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## PAPER

# Real-space effects of a quench in the Su–Schrieffer–Heeger model and elusive dynamical appearance of the topological edge states

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E-mail: [lorenzo.rossi@polito.it](mailto:lorenzo.rossi@polito.it) and [fabrizio.dolcini@polito.it](mailto:fabrizio.dolcini@polito.it)**Keywords:** quantum quench, Su–Schrieffer–Heeger model, symmetry protected topological states, charge conjugation symmetry

## Abstract

The topological phase of the Su–Schrieffer–Heeger (SSH) model is known to exhibit two edge states that are topologically protected by the chiral symmetry. We demonstrate that, for any parameter quench performed on the half-filled SSH chain, the occupancy of each lattice site remains locked to  $1/2$  at any time, due to the additional time-reversal and charge conjugation symmetries. In particular, for a quench from the trivial to the topological phase, no signature of the topological edge states appears in real-space occupancies, independently of the quench protocol, the temperature of the pre-quench thermal state or the presence of chiral disorder. However, a suitably designed local quench from/to a SSH ring threaded by a magnetic flux can break these additional symmetries while preserving the chiral one. Then, real-space effects of the quench do appear and exhibit different dynamical features in the topological and in the trivial phases. Moreover, when the particle filling is different from a half and the pre-quench state is not insulating, the dynamical appearance of the topological edge states is visible already in a chain, it survives time averaging and can be observed also in the presence of chiral-breaking disorder and for instantaneous quenches.

## 1. Introduction

The existence of topological edge states protected by some symmetry is perhaps the most striking feature characterizing topological insulators and superconductors. Evidence of the conducting channels in 2D topological insulators and of Dirac surface states in 3D topological insulators has been found in a number of experiments through transport, magnetotransport and photoemission spectroscopy measurements [1–4]. In one-dimensional (1D) topological superconductors, the interpretation in terms of Majorana quasi-particles [5, 6] of the robust zero-bias peak observed in the conductance of spin–orbit coupled nanowires [7, 8] has led to some controversy [9]. A more direct evidence of these exotic quasi-particles seems to be provided by spatially resolved spectroscopic techniques applied to ferromagnetic atom chains [10]. In fact, in last years an increasing number of works have been devoted to the search for a real-space imaging of topological phases and states [10–18].

After a decade characterized by a remarkable effort to find signatures of these edge states in various materials, presently one of the most fascinating challenges in physics is the possibility to manipulate these states and to possibly encode information therein [19–21]. To this purpose, the implementation with cold atoms in optical lattices [22–39] offers a twofold advantage, namely a pretty reliable system isolation from the environmental decoherence, and an extremely precise control of the system Hamiltonian. In particular, it is possible to realize quantum quenches of the Hamiltonian parameters [40–43], both over the entire system and on a spatially localized portion. These experimental advances thus also bring up new interesting questions about topological systems. Consider, for instance, a topological insulator characterized by some symmetry and suppose that, by a quantum quench preserving such symmetry, the system is dynamically

brought from the trivial to the topological phase, passing through a gap closing. Can one observe the topological states *dynamically* appear in real-space at its edges? Conversely, how do they evolve and possibly disappear when the quench is towards the trivial phase? In this paper we aim to answer these questions, focussing on a prototypical case, namely the Su–Schrieffer–Heeger (SSH) model [44, 45].

The SSH model describes spinless fermions in a bipartite 1D lattice through the following tight-binding Hamiltonian

$$\hat{\mathcal{H}}_{\text{SSH}} = v \sum_j \left( \hat{c}_{j,A}^\dagger \hat{c}_{j,B} + \hat{c}_{j,B}^\dagger \hat{c}_{j,A} \right) + w \sum_j \left( \hat{c}_{j,B}^\dagger \hat{c}_{j+1,A} + \hat{c}_{j+1,A}^\dagger \hat{c}_{j,B} \right), \quad (1)$$

where  $\hat{c}_{j,s}^\dagger$  and  $\hat{c}_{j,s}$  denote the fermionic creation/annihilation operators for electrons localized at atom  $s = A, B$  within the  $j$ th cell of the lattice, whereas  $v$  and  $w$  indicate the intra- and inter-cell tunneling amplitudes, respectively. The model, first introduced in the description of opto-electronic properties of polyacetylene [44–46], is considered as a paradigmatic example of 1D topological insulators [47, 48]. Indeed at half filling (one electron per unit cell) the SSH model describes a band insulator characterized by a band gap  $2\varepsilon_g$  with  $\varepsilon_g = ||v| - |w||$  and by a sublattice symmetry called chiral symmetry, which identifies for  $|v| < |w|$  and for  $|v| > |w|$  two topologically different phases that cannot be connected to each other without closing the gap. In the topologically non-trivial phase, a SSH chain exhibits at its edges localized states that are protected by the chiral symmetry. Recently, soliton states and topological indices of the SSH model have been experimentally observed in implementations with cold atoms [49, 50]. Moreover, the effects of time-dependent perturbations to the SSH hopping amplitudes have been analyzed in the context of topologically protected quantum gates [51] and Floquet nonequilibrium states generated by periodic drives [52–55].

Consider now a SSH chain-lattice, initially in the ground state of the topologically trivial phase ( $|v| > |w|$ ), at half filling, and perform a quench of the hopping amplitudes to the topological phase ( $|v| < |w|$ ). If the quench remains within the chiral symmetry class, the gap closes at some time and, by inspecting the dynamical evolution of the occupancy at each chain site, one would expect localized topological states to gradually emerge at the chain edges. Here we show that this is not the case: the site occupancy remains exactly equal to the one of the trivial pre-quench state at any time and any site, including at the chain edges, regardless of the quench protocol (fast or slow) and even in the presence of chiral disorder. As we shall demonstrate, the reason boils down to the charge conjugation or time-reversal symmetries of the pre-quench state and of the quenching Hamiltonian. In an open chain, these additional symmetries are typically present and completely mask any effect of the quench in real-space, including the appearance of the topological states. Effects of a quench can be observed in real-space occupancies only when such symmetries are broken, which can be done in two ways: (i) remaining within the topological insulator framework, i.e. preserving the chiral symmetry and the half-filling condition; (ii) by ‘brute force’, i.e. by breaking the chiral symmetry and/or by moving away from an insulating state.

We shall first explore the first option and propose two ways to observe the dynamical effects of the quench in real-space. The quench protocols are based on a local quench, where a ring lattice is cut into a chain or, vice versa, two edges of a chain are bridged to form a ring. In both cases the presence of a magnetic flux threading the ring is crucial to induce a real-space dynamical response to the quench, which is different depending on whether the involved chain is in the trivial or in the topological phase.

Then, we shall explore the second option and analyze the effects of quenches beyond the framework of topological insulator, i.e. by breaking the chiral symmetry and by considering filling values different from  $1/2$ , where the SSH model describes a metallic state. We find that the optimal way to observe the dynamical appearance of the edge states characterizing the topological insulator is to have a slightly metallic system. Then, the dynamical appearance of these states is robust even in the limit of short quench time and in the presence of chiral breaking disorder.

The paper is organized as follows. In section 2, after briefly summarizing the aspects of the SSH model that are needed to illustrate our results, including its symmetries, we shall describe the method we used to compute the dynamical evolution. In section 3 we present a general theorem ensuring that the site occupancy remains locked to  $1/2$  when charge conjugation symmetry is present. In particular, this explains the case of a quenched half-filled SSH chain. Then, in section 4 we show how to violate the hypotheses of the theorem and observe real-space effects of the quench without breaking the chiral symmetry. Finally, after analyzing the effects of chiral breaking terms and of a filling different from  $1/2$  in section 5, we discuss our results and draw our conclusions in section 6.

## 2. Model, symmetries and method

### 2.1. Generalized SSH model and symmetries

In this paper we consider a generalized SSH model

$$\hat{\mathcal{H}}_{\text{SSH},\chi} = \sum_j \left( v_j \hat{c}_{j,A}^\dagger \hat{c}_{j,B} + v_j^* \hat{c}_{j,B}^\dagger \hat{c}_{j,A} \right) + \sum_j \left( w_j \hat{c}_{j,B}^\dagger \hat{c}_{j+1,A} + w_j^* \hat{c}_{j+1,A}^\dagger \hat{c}_{j,B} \right) \quad (2)$$

extending the SSH Hamiltonian (1) to the case where the tunneling amplitudes  $\{v_j, w_j\}$  are possibly complex and site-dependent. Here  $j = 1, 2, \dots, M$ , where  $M$  is the number of cells in the lattice.

Furthermore, because symmetries play an important role in the dynamical effects that we aim to discuss, it is worth briefly recalling the behavior of the Hamiltonian (2) under three transformations that are local on the lattice site operators. The first one is *charge-conjugation*  $\mathcal{C}$ , a linear and unitary transformation mapping the lattice site creation/annihilation operators as follows

$$\begin{cases} \mathcal{C} \hat{c}_{j,A} \mathcal{C}^{-1} = \hat{c}_{j,A}^\dagger \\ \mathcal{C} \hat{c}_{j,B} \mathcal{C}^{-1} = -\hat{c}_{j,B}^\dagger \end{cases}, \quad (3)$$

and fulfilling  $\mathcal{C}^{-1} = \mathcal{C}^\dagger = \mathcal{C}$ . The second one is the *chiral transformation*  $\mathcal{S}$ . Despite acting on the lattice site operators in the same way as  $\mathcal{C}$

$$\begin{cases} \mathcal{S} \hat{c}_{j,A} \mathcal{S}^{-1} = \hat{c}_{j,A}^\dagger \\ \mathcal{S} \hat{c}_{j,B} \mathcal{S}^{-1} = -\hat{c}_{j,B}^\dagger \end{cases}, \quad (4)$$

it is by definition anti-linear ( $\mathcal{S}i\mathcal{S} = -i$ ) and anti-unitary  $\langle \mathcal{S}\Psi_1 | \mathcal{S}\Psi_2 \rangle = \langle \Psi_1 | \Psi_2 \rangle^*$ . Finally, the *time-reversal* transformation, which leaves lattice site operators unaltered

$$\begin{cases} \mathcal{T} \hat{c}_{j,A} \mathcal{T}^{-1} = \hat{c}_{j,A} \\ \mathcal{T} \hat{c}_{j,B} \mathcal{T}^{-1} = \hat{c}_{j,B} \end{cases} \quad (5)$$

but is also anti-linear and anti-unitary, with  $\mathcal{T}^2 = \mathbb{I}$ , as is the case for spinless fermions. In fact, only two of these transformations are independent because the chiral symmetry  $\mathcal{S}$  can be obtained as the product  $\mathcal{S} = \mathcal{T}\mathcal{C}$ .

The Hamiltonian (2) exhibits the chiral symmetry (4)

$$\mathcal{S} \hat{\mathcal{H}}_{\text{SSH},\chi} \mathcal{S}^{-1} = \hat{\mathcal{H}}_{\text{SSH},\chi} \Leftrightarrow [\hat{\mathcal{H}}_{\text{SSH},\chi}, \mathcal{S}] = 0, \quad (6)$$

and the subscript  $\chi$  stands in fact for ‘chiral’. In chiral-symmetric models like  $\hat{\mathcal{H}}_{\text{SSH},\chi}$ ,  $\mathcal{T}$  and  $\mathcal{C}$  are intimately related. Indeed, because  $\mathcal{C} = \mathcal{T}\mathcal{S}$ , time-reversal and charge conjugation transformations are either both preserved or both broken. In particular, for  $\{v_j, w_j\} \in \mathbb{R}$ , the Hamiltonian (2) also commutes with time-reversal  $\mathcal{T}$  and charge conjugation  $\mathcal{C}$ . However, when  $v_j = |v_j|e^{i\phi_j^v}$  and  $w_j = |w_j|e^{i\phi_j^w}$  have non-vanishing complex phases, the preservation of  $\mathcal{T}$  and  $\mathcal{C}$  heavily depends on the geometric boundary conditions. In particular, in a chain, i.e. a lattice with open boundary conditions (OBCs),  $\mathcal{T}$  and  $\mathcal{C}$  are always preserved, since such complex phases can be eliminated through a canonical transformation onto the lattice operators  $c_{j,A} \rightarrow \tilde{c}_{j,A} = e^{i\alpha_j} c_{j,A}$  and  $c_{j,B} \rightarrow \tilde{c}_{j,B} = e^{i(\alpha_j + \phi_j^v)} c_{j,B}$ , where  $\alpha_1 = 0$  and  $\alpha_j = \sum_{i=1}^{j-1} (\phi_i^v + \phi_i^w)$  for  $j = 2, \dots, M$ , recasting the Hamiltonian into the case of real and positive tunneling amplitudes [47]. In contrast, in a ring-shaped lattice, the periodic boundary conditions (PBCs) prevent the elimination of the phases of  $v_j$  and  $w_j$ . Physically, this can be understood in terms of the Peierls substitution [56, 57], where the complex phases of the tunneling amplitudes describe the integral of a vector potential from one lattice site to next one. While in a chain the vector potential can always be gauged out, in a ring this is not possible, for its circulation yields the magnetic flux  $\Phi$  threading the ring, and one has  $\sum_{j=1}^M (\phi_j^v + \phi_j^w) = 2\pi\Phi/\Phi_0 \neq 0$ , where  $\Phi_0 = h/e$  is the flux quantum. Thus, in the SSH ring with a flux  $\Phi \neq p\Phi_0/2$  (with  $p \in \mathbb{Z}$ ),  $\mathcal{T}$  and  $\mathcal{C}$  are broken.

To perform our time-dependent analysis, we represent the second-quantized Hamiltonian (2) in the real-space basis as follows

$$\hat{\mathcal{H}}_{\text{SSH},\chi} = \sum_{j_1, j_2=1}^M \sum_{s_1, s_2=A,B} \hat{c}_{j_1, s_1}^\dagger H_{j_1 s_1, j_2 s_2} \hat{c}_{j_2, s_2}, \quad (7)$$



levels in the spectrum near  $\varepsilon = 0$ , which correspond to states localized at the edges. For the sake of completeness, a short summary of these aspects is given in the [appendix](#).

## 2.2. Quenches, density matrix approach and observables

In the following, we shall investigate the dynamical effects of a quench in the parameters  $\{v_j, w_j\}$  of the Hamiltonian (2). Specifically, the system is prepared in an initial state  $\rho^{\text{pre}}$ , typically the ground state or the thermal equilibrium state of a pre-quench Hamiltonian  $\hat{\mathcal{H}}^{\text{pre}} = \hat{\mathcal{H}}_{\text{SSH},\chi}(t < t_0)$ . Then, at  $t = t_0$  the system is disconnected from the environment and the dynamics is unitarily governed by the Hamiltonian  $\hat{\mathcal{H}}_{\text{SSH},\chi}(t)$ , which varies until a time  $t_f$  from  $\hat{\mathcal{H}}^{\text{pre}}$  to a post-quench Hamiltonian  $\hat{\mathcal{H}}^{\text{post}} = \hat{\mathcal{H}}_{\text{SSH},\chi}(t > t_f)$ . The quench protocol specifies the way the parameters  $\{v_j(t), w_j(t)\}$  in equation (2) are varied during the quench time  $\tau_q = t_f - t_0$ .

Although in our analysis we shall mainly focus on short quench time limit ( $\tau_q \rightarrow 0$ ), we shall keep the parameter time dependence arbitrary because, as we shall see, some results are independent of the specific quench protocol. Moreover, we shall deal with both global and local quenches. A global quench involves a change in a significant number (scaling like the number  $M$  of cells) of hopping amplitudes along the chain. This occurs, for instance, when all the hopping amplitudes of a homogeneous chain ( $v_j \equiv v$  and  $w_j \equiv w$ ) are brought from the trivial to the topological phase. In contrast, a local quench only involves a limited number of hopping amplitudes. For instance, the cutting of a ring into a chain is described by quenching to zero the hopping amplitude of one single bond. Note that, because of equation (6), the chiral symmetry is preserved at any time, so that the quench occurs within the chiral symmetry class. Yet, the result of a quench depends not only on the quenching Hamiltonian, but also on the pre-quench state and its symmetries, as we shall see. Furthermore, in section 5, we shall also analyze the quench in chiral symmetry broken cases.

We shall be interested in one-body observables  $\hat{\mathcal{A}} = \sum_{j_1 s_1, j_2 s_2} \hat{c}_{j_1 s_1}^\dagger A_{j_1 s_1, j_2 s_2} \hat{c}_{j_2 s_2}$ , whose expectation values are straightforwardly evaluated in terms of the single-particle density matrix  $\rho_{j_1 s_1, j_2 s_2}(t) = \text{Tr}\{\hat{c}_{j_2 s_2}^\dagger \hat{c}_{j_1 s_1} \hat{\rho}(t)\}$ , where  $\hat{\rho}(t)$  denotes the dynamical evolution of the full system density matrix and  $\text{Tr}$  the trace over the Fock space. Due to the quadratic structure of equation (7), the Liouville–von Neumann equation for  $\hat{\rho}$  straightforwardly implies the dynamical equation for  $\rho$ , which reads

$$i\hbar \frac{d\rho}{dt} = [H(t), \rho]. \quad (15)$$

We numerically solve equation (15) with the initial condition  $\rho(t_0) = \rho^{\text{pre}}$  corresponding to the single-particle density matrix of the pre-quench state, typically the ground state of the pre-quench Hamiltonian. Then, the expectation values of an observable  $\hat{\mathcal{A}}$  are obtained as

$$\langle \hat{\mathcal{A}} \rangle(t) = \text{tr}\{A\rho(t)\}, \quad (16)$$

where ‘tr’ denotes the trace over the single-particle Hilbert space. In particular, we shall henceforth focus on the site occupancy, evaluated as

$$N_{j,s}(t) = \langle \hat{n}_{j,s} \rangle(t) = \rho_{j,s;j,s}(t), \quad (17)$$

and on the cell polarization

$$P_j(t) = \langle \hat{n}_{j,A} \rangle(t) - \langle \hat{n}_{j,B} \rangle(t), \quad (18)$$

obtained as  $P_j = \rho_{jA,jA} - \rho_{jB,jB}$ . The total number of electrons  $N_e = \sum_{j,s} N_{j,s}$  is simply given by  $N_e = \text{tr} \rho$  and is constant as a consequence of equation (15). In section 5 we shall also discuss the non-equilibrium energy distribution in the post-quench eigenbasis  $\{\lambda\}$ , where  $\hat{\mathcal{H}}^{\text{post}} = \sum_{\lambda} \varepsilon_{\lambda} \hat{n}_{\lambda}$  is diagonal. The energy distribution is obtained as

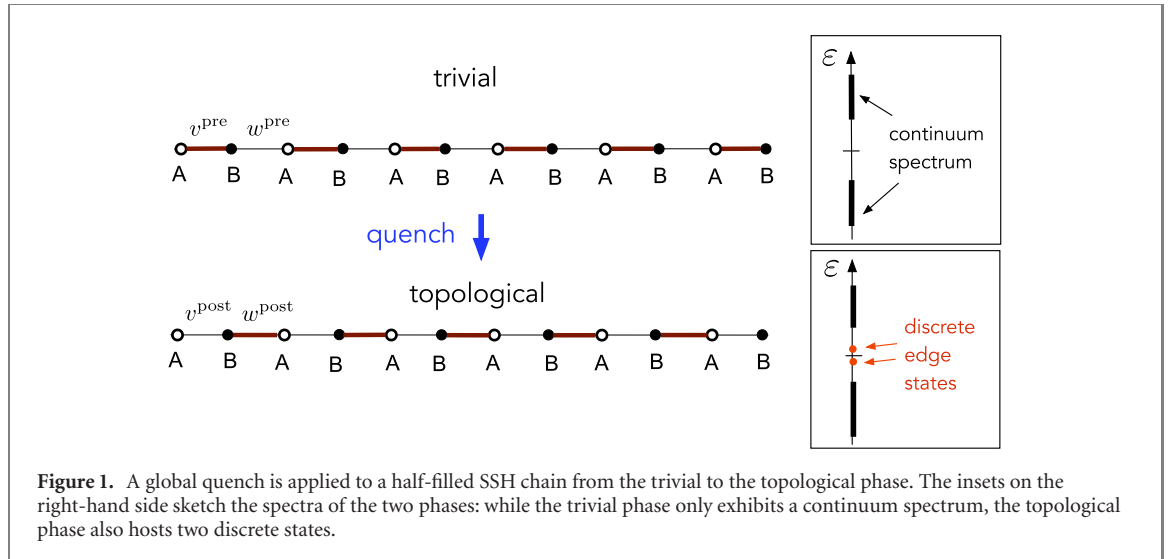
$$\langle \hat{n}_{\lambda} \rangle(t) = \sum_{j_1, j_2=1}^M \sum_{s_1, s_2=A,B} U_{\lambda, j_2 s_2} U_{\lambda, j_1 s_1}^* \rho_{j_2 s_2, j_1 s_1}(t), \quad (19)$$

where  $U_{\lambda, j s} = \langle \lambda | j s \rangle$  is the unitary matrix determining the single-particle change of basis from the real-space basis to the post quench eigenbasis.

## 3. Quenches in half-filled SSH models: the locking of site occupancy

We start by considering a chain of the customary SSH model equation (1) with homogeneous hopping amplitudes, which can be assumed to be positive ( $v, w > 0$ ). Let the pre-quench state be the half-filled ground state of the chain in the trivial phase, so that there is one electron per cell ( $N_e = M$ ), i.e. half an electron per site on average, and

$$w^{\text{pre}} < v^{\text{pre}}, \quad (20)$$



where  $w^{\text{pre}} \equiv w(t < 0)$  and  $v^{\text{pre}} \equiv v(t < 0)$ . No edge state is present. At  $t = 0$  we start to quench the Hamiltonian parameters towards the topological phase, as sketched in figure 1. This means that, within a quench time  $\tau_q$ , the values of the hopping amplitudes are brought to

$$w^{\text{post}} > v^{\text{post}}, \quad (21)$$

where  $w^{\text{post}} \equiv w(t > \tau_q)$  and  $v^{\text{post}} \equiv v(t > \tau_q)$ .

At first, one would expect the discrete states characterizing the chain spectrum in the topological phase to gradually appear in real-space, causing an occupancy increase localized at the two edges. However, this is not the case: we find that the site occupancy is locked at  $1/2$  at any time and at any site, including the chain edges

$$N_{j,s}(t) = 1/2 \quad \forall t \forall j, s \quad (22)$$

just like in the trivial pre-quench phase. Notably, such a locking of the real-space occupancy occurs for any quench duration  $\tau_q$ , regardless of the specific way one changes the hopping amplitudes from  $(v^{\text{pre}}, w^{\text{pre}})$  to  $(v^{\text{post}}, w^{\text{post}})$ . Furthermore, it also holds in the presence of chiral disorder and/or if the pre-quench state is a thermal state at finite temperature. Indeed the result (22) is a consequence of a general theorem that we shall prove here below. Before doing that, it is worth emphasizing that the quench does affect the system, though. For instance, the energy distribution of the post-quench Hamiltonian strongly differs from the pre-quench equilibrium distribution and exhibits a striking band population inversion, as has been proven in reference [58] for a SSH ring exposed to a sudden quench.

### 3.1. General theorem about site occupancy

The following general result can be proven: (i) if the pre-quench state ( $t = t_0$ ) is invariant under charge-conjugation

$$\mathcal{C} \hat{\rho}^{\text{pre}} \mathcal{C}^{-1} = \hat{\rho}^{\text{pre}} \quad (23)$$

and (ii) if the time-dependent Hamiltonian  $\hat{\mathcal{H}}(t)$  characterizing the quench ( $t > t_0$ ) commutes with charge-conjugation transformation

$$\left[ \hat{\mathcal{H}}_{\text{SSH},\chi}(t > t_0), \mathcal{C} \right] = 0, \quad (24)$$

then equation (22) holds. The proof starts by recalling that the pre-quench state  $\hat{\rho}^{\text{pre}}$  evolves as  $\hat{\rho}(t) = \mathcal{U}(t) \hat{\rho}^{\text{pre}} \mathcal{U}^\dagger(t)$ , where the evolution operator is

$$\mathcal{U}(t) = \overleftarrow{\mathbb{T}} \left[ \exp \left( -\frac{i}{\hbar} \int_{t_0}^t \hat{\mathcal{H}}(t') dt' \right) \right] \quad \forall t > t_0 \quad (25)$$

and  $\overleftarrow{\mathbb{T}}$  denotes the time-ordering. Moreover, the property (24) and the linearity of  $\mathcal{C}$  imply that

$$[\mathcal{C}, \mathcal{U}(t)] = 0 \quad \forall t > t_0. \quad (26)$$

By using equations (23) and (26) the time evolution of the site occupancy  $N_{j,s} = \langle \hat{n}_{j,s} \rangle$  is then computed as

$$\begin{aligned}
N_{j,s}(t) &= \text{Tr} \{ \hat{\rho}(t) \hat{n}_{j,s} \} = \text{Tr} \{ \mathcal{U}(t) \hat{\rho}^{\text{pre}} \mathcal{U}^\dagger(t) \hat{n}_{j,s} \} = \text{Tr} \{ \mathcal{U}(t) \mathcal{C} \hat{\rho}^{\text{pre}} \mathcal{C}^{-1} \mathcal{U}^\dagger(t) \hat{n}_{j,s} \} \\
&= \text{Tr} \{ \mathcal{C} \mathcal{U}(t) \hat{\rho}^{\text{pre}} \mathcal{U}^\dagger(t) \mathcal{C}^{-1} \hat{n}_{j,s} \} = \text{Tr} \{ \mathcal{U}(t) \hat{\rho}^{\text{pre}} \mathcal{U}^\dagger(t) \mathcal{C} \hat{n}_{j,s} \mathcal{C}^{-1} \} = \text{Tr} \{ \hat{\rho}(t) (1 - \hat{n}_{j,s}) \} \\
&= 1 - N_{j,s}(t),
\end{aligned} \tag{27}$$

where we have used  $\mathcal{C} = \mathcal{C}^{-1}$  and  $\mathcal{C} \hat{n}_{j,s} \mathcal{C}^{-1} = 1 - \hat{n}_{j,s}$ . The result equation (22) follows from equation (27), and shows that the site occupancy remains locked to its trivial phase value 1/2. We also observe that, by a very similar argument, the hypotheses of the theorem also imply that the off-diagonal single-particle density matrix entries are always either real or purely imaginary, at any time. Specifically  $\rho_{iA,jB}(t) = \langle c_{jB}^\dagger c_{iA} \rangle$  is real  $\forall i, j$ , while  $\rho_{is,js}(t) = \langle c_{js}^\dagger c_{is} \rangle$  is purely imaginary  $\forall i \neq j$  and  $s = A, B$ .

### 3.2. Global quench in a SSH chain

We shall now show that a quench of the half-filled SSH chain satisfies the hypotheses equations (23) and (24) of the above theorem, whence one straightforwardly deduces the locking of the site occupancy, equation (22). Indeed equation (24) is satisfied by  $\hat{\mathcal{H}}_{\text{SSH},\chi}(t)$  because, as observed in section 2.1, in a chain with OBCs the SSH model equation (2) preserves both charge-conjugation  $\mathcal{C}$  and time-reversal symmetry  $\mathcal{T}$ . Furthermore, if the pre-quench state is the thermal equilibrium state at half-filling ( $\mu = 0$ ) of the pre-quench SSH Hamiltonian  $\hat{\mathcal{H}}^{\text{pre}}$

$$\hat{\rho}^{\text{pre}} = \frac{e^{-\beta \hat{\mathcal{H}}^{\text{pre}}}}{\text{Tr}[e^{-\beta \hat{\mathcal{H}}^{\text{pre}}}]}, \tag{28}$$

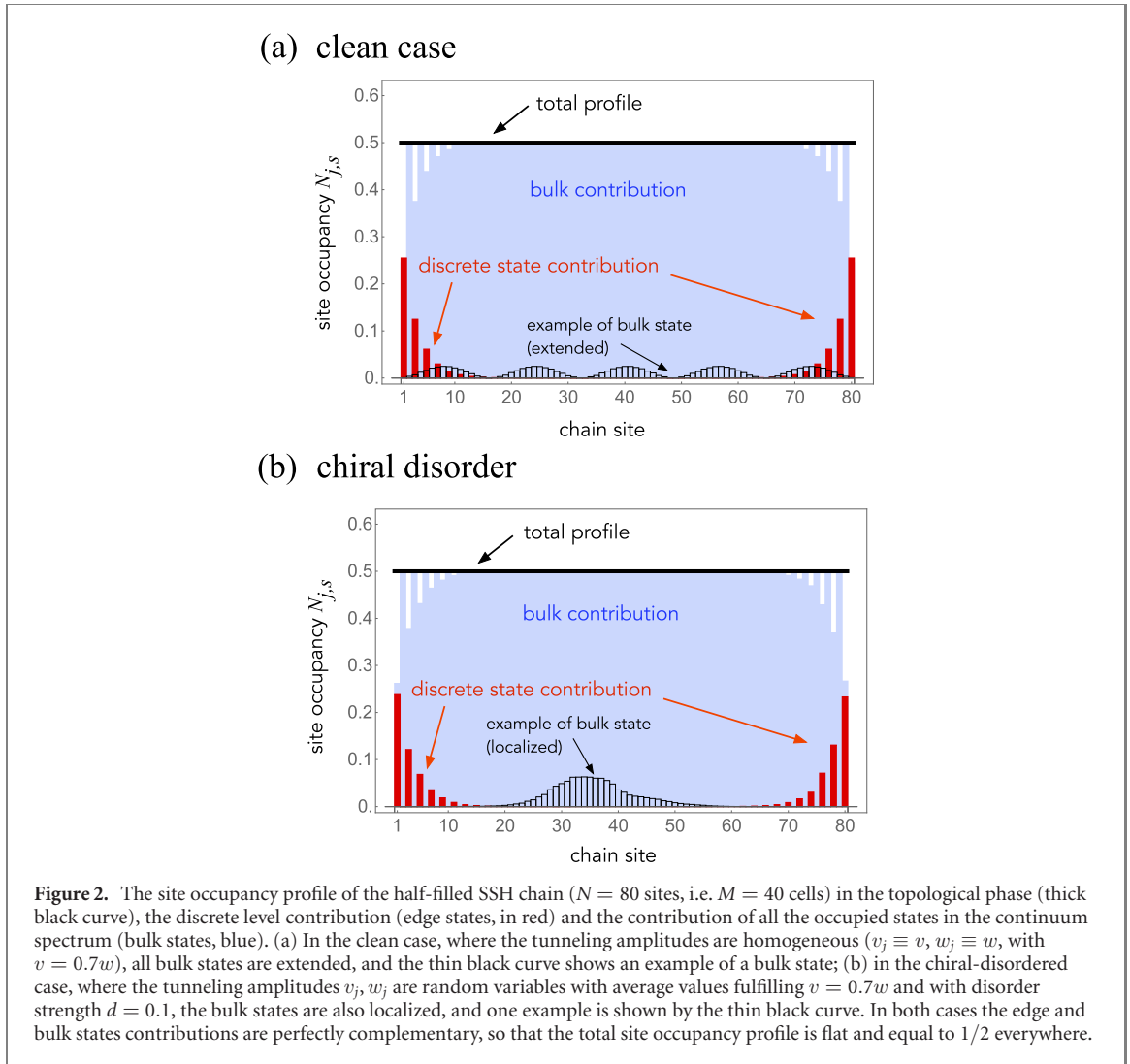
where  $\beta = 1/k_B T$  is the inverse temperature, the symmetry  $[\hat{\mathcal{H}}^{\text{pre}}, \mathcal{C}] = 0$  straightforwardly implies equation (23). In particular, this is true for the half-filled ground state  $\hat{\rho}^{\text{pre}} = |\text{H.F.}\rangle \langle \text{H.F.}|$ , where  $|\text{H.F.}\rangle$  is constructed by occupying all the negative energy states of  $\hat{\mathcal{H}}^{\text{pre}}$   $|\text{H.F.}\rangle = \prod_{\alpha=1}^M \gamma_{\alpha,-}^\dagger |0\rangle$  and is non-degenerate.

This explains why in a half-filled SSH chain the quench does not lead to any change of the site occupancy, which remains uniform and constant regardless of (i) the specific quench time and protocol, (ii) the presence of chiral disorder in the tunneling amplitudes  $\{v_j, w_j\}$ , and (iii) finite temperature of the pre-quench thermal state. In particular, in a chain it is impossible to observe the appearance of the topological states or any other difference between the trivial and the topological phase in real-space occupancies. In section 4 we shall propose a different setup where real-space effects of a quench can be observed. However, we wish to first provide a more physical justification for the result equation (22).

#### 3.2.1. The case of infinitely slow quench: comparison between the trivial and topological half-filled ground states

Because the result equation (22) is valid for any quench protocol, it holds in particular for an infinitely slow quench ( $\tau_q \rightarrow \infty$ ), where the pre-quench ground state evolves into the post-quench ground state. In this particular limit, the result equation (22) can thus be understood by comparing the site occupancy profile of the trivial and topological half-filled ground states. In the trivial phase, where the spectrum is purely continuum, the uniform pattern  $N_{j,s}^{\text{pre}} \equiv 1/2$  is expected from the contribution of the bulk states extending over the entire chain. In the topological phase, where the additional discrete levels  $\pm \varepsilon^{\text{edge}}$  near  $\varepsilon = 0$  are present, the site occupancy profile results from two types of contributions. The red curve in figure 2(a) shows the discrete state contribution localized at the chain edges (see equation (A.5)), while the thin black curve displays, as an illustrative example, the contribution of one bulk state, whose wavefunction extends over the entire chain (see equation (A.4)). Notably, the blue curve, describing the contribution of *all* the occupied bulk states of the chain, features two dips at the edges, which are perfectly complementary to the edge state contribution: the bulk states ‘feel’ the presence of the edge states and make room for them by slightly modifying their behavior near the boundaries with respect to the trivial phase. This can be considered as a real-space imaging of the bulk-boundary correspondence. The thick black curve is the sum of the two contributions and uniformly takes the value  $N_{j,s}^{\text{post}} \equiv 1/2$ . Thus the half-filled ground state of the chain in the topological phase does not show any different feature in real-space occupancy with respect to the trivial phase, despite the presence of the edge states in the spectrum.

Such a lack of difference seems at first to contradict the argument that is customarily invoked to illustrate the emergence of the edge states in the topological phase, based on the dimerized limit of the chain: when the extremal links of the chain are very weak,  $v/w \rightarrow 0$ , the outmost chain sites host a localized electron. However, this can only hold when the number  $N_e$  of electrons in the chain is  $M + 1$ . At half filling,  $N_e = M$ , only one electron can be accommodated in the two edge sites and, in fact, each of them hosts ‘half an electron’. Indeed the red curve of figure 2, peaked at *both* chain edges, describes the contribution of only *one* discrete state, namely the one at energy  $-\varepsilon^{\text{edge}}$ , which is occupied in the half-filled ground state.



We also emphasize that such uniform site occupancy profile is not merely due to the accidental spatial parity [59] of the homogeneous SSH model (1). The very chiral symmetry forbids disorder to localize the two discrete states on opposite sides of the chain: the two wavefunctions  $\psi_{\pm}^{\text{edge}}$  with opposite energies  $\pm \varepsilon^{\text{edge}}$  are mapped into each other by a mere sign change in the  $B$ -sites through the chiral transformation  $S$  (see section 2.1), so that their square moduli have to coincide, even in the presence of chiral disorder. This is illustrated in figure 2(b), which refers to a disordered SSH model realized by taking  $w_j = w(1 + \xi_j d)$  and  $v_j = v(1 + \eta_j d)$  in equation (2), where  $\{\xi_j, \eta_j\}$  are sets of random variables uniformly distributed in  $[-1/2, 1/2]$ ,  $v > 0$  and  $w > 0$  are the average tunneling amplitudes, and  $d < |v - w|/\max(v, w)$  is the disorder strength. For  $v < w$  the spectrum of the disordered SSH chain still consists of a continuum branch and of two additional discrete levels. The red curve describes the only occupied discrete level and again is localized on both edges. In fact, disorder has a stronger impact on the bulk states, which get localized too, as shown by the thin black curve in figure 2(b), in agreement with Anderson localization [60, 61]. However, their total contribution to the site occupancy profile (blue curve in figure 2(b)) is still uniformly flat in the bulk and exhibits two dips by the edges, just like in the clean case of figure 2(a). Again, at half filling, the sum of bulk and edge state contributions yields a perfectly uniform occupancy profile  $N_{j,s} \equiv 1/2$  (black thick line in figure 2(b)).

It is also straightforward to understand why such uniform profile is unaltered by finite temperatures: despite the energy separation between the two discrete levels is tiny, the partial occupancy of the level at energy  $+\varepsilon^{\text{edge}}$  induced by thermal excitations is perfectly balanced by the corresponding depletion of the level at energy  $-\varepsilon^{\text{edge}}$ .

#### 4. Breaking charge conjugation in chiral symmetric models: a local quench

In order to observe some effects of the quench in the real-space occupancy, and possibly the appearance of the topological edge states, a necessary condition is that at least one of the two crucial hypotheses of the theorem, equations (23) and (24), is violated. Here below we show how this is possible while still *preserving*

the chiral symmetry and while operating at half-filling, i.e. under the conditions where the SSH model is rigorously characterized as a topological insulator.

#### 4.1. Ring-to-chain quench

The first option is to violate the hypothesis of charge-conjugation invariance of the pre-quench state, equation (23). This can be achieved by choosing as  $\hat{\rho}^{\text{pre}}$  a thermal equilibrium state of the homogeneous half-filled SSH model, like in equation (28), where  $\hat{\mathcal{H}}^{\text{pre}}$  is equation (2) defined on a ring-shaped lattice threaded by a magnetic flux  $\Phi$ . In this case both  $\hat{\mathcal{H}}^{\text{pre}}$  and  $\hat{\rho}^{\text{pre}}$  break time-reversal symmetry  $\mathcal{T}$  and hence charge conjugation  $\mathcal{C}$  symmetry, so that the condition equation (23) is violated. Note that, nevertheless, the pre-quench site occupancy still equals exactly 1/2. Indeed, since  $\hat{\mathcal{H}}^{\text{pre}}$  commutes with the chiral symmetry  $\mathcal{S}$ , the relation

$$N_{j,s}^{\text{pre}} = \text{Tr} [\hat{\rho}^{\text{pre}} \hat{n}_{j,s}] = \frac{\text{Tr} [\mathcal{S} e^{-\beta \hat{\mathcal{H}}^{\text{pre}}} \mathcal{S}^{-1} \hat{n}_{j,s} \mathcal{S}^{-1}]}{\text{Tr} [e^{-\beta \hat{\mathcal{H}}^{\text{pre}}}] } = \frac{\text{Tr} [e^{-\beta \hat{\mathcal{H}}^{\text{pre}}} (1 - \hat{n}_{j,s})]}{\text{Tr} [e^{-\beta \hat{\mathcal{H}}^{\text{pre}}}] } = 1 - N_{j,s}^{\text{pre}} \quad (29)$$

implies that  $N_{j,s}^{\text{pre}} = 1/2 \forall j, s$ . For definiteness, we take for  $\hat{\rho}^{\text{pre}}$  the ground state of the SSH ring.

Then, after isolating the system from the environment, at  $t = 0$  we perform a *local* quench, i.e. we bring one single ring bond, e.g.  $w_M$ , from  $w_M^{\text{pre}} = w$  to  $w_M^{\text{post}} = 0$ , leaving all the other bonds  $v$  and  $w$  unaltered. As a consequence, the ring gets cut into a chain, as illustrated in figure 3(a). The post-quench Hamiltonian, being defined on a chain lattice, preserves  $\mathcal{T}$  and  $\mathcal{C}$  and the second hypothesis equation (24) of the theorem is satisfied. Depending on whether the cut bond is weak ( $|w_M^{\text{pre}}| < |v|$ ) or strong ( $|w_M^{\text{pre}}| > |v|$ ), the post quench chain is in the trivial or in the topological phase, respectively. For simplicity we shall consider the limit of an instantaneous quench. Still, two timescales characterize the post-quench evolution, namely

$$\tau_g = \frac{\hbar}{||v| - |w||}, \quad (30)$$

which is the timescale related to the inverse half-gap, and

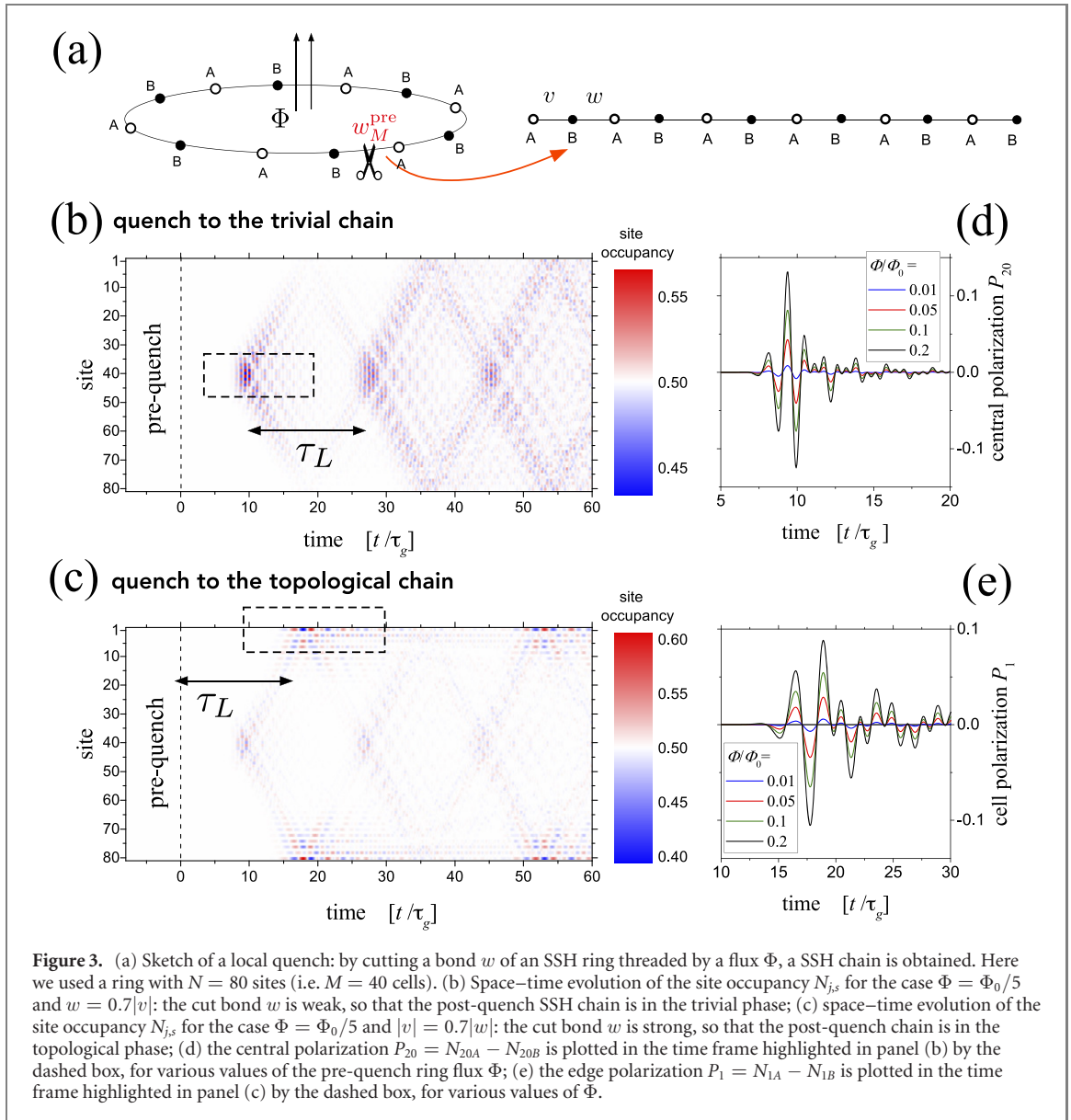
$$\tau_L = \frac{\hbar M}{\min(|v|, |w|)}, \quad (31)$$

corresponding to the typical time an electron wavepacket takes to travel the system length  $L = Ma$  (see the appendix).

The space–time evolution of the site occupancy induced by the quench is depicted in figure 3 for a ring of  $N = 80$  sites ( $M = 40$  cells), initially threaded by a magnetic flux  $\Phi = \Phi_0/5$ , where the  $w$ -bond between sites 1 and 80 is cut by the local quench. In the plot, the red (blue) color characterizes a positive (negative) fluctuation  $N_{j,s} - 1/2$  from the pre-quench occupancy 1/2 (white color), and time is expressed in units of  $\tau_g$ . Panels (b) and (c) refer to the cases of quench from the ring to the trivial and to the topological chain, respectively. After the bond is cut ( $t > 0$ ), we observe in both cases that the site occupancy remains roughly equal to 1/2 everywhere until a time  $\tau_L/2$ , which corresponds to the timescale needed by the quench-induced electron waves propagating in opposite directions to meet again and interfere in the middle of the chain, i.e. at the opposite site of the cut bond.

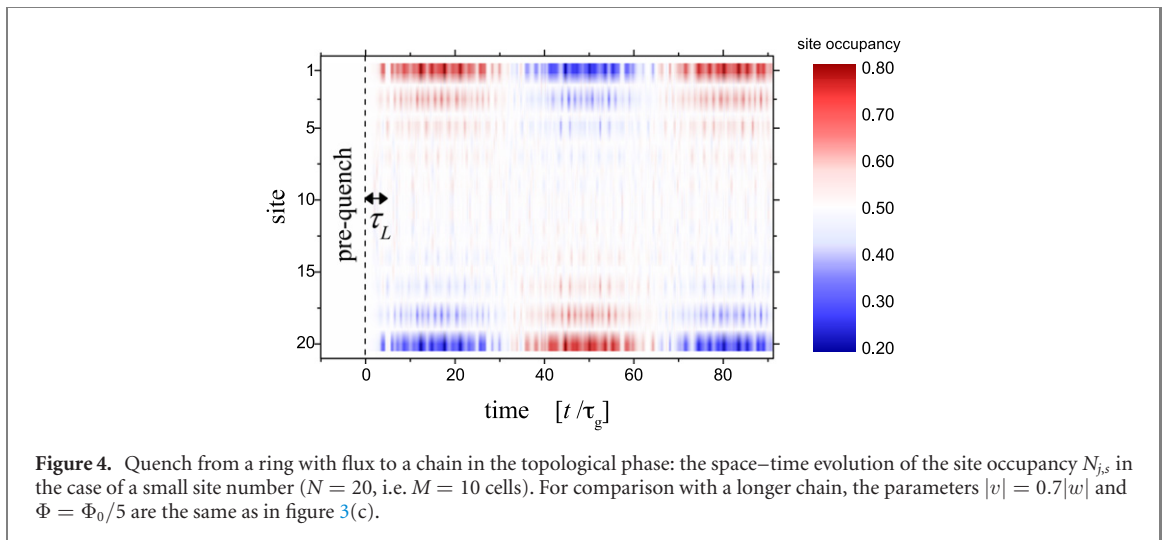
After such time, the two panels feature qualitatively different behaviors. Indeed for a quench to the trivial chain (see figure 3(b)), the fluctuations  $N_{j,s} - 1/2$  from the pre-quench occupancy are more pronounced near the center of the chain and occur at times  $t \simeq \tau_L(m + 1/2)$  corresponding to half-integer values of the typical time related to the chain length, equation (31). In contrast, for a quench to the topological chain (figure 3(c)), the largest occupancy fluctuations are observed at the chain edges and dynamically appear for the first time at  $t \simeq \tau_L$ , and then again at odd integer multiples  $(2m + 1)\tau_L$ . Note that, at each appearance, the fluctuations take opposite signs at the two boundaries, since the total charge is conserved.

We emphasize that the pre-quench flux  $\Phi$  is crucial in determining the magnitude of fluctuations from the occupancy value 1/2, for both quenches to the trivial and to the topological chain. For the quench to the trivial chain this is clearly illustrated in panel (d), which displays the bulk polarization  $P_{20} = N_{20,A} - N_{20,B}$ , i.e. the polarization of the central cell  $j = 20$ , for the time range highlighted by the dashed box of panel (b), for various values of the flux  $\Phi$  through the pre-quench ring. Similarly, for the case of quench to the topological chain, panel (e) shows the edge polarization  $P_1 = N_{1,A} - N_{1,B}$ , i.e. the polarization of the cell  $j = 1$  in the time frame highlighted in panel (c). When  $\Phi$  is vanishing or equal to  $\Phi = p\Phi_0/2$ , with  $p \in \mathbb{Z}$ , the site occupancy remains locked to  $N_{j,s} \equiv 1/2$  at any time.



Furthermore, for a given flux  $\Phi$ , the magnitude of the fluctuations also depends on the gap  $2\varepsilon_g$  through the ratio  $r = \varepsilon_g / \max(|w|, |v|)$ . In particular, when  $r \rightarrow 1$ , the model tends to the dimerized limit where the flux plays no role and the fluctuations from occupancy  $1/2$  vanish everywhere. In general, both for quenches to the trivial and to the topological phase, a decrease in the value of  $r$  implies an increase in the fluctuation magnitude. For instance, for  $N = 80$  and  $\Phi = \Phi_0/5$ , when  $r$  is decreased from  $r = 0.5$  down to  $r = 0.1$ , the fluctuation magnitude at the edge sites of the topological chain increases from 14% to 30% of the pre-quench site occupancy value  $1/2$ . However, when  $r \rightarrow 0$  the model tends to the gapless metallic tight-binding model, and a difference between topological and trivial phase emerges. Indeed if such limit is taken from the trivial phase ( $|v| > |w|$ ) the magnitude of fluctuations located near the center of the chain (see figure 3(b)) tends to a finite value and survive even in the metallic case. In contrast, when the gap is decreased from the topological phase side ( $|v| < |w|$ ), the magnitude of the fluctuations located at the chain edges (see figure 3(c)) eventually drops to zero for  $r < 10^{-2}$  and the effect is completely suppressed in the metallic case  $r = 0$ , in agreement with the fact that edge states disappear in such a case.

To conclude this subsection, we analyze how the site occupancy fluctuations induced by the quench depend on the chain length  $L = Ma$ . The cases analyzed so far in figure 3 correspond to the regime  $\tau_L \gg \tau_g$  of a long chain. In this regime the magnitude of site occupancy fluctuations does not significantly change with the number  $N = 2M$  of sites, while the occurrence timescale  $\tau_L$  does of course depend on  $M$  (see equation (31)). For fixed values of  $v$  and  $w$ , when the number of lattice sites is reduced one reaches the regime  $\tau_L \sim \tau_g$ , where the typical energy separation  $\Delta$  between the bulk states becomes comparable with



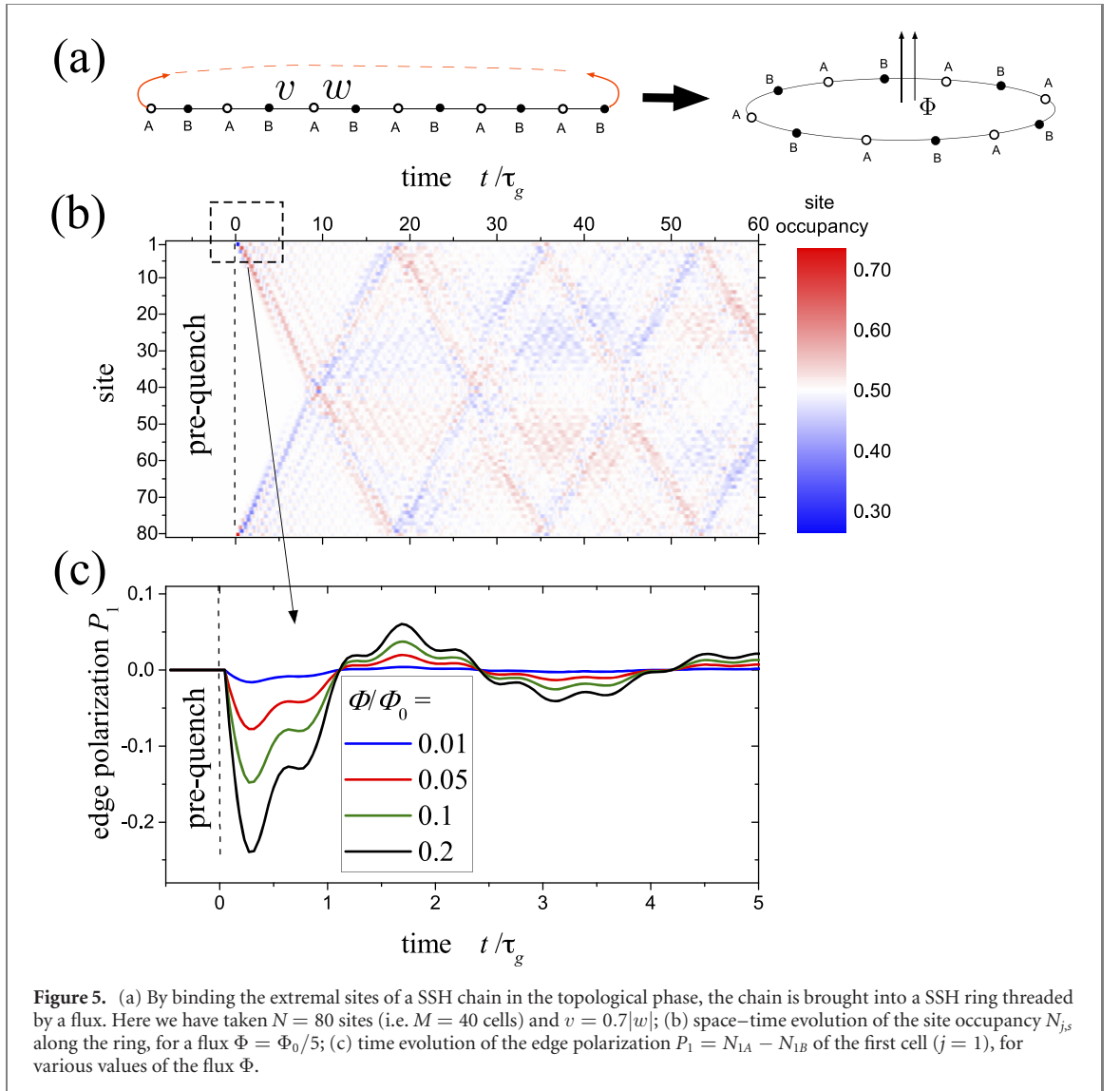
the gap  $2\varepsilon_g$ , so that the very notion of bulk gap becomes somewhat questionable. Yet, two discrete energy levels  $\pm\varepsilon^{\text{edge}}$  near  $\varepsilon = 0$  are still present in the topological phase. In figure 4 we have plotted the space–time evolution of the site occupancy for a short lattice with  $N = 20$  sites (i.e.  $M = 10$  cells), when the ring with flux is cut into a chain in the topological phase, keeping all the other parameters unchanged with respect to the case of figure 3(c). The comparison shows two interesting effects of the reduced system size on the dynamics. First, the magnitude of the fluctuations at the edges is bigger in the shorter chain (figure 4) than in the longer chain (figure 3(c)), highlighting the dynamical alternation of excess and depletion of occupancy at the two edges. Second, along with the short timescale  $\tau_L$  determining the roughly periodic occurrence described above, we observe a second longer period that further modulates the fluctuation magnitude. Such a timescale is associated to the small energy splitting  $2\varepsilon^{\text{edge}}$  between the two discrete edge states. Indeed such energy separation increases when reducing the system size and, despite being much smaller than the gap, it becomes visible through this time-dependent modulation.

The local quench cutting the ring thus leads to qualitatively different behaviors in real-space, depending on whether the post-quench chain is in the trivial or in the topological phase. In particular, in the quench to a topological chain the fluctuations of site occupancy are localized at the edges, and alternate in time from excess to depletion. A time-resolved measurement is thus needed to observe such real-space signatures, while a time-average would vanish, just like in any site of the bulk. This is typical of a half-filled system. In section 5 we shall discuss the case of different filling values.

#### 4.2. Chain-to-ring quench

The second possibility to tackle the theorem of section 3.1 is to break the hypothesis equation (24). We keep the first hypothesis equation (23) by choosing as a pre-quench state the ground state of a homogeneous half-filled SSH chain. Then, one can perform a local quench binding the first and last site of the chain, i.e. bringing the tunneling amplitude  $w_M$  from  $w_M^{\text{pre}} = 0$  to  $w_M^{\text{post}} = w$ , thereby enclosing the chain into a ring, as illustrated in figure 5(a). If the ring is threaded by a magnetic flux, the quenching Hamiltonian breaks  $\mathcal{T}$  and hence  $\mathcal{C}$  symmetries, and the condition equation (24) is violated, opening up the possibility to observe real-space signatures of the quench.

Figure 5(b) displays the space–time evolution of the site occupancy when the pre-quench state is the ground state of a 80-sites half-filled SSH chain in the topological phase ( $|v| < |w|$ ) and the post-quench ring is threaded by a flux  $\Phi = \Phi_0/5$ . Note that, although the initial chain is in the topological phase, before the quench ( $t < 0$ ) the site occupancy is locked to  $1/2$  everywhere, in agreement with the theorem proven above, and no signature of the edge state emerges in real-space. However, after the quench ( $t > 0$ ), two occupancy fluctuations  $N_{j,s} - 1/2$  of opposite signs depart from the bridged link and propagate along the ring in opposite directions, determined by the sign of the flux  $\Phi$ . When the SSH chain is initially in the trivial phase, such an effect is absent, the occupancy evolution is quite similar to the case of a ring-to-trivial chain quench already shown in figure 3(b) and is not reported here. The evolution of the edge polarization at the cell  $j = 1$  is shown in figure 5(c) in the early time range highlighted by the dashed circle of panel (b), for various flux values. Again the presence of the flux is crucial to observe real-space signatures of the quench.



## 5. Quenches in chiral-symmetry broken models and effects of band filling

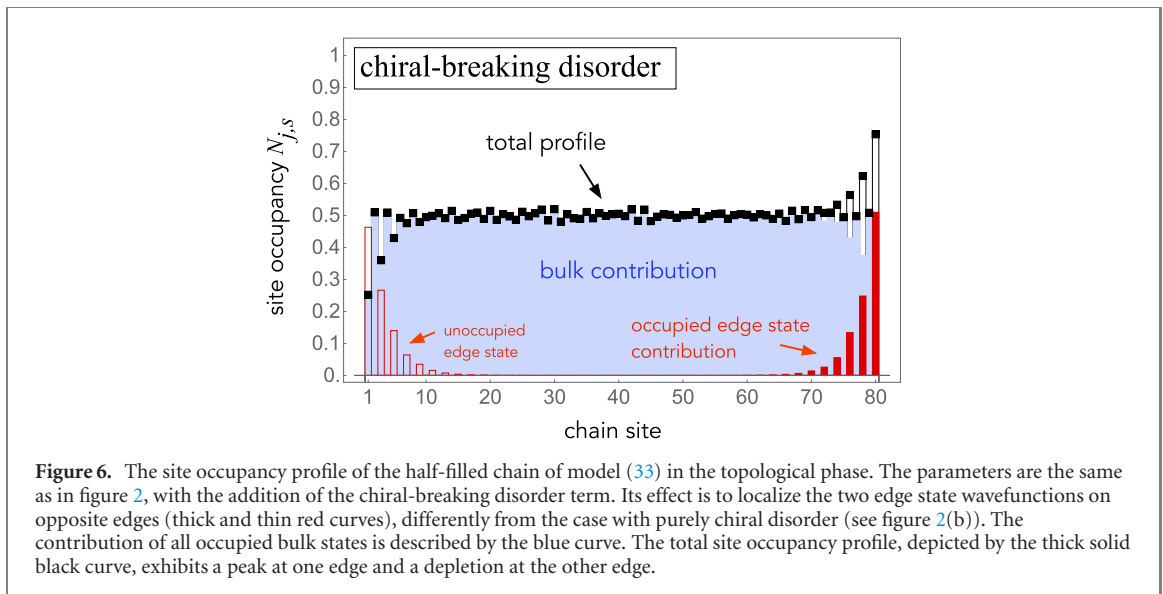
In the previous section we have shown how to violate the conditions equations (23) and (24), while preserving the chiral symmetry  $\mathcal{S}$  in the half-filled SSH model. Here we want to explore the dynamical effect of quenches when the chiral symmetry is broken and the filling is not necessarily equal to  $1/2$ , i.e. beyond the framework where the model can be classified as a topological insulator. As a matter of fact, in a realistic electron model on a bipartite lattice, the chiral symmetry is fragile. A difference  $\delta_j$  between the on-site energies of  $A$  and  $B$  sites is likely to exist, leading to an additional Hamiltonian term

$$\hat{\mathcal{H}}_{\chi b} = \sum_j \delta_j (\hat{n}_{j,A} - \hat{n}_{j,B}), \quad (32)$$

which breaks the chiral symmetry because  $\mathcal{S}\hat{\mathcal{H}}_{\chi b}\mathcal{S}^{-1} = -\hat{\mathcal{H}}_{\chi b}$  [62]. On the one hand, in the absence of chiral symmetry the very topological classification is not well defined since, for instance, one could go from the range  $|v| > |w|$  to the range  $|v| < |w|$  without closing the gap. *A priori*, there is no guarantee that the topological states exist at all. On the other hand, numerical analysis shows that, if the values of the  $\delta_j$  are small compared to the band gap  $2\varepsilon_g$ , edge states still persist. Specifically, we shall consider a Hamiltonian

$$\hat{\mathcal{H}} = \hat{\mathcal{H}}_{\text{SSH},\chi} + \hat{\mathcal{H}}_{\chi b}, \quad (33)$$

where the first term, equation (2), contains a *chiral disorder*  $w_j = w(1 + \xi_j d)$  and  $v_j = v(1 + \eta_j d)$ , while the second term, equation (32), contains a *chiral-breaking disorder*  $\delta_j = \zeta_j \max(v, w)d$ . Here we have assumed



**Figure 6.** The site occupancy profile of the half-filled chain of model (33) in the topological phase. The parameters are the same as in figure 2, with the addition of the chiral-breaking disorder term. Its effect is to localize the two edge state wavefunctions on opposite edges (thick and thin red curves), differently from the case with purely chiral disorder (see figure 2(b)). The contribution of all occupied bulk states is described by the blue curve. The total site occupancy profile, depicted by the thick solid black curve, exhibits a peak at one edge and a depletion at the other edge.

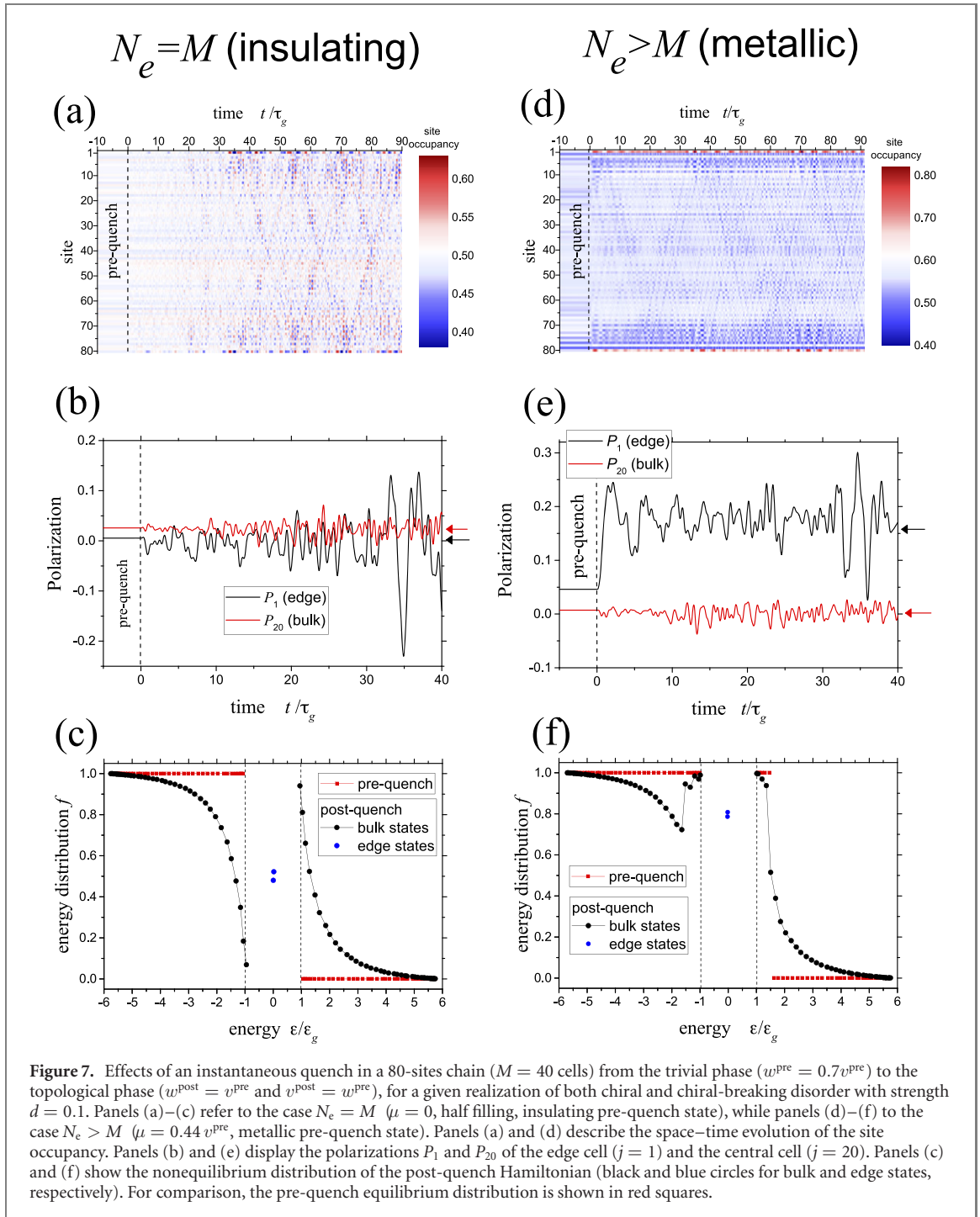
$v, w > 0$  and  $\{\xi_j, \eta_j, \zeta_j\}$  denote real random variables uniformly distributed in  $[-1/2, +1/2]$  with a disorder strength  $d < |v - w|/\max(v, w)$ . Note that, although for each disorder realization the chiral symmetry is broken by equation (33), the disorder-averaged Hamiltonian equation (33) still preserves the chiral symmetry, so that the trivial and topological phases can still be defined in the sense of the average values  $v$  and  $w$ . While reference [63] analyzed the Hamiltonian equation (33) in the case of local quenches performed over a long quench time ( $\tau_q \gg \tau_g$ ) and at half filling, here we shall focus on the complementary situation of a global quench in the short quench time limit ( $\tau_q \ll \tau_g$ ) and consider also filling values different from a half, which turns out to be important for the effects in real-space.

Let us thus go back to the original problem illustrated in figure 1, and analyze a global quench from the trivial to the topological chain, where now the Hamiltonian equation (33) includes the chiral-breaking term equation (32).

### 5.1. The case of half-filling

At first, one might even expect that  $\hat{\mathcal{H}}_{\chi b}$  may favor the appearance of the edge states already at half-filling. Indeed without such term equation (32) the site occupancy would always remain strictly locked to  $1/2$ , due to the theorem proven in section 3.1. In contrast, because the term (32) also breaks the charge conjugation symmetry,  $\mathcal{C}\hat{\mathcal{H}}_{\chi b}\mathcal{C}^{-1} = -\hat{\mathcal{H}}_{\chi b}$ , the hypotheses of the theorem are violated, opening up the possibility to observe fluctuations of the site occupancy, possibly at the edges. This expectation seems to be confirmed when analyzing how the edge state wavefunctions are modified by the term  $\hat{\mathcal{H}}_{\chi b}$ . While in the purely chiral SSH model  $\hat{\mathcal{H}}_{\text{SSH},\chi}$  each wavefunction is localized on *both* edges even in the presence of *chiral disorder*  $\{v_j, w_j\}$  (see figure 2), the addition of the chiral-breaking disorder  $\hat{\mathcal{H}}_{\chi b}$  localizes each discrete state of the Hamiltonian (33) only on *one single* edge of the chain, as shown by the red curves of figure 6. This is the hallmark of the break-up of the topological protection. Depending on the specific disorder realization, one of the two edge wavefunctions is energetically slightly more favored than the other, so that the ground state of the half-filled topological chain with chiral-breaking disorder exhibits a site occupancy with an enhancement at one edge, a depletion on the other edge and a value roughly equal to  $1/2$  in the bulk (see black curve of figure 6). In principle, such real-space signature of the edge state should appear by performing an infinitely slow quench from the trivial to the topological chain, where the pre-quench trivial ground state should evolve into the post-quench topological ground state. However, if the chiral-breaking disorder term is weak, the energy separation between the two localized states is very small. In practice, at half filling, any finite temperature in the pre-quench state leads the post-quench state to exhibit only *half-occupancy* of *both* discrete states, quite similarly to what happens in the chiral SSH model  $\hat{\mathcal{H}}_{\chi b}$ .

The same effect occurs when the duration of the quench is short, as shown in figure 7(a), which illustrates the space–time evolution of the site occupancy of a 80-sites chain that is suddenly quenched from the ground state of the trivial phase ( $w^{\text{pre}} = 0.7v^{\text{pre}}$ ) to the topological phase ( $w^{\text{post}} = v^{\text{pre}}$  and  $v^{\text{post}} = w^{\text{pre}}$ ), for given pre-quench and post-quench realizations of chiral and chiral-breaking disorder with strength  $d = 0.1$ . Due to the chiral-breaking term equation (32), static deviations from the site occupancy  $1/2$  are present even before the quench, while after the quench these deviations fluctuate in time as well. In



**Figure 7.** Effects of an instantaneous quench in a 80-sites chain ( $M = 40$  cells) from the trivial phase ( $w^{\text{pre}} = 0.7v^{\text{pre}}$ ) to the topological phase ( $w^{\text{post}} = v^{\text{pre}}$  and  $v^{\text{post}} = w^{\text{pre}}$ ), for a given realization of both chiral and chiral-breaking disorder (with strength  $d = 0.1$ ). Panels (a)–(c) refer to the case  $N_e = M$  ( $\mu = 0$ , half filling, insulating pre-quench state), while panels (d)–(f) to the case  $N_e > M$  ( $\mu = 0.44v^{\text{pre}}$ , metallic pre-quench state). Panels (a) and (d) describe the space–time evolution of the site occupancy. Panels (b) and (e) display the polarizations  $P_1$  and  $P_{20}$  of the edge cell ( $j = 1$ ) and the central cell ( $j = 20$ ). Panels (c) and (f) show the nonequilibrium distribution of the post-quench Hamiltonian (black and blue circles for bulk and edge states, respectively). For comparison, the pre-quench equilibrium distribution is shown in red squares.

figure 7(b), the corresponding edge polarization  $P_1 = N_{1,A} - N_{1,B}$  (black curve) is compared to the polarization at the central chain cell  $P_{20} = N_{20,A} - N_{20,B}$  (red curve). As one can see, fluctuations do have larger amplitudes at the chain edges than in the chain bulk. However, at each edge, the site occupancy experiences an alternation of depletion and excess (blue and red colors in panel (a)), just like for the quenches preserving the chiral symmetry discussed above (see e.g. figures 3(b) or 4). This quantitatively shows that the chiral-breaking term does not really improve the observability of real-space effects of the quench. In particular, the time-average of the fluctuations at the edge and in the bulk is essentially the same, as highlighted by the arrows in figure 7(b).

In energy-space, however, the effects of the quench are seizable. Indeed the energy distribution of the post-quench Hamiltonian displayed in figure 7(c) strongly differs from the pre-quench equilibrium one, depicted in red for comparison. In particular, as far as the continuum spectrum is concerned (black symbols), the non-equilibrium distribution that we obtain for the disordered chain-to-chain quench is

quite similar to the result obtained for a quench in the clean bulk SSH model analyzed in reference [58], and describes the population inversion effect found upon quenching from one phase to the other. This effect can open up the possibility that, when the SSH model is coupled to a radiation [64], a stimulated emission occurs due to transitions from the (almost filled) continuum states near the bottom of the conduction band to the (almost depleted) states near the top of the valence band, with a radiation frequency corresponding to the band gap. Apart from the presence of disorder, the major difference from the ring case arises from the presence of the topological edge states in the post-quench spectrum of the chain, highlighted by the blue symbols. Note that their occupancy is roughly  $1/2$ . On the one hand, this is precisely what disguises these states in the real-space occupancy at the chain edges, as argued above (see figure 7(a)). On the other hand, differently from a purely bulk SSH system, in a chain quenched to the topological phase the presence of half-occupied discrete levels near  $\varepsilon = 0$  causes an additional emission process, characterized by a frequency corresponding to a half of the gap, similar to the phenomenon described in reference [65] of a quenched quantum well potential.

## 5.2. Away from half-filling

The results obtained above at half filling ( $N_e = M$ ), where the model (33) describes a band insulator, indicate that a quench from a trivial to the topological phase of a chain does lead to the appearance of site occupancy fluctuations that are larger at the chain edges than in the bulk. However, the alternation of depletion and excess yields a vanishing result upon time-average, both at the edges and in the bulk. In the short quench time limit this holds for any temperature of the pre-quench state. We now analyze the effects of a quench in a non half-filled chain, where the number  $N_e$  of electrons differs from the number  $M$  of lattice cells. Note that in such a case the pre-quench ground state is *metallic*, with an excess of electrons (holes) in the conduction (valence) band for  $N_e > M$  ( $N_e < M$ ). At finite temperature this is described by

$$\hat{\rho}^{\text{pre}} = \frac{e^{-\beta(\hat{H}^{\text{pre}} - \mu\hat{N}_e)}}{\text{Tr}[e^{-\beta(\hat{H}^{\text{pre}} - \mu\hat{N}_e)}]}, \quad (34)$$

where  $\hat{N}_e = \sum_{j,s} \hat{n}_{j,s}$  is the total electron number operator and  $\mu$  the chemical potential. While in the insulator  $\mu = 0$ , the metallic state is described by  $\mu \neq 0$ . Moreover, because  $\hat{N}_e$  transforms under charge conjugation as  $\mathcal{C}\hat{N}_e\mathcal{C}^{-1} = 2M - \hat{N}_e$  the condition (23) of the theorem proven in section 3.1 is violated, opening up the way to observe effects of the quench in real-space.

In principle, if one takes the standard disorder-free SSH model equation (1), the ideal situation to observe the appearance of the topological edge states is a Gedankenexperiment where the pre-quench state is the ground state of the trivial SSH chain ( $v^{\text{pre}} > w^{\text{pre}}$ ) with exactly  $N_e = M + 1$  electrons, with the extra electron lying in the conduction band. By performing an infinitely slow quench ( $\tau_q \rightarrow \infty$ ) to the topological chain ( $w^{\text{post}} > v^{\text{post}}$ ), the ground state evolves to the post-quench ground state of the topological chain (see figure 1), where now *both* discrete levels will be occupied, instead of only one like in the half-filled case. The extra electronic level, delocalized on both edges, causes the gradual appearance of peaks localized at the chain edges in the site occupancy profile, over a value of  $N_{j,s} = 1/2$  in the bulk of the chain. In practice, however, such ideal conditions are not necessarily easy to realize and/or useful. First, chiral-breaking disorder equation (32) is typically present as well. Second, in view of technological applications, one typically wants these operations to be performed sufficiently fast. For a finite and possibly short quench time  $\tau_q$  the post-quench state may differ from the slow-quench scenario. Third, in a metallic system the filling would deviate from  $1/2$  not just by one single electron. A finite fraction of the (say) conduction band is occupied and, even in the adiabatic quench limit, these extra conduction states may mask the localized peaks due to the edge states. This is certainly the case, for instance, when the filling approaches 1. The question is thus whether the edge states dynamically appear in real-space when these aspects are taken into account.

For definiteness, we shall analyze the case  $N_e > M$ , where before the quench a small fraction of the conduction band of the trivial phase is occupied, e.g.  $1/10$  of the conduction bandwidth from its band bottom. For the values  $w^{\text{pre}} = 0.7v^{\text{pre}}$  this corresponds to setting the chemical potential to  $\mu = 0.44v^{\text{pre}}$ . Then, we consider a quench to the topological phase of the chain ( $w^{\text{post/pre}} = v^{\text{pre/post}}$ ), in the short quench time limit ( $\tau_q \rightarrow 0$ ). The resulting time evolution of the site occupancy is shown in figure 7(d). As one can see, while in the pre-quench state the edge states are absent, after the quench they start to become visible and stable. This can be seen explicitly in figure 7(e), where the polarization of the edge cell (black curve) and of the central chain cell (red curve) are compared. Differently from the half-filling case [panel(b)], the edge occupancy oscillates around an average value that is finite and thus differs from the small one obtained in the bulk of the chain, as highlighted by the arrows. Finally figure 7(f) shows the corresponding nonequilibrium post-quench energy distribution. Note that the occupancy of the discrete states near  $\varepsilon = 0$

significantly differs from  $1/2$ . On the one hand, this is the reason for their appearance in real-space at the chain edges. On the other hand, this reduces the occupancy difference from the (almost filled) states near the bottom of the conduction band, thereby reducing the spontaneous emission effect as compared to the half-filling case shown panel (c).

## 6. Discussion and conclusion

In this article we have analyzed how a quantum quench applied to the SSH model (2) impacts on observables that are local in real-space, namely the site occupancy and the cell polarization.

In section 3 we have proven a general theorem ensuring that, when the pre-quench state and the quenching Hamiltonian fulfill the charge conjugation symmetry  $\mathcal{C}$ , the occupancy of each lattice site remains firmly locked to the value  $1/2$ , at any time. These symmetries are always satisfied in the customary case of a half-filled SSH chain. Indeed, because  $\mathcal{C} = \mathcal{S}\mathcal{T}$  and the chiral symmetry  $\mathcal{S}$  is preserved by the SSH model, the breaking of  $\mathcal{C}$  requires also the breaking of time-reversal symmetry  $\mathcal{T}$ , which is not possible for spinless electrons in a chain lattice with OBCs. As a consequence of the proven theorem, a quench from the trivial to the topological phase performed on a SSH chain has no effect whatsoever on the site occupancies. In particular, no signature of the topological edge states appears locally in real-space, independently of the quench protocol, of the temperature of the pre-quench thermal state and also of the presence of chiral disorder. This is strikingly different from what is known to happen in  $k$ -space. Indeed when a quench between two topologically different phases is performed, a dynamical quantum phase transition [66] is known to arise, and the momentum distribution exhibits a band population inversion related to a dynamical topological invariant [58, 67, 68]. Our result thus implies that these out of equilibrium phenomena can be detected in real space only through *non-local* quantities, such as correlation functions.

The effects of the quench can become visible in observables that are local in real space only when charge conjugation symmetry is broken. This can be done either remaining within the framework of the topological insulator characterization, i.e. by preserving the chiral symmetry  $\mathcal{S}$  and the half-filling condition, or going out of such framework. The first case requires suitably engineered setups. In particular, in section 4 we have shown that a *local* quench cutting a SSH ring threaded by a flux into a SSH chain (or vice versa), violates the hypotheses of the above theorem while still remaining in the topological insulator framework. Real-space effects of the quench then become observable and quite distinct dynamical features appear in the two phases. In particular, when the pre-quench ring is cut into a trivial chain the site occupancy fluctuations appear near the center of the chain, while when the ring is cut into a topological chain these fluctuations appear at the chain edges after a time  $\tau_L$  (see figure 3), and then repeat with a dynamical alternation of excess and depletion at each edge. Such effect at the edges appears even more clearly in a lattice with smaller number of sites (see figure 4). Conversely, when the local quench bridges a chain to form a ring with flux, the site occupancy fluctuations propagate towards the chain center if the pre-quench chain is in the topological phase (see figure 5), whereas such effect is absent if the chain is in the trivial phase. In all such dynamical effects the presence of the flux  $\Phi$  threading the ring is crucial. For vanishing flux or for  $\Phi = p\Phi_0/2$ , where time-reversal and charge conjugation hold, the site occupancy remains locked to the pre-quench value  $1/2$ .

In section 5 we have explored the effects of the quench beyond the framework of the topological insulator. By adding a disordered on-site potential term equation (32) both the chiral and the charge conjugation symmetries get broken. Although for a disorder realization the topological classification is in principle not well defined and the existence of the edge states is not guaranteed, the disorder-averaged Hamiltonian (33) still preserves  $\mathcal{S}$ , and the topological phases can still be considered to hold for weak enough disorder. We have thus analyzed the effects of a quench from a trivial to a topological chain. Our results show that, although the chiral-breaking disorder localizes each edge state wavefunction on one edge only (see figure 6), in practice such term does not lead to any improvement in terms of their observability in real-space as compared to the purely chiral SSH model. In particular, for a half-filled system, while the edge occupancy exhibits much larger fluctuations than the bulk, its time average is roughly equal to the bulk one (see figure 7(b)). Thus, real-space effects of the quench do exist, but time-resolved measurements are needed to probe the quench-induced appearance of the edge states. In contrast, for filling values different from  $1/2$ , where the model is slightly metallic, the edge site occupancy fluctuates around a value that is different from the bulk. In this case the dynamical signature of the topological states survive both time-average and the presence of chiral-breaking disorder, and persist even in the short quench time limit (see figures 7(d)–(e)).

In conclusion our analysis points out that, when a topological insulator is driven out of equilibrium by a quantum quench, the presence of additional symmetries (such as charge conjugation or time-reversal) in the quenching Hamiltonian and in the pre-quench state can completely mask the impact of the quench in

real-space occupancies, even in customary cases where the energy distributions are typically strongly affected. Only when such additional symmetries are suitably broken, like in the setups and protocols proposed here in sections 4 and 5, real-space effects do emerge in local observables, and exhibit distinct dynamical behavior in the topological and trivial phases. The huge advances in realizing topological models with cold atoms in optical lattices, which nowadays also enable one to effectively implement a Peierls substitution in tunneling amplitudes [69–71], represent a promising perspective to test the predicted quench effects.

## Data availability statement

The data that support the findings of this study are available upon reasonable request from the authors.

## Appendix

Here we briefly summarize some aspects of the standard *homogeneous* SSH model, which corresponds to taking  $v_j \equiv v \in \mathbb{C}$  and  $w_j \equiv w \in \mathbb{C}$  in equation (2). Let us now recall the cases of a ring and of a chain.

### A.1. Ring (periodic boundary conditions) and topological classification

Let  $M$  denote the number of cells and by  $N = 2M$  the number of lattice sites of a ring-shaped lattice. By re-expressing the site operators as  $c_{j,s} = M^{-1/2} \sum_k e^{ikja} c_{k,s}$ , the PBC  $c_{M+1,A} = c_{1,A}$  enables one to straightforwardly rewrite the Hamiltonian equation (2) as a decoupled set of  $k$ -dependent Hamiltonians

$$\hat{\mathcal{H}}_{\text{SSH},\chi} = \sum_k \left( \hat{c}_{k,A}^\dagger, \hat{c}_{k,B}^\dagger \right) H(k) \begin{pmatrix} c_{k,A} \\ c_{k,B} \end{pmatrix}. \quad (\text{A.1})$$

Here  $a$  denotes the cell size,  $k = 2\pi n/Ma$  the wavevectors (with  $n = -\frac{M}{2}, -\frac{M}{2} + 1, \dots, \frac{M}{2} - 1$  for even  $M$  and  $n = -[\frac{M}{2}], -[\frac{M}{2}] + 1, \dots, +[\frac{M}{2}]$  for odd  $M$ ), and  $c_{k,s}$  are the Fourier mode operators [72]. In equation (A.1)  $H(k) = \boldsymbol{\sigma} \cdot \mathbf{b}(k)$  is the first-quantized SSH Hamiltonian in  $k$ -space, with  $\boldsymbol{\sigma}$  denoting the set of Pauli matrices and  $\mathbf{b}(k) = (\text{Re}(v) + \text{Re}(w e^{ika}), -\text{Im}(v) + \text{Im}(w e^{ika}), 0)$  a vector lying in the plane. The absence of the  $b_z$  component is the hallmark of the chiral symmetry, which is expressed by the property  $\sigma_z H(k) \sigma_z = -H(k)$  in terms of the first-quantized Hamiltonian. In terms of the tenfold classification scheme [73, 74], the SSH model is in the AIII symmetry class. If  $w, v \in \mathbb{R}$  the Hamiltonian  $H(k)$  also fulfills the properties  $H^*(k) = H(-k)$  and  $\sigma_z H^*(k) \sigma_z = -H(-k)$  encoding the time-reversal and charge conjugation symmetries, respectively, and the model is in the BDI symmetry class. The spectrum consists of two bands

$$\varepsilon_{\pm}(k) = \pm |\mathbf{b}(k)| = \pm \sqrt{|w|^2 + |v|^2 + 2|vw| \cos(ka + \arg(v) + \arg(w))}, \quad (\text{A.2})$$

separated by the bandgap  $2\varepsilon_g = 2||v| - |w||$ . The maximal value of the group velocity  $v(k) = \hbar^{-1} \partial \varepsilon / \partial k$  is  $v_{\text{max}} = a \min(|v|, |w|) / \hbar$  and determines the minimal timescale equation (31) an electron wavepacket takes to travel across the ring length  $L = Ma$ .

The single-particle eigenvectors related to the two bands  $\varepsilon_{\pm}$  are  $|u_{-}(k)\rangle = (1, -e^{i\varphi(k)})^T / \sqrt{2}$  and  $|u_{+}(k)\rangle = (e^{-i\varphi(k)}, 1)^T / \sqrt{2}$ , with  $\varphi(k) = -\varphi(-k)$  denoting the polar angle of the  $\mathbf{b}(k)$  vector, so that  $\tan \varphi(k) = b_y(k) / b_x(k)$ . In particular, in the thermodynamic limit, where  $k$  becomes a continuous variable spanning the Brillouin zone  $[-\pi, +\pi] / a$ ,  $\mathbf{b}(k)$  draws a circle centered at  $(\text{Re}(v), -\text{Im}(v))$  and with radius  $|w|$ . As is well known, this enables one to identify two topological classes of the insulator, depending on whether such circle encloses or not the origin (corresponding to the gap closing). Correspondingly, the winding number of the fully occupied lower band of the insulator

$$\nu = -\frac{i}{\pi} \oint \langle u_{-} | \partial_k u_{-} \rangle dk = \frac{1}{2\pi} \oint \frac{d\varphi}{dk} dk \quad (\text{A.3})$$

takes two different integer values  $\nu = 1$  (for  $|v| < |w|$ ) and  $\nu = 0$  (for  $|v| > |w|$ ). We emphasize that, while the two phases are topologically distinct, labeling one phase as ‘topological’ and the other one as ‘trivial’ is in fact unphysical as long as the model is defined on a ring. This is because the topological classification is defined once the unit cell is identified, which is completely arbitrary in a system with PBCs, though. Indeed the very Hamiltonian  $\hat{\mathcal{H}}_{\text{SSH}}$ , written in equation (1) by adopting  $(A, B)$  as unit cell, could be equivalently rewritten choosing  $(B, A)$  as unit cell, which would amount to exchanging the role of intra- and inter-cell hopping amplitudes ( $v \leftrightarrow w$ ), so that the ‘topological’ phase for the choice  $(A, B)$  corresponds to the ‘trivial’ phase for the choice  $(B, A)$  and vice versa. The emergence of topological edge states, which are

perhaps the most striking hallmark distinguishing the topological character of the two phases, requires the breaking of the PBCs.

## A.2. Chain (open boundary conditions)

The customary way to break the PBCs is to cut the ring into a finite chain, thereby interfacing the SSH model with vacuum. In turn, this lifts the degeneracy about the choice of the unit cell: if (say)  $(A, B)$  is the unit cell of the chain with an even number of sites  $N = 2M$ , the OBCs of the chain impose  $c_{M+1,A} = 0 = c_{0,B}$ , where  $M$  again denotes the number of cells. As argued above, in a chain the hopping amplitudes can be taken as real and positive,  $v, w \in \mathbb{R}^+$ , without loss of generality. The OBCs modify the spectrum and enable one to identify the actual topological and trivial phases. Indeed, besides a continuum spectrum similar to the ring, when  $v < w$  the chain also features two additional discrete levels (topological phase), which are absent for  $v > w$  (trivial phase) instead. The difference between the two phases becomes apparent in the so called dimerized limit, where one of the two hopping amplitude is set to zero. The SSH chain eigenstates resulting from the OBCs are non-degenerate and can be given an analytic expression [75, 76]. In particular, the continuum eigenstates extend over the entire bulk of the chain and can formally be built by linearly combining the  $|u_{\pm}(k)\rangle$  and  $|u_{\pm}(-k)\rangle$  of the ring

$$|v_{\pm}^{\text{bulk}}(k)\rangle = \frac{1}{\sqrt{\mathcal{N}_k}} \sum_{j=1}^M \begin{pmatrix} \sin[kaj - \varphi(k)] \\ \pm \sin[kaj] \end{pmatrix}, \quad (\text{A.4})$$

where  $\tan \varphi(k) = w \sin(ka)/(v + w \cos(ka))$  and  $\mathcal{N}_k = M + v(v + w \cos(ka))/(v^2 + w^2 + 2vw \cos(ka))$  is a normalization constant. However, the quantization rule of  $k$ 's differs from the customary  $kaM = 2\pi n$  in the ring and fulfill the transcendental equation  $ka(M + 1) = \pi n + \varphi(k)$  with  $n = 1, 2, \dots, M$ . In contrast, the topological edge eigenstates read [75, 76]

$$|\psi_{\pm}^{\text{edge}}\rangle = \frac{1}{\sqrt{\mathcal{N}_0}} \sum_{j=1}^M (-1)^{j+1} \begin{pmatrix} \sinh[\kappa a(M + 1 - j)] \\ \pm \sinh[\kappa aj] \end{pmatrix}, \quad (\text{A.5})$$

where  $\kappa$  fulfills  $v \sinh[\kappa(M + 1)a] = w \sinh[\kappa Ma]$  and  $\mathcal{N}_0 = (w \sinh(2\kappa aM)/2v \sinh(\kappa a)) - (M + 1)$  is a normalization constant. They are localized mainly on  $A$  sites on the left edge and on  $B$  sites on the right edge. Their energies are  $\epsilon_{\pm}^{\text{edge}} = \pm w \sinh(\kappa a)/\sinh(\kappa(M + 1)a)$ , whose difference decreases exponentially as  $\sim \exp[-\kappa Ma]$  with the number of cells.

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## References

- [1] Hasan M Z and Kane C L 2010 *Rev. Mod. Phys.* **82** 3045
- [2] Qi X-L and Zhang S-C 2011 *Rev. Mod. Phys.* **83** 1057
- [3] Ando Y 2013 *J. Phys. Soc. Japan* **82** 102001
- [4] Hasan M Z and Moore J E 2011 *Annu. Rev. Condens. Matter Phys.* **2** 55
- [5] Alicea J 2012 *Rep. Prog. Phys.* **75** 076501
- [6] Aguado R 2017 *Riv. Nuovo Cimento* **40** 523
- [7] Mourik V, Zuo K, Frolov S M, Plissard S R, Bakkers E P A M and Kouwenhoven L P 2012 *Science* **336** 1003
- [8] Gül Ö et al 2018 *Nat. Nanotechnol.* **13** 192
- [9] Yu P, Chen J, Gomanko M, Badawy G, Bakkers E P A M, Zuo K, Mourik V and Frolov S M 2021 *Nat. Phys.* **17** 482
- [10] Nadj-Perge S, Drozdov I K, Li J, Chen H, Jeon S, Seo J, MacDonald A H, Bernevig B A and Yazdani A 2014 *Science* **346** 602
- [11] König M et al 2013 *Phys. Rev. X* **3** 021003
- [12] Nowack K C et al 2013 *Nat. Mater.* **12** 787
- [13] Goldman N, Dalibard J, Dauphin A, Gerbier F, Lewenstein M, Zoller P and Spielman I B 2013 *Proc. Natl Acad. Sci.* **110** 6736
- [14] Pauly C, Saunus C, Liebmann M and Morgenstern M 2015 *Phys. Rev. B* **92** 085140
- [15] Peng L, Yuan Y, Li G, Yang X, Xian J-J, Yi C-J, Shi Y-G and Fu Y-S 2017 *Nat. Commun.* **8** 659
- [16] Tiwari K L, Coish W A and Pereg-Barnea T 2017 *Phys. Rev. B* **96** 235120
- [17] Kaku S, Ando T and Yoshino J 2019 *ACS Nano* **13** 12980
- [18] Morgenstern M et al 2021 *Phys. Status Solidi b* **258** 2000060
- [19] Lian B, Sun X-Q, Vaezi A, Qi X-L and Zhang S-C 2018 *Proc. Natl Acad. Sci. USA* **115** 10938
- [20] He M, Sun H and He Q L 2019 *Front. Phys.* **14** 43401
- [21] Guo X, Hu G, Zhang Y, Liu R, Dan M, Li L and Zhang Y 2019 *Nano Energy* **60** 36
- [22] Scarola V W and Das Sarma S 2008 *Phys. Rev. A* **77** 023612

- [23] Goldman N, Urban D F and Bercioux D 2011 *Phys. Rev. A* **83** 063601
- [24] Mei F, Zhu S-L, Feng X-L, Zhang Z-M and Oh C H 2011 *Phys. Rev. A* **84** 023622
- [25] Mei F, Zhu S-L, Zhang Z-M, Oh C H and Goldman N 2012 *Phys. Rev. A* **85** 013638
- [26] Atala M, Aidelsburger M, Barreiro J T, Abanin D, Kitagawa T, Demler E and Bloch I 2013 *Nat. Phys.* **9** 795
- [27] Liu X-J, Law K T, Ng T K and Lee P A 2013 *Phys. Rev. Lett.* **111** 120402
- [28] Nonne H, Moliner M, Capponi S, Lecheminant P and Totsuka K 2013 *Europhys. Lett.* **102** 37008
- [29] Liu X-J, Law K T and Ng T K 2014 *Phys. Rev. Lett.* **112** 086401
- [30] Laflamme C, Baranov M A, Zoller P and Kraus C V 2014 *Phys. Rev. A* **89** 022319
- [31] Deng D-L, Wang S-T and Duan L-M 2014 *Phys. Rev. A* **90** 041601
- [32] Scheurer M S, Rachel S and Orth P P 2015 *Sci. Rep.* **5** 8386
- [33] Mugel S, Celi A, Massignan P, Asbóth J K, Lewenstein M and Lobo C 2016 *Phys. Rev. A* **94** 023631
- [34] Zhai H, Rechtsman M, Lu Y-M and Yang K 2016 *New J. Phys.* **18** 080201
- [35] Mai X-Y, Zhang D-W, Li Z and Zhu S-L 2017 *Phys. Rev. A* **95** 063616
- [36] Potirniche I-D, Potter A C, Schleier-Smith M, Vishwanath A and Yao N Y 2017 *Phys. Rev. Lett.* **119** 123601
- [37] Ye X-S, Liu Y-J, Zhang X Y and Wu G 2017 *Sci. Rep.* **7** 13541
- [38] Zhang D-W, Zhu Y-Q, Zhao Y X, Yan H and Zhu S-L 2018 *Adv. Phys.* **67** 253
- [39] Liu H, Xiong T-S, Zhang W and An J-H 2019 *Phys. Rev. A* **100** 023622
- [40] Calabrese P and Cardy J 2006 *Phys. Rev. Lett.* **96** 136801
- [41] Polkovnikov A, Sengupta K, Silva A and Vengalattore M 2011 *Rev. Mod. Phys.* **83** 863
- [42] Eisert J, Friesdorf M and Gogolin C 2015 *Nat. Phys.* **11** 124
- [43] Mitra A 2018 *Annu. Rev. Condens. Matter Phys.* **9** 245
- [44] Su W P, Schrieffer J R and Heeger A J 1979 *Phys. Rev. Lett.* **42** 1698
- [45] Su W P, Schrieffer J R and Heeger A J 1980 *Phys. Rev. B* **22** 2099
- [46] Barford W 2005 *Electronic and Optical Properties of Conjugated Polymers* (Oxford: Clarendon)
- [47] Asbóth J K, Oroszlány L and Pályi A 2016 *A Short Course on Topological Insulators* (Berlin: Springer)
- [48] Shen Q-C 2012 *Topological Insulators* (Heidelberg: Springer)
- [49] Meier E J, An F A and Gadway B 2016 *Nat. Commun.* **7** 13986
- [50] Xie D, Gou W, Xiao T, Gadway B and Yan B 2019 *npj Quantum Inf.* **5** 55
- [51] Boross P, Asbóth J K, Széchenyi G, Oroszlány L and Pályi A 2016 *Phys. Rev. B* **100** 045414
- [52] Dal Lago V, Atala M and Foa Torres L E F 2015 *Phys. Rev. A* **92** 023624
- [53] Bandyopadhyay S, Bhattacharya U and Dutta A 2019 *Phys. Rev. B* **100** 054305
- [54] Bandyopadhyay S and Dutta A 2019 *Phys. Rev. B* **100** 144302
- [55] Lü X-L and Xie H 2019 *J. Phys.: Condens. Matter.* **31** 495401
- [56] Peierls R 1933 *Z. Phys.* **80** 763
- [57] Graf M and Vogl P 1995 *Phys. Rev. B* **51** 4940
- [58] Porta S, Ziani N T, Kennes D M, Gambetta F M, Sassetti M and Cavaliere F 2018 *Phys. Rev. B* **98** 214306
- [59] Specifically, equation (1) commutes with  $\mathcal{P}$ , defined through  $\mathcal{P}c_{j,A}\mathcal{P}^{-1} = c_{M-j+1,B}$  and  $\mathcal{P}c_{j,B}\mathcal{P}^{-1} = c_{M-j+1,A}$
- [60] Anderson P W 1958 *Phys. Rev.* **109** 1492
- [61] Thouless D J 1972 *J. Phys. C: Solid State Phys.* **5** 77
- [62] The first-quantized version of equation (32) is  $H_{\chi b} = \bigoplus_{j=1}^M \delta_j(\sigma_z)_j$  and fulfills  $SH_{\chi b}S^{-1} = +H_{\chi b}$ , causing the breaking of equation (9).
- [63] He Y and Chien C-C 2016 *Phys. Rev. B* **94** 024308
- [64] Gebhard F, Bott K, Scheidler M, Thomas P and Koch S W 1997 *Phil. Mag. B* **75** 1
- [65] Rossi L, Dolcini F, Cavaliere F, Traverso Ziani N, Sassetti M and Rossi F 2021 *Entropy* **23** 220
- [66] Heyl M, Polkovnikov A and Kehrein S 2013 *Phys. Rev. Lett.* **110** 135704
- [67] Vajna S and Dóra B 2015 *Phys. Rev. B* **91** 155127
- [68] Yang C, Li L and Chen S 2018 *Phys. Rev. B* **97** 060304
- [69] Jiménez-García K, LeBlanc L J, Williams R A, Beeler M C, Perry A R and Spielman I B 2012 *Phys. Rev. Lett.* **108** 225303
- [70] Struck J, Ölschläger C, Weinberg M, Hauke P, Simonet J, Eckardt A, Lewenstein M, Sengstock K and Windpassinger P 2012 *Phys. Rev. Lett.* **108** 225304
- [71] Velasco C G and Paredes B 2017 *Phys. Rev. Lett.* **119** 115301
- [72] The  $k$ -mode operators  $c_{k,s}$  (with  $s = A, B = \pm$ ) transform under  $\mathcal{C}$ ,  $\mathcal{S}$  and  $\mathcal{T}$  as follows:  $\mathcal{C}c_{k,s}\mathcal{C}^{-1} = (-1)\hat{c}_{-k,s}^\dagger$ ,  $\mathcal{S}c_{k,s}\mathcal{S}^{-1} = (-1)\hat{c}_{k,s}^\dagger$ , and  $\mathcal{T}c_{k,s}\mathcal{T}^{-1} = \hat{c}_{-k,s}$ .
- [73] Ryu S, Schnyder A P, Furusaki A and Ludwig A W W 2010 *New J. Phys.* **12** 065010
- [74] Chiu C-K, Teo J C Y, Ryu S and Schnyder A P 2016 *Rev. Mod. Phys.* **88** 035005
- [75] Delplace P, Ullmo D and Montambaux G 2011 *Phys. Rev. B* **84** 195452
- [76] Zaimi M, Boudreault C, Baspin N, Delnour N, Eleuch H, MacKenzie R and Hilke M 2021 *Phys. Lett. A* **388** 127035