \in \mathbb{R}^{n_u}$, and $\rho_{(r)}$ is the maximum allowed level of the signal $u_{(r)}$ due to the saturation of the r^{th} actuator. The exogenous signal $w \in \mathbb{R}^{n_w}$ is energy-bounded belonging to the set

$$\mathscr{W} = \left\{ w \in \mathbb{R}^{n_w} : \|w\|_2^2 \leq \delta^{-1} \right\}, \tag{3}$$

with $\|w\|_2^2 = \int_0^\infty w^\top(\tau)w(\tau)d\tau$. The matrices $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times n_u}$, $B_w \in \mathbb{R}^{n \times n_w}$, $C \in \mathbb{R}^{n_y \times n}$, and $D_w \in \mathbb{R}^{n_y \times n_w}$ depend linearly on α. All of these matrices can be generically written as $Q(\alpha) = \sum_{i=1}^N \alpha_i Q_i$, $\alpha \in \Lambda_N$, where Q_i, $i = 1, \dots, N$, are the vertices of the polytope and Λ_N is the unit simplex:

$$\Lambda_N = \left\{ \alpha \in \mathbb{R}^N : \sum_{i=1}^N \alpha_i = 1,\ \alpha_i \geq 0,\ i = 1, \dots, N \right\}. \tag{4}$$

As usual in LPV literature, we suppose the on-line availability of the time-varying parameter α, which has no restriction imposed on its variation rates. Moreover, it may even present discontinuities. These assumptions match some realistic conditions, such as allowing one to handle systems with jumps in the parameters, as it occurs in the class of switching systems. In the sequel, we formalize the main problem addressed.

Definition 2.1. A closed-loop system is said input-to-state stable (ISS), if for all $w \in \mathscr{W}$ and $x(0) \in \mathscr{R}_0$, the corresponding trajectories are bounded, i.e., they are confined in $\mathscr{R}_{\mathscr{A}}$ and if the disturbing is vanishing, then the trajectories converge asymptotically to the origin [7].

Problem 1. Consider the continuous-time system under saturating actuators (1)-(4), that depend linearly on the time-varying parameter α. Design a state-feedback controller depending linearly on smooth version of α, ensuring that:

1. the closed-loop system, which will be presented in the following section, is said ISS.
2. The value of $\sqrt{\gamma}$, $\gamma > 0$, is an $\mathscr{L}_2$-gain between the exogenous input and the system output, thus verifying

$$\|y\|_2 = \sqrt{\gamma}(\|w\|_2 + \mathfrak{b}), \tag{5}$$

for any $w \neq 0 \in \mathscr{W}$, and where the bias term $\mathfrak{b}$ is due to the non-null initial condition.

Associated with the items 1-2 of Problem 1, some optimization procedures, presented in Section 4, can be derived.

3 | PRELIMINARY RESULTS

Due to the assumptions on the time-varying parameter α, we may expect discontinuities and fast changes that could affect the performance of the actuators. Moreover, the discontinuities on α impact the search for a Lyapunov candidate function, since unbound terms may arise in its time derivative. To overcome such issues, we propose to use a filtered version of α, leading to a smoother parameter signal which will be used to certify the stability. By hypothesis, we assume α is (at least) piecewise continuous over the interval $[t - \beta, t]$, allowing to write for $\beta > 0$

$$\xi(t) = \beta^{-1} \int_{t-\beta}^{t} \alpha(\tau)d\tau, \quad \forall t > 0, \tag{6}$$

where ξ holds the convex sum property [5]: $\sum_{i=1}^N \xi_i = \sum_{i=1}^N \beta^{-1} \int_{t-\beta}^{t} \alpha(\tau)d\tau$. By using the linearity property, the integral and sum operators are interchangeable, leading to $\sum_{i=1}^N \xi_i = \beta^{-1} \int_{t-\beta}^{t} \sum_{i=1}^N \alpha_i(\tau)d\tau = \beta^{-1} \int_{t-\beta}^{t} d\tau = 1$. Moreover, from (6), the fundamental theorem of calculus allows us to write

$$\dot{\xi}(t) = \beta^{-1}\left(\alpha(t) - \alpha(t - \beta)\right). \tag{7}$$

Even for piecewise continuous functions, it is possible to see that $\dot{\xi}(t)$ is always finite and bounded for all $t \in [-\beta, +\infty]$. Another important aspect is that, for strictly positive values of β, $\xi(t)$ can be seen as a smoothed approximation of $\alpha(t)$. Furthermore, the bigger β, the smoother $\xi(t)$, as shown in [9].

To facilitate the following manipulations, the saturation function is rewritten in terms of the dead zone function [26] which is defined by $\Psi(u) = u - \mathtt{sat}\,(u)$. By considering a static feedback control law depending linearly on the parameter ξ, $u = K(\xi)x$, with $K(\xi) \in \mathbb{R}^{n_u \times n}$, and the dead-zone function $\Psi(u)$, it is possible to rewrite system's equation (1) as

$$\dot{x} = \mathscr{A}_{cl}x - B(\alpha)\Psi(u) + B_w(\alpha)w, \tag{8}$$

where $\mathscr{A}_{cl} = A(\alpha) + B(\alpha)K(\xi)$ is the closed-loop dynamic matrix, whenever $|u_{(r)}| \leq \rho_{(r)}$, $r = 1, \ldots, n_u$.

System (8) is nonlinear because of $\Psi(u)$ hence, the local stability is required to characterize the allowed initial conditions ensuring their respective trajectories go to the origin. The set of all initial conditions resulting in trajectories that converge to the origin is denoted by $\mathscr{R}_{\mathscr{A}} \subseteq \mathbb{R}^n$, being called the region of attraction. Such a region can be non-convex, open, and even unbounded in some directions, mading its characterization non trivial [26]. Therefore, we provide an estimate region, $\mathscr{R}_{\mathscr{E}} \subseteq \mathscr{R}_{\mathscr{A}}$, as large as possible.

Moreover, because of Problem 1, we are also interested in a subset $\mathscr{R}_0 \subseteq \mathscr{R}_{\mathscr{E}}$, as large as possible, consisting of the initial conditions such that the respective closed-loop trajectories converge to the origin without leaving $\mathscr{R}_{\mathscr{E}}$, for a given disturbance w with bounded energy δ^{-1}, i.e., for $w \in \mathscr{W}$.

We employ the sector nonlinearity to handle the saturating actuator in (8). Thus, if we provide that the control signal u and an auxiliary signal $v \in \mathbb{R}^{n_u}$ belonging to the set

$$\mathbb{S}(u - v, \rho) = \left\{ u \in \mathbb{R}^n, v \in \mathbb{R}^n \,:\, |u_{(r)} - v_{(r)}| \leq \rho_{(r)} \right\}, \tag{9}$$

$\forall r = 1, \ldots, n_u$, then, the following generalized sector condition [26, 12] concerning with the dead-zone nonlinearity:

$$\Psi(u)^\top S(\xi)\Big(\Psi(u) - v\Big) \leq 0 \tag{10}$$

is verified for any positive definite diagonal matrix $S(\xi) \in \mathbb{R}^{n_u \times n_u}$. We choose v with similar structure to the signal u, i.e., we consider $v = G(\xi)x$, with G depending linearly on ξ is an extra degree of freedom in the design conditions. We consider a non-quadratic Lyapunov function candidate, depending linearly on the parameter ξ, given by

$$V(x, \xi) = x^\top P(\xi)^{-1}x. \tag{11}$$

Such a choice of structure for the Lyapunov candidate function, allows us to keep a relatively low computational requirement while providing a solution for Problem 1 for a large class of systems. Moreover, taking a parameter independent Lyapunov candidate function leads to a controller that cannot be adapted to parameter changes, yielding to more conservative results.

To estimate the region of attraction, $\mathscr{R}_{\mathscr{E}}$, we use a level set of the Lyapunov function associated with the closed-loop system:

$$\mathscr{L}_{\mathscr{V}}(\mu) = \bigcap_{\forall\, \xi \in \Lambda} \mathscr{E}(P(\xi)^{-1}, \mu) \tag{12}$$

with $\mu > 0$ and the ellipsoidal set

$$\mathscr{E}(P(\xi), \mu) = \{x \in \mathbb{R}^n \mid x^\top P(\xi)^{-1}x \leq \mu^{-1}\}. \tag{13}$$

The level set presented in (12) remains as an infinite dimensional condition, since all values of ξ in the continuous set Λ must be verified. The following lemma, developed in [16], where the proof can be found, presents an equivalent form to compute such a set through a finite dimensional condition.

Lemma 3.1. Suppose that $V(x, \xi)$ is a Lyapunov function for system (8). Then, the level set (12) can be equivalently computed by using the finite dimensional condition

$$\mathscr{L}_{\mathscr{V}}(\mu) = \bigcap_{\forall\, \xi \in \Lambda} \mathscr{E}(P(\xi)^{-1}, \mu) = \bigcap_{i=1}^{N} \mathscr{E}(P_i^{-1}, \mu) \tag{14}$$

for $\mu > 0$ and $\mathscr{E}(P_i, \mu) = \{x \in \mathbb{R}^n; x^\top P_i^{-1}x \leq \mu^{-1}\}$.

4 | MAIN RESULTS

The main contribution of this work is the proposition of new design conditions of an LPV state-feedback gain stabilizing systems depending linearly on time-varying parameters and under saturating actuators. Because of the use of smoothed parameter ξ given in (6), the systems' parameter may have jumps.

Theorem 4.1. Consider a continuous-time system described by (1)-(4). If there exist symmetric positive-definite matrices $P_i \in \mathbb{R}^{n \times n}$, matrices $U_i \in \mathbb{R}^{n_u \times n}$, $Z_i \in \mathbb{R}^{n_u \times n}$, diagonal positive-definite matrices $S_i \in \mathbb{R}^{n_u \times n_u}$, with $i = 1, \ldots, N$, and given positive real scalars μ, δ, and γ such that

$$\begin{bmatrix} \mathrm{He}\big(A_jP_i + B_jZ_i\big) - \frac{1}{\beta}(P_j - P_k) & -B_jS_i + U_i^\top & B_{wj} & P_iC_j^\top \\ \star & -S_i - S_i^\top & \mathbf{0} & \mathbf{0} \\ \star & \star & -\mathbf{I} & D_{wj}^\top \\ \star & \star & \star & -\gamma\mathbf{I} \end{bmatrix} < \mathbf{0}, \tag{15}$$

$$\begin{bmatrix} P_i & Z_i^\top - U_i^\top \\ \star & \rho_{(r)}^2\mu \end{bmatrix} \geq \mathbf{0}, \tag{16}$$

$$\delta - \mu \geq 0, \tag{17}$$

hold, for all $i,\ j,\ k = 1, \ldots, N$, and $r = 1, \ldots, n_u$, then the time-varying gains given by

$$K_i = Z_iP_i^{-1}, \tag{18}$$

are such that the control law $u = K(\xi)x$ with $K(\xi)$ computed as $K(\xi) = \sum_{i=1}^{N} \xi_iK_i$ ensures a ISS closed-loop system, providing a solution to Problem 1, that is:

1. for $w = 0$, the trajectories of the closed-loop system initiated in the set $\mathscr{R}_{\mathscr{E}} = \mathscr{L}_{\mathscr{V}}(\mu)$, with $\mathscr{L}_{\mathscr{V}}$ given in (12) converge to the origin without leaving this set;
2. for any $w \in \mathscr{W}$, $w \neq 0$, the trajectories of (8) remain confined in $\mathscr{R}_{\mathscr{E}} = \mathscr{L}_{\mathscr{V}}(\mu)$ for all initial conditions belonging to $\mathscr{R}_0 = \mathscr{L}_{\mathscr{V}}(\lambda)$, with $\lambda = (\mu^{-1} - \delta^{-1})^{-1}$;
3. for any $w \in \mathscr{W}$, $w \neq 0$, and $x(0) \in \mathscr{L}_{\mathscr{V}}(\mu)$, $\sqrt{\gamma}$ is a guaranteed $\mathscr{L}_2$-gain between the exogenous input w and the output y, ensuring (5) with $b = x(0)^\top P(\xi)^{-1}x(0)$.

Proof. The proof is given in two parts: one for the local ISS and $\mathscr{L}_2$-gain of the closed loop system, and another for the inclusion of the level set into $\mathbb{S}$. Stability part: Multiply (15) by $\xi_i(t)$, $i = 1, \dots, N$, and sum it up to get the terms depending in i expressed in terms of $\xi(t)$. Additionally, replace $Z(\xi) = K(\xi)P(\xi)$, $U(\xi) = G(\xi)P(\xi)$, and $S(\xi) = T(\xi)^{-1}$, multiply the obtained equation by $\alpha_k(t-\beta)$, for $k = 1, \dots, N$, $\alpha_j(t)$, for $j = 1, \dots, N$, and sum it up to get,

$$\begin{bmatrix} \mathscr{R} & -B(\alpha)T(\xi)^{-1} + P(\xi)G(\xi)^\top & B_w(\alpha) & P(\xi)C(\alpha)^\top \\ \star & -T(\xi)^{-1} - T(\xi)^{-T} & \mathbf{0} & \mathbf{0} \\ \star & \star & -\mathbf{I} & D_w(\alpha)^\top \\ \star & \star & \star & -\gamma\mathbf{I} \end{bmatrix} < \mathbf{0}.$$

with $\mathscr{R} = \mathscr{A}_{cl}P(\xi) + P(\xi)\mathscr{A}_{cl}^\top - \beta^{-1}\Big(P(\alpha) - P\big(\alpha(t-\beta)\big)\Big)$, and $\mathscr{A}_{cl}$ as in (8). By employing (7) and the convexity properties of the polytopic representation, one has $-\beta^{-1}\Big(P(\alpha) - P\big(\alpha(t-\beta)\big)\Big) = -\sum_{i=1}^{N}\beta^{-1}\big(\alpha_i(t) - \alpha_i(t-\beta)\big)P_i = \sum_{i=1}^{N}\dot{\xi}_i P_i = \dot{P}(\xi)$. Since $T(\xi)$ and $P(\xi)$ are regular matrices, we pre- and post-multiply the last inequality by $\mathrm{diag}\big(P(\xi)^{-1}, T(\xi), \mathbf{I}, \mathbf{I}\big)$. Moreover, we use the fact $P(\xi)P(\xi)^{-1} = I$ to write $\dot{P}(\xi)P(\xi)^{-1} + P(\xi)\dot{P}(\xi)^{-1} = \mathbf{0}$, or simply $-P(\xi)^{-1}\dot{P}(\xi)P(\xi)^{-1} = \dot{P}(\xi)^{-1}$, yielding

$$\begin{bmatrix} \mathscr{W} & \mathscr{F} & P(\xi)^{-1}B_w(\alpha) & C^\top(\alpha) \\ \star & -T(\xi)^\top - T(\xi) & \mathbf{0} & \mathbf{0} \\ \star & \star & -\mathbf{I} & D_w{}^\top(\alpha) \\ \star & \star & \star & -\gamma\mathbf{I} \end{bmatrix} < \mathbf{0},$$

with $\mathscr{W} = \mathrm{He}(P(\xi)^{-1}\mathscr{A}_{cl}) + \dot{P}(\xi)^{-1}$ and $\mathscr{F} = -P(\xi)^{-1}B(\alpha) + G(\xi)^\top T(\xi)^\top$. By applying the Schur complement, and pre- and post-multiplying the resulting inequality by $\begin{bmatrix} x^\top & \Psi(u)^\top & w^\top \end{bmatrix}$ and its transpose, it is possible to get $\dot{V}(x) + \gamma^{-1}y^\top y - w^\top w - 2\Psi(u)^\top S(\xi)\Big(\Psi(u) - G(\xi)x\Big) < 0$, where $V(x) = x^\top P(\xi)^{-1}x$, and $\dot{V}(x) = \dot{x}^\top P(\xi)^{-1}x + x^\top P(\xi)^{-1}\dot{x} + x^\top \dot{P}(\xi)^{-1}x$. With $u = K(\xi)x$ and $v = G(\xi)x$, and supposing that these signals belong to the set $\mathbb{S}(u - v, \rho)$ given in (9), then the generalized sector condition (10) is ensured and it is possible to guarantee that $\dot{V}(x) + \gamma^{-1}y^\top y - w^\top w < 0$, meaning that with $\epsilon_1 = \max_{\xi\in\Lambda}\lambda\big(P(\xi)\big)$, $\epsilon_2 = \min_{\xi\in\Lambda}\lambda\big(P(\xi)\big)$ we verify $\epsilon_1\|x\|^2 \le V(x) \le \epsilon_2\|x\|^2$. Moreover, with $w = 0$ for $t \ge t_0$, we have $\dot{V}(x) \le -\gamma^{-1}y^\top y + 2\Psi(u)^\top S(\xi)\Big(\Psi(u) - G(\xi)x\Big) \le -\epsilon_3\|x\|^2 < 0$ for a small enough $\epsilon_3 > 0$. Thus, $V(x) = x^\top P(\xi)^{-1}x$ given in (11) is a Lyapunov function for the closed-loop system (8), allowing us to conclude on the local ISS stability of such a system.

Inclusion part: In the previous part, we required that signals u and v belong to the set $\mathbb{S}$. In this part of the proof, we demonstrate that the choice of initial conditions $x(0) \in \mathscr{L}_{\mathscr{V}}(1)$ ensures such a property.

If additionally to (15), the inequality (16) is also verified, multiply it by $\xi_i(t)$, $i = 1, \dots, N$, add it up, and replace $Z(\xi)$ and $U(\xi)$ by $K(\xi)P(\xi)$ and $G(\xi)P(\xi)$, respectively, pre- and post-multiply the obtained inequality by $\mathrm{diag}\big(P(\xi)^{-1}, 1\big)$ and its transpose, and apply Schur complement, to get $P(\xi)^{-1} - \big[K(\xi)_{(r)} - G(\xi)_{(r)}\big]^\top \rho_{(r)}^{-2}\mu^{-1}\big[K(\xi)_{(r)} - G(\xi)_{(r)}\big] \ge \mathbf{0}$ for all $r = 1, \dots, n_u$. This last inequality can be pre- and post-multiplied by $x^\top$ and its transpose, respectively, yielding $x^\top P(\xi)^{-1}x \ge x^\top\big[K(\xi)_{(r)} - G(\xi)_{(r)}\big]^\top \rho_{(r)}^{-2}\mu^{-1}\big[K(\xi)_{(r)} - G(\xi)_{(r)}\big]x$ for all $r = 1, \dots, n_u$. With $u = K(\xi)x$, $v = G(\xi)x$, and taking the initial condition $x(0) \in \mathscr{L}_{\mathscr{V}}(\mu)$, we have that, due to the stability of the system, $x^\top P(\xi)^{-1}x \le V(x(0)) = x(0)^\top P(\xi)^{-1}x(0) \le \mu^{-1}$. Therefore, it is possible to write $1 \ge \mu V(x(0)) \ge \mu x^\top P(\xi)^{-1}x \ge |u_{(r)} - v_{(r)}|^2/\rho_{(r)}^2$, which allows us to conclude that $|u_{(r)} - v_{(r)}| \le \rho_{(r)}$. Thus, inequality (16) ensures that the generalized sector condition is verified with initial conditions belonging to $\mathscr{L}_{\mathscr{V}}(\mu)$. As a consequence, the level set contractiveness is also guaranteed, providing $\mathscr{R}_{\mathscr{E}} = \mathscr{L}_{\mathscr{V}}(\mu)$, meaning that the set $\mathbb{S}$ includes the contractive set $\mathscr{R}_{\mathscr{E}} = \mathscr{L}_{\mathscr{V}}$. As a consequence, the Lemma 3.1 is valid for any trajectory of the closed-loop system starting $\mathscr{L}_{\mathscr{V}}$ remains in $\mathbb{S}$. □

Remark 1. The numerical complexity of Theorem 4.1 is associated with the number of scalar decision variables N_V, and the number of LMI rows N_R, that can be formulated as $N_V = N\left[\frac{n(n+1)+2n_u(2n+1)}{2}\right]$, $N_R = N\left[n_u(n+1) + N^2(n + n_u + n_w + n_y)\right]$, where N is the number of vertices of the system.

When $w = 0$ in (1) and (8), Theorem 4.1 can be adapted to deal with the stabilization problem.

Corollary 4.2. Consider a continuous-time system described by (1)-(4). If there exist symmetric positive-definite matrices $P_i \in \mathbb{R}^{n\times n}$, matrices $U_i \in \mathbb{R}^{n_u\times n}$, $Z_i \in \mathbb{R}^{n_u\times n}$, diagonal positive-definite matrices $S_i \in \mathbb{R}^{n_u\times n_u}$, with $i = 1, \dots, N$, and a given positive scalar μ, such that (16) and

$$\begin{bmatrix} \mathrm{He}\big(A_jP_i + B_jZ_i\big) - \frac{1}{\beta}(P_j - P_k) & -B_jS_i + U_i^\top \\ \star & -S_i - S_i^\top \end{bmatrix} < \mathbf{0},$$

hold, for all i, j, $k = 1, \dots, N$, $r = 1, ..., n_u$, then the time-varying controller given by (18) ensures that the closed-loop system is asymptotically stable for all initial conditions belonging to $\mathscr{R}_{\mathscr{E}} = \mathscr{L}_{\mathscr{V}}(\mu)$.

Note that if there is no information concerning the time-varying parameter is available, one may use $S_i = S$, $Z_i = Z$, and $P_i = P$, leading to a robust controller design.

Remark 2. In case no information on the time-varying parameter and its derivative is available, the quadratic approach can be recovered from Theorem 4.1 by imposing $P_i = P$, $Z_i = Z$, $S_i = S$, $U_i = U$, $i = 1, \dots, N$, leading to quadratic input-to-state stabilizing control law $u = Kx$, with $K = ZP^{-1}$.

Remark 3. Formulations presented in Theorem 4.1 and Corollary 4.2 heritages some conservatism from the structure chosen for the Lyapunov candidate function. Less conservative formulations may be obtained with homogeneous polynomial functions [23] or even with polynomial dependency on the state [27, 29]. However, the computational burden increases due to the Sum of Squares (SOS) requirements. Also, slack variables could be used following the approach in [5], leading to a controller design that does not depend directly on the Lyapunov matrix. Also, despite inequality (16) may have some conservatism due to the generalized sector condition, it is a key feature to handle control input saturation allowance, which leads to better results than saturation avoidance methods.

4.1 | Optimization design procedures

The conditions proposed in Section 4 work with different scalar variables, μ, δ, γ, and ρ, that can be optimized to specific performances of the closed-loop system. For instance, it is possible designing controllers that maximize $\mathscr{R}_{\mathscr{E}}$ or the maximal tolerable energy disturbance, δ^{-1}. Another design objective may be the minimization of the $\mathscr{L}_2$ gain, $\sqrt{\gamma}$, or the minimal saturating bound of the control signal that ensures the regional stability or a desired $\mathscr{L}_2$. Such objectives can be pursued with the aid of the convex optimization procedures.

Maximization of $\mathscr{R}_{\mathscr{E}}$: The design of the parameter-dependent control gain can be optimized to achieve a larger set of allowed initial conditions. Thus, the goal is to maximize the estimated region of attraction considering $w = 0$.

For maximizing the region of attraction, we consider the optimization problem given by maximizing an ellipsoid given by $\mathscr{E}(H, \mu)$, $H \in \mathbb{R}^{n\times n}$, such that $\mathscr{E}(H, \mu) \subset \mathscr{L}_{\mathscr{V}}(\mu)$, which can ensured by

$$\mathscr{T}_1 : \begin{cases} \min\limits_{P_i, U_i, Z_i} \ \texttt{trace}(H) \\ \text{subject to:} \quad \text{Theorem 4.1 or Corollary 4.2 and} \\ \qquad \begin{bmatrix} P_i & \mathbf{I} \\ \star & H \end{bmatrix} > \mathbf{0}. \end{cases} \tag{19}$$

Optimization procedure $\mathscr{T}_1$ explores an indirectly way to maximize the estimate of the region of attraction by minimizing the $\texttt{trace}(H)$: from Schur complement in the last constraint of (19), we have $H > P_i^{-1}$ that, with the objective function, shrinks P_i enlarging the region of attraction. The other optimization procedures can be proposed in the form of

$$\mathscr{T}_2 : \begin{cases} \min f \\ \text{subject to:} \quad \text{Theorem 4.1 or Corollary 4.2,} \end{cases} \tag{20}$$

where the objective function f can be adequately chosen with the intents of the designer. For instance, consider the following three criteria:

1. Choose $f = \rho$ to search for the smallest amount of control signal necessary to stabilize the system and reject the disturbance.
2. Choose $f = \gamma$ to minimize the $\mathscr{L}_2$-gain between the disturbance w and the output y. Note that, if null initial conditions are employed, i.e., $x(0) = 0$, then $\delta^{-1} = \mu^{-1}$.
3. Choose $f = \mu$ to maximize the energy bound of the disturbance, δ^{-1}, for initial conditions $x(0) \in \mathscr{R}_0$. Note that if $x(0) = 0$, $\delta^{-1} = \mu^{-1}$.

Corollary 4.2 cannot be used in cases 2 and 3 because it is assumed there is no external disturbance, in such a case.

Remark 4. If the optimization procedures are not used, one can choose the parameters according to the following: ρ is the maximum control signal amplitude available to the system, so it should be chosen such that $\rho \leq \texttt{sat}\left(u_r\right)$; choose $\sqrt{\gamma}$ as a desirable bound for the $\mathscr{L}_2$-gain between the disturbance w and the output y; select μ according to the energy bound of the disturbance applied on the system, δ^{-1}, and the initial conditions $x(0)$, as (17). Concerning the filtering parameter $\beta > 0$, any small enough value is sufficient to handle discontinuities in the time-varying parameter since its time-derivative (7) will be finite. Such a β can also be chosen as a filtering tax, so one may tune it to mitigate undesired frequencies and oscillations of the time-varying parameter signal. The greater the β value, the smoother the ξ time-behavior. See also Example 2 where the effect of parameters ρ, γ, and β on control performance is discussed.

5 | NUMERICAL EXPERIMENTS

This section illustrates the results for the design of stabilizing controllers for LPV continuous-time systems with saturating actuators. The effects of the filtered time-varying parameter ξ in the control design and some optimization design procedures will be explored. The routines were implemented in MATLAB, version 8.5.0.197613 (R2015a), Windows 10 (Intel Celeron N2940, 1.83 GHz, x64), using Yalmip and SeDuMi.

Example 1. In this example, it is discussed the influence of the filtering tax β on both time-varying parameter and state-feedback gains. It is shown that even when the time-varying parameter derivative is unbounded, LPV controllers can be calculated. Besides, the smoothest behavior of the state-feedback gains is obtained for higher values of β.

Consider the inverted pendulum system investigated in [8] where the pendulum varies between $-30°$ and $+30°$ and the control signal is constrained by the voltage applied to the armature motor which cover a range from -13V up to $+13$V. Such a system is described by (1), where the polytope vertices are given by matrices:

$$A_1 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 2.2643 & -15.148 & -0.0073 \\ 0 & 27.8203 & -36.6044 & -0.0896 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 0 \\ 2.2772 \\ 5.2470 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 2.2643 & -16.664 & -0.0073 \\ 0 & 27.8203 & -36.6044 & -0.0896 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix},$$

$B_1 = B_2 = B$, $B_{w1} = \begin{bmatrix} 0 & 0 & -1.2497 & -2 \end{bmatrix}^\top$, $B_{w2} = \begin{bmatrix} 0 & 0 & -1.3748 & -2 \end{bmatrix}^\top$, $D_w = 0$. The system input, which is the force applied by a DC servomotor, saturates at $|\rho| = 13$.

Our goal in this experiment is to design state-feedback gains for different values of the filtering tax β, and illustrate its impact on both filtering parameter and control law. With the obtained designs, we simulated the system with the same noise affecting the measured parameter α. Figure 1 shows the influence of β on the time-varying parameter α_1, where the great values of β clearly lead to smooth behavior of parameter ξ used in the control law. The green bands indicate instants where the parameter α is discontinuous. Therefore, it is clear that the approach in [25] can not handle this case because of the Lipschitz assumption. Additionally the parameter dependent Lyapunov function proposed in [21] cannot be computed due to the discontinuities in α.

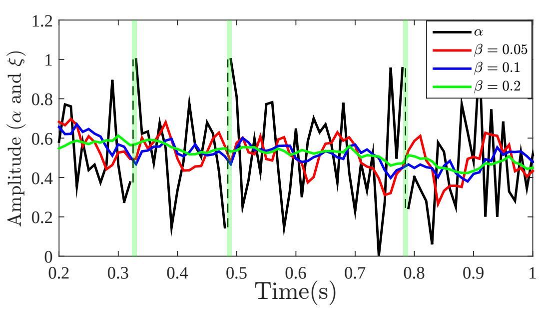

FIGURE 1 Measured parameter α and its filtered version, ξ, for different values of β.

Therefore, the designer can tune β accordingly with the noise level in the measured parameter, mitigating unnecessary oscillations in the control signal and reducing maintenance costs. Such an aspect can be noted by the gains variation as shown in Figure 2. The design is performed by Theorem 4.1 with $\gamma = 1$, $x(0) = 0$ and $\delta = \mu = 1$. It is clear from Figure 2 that the more the parameter is filtered, the more the state-feedback gains have a smooth behavior.

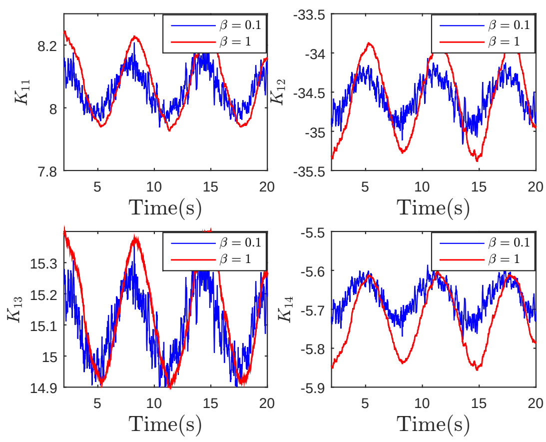

FIGURE 2 Parameter-dependent state-feedback gains behavior for different filtering values β.

Example 2. This example discusses the relation among the saturation level ρ, the $\mathscr{L}_2$ gain γ, and the disturbance energy bound δ^{-1}.

Consider the following system, adapted from [20], that represents a helicopter in a vertical flight subject to airspeed changes, given by

$$A = \begin{bmatrix} -0.0366 & 0.0271 & 0.188 & -0.4555 \\ 0.0482 & -1.0100 & 0.0024 & -4.0208 \\ 0.1002 & a_{32}(t) & -0.7070 & a_{34}(t) \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

$$B = \begin{bmatrix} 0.4422 & 0.1761 \\ b_{12}(t) & -7.5922 \\ -5.5200 & 4.9900 \\ 0 & 0 \end{bmatrix}, C = \begin{bmatrix} 0.1 & 0 & 0 & 0 \\ 0 & 0 & 0.1 & 0 \end{bmatrix},$$

with $B_w = \mathbf{I}_4$, $D_w = \mathbf{0}$, and $-0.1134 \leq a_{32}(t) \leq 0.6844$, $-0.8667 \leq a_{34}(t) \leq 3.5125$ and $-0.7178 \leq b_{12}(t) \leq 6.8072$. The system has three time-varying parameters, hence a polytope of $N = 8$ vertices will be used to describe it.

Because we have three interconnected variables, γ, δ, and ρ, the optimization procedure $\mathscr{T}_2$ can be exploited in different ways. In the sequel, we present two sets of tests illustrating possible uses of it. In all cases, the optimized values are investigated as a function of the actuator's saturation level, and the initial condition is supposed null ($x(0) = 0$). Such an approach allows, for instance, the correct specification of an actuator to obtain the desired performance. In the first set, we use $\mathscr{T}_2$ with $f = \gamma$ and a given disturbance energy, δ^{-1}, to seek for the minimal $\mathscr{L}_2$-gain, $\sqrt{\gamma}$. In the second set of tests, we use $\mathscr{T}_2$ with $f = \mu$ and a given $\mathscr{L}_2$-gain, $\sqrt{\gamma}$, to maximize the tolerable energy disturbance, δ^{-1}. The achieved results are shown in Figure 3, where the left-hand side ordinate axis refers to the minimal $\mathscr{L}_2$-gain achieved (green lines) and the right-hand ordinate axis refers to the maximal tolerable energy allowed (blue lines). The solid lines correspond to better achievements.

Concerning the $\mathscr{L}_2$-gain, green lines in Figure 3, it is clear that smaller values of γ are obtained when ρ increases, thanks to more control signal available to reject the input disturbance (see the left-hand side ordinate axis). Moreover, the $\mathscr{L}_2$-gain increases as the disturbance energy augments: the values of the solid green line, $\delta^{-1} = 0.2$, are more significant than those of the dashed green line, $\delta^{-1} = 0.1$, by 5% to 10%, approximately.

On the other hand, the maximization of the tolerable energy for a given $\mathscr{L}_2$-gain, right-hand ordinate axis, and blue lines shows that the greater the control signal, the bigger the tolerable energy is. Furthermore, more significant $\mathscr{L}_2$-gain allows more disturbance energy, as we can compare the dashed blue line ($\gamma = 5$) achieves smaller values of tolerable energy than those from the solid blue line ($\gamma = 6$). With the results

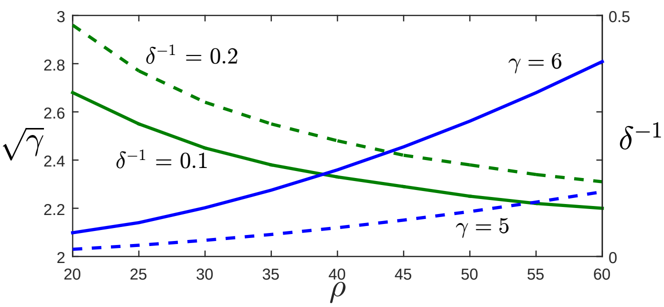

FIGURE 3 Behavior of the $\mathscr{L}_2$-gain and maximum tolerable energy δ^{-1} as a function of the saturation limit.

presented, one can see that:

1. For a fixed disturbance energy bound (δ^{-1}), bigger values of saturation level (ρ) increase the closed-loop system capability of mitigating disturbances, that is, smaller values of $\mathscr{L}_2$ gain (γ) are obtained.
2. Fixing the saturation level (ρ), disturbances with more energy (δ^{-1}) lead to bigger values of γ.
3. The disturbance tolerance is higher when more control signal is available. Besides, if $\mathscr{L}_2$ gain (γ) grows, so does the disturbance energy bound (δ^{-1}) such that the system is ISS.

Example 3. In this example, it is discussed the relation between the estimation of the region of attraction and the saturation level ρ. We intend to show that, when more control signal is available, the region of attraction gets bigger.

Consider the following system, borrowed from [17], with two time-varying parameters described by

$$A(\theta) = \begin{bmatrix} 0 & 1 \\ -(1+\theta_1) & -(1+\theta_2) \end{bmatrix}, \; B(\theta) = \begin{bmatrix} 0 \\ -1+\theta_2 \end{bmatrix},$$

where $-0.04 \leq \theta_1 \leq 5$, and $-5 \leq \theta_2 \leq 0.04$. In this scenario, there is no disturbance ($w = 0$) and our goal is, by using optimization procedure $\mathscr{T}_1$, design state-feedback controllers that maximizes the estimations of the region of attraction for different values of control signal saturation ρ. Consider that $\beta = 0.1$, $\mu = 1$ and $\gamma = 0.5$. In Figure 4, it is possible to see that the regions of attraction get bigger when more control energy is available. Response to initial conditions are shown and it is important to notice that the closed-loop system is asymptotically stable when the initial conditions belong to the region of attraction. For the case $\rho = 5$, it is also possible to see trajectories beginning outside the estimation of the region of attraction that make the closed-loop system unstable. This fact highlights that the obtained estimation of the region of attraction could be improved, however, it also provides a good measurement of the real region of attraction of the system.

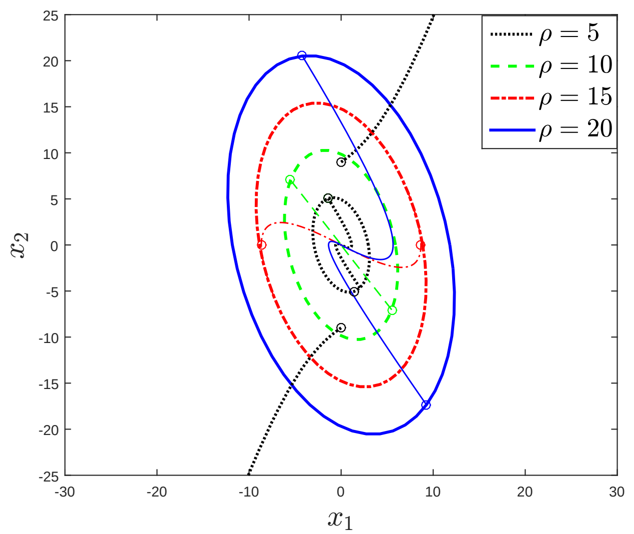

FIGURE 4 Estimated regions of attraction for different values of saturation ρ and closed-loop system response to initial conditions.

Example 4. In this example, it is shown an extension of the proposed conditions for the case of reference tracking. Hence, the designed controller is capable of following piecewise constant references and rejecting exogenous disturbances.

Consider the nonlinear plant borrowed from [18], where a discrete-time version of this model is addressed. The system consists of two tanks coupled, where one of them has a nonlinear solid inside. The input of the system is the pump power $0\% \leq u \leq 100\%$, which affects q_{in} and the output is the level (meters) in TQ-01, shown in Figure 5.

For $0.28m \leq h_1 \leq 0.48m$, the system can be written as a polytope, whose vertices are

$$A_1 = \begin{bmatrix} -0.0111 & 0.0111 \\ 0.0155 & -0.0193 \end{bmatrix},\; A_2 = \begin{bmatrix} -0.0111 & 0.0111 \\ 0.0238 & -0.0282 \end{bmatrix},$$

$B_{u1} = B_{u2} = \begin{bmatrix} 0.5432 \times 10^{-4} & 0 \end{bmatrix}^\top$, $C_1 = C_2 = \begin{bmatrix} 0 & 1 \end{bmatrix}$, $D_1 = D_2 = 0$, and matrices of disturbance are $B_{w1} = \begin{bmatrix} 0.5432 \times 10^{-3} & 0 \end{bmatrix}^\top$, $B_{w2} = \begin{bmatrix} 0.7432 \times 10^{-3} & 0 \end{bmatrix}^\top$.

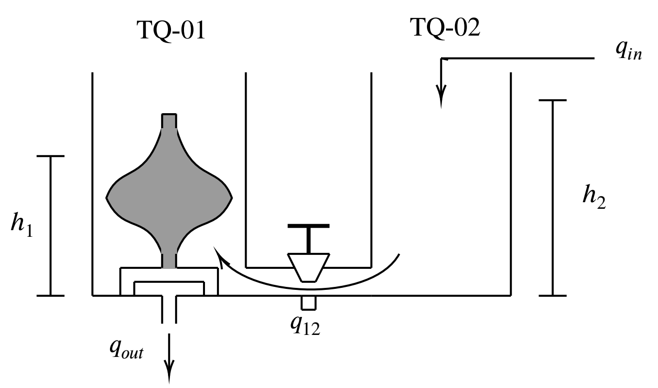

FIGURE 5 Diagram of the nonlinear coupled tanks system.

Similarly to [18], we use an augmented version of the controlled systems to include both the state feedback control action and an integrator over the tracking error, to ensure perfect tracking under piecewise constant references. The augmented system, due to the added integrator, is described by

$$\begin{aligned} \dot{x} &= \mathscr{A}(\alpha)x + \mathscr{B}(\alpha)\mathtt{sat}\,(u) + \mathscr{B}_w(\alpha)w \\ y &= \mathscr{C}(\alpha)x + D_w(\alpha)w \end{aligned} \tag{21}$$

where $\mathscr{A}(\alpha) = \begin{bmatrix} A(\alpha) & 0 \\ -C(\alpha) & 0 \end{bmatrix}$, $\mathscr{B}(\alpha) = \begin{bmatrix} B(\alpha) \\ 0 \end{bmatrix}$, $\mathscr{B}_w(\alpha) = \begin{bmatrix} B_w(\alpha) \\ 0 \end{bmatrix}$, $\mathscr{C}(\alpha) = \begin{bmatrix} C(\alpha) & 0 \end{bmatrix}$, and the control law is given by $u(\xi) = K(\xi)\tilde{x}$, where $K(\xi) = \begin{bmatrix} K_p(\xi) & K_I(\xi) \end{bmatrix}$ and $\tilde{x} = \begin{bmatrix} x^\top & e \end{bmatrix}^\top$.

Our goal is designing, by using Theorem 4.1, state-feedback gains for system (21) when the $\mathscr{L}_2$-gain between the disturbance w and the output y is $\gamma = 5$. Consider that $\beta = 0.1$, $\mu = 0.1$, and $x(0) = 0$ for $\rho = 40$ and $\rho = 60$.

Figure 6 shows the closed-loop system step response for two different levels of control signal saturation. The input reference is changed $0.28m$ and $0.30m$ when time $t = 100s$ and $t = 1800s$, respectively. As one may see, when more control signal is available ($\rho = 60$), the system output reaches the input reference with a smaller setting time. It is important to emphasize that, because of the integrator, there is no steady-state error in both cases. Morover, the closed-loop system disturbance rejection is shown in this figure: from $t = 3200s$ to $t = 3450s$, a disturbance of amplitude $w = 0.2$, leading to $\delta^{-1} = 10$, is applied on the system. Both controllers reject the disturbance and drive the system to the input reference with an $\mathscr{L}_2$ gain $\gamma \leq 5$, as proposed.

To simulate the real plant, it is considered an accuracy of $0.5cm$ on both states measurement. Hence, an aleatory value $\kappa \in [-0.5,\ 0.5]$ is added to each x_1 and x_2. Such a measurement noise may represent an instantaneous variation on the parameter compromising the assumptions in [25, 21].

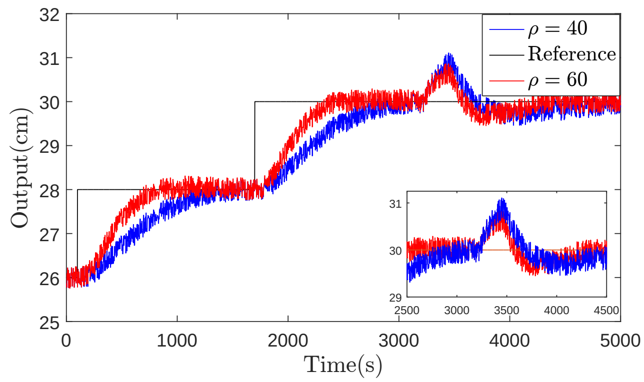

FIGURE 6 Closed-loop system input reference tracking and disturbance rejection.

State-feedback gains for a real plant model of two coupled tanks were designed. With the simulations presented, it is possible to certify the effectiveness of the controller, since the closed-loop system is capable of tracking the reference, rejecting the disturbance and when subject to initial conditions, the states of the closed-loop system converged to the origin.

Example 5. Consider the following system, adapted from [4, 15] described by

$$A(\theta) = \begin{bmatrix} 1.5 + 0.5\theta & 3 & \theta \\ -2.2 + \theta & -1.8 + 0.5\theta & 0.2\theta \\ 0.1 & 0.5 & -\theta \end{bmatrix},\; B(\theta) = \begin{bmatrix} 2\theta \\ 0.1 + \theta \\ 0.2 \end{bmatrix},$$

$$B_w = \begin{bmatrix} 0.2 & 0.02 & 0.1 \end{bmatrix}^\top,\; C = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},\; D_w = \begin{bmatrix} 0 & 0 \end{bmatrix}^\top,$$

where $\theta = 0.5\sin(0.2t) + 0.5$ is a time-varying parameter, $\dot{\theta}(t) \leq 0.1$ and the exogenous disturbance input is $w = \exp(-t)\sin(0.5t)$ as given in [4]. Under these circumstances, it is possible design state-feedback controllers by using Theorem 4.1 and Theorem 7 proposed in [4]. In both approaches, the optimization problem was solved in order to minimize γ. Considering $x_0 = 0$ and w as defined before, the $\mathscr{L}_2$-gain between the exogenous input and the system outputs are $\gamma = 0.0564$ for Theorem 4.1 and $\gamma = 0.0524$ for Theorem 7 (obtained from the simulations). The closed-loop response, w, and θ are shown in Figure 7.

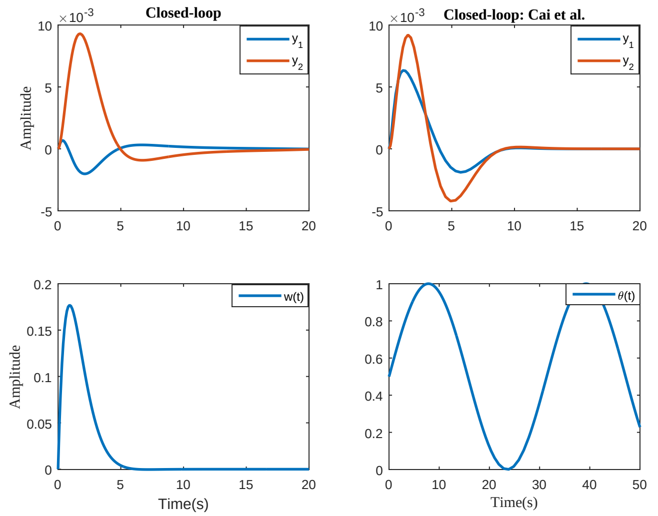

FIGURE 7 Disturbance rejection for the closed-loop systems obtained by using Theorem 4.1 and Theorem 7 in [4].

However, when θ presents discontinuities, the approaches proposed in [4, 15] cannot be applied since $\dot{\theta}(t) \to \infty$, whereas Theorem 4.1 overcomes this issue. The closed-loop response for this situation, θ and its filtered version, ξ, are presented in Figure 8. Note that we assume $\theta = 0$ for $t < 0$. Therefore, the first point in the simulation is already a discontinuity in the parameter behavior. During the first second, $\beta = 1$, ξ value is affected by null values of θ from the time interval $t < 0$. Despite the discontinuity of θ (red line), the parameter used by the controller evolves smoothly (blue line). Finally, it is worth to say that our method requires $N_V = 26$ scalar decision variables and $N_R = 64$ LMI rows, while the one in [4] requires $N_V = 170$ and $N_R = 71$, respectively. Therefore, besides overcoming the discontinuity of the system's parameter, our method is considerable less computing demanding.

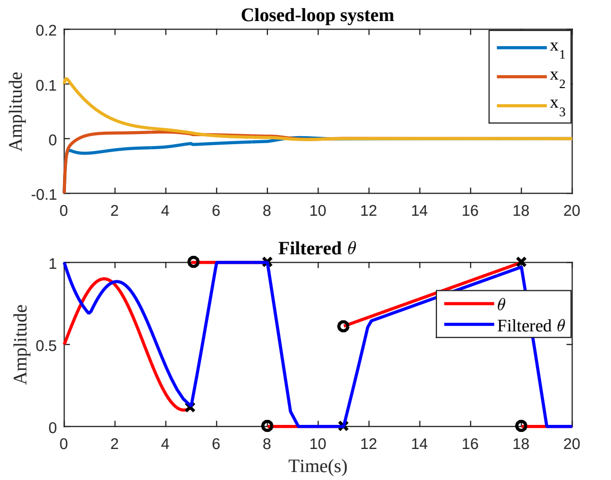

FIGURE 8 Response for $x_0 = [-0.1 \quad -0.1 \quad 0.1]^\top$, $\gamma = 0.04$, $\rho = 10$, $\mu = 0.1$, and $\beta = 1$.

6 | CONCLUSIONS

This paper proposed new conditions to ISS stabilization for a class of systems that depend on time-varying parameters that may vary arbitrarily fast and present discontinuities in its behavior. Moreover, the system is subject to actuator saturation, making it nonlinear. The state-feedback controller makes use of a smoothed approximation of the real time-varying parameter (the filtered parameter ξ), reducing actuator stress. The controller depends rationally on the parameter ξ. It has been shown that higher values of ρ have provided larger estimations for the region of attraction and smaller $\mathscr{L}_2$-gain between the output and input disturbance. Furthermore, the proposed method can be used to tackle LPV systems under constraints, enhancing the robustness associated with the state-feedback controllers. Besides, the conditions were extended to the problem of designing LPV state-feedback controllers, such that the closed-loop system output is able to follow input step references. Future works include the research of more general structures for filtered Lyapunov candidate functions and the extension of the proposed conditions to state polynomial systems. In such a case, the Lyapunov candidate matrix should include a polynomial dependency on the states, leading to more involved conditions to ensure the local stability while enlarging the estimate of the region of attraction.

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