On Optimal Clearing Payments in Financial Networks

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***Abstract*— Modern financial networks are characterized by complex structures of mutual obligations. Such interconnections may propagate and amplify individual defaults, leading in some cases to financial disaster. For this reason, mathematical models for the study and control of systemic risk have attracted considerable research attention in recent years. One important line of research is concerned with mechanisms of *clearing*, that is, the mechanism by which mutual debts are repaid, in the regular regime, or in a default regime. One of the first models of a clearing mechanism was proposed by Eisenberg and Noe in [1], which introduced the concept of clearing vector of payments. In this paper, we propose a necessary and sufficient condition for the uniqueness of the clearing vector applicable to an arbitrary topology of the financial network. Further, we show that the overall system loss can be reduced if one relaxes the pro-rata rule and replaces the clearing vector by a matrix of clearing payments. This approach shifts the focus from the individual interest to the system, or social, interest, in order to control and contain the adverse effects of cascaded failures.**

I. Introduction

Globalization has led to highly interconnected financial systems, where organizations are densely linked to each other with an intricate structure of obligations. The behavior of such financial interconnected system has been extensively studied over the past years [2], [3]. Interconnections among financial institutions create potential channels of contagion, where a failure of a single entity can result in a threat to the stability of the entire financial system. Recent examples of such a behaviour include the collapse of Lehman Brothers, being one of the reasons of the global-wide financial crisis in 2008, the government bailout of the giant insurance company AIG in order to prevent a failure cascade, and the exposure of European banks to potential defaults by some European countries. For this reason, much effort has been invested in understanding effects of systemic risk: how stresses, such as bankrupts and failures, to one part of the system can spread to others and lead to avalanche breakdown [1], [4]–[6].

An important line of research pursued in systemic risk theory focuses on development of realistic models of *clearing* procedures between financial institutions. Clearing allows to reduce the absolute liabilities by a full or partial reimbursement of credits in order to diminish eventual consequences of defaults [7]. The seminal work in [1] introduced a simple model of clearing in a financial network, in which financial institutions have two types of assets: the external assets (e.g., incoming cash flows) and the internal assets (e.g., some funds that banks lend to one another). The model from [1] assumes that the obligations of all entities (nodes) within the financial system are paid simultaneously and are determined by three fundamental rules: 1) limited liability, that is, the total payment of each node can not exceed its available cash flow; 2) the priority of the debt claims, that is, stockholders receive no value until the node is able to completely pay off all of its outstanding liabilities; 3) the proportionality, or pro-rata rule, that is, all debts have equal priority, so that all claimant institutions are paid proportionally to their nominal claims. Under these assumptions, the matrix of mutual interbank payments is determined by the so-called *clearing vector* that is found from a nonlinear equation. Using the Knaster-Tarski fixed-point theorem, it was shown in [1] that a clearing vector always exists. Furthermore, such vector is unique under certain regularity assumptions [1], [8].

The basic model offered in [1] has been later extended in various directions, incorporating non-trivial features of real-world financial networks. The models from [9]–[11], for instance, take into account cross-holdings, cross-ownership of equities and liabilities and seniorities of liabilities. The papers [12], [13] introduce measures of liquidity risk. Other works considered illiquid assets [14], decentralized clearing processes [15] and clearing with multiple maturity dates [16].

The contribution of the present work is twofold. First, we address the problem of the uniqueness of the clearing vector. The first sufficient graph-theoretical condition for its uniqueness was obtained in [1] (see also [7]): the clearing vector is unique if the financial network is *regular*, which means that every bank either has an outside asset, or has a (direct or indirect) creditor with outside assets. Another sufficient condition for uniqueness is formulated in [6]: the clearing vector is unique if each node of the network has a chain of liability to the external sector. Both conditions, even though they hold in the generic situation, are only sufficient yet not necessary. To the best of our knowledge, the only necessary and sufficient condition for the clearing vector's uniqueness applicable to an arbitrary financial network available in the literature is the very general result from [17], which examines the uniqueness of equilibria in a dynamical flow network with saturations. As it will be discussed below in Section IV, this criterion appears to be superfluous for the classical Eisenberg-Noe model, being primarily motivated by the more general models from [18]. In this paper, we give an alternative necessary and sufficient criterion, which is simpler for validation than the criterion in [17] (see Section IV-D). Similar to the algorithm from [17], our method in fact allows to find the polytopic set of all clearing vectors. We also derive some properties of the maximal (*dominant*) clearing vector.

The second aspect addressed in this paper is the relaxation

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of the proportionality (pro-rata) rule. Altough this rule seems fair at the "local" level of a node (all debts have equal priority) and allows in general to determine the matrix of payments uniquely, we show that its abolishment allows to reduce the total system loss, intended as the deviation between the nominal and actual payment vector, in some norm [6]. A natural question thus arises: what is the "price" of a pro-rata restriction in terms of the system loss? Although analytic comparison between the optimal proportional and non-proportional clearing policies is difficult, we provide a numerical experiment using a synthetic random network model inspired by a standard "testbench" network from [19]. One downside of relaxing the pro-rata rule is the loss of the clearing matrix uniqueness; furthermore, even the *optimal* clearing matrix can be non-unique [20, Example 2].

The paper is organized as follows. Section II defines the notation used in the paper. Section III introduces the Eisenberg-Noe model and related concepts. Section IV offers a necessary and sufficient condition for the clearing vector's uniqueness, in the situation where the pro-rata constraint is adopted. In Section V, we consider clearing matrices that do not satisfy the pro-rata rule and show that one such matrix can always be found by solving a convex optimization problem aimed at minimizing the system loss. Section VI presents numerical simulations to compare the system losses in the cases where the pro-rata rule is adopted and where it is discarded. Section VII concludes the paper. For space reasons, all proofs are omitted; they are available in [20].

II. Preliminaries and notation

Given a finite set $\mathcal{V}$, the symbol $|\mathcal{V}|$ stands for its cardinality. For two families of real numbers $(a_\xi)_{\xi\in\Xi}, (b_\xi)_{\xi\in\Xi}$, the symbol $a \leq b$ (b dominates a, or a is dominated by b) denotes the element-wise relation $a_\xi \leq b_\xi\,\forall\xi\in\Xi$. We write $a \lneq b$ if $a \leq b$ and $a \neq b$. The operations $\min, \max$ are also defined elementwise, e.g., $\min(a, b) \doteq (\min(a_\xi, b_\xi))_{\xi\in\Xi}$. These symbols apply to both vectors and matrices (usually, $\Xi = \{1, \ldots, n\}$ or $\Xi = \{1, \ldots, n\} \times \{1, \ldots, n\}$ respectively).

Every nonnegative square matrix $A = (a_{ij})_{i,j\in\mathcal{I}}$ corresponds to a weighted digraph $\mathcal{G}[A] = (\mathcal{I}, \mathcal{E}[A], A)$ whose nodes are indexed by $\mathcal{I}$ and whose set of arcs is defined as $\mathcal{E}[A] = \{(i, j) : a_{ij} > 0\}$. The value a_{ij} can be interpreted as a weight of arc (i, j). A sequence of arcs $i_0 \to i_1 \to \ldots \to i_{s-1} \to i_s$ constitute a *walk* between nodes i_0 and i_s. Set of nodes $J \subseteq \mathcal{I}$ is *reachable* from node i if $i \in J$ or a walk from i to some element $j \in J$ exists; J is *globally reachable* if it is reachable from every node.

A graph is *strongly connected* (strong) if every two nodes i, j are mutually reachable. Otherwise, the graph has several strongly connected components (for brevity, we call them simply *components*). A component is said *non-trivial* if it contains more than one node. A component is a *sink* component if no arc leaves it.

A nonnegative matrix $A \in \mathbb{R}^{\mathcal{I}\times\mathcal{I}}$ is *stochastic* if all its rows sum to 1: $\sum_{j\in\mathcal{I}} a_{ij} = 1\,\forall i \in \mathcal{I}$ (that is, $A\mathbf{1} = \mathbf{1}$) and *substochastic* if $\sum_{j\in\mathcal{I}} a_{ij} \leq 1\,\forall i \in \mathcal{I}$ (i.e, $A\mathbf{1} \leq \mathbf{1}$). Here $\mathbf{1} \in \mathbb{R}^{\mathcal{I}}$ denotes the column vector of ones.

III. Financial Networks

We henceforth use the notation introduced in [6], except for a few minor changes. A *financial network* may be represented as a weighted graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \bar{\mathcal{P}})$ whose nodes stand for financial institutions (banks, funds, insurance companies etc.) and whose weighted adjacency matrix $\bar{P} = (\bar{p}_{ij})$ represents the mutual liabilities of the institutions. Namely, entry $\bar{p}_{ij} \geq 0$ stands for the obligation of payment from node i to node j at the end of the current time period, and arc $(i, j) \in \mathcal{E}$ from node i to node j exists if and only if $\bar{p}_{ij} > 0$. By definition, $\bar{p}_{ii} = 0\,\forall i$, so the graph contains no self-arcs.

Along with mutual liabilities, the banks have *outside assets*. The outside *asset* $\bar{c}_i \geq 0$ is the total payment due from non-financial entities (the external sector) to node i; these numbers constitute vector $\bar{c} = (\bar{c}_i)_{i\in\mathcal{V}}$. Similarly, one can consider the outside liability of node i is the total payment $\bar{b}_i \geq 0$ from node i to the non-financial sector. Often the outside liabilities are replaced by payments to an additional "fictitious" node, representing the external sector [1]. Adding this "virtual" node to $\mathcal{V}$, we may assume that $\bar{b} = 0$.

The nominal in-flow and out-flow, referred to also as the *asset/liability* sides of the node i's balance sheet, are

$$\bar{\phi}_i^{\mathrm{in}} \doteq \bar{c}_i + \sum_{k\neq i} \bar{p}_{ki}, \quad \bar{p}_i \doteq \bar{\phi}_i^{\mathrm{out}} \doteq \sum_{k\neq i} \bar{p}_{ik}. \tag{1}$$

The nodes with $\bar{p}_i = 0$ have no outgoing arcs and, according to the graph-theoretical terminology, they are called *sinks*. As mentioned before, one such node can be fictiously defined for the purpose of collecting the debts to the external sector. In general, however, other sinks may exist in the network.

Normally, $\bar{\phi}_i^{\mathrm{in}} \geq \bar{\phi}_i^{\mathrm{out}}$, that is, each bank is able to pay its debts at the end of the current period. The main concern of systemic risk theory is the situation where some banks suffer financial shocks, and their outside assets drop to smaller values $c_i \in [0, \bar{c}_i)$. In this situation, it may happen that

$$c_i + \sum_{k\neq i} \bar{p}_{ki} < \bar{p}_i,$$

in which situation node i is unable to fully meet its payment obligations and hence *defaults*. When in default, a node pays out according to its capacity, thus reducing the amounts paid to the adjacent nodes, which in turn, for this reason, may also default and reduce their payments, etc. As a result of default, the *actual* payment $p_{ij} \in [0, \bar{p}_{ij}]$ from node i to node j, in general, may be less than the nominal due payment $\bar{p}_{ij}$. A natural question arises: which matrices of actual payments $P = (p_{ij}) \leq \bar{P}$ may be considered as "fair" in the case of default? We shall see that the pro-rata rule is a commonly accepted rule for allocating payments in the case of default, but we shall also explore an alternative approach that aims at minimizing the overall loss over the financial system.

Denote the vectors of actual in-flows and out-flows by

$$\phi^{\mathrm{in}} \doteq c + P^\top\mathbf{1}, \quad p \doteq \phi^{\mathrm{out}} \doteq P\mathbf{1}. \tag{2}$$

The minimal requirements to the matrix of actual payments $P = P(c)$ are as follows [1]:

i) **(limited liability)** The total payment of node i does not exceed the in-flow, that is, $\phi^{\text{in}} \geq \phi^{\text{out}}$;

ii) **(absolute priority of debt claims)** Either node i pays its obligations in full ($p_i = \bar{p}_i$), or it pays all its value to the creditors ($p_i = \phi_i^{\text{in}}$).

Recalling that $P \leq \bar{P}$ and thus $p = P\mathbf{1} \leq \bar{p}$, conditions i)-ii) may be reformulated in the following compact form

$$P\mathbf{1} = \min(\bar{p}, c + P^\top \mathbf{1}). \tag{3}$$

Definition 1: A matrix P is called a *clearing matrix* (or matrix of clearing payments) corresponding to the vector of outside assets c, if $0 \leq P \leq \bar{P}$ and (3) holds.

Notice that (3) is a system of $n \doteq |\mathcal{V}|$ nonlinear equations with n^2 variables p_{ij}. Hence, one cannot expect to find a unique solution. Often, the third requirement is introduced [1] known the *proportionality* or *pro-rata* rule, which expresses the requirement that all debts have equal priority and must be paid in proportion to the initial claims. As discussed in [15], [21] the proportionality rule is implemented in the bankruptcy law across the globe. Mathematically, this requirement reduces the number of unknowns to $n = |\mathcal{V}|$ and replacing the clearing matrix by the clearing *vector*. The pro-rata is also related to other mathematical properties of the clearing matrix, e.g. the invariance to "mitosis" [21] (splitting of a financial institution into multiple agents or merging of a group of agents into a single institution).

The clearing vector always exists [1], also, one such vectors can be found by solving a convex optimization problem with n variables. In Section IV, a necessary and sufficient criterion for such a vector's *uniqueness* is offered along with an algorithm that find the set of all possible clearing vectors, which is the first contribution of this work.

Also, an idea we propose in this work is the possibility of turning away from the pro-rata rule, and compute instead an optimal clearing matrix which guarantees the best possible system-level performance. Such clearing matrix can be computed by solving a convex optimization problem in n^2 variables, as discussed in Section IV. The underlying philosophy is that while the pro-rata rule is enforced in order to be fair locally for the individual nodes involved in the payment obligations, the optimal clearing matrix approach aims at being fair globally, by devising clearing payments that minimize the overall impact of defaults on the whole system. This second approach indeed outperforms the method of pro-rata clearing vector in terms of the overall system loss, as will be demonstrated in Section VI by numerical experiments.

IV. PRO-RATA RULE AND CLEARING VECTORS

One standard approach to determine the clearing payments is based on imposing additional restrictions on the payments p_{ij}, stating that the payments of node i to the claimants should be proportional to the nominal payments $\bar{p}_{ij}$. Introduce the *stochastic* matrix of relative liabilities

$$A = (a_{ij}), \quad a_{ij} = \begin{cases} \frac{\bar{p}_{ij}}{\bar{p}_i}, & \bar{p}_i > 0, \\ 1, & \bar{p}_i = 0 \wedge i = j, \\ 0, & \text{otherwise}, \end{cases} \tag{4}$$

the pro-rata (known also as the proportionality or the equal priority) rule can be formulated as

$$p_{ij} = p_i a_{ij}, \quad \forall i, j \in \mathcal{V}. \tag{5}$$

If this condition holds, then $P^\top \mathbf{1} = A^\top p$, which allows to rewrite (3) in the equivalent vector form

$$p = \min(\bar{p}, c + A^\top p). \tag{6}$$

Definition 2: Under the assumption (5), the vector p is said to be a *clearing* vector if $p \geq 0$ and (6) holds.

The existence of a clearing vector is usually proved by appealing to the general Knaster-Tarski fixed-point theorem [1], [7]. This theorem implies that the set of clearing vectors is non-empty and, furthermore, constitutes a complete lattice (with respect to the relation $\leq$), therefore, the *minimal* and *maximal* clearing vectors do exist. This monotonicity-based approach allows to prove the existence of clearing vectors in more complicated models [7], [22].

Below, we examine the properties of the maximal (or *dominant*) clearing vector, which can be found in several ways. One way for finding it is to solve a convex optimization (e.g., LP or QP) problem with n variables and $2n$ linear constraints (we use this approach in our numerical experiments). A modification of the fictitious default algorithm proposed in [8] finds the maximal clearing vector in no more than n steps, and at each step one has to solve a non-degenerate system of linear equations of the dimension $O(n)$.

In degenerate situations, the clearing vector may be non-unique. For instance, if $A = I_n$, $\bar{p} = 0$ and $c = 0$, every vector p such that $0 \leq p \leq \bar{p}$, obviously, satisfies (6). We will see that the existence of such a "closed" subgroup of banks that are unaffected by the remaining network and external sector is in fact the only reason for non-uniqueness (Theorem 2 below). Below we find necessary and sufficient conditions for the clearing vector's uniqueness and show that, even if this condition does not hold, some of the clearing vector's elements are uniquely determined by A and c.

A. The dominant clearing vector – Extremal properties

Although the existence of a maximal clearing vector is usually proved via the Knaster-Tarski fixed-point theorem, we consider an alternative construction, which also clarifies the geometrical meaning of this vector.

Consider the convex polyhedron

$$\mathcal{D} = [0, \bar{p}] \cap \{p : c + A^\top p \geq p\}. \tag{7}$$

Obviously, the set $\mathcal{D}$ is non-empty (containing, e.g., the null vector), being thus a convex hull spanned by several extreme points (or vertices). The following lemma shows that one of the extreme points is the maximal (with respect to $\leq$ relation) element of $\mathcal{D}$, and also the maximal clearing vector.

Definition 3: Function $F : [0, \bar{p}] \to \mathbb{R}$ is non-increasing (respectively, decreasing) if $F(p^1) \geq F(p^2)$ ($F(p^1) > F(p^2)$) whenever $p^1, p^2 \in [0, \bar{p}]$ and $p^1 \leq p^2$ ($p^1 \lneq p^2$).

Lemma 1: [20, Lemma 2] Convex polyhedron $\mathcal{D}$ is featured by the following properties:

1) a *maximal* point $p^* \in \mathcal{D}$ exists that dominates all other points $p^* \geq p \, \forall p \in \mathcal{D}$;
2) p^* is a global minimizer in the optimization problem

$$\min F(p) \quad \text{subject to} \quad p \in \mathcal{D} \tag{8}$$

whenever function $F : [0, \bar{p}] \to \mathbb{R}$ is non-increasing. If F is decreasing, then p^* is the *unique* minimizer in (8).
3) p^* is a clearing vector for the financial network.

The clearing vector p^* from Lemma 1 is henceforth referred to as the *dominant* clearing vector, because it dominates all elements of $\mathcal{D}$ (and, in particular all clearing vectors). Lemma 1 implies that one of the clearing vectors can be found by solving the convex QP problem (8) with $F(p) = \|\bar{\phi}^{\text{in}} - \phi^{\text{in}}\|_2^2$ or the LP problem (8) with $F(p) = \sum_{i=1}^n (\bar{\phi}_i^{\text{in}} - \phi_i^{\text{in}})$. It can be easily shown that both of these functions $\|\phi^{\text{in}} - \bar{\phi}^{\text{in}}\|_2^2$ and $\|\phi^{\text{in}} - \bar{\phi}^{\text{in}}\|_1 = \mathbf{1}^\top(\bar{\phi}^{\text{in}} - \phi^{\text{in}})$ are decreasing on $[0, \bar{p}]$, and hence, the unique minimizer $p = p^*$ exists. It should be noted that the optimization problems to find the dominant clearing vector admit various reformulations; some of them were used in [23] to study the sensitivity of the clearing vector to system's parameters.

The extremal property of the dominant clearing vector p^* allows to prove the following lemma.

Lemma 2: [20, Lemma 3] The element p_i^* of the dominant clearing vector is positive if and only if i is not a sink node ($\bar{p}_i > 0$) and one of the following conditions holds:

1) i has outside assets, that is, $c_i > 0$;
2) i is reachable from some node $j \neq i$ with $c_j > 0$;
3) the strongly connected component of graph $\mathcal{G}$ to which i belongs is a sink component (has no outcoming arcs).

B. Uniqueness of the clearing vector: a sufficient condition

In this subsection, we offer a *sufficient* condition ensuring that no clearing vectors other than the dominant clearing vector p^* exist. Furthermore, we show that *some* elements of the clearing vectors are always determined uniquely.

We start with introducing some auxiliary notation. Let $C^+ \doteq \{i : c_i > 0\}$ stand for the set of nodes that have outside assets and $S \doteq \{i : \bar{p}_i = 0\}$ stand for the set of sink nodes that owe no liability payments (and thus have no outcoming arcs in the graph). We introduce the set

$$I_0 \doteq C^+ \cup S = \{i : c_i > 0 \vee \bar{p}_i = 0\}. \tag{9}$$

The following lemma establishes a sufficient condition for uniqueness of the clearing vector.

Lemma 3: [20, Lemma 4] If set I_0 is *globally* reachable in the graph $\mathcal{G}$, then the dominant clearing vector p^* is the only clearing vector corresponding to the vector of outside assets c. More generally, let $I'_0 \supseteq I_0$ stand for the set of all nodes in the graph $\mathcal{G}$, from where I_0 can be reached. Then, for *every* clearing vector p we have $p_i = p_i^* \quad \forall i \in I'_0$.

Remark 1: Since each (simple) path in the graph ends in one of the *sink* components, it can be easily proved that the condition from Lemma 3 admits the following equivalent reformulation: each sink component of graph $\mathcal{G}$ is either trivial (contains the only node) or has node i such that $c_i > 0$.

C. Uniqueness of the clearing vector: the general case

In this subsection, we derive two necessary and sufficient criteria of the clearing vector's uniqueness. The first of them (Theorem 1) assumes that the dominant clearing vector p^* is known. This result and the algorithm on which it relies allow to describe the whole set of admissible clearing vectors. If, however, one is interested only in the uniqueness of a clearing vector, a simpler graph-theoretic criterion can be used (Theorem 2) that does not require knowledge of p^*.

Assume that the condition from Lemma 3 does not hold, that is, $I'_0 \neq \mathcal{V}$. The banks corresponding to nodes from $\mathcal{V}_1 \doteq \mathcal{V} \setminus I'_0$ neither have outside assets ($\mathcal{V}_1 \cap C^+ = \emptyset$) nor pay to nodes from I'_0 (otherwise, they would also belong to I'_0). Hence, matrix $A^1 \doteq (a_{ij})_{i,j \in \mathcal{V}_1}$ is stochastic.

At the same time, nodes from I'_0 *can* have liability payments to nodes from $\mathcal{V}_1$, which payments depend only on the dominant vector p^* and constitute the vector

$$c^{(1)} \doteq (c_i^{(1)})_{i \in \mathcal{V}_1}, \;\; c_i^{(1)} = c_i^{(1)}(p^*) \doteq \sum\nolimits_{k \in I'_0} a_{ki} p_k^*, \;\; i \in \mathcal{V}_1.$$

We can now apply Lemma 3 to a reduced financial network $\mathcal{G}_1$ with the node set $\mathcal{V}_1$, the normalized payment matrix A_1 and the vector of external assets $c^{(1)}$. Introducing the set[1]

$$I_1 = \{i \in \mathcal{V}_1 : c_i^{(1)} > 0\}$$

and denoting $I'_1 \supseteq I_1$ all nodes from which set I_1 is reachable (banks that are connected by chains of liability to nodes from I_1), Lemma 3 ensures that the elements of the reduced network's clearing vector $p_i, i \in I'_1$, are determined *uniquely*. The definition (6) entails that for if p is a clearing vector for the original network, then its subvector $p^1 = (p_i)_{i \in \mathcal{V}_1}$ is a clearing vector for the reduced network $\mathcal{G}_1$. This also applies to p^*. Lemma 3 entails now that for each clearing vector p (in the original network) one has $p_i = p_i^* \, \forall i \in I'_1$.

If $I'_0 \cup I'_1 = \mathcal{V}$, we have uniqueness of the clearing vector. Otherwise, we have a group of banks $\mathcal{V}_2 = \mathcal{V} \setminus (I'_0 \cup I'_1)$ that are not in debt to any node from $I'_1 \cup I'_0$, however, can receive liability payments from I'_1. For the group $\mathcal{V}_2$, these payments may be considered as outside assets. Denote

$$c^{(2)} \doteq (c_i^{(2)})_{i \in \mathcal{V}_2}, \quad c_i^{(2)} \doteq \sum\nolimits_{k \in I_1} a_{ki} p_i^*.$$

If the set $I_2 = \{i \in \mathcal{V}_2 : c_i^{(2)} > 0\}$ is non-empty, one can consider the set $I'_2 \supseteq I_2$ of all nodes from where I_2 can be reached. Lemma 1 implies that the elements $p_i, \, , i \in I'_2$ of the clearing vector are uniquely determined: $p_i = p_i^* \, \forall i \in I'_2$.

We arrive at the following iterative procedure, which allows to test the clearing vector's uniqueness (and, in fact, even to find the whole state of clearing vectors).

Example 1. Algorithm 1 is illustrated by Fig. 1, which displays a network with $n = 15$ nodes that contains only one sink node ($S = \{0\}$) and three nodes with outside assets ($C^+ = \{1, 2, 3\}$), which four nodes constitute the set I_0. Lemma 2 entails that in this situation $p_i^* > 0 \, \forall i \neq 0, 9$ (notice that nodes 13-15 constitute a sink component,

[1] Notice that unlike I_0, the set I_1 contains no sink nodes. By construction, all sink nodes of the graph $\mathcal{G}$ belong to I_0.

Algorithm 1 The clearing vector's uniqueness test.

Initialization. Compute the dominant clearing vector p^*. Set $q \leftarrow 0$, $I_0 \leftarrow C^+ \cup S = \{i : c_i > 0 \vee \bar{p}_i = 0\}$. Find the set $I_0' \supseteq I_0$ of all nodes, from which I_0 is reachable.
repeat
1) $q \leftarrow q+1$;
2) $\mathcal{V}_q \leftarrow \mathcal{V} \setminus (I_0' \cup I_1 \ldots \cup I_{q-1}')$;
3) compute the vector of payments from I_{q-1}' to $\mathcal{V}_q$

$$c^{(q)} = (c_i^{(q)})_{i \in \mathcal{V}_q}, \quad c_i^{(q)} \doteq \sum\nolimits_{k \in I_{q-1}'} a_{ki} p_k^* \quad \forall i \in \mathcal{V}_q;$$

4) find the set $I_q = \{i \in \mathcal{V}_q : c_i^{(q)} > 0\}$;
5) find the set $I_q' \supseteq I_q$ of nodes from $\mathcal{V}_q$, from where I_q can be reached in $\mathcal{G}$.
until $\mathcal{V}_q = \emptyset$ or $c^{(q)} = 0$.

satisfying thus condition 3) from Lemma 2). The set I_0' contains I_0 and two nodes $4, 5$ that owe liability payments to nodes 0 and 2. The set I_1 contains nodes that have no liability payments to I_0', however, receive the liability payments from 4 and 5. Hence, $c_6^{(1)}, c_7^{(1)}, c_8^{(1)} > 0$. The nodes $6, 7, 8$ constitute the set I_1; the set I_1' is obtained by adding node 9 paying liability to one of them. On the next iteration of the algorithm, one computes the sets $I_2 = \{10, 11\}$ and $I_2' = \{12\} \cup I_2$. The nodes of the graph are not exhausted, however, the next vector $c^{(3)}$ will be zero, because the remaining nodes of the graph constitute an isolated group. Hence, the clearing vector is not unique, however, for each clearing vector p one has $p_i = p_i^*$, $\forall i = 0, \ldots, 12$.

Theorem 1: [20, Theorem 1] Algorithm 1 stops in a finite number of steps $s \geq 0$. The elements of a clearing vector, corresponding to indices $i \in I_0' \cup I_1' \cup \ldots \cup I_s'$, are uniquely determined: $p_i = p_i^*$. The clearing vector is unique if and only if $\mathcal{V}_s = \emptyset$, otherwise, there are infinitely many clearing vectors. Precisely, p is a clearing vector if and only if

$$p_i = \begin{cases} p_i^*, & i \in I_0' \cup I_1 \ldots \cup I_s', \\ \xi_i, & i \in \mathcal{V}_s, \end{cases} \tag{10}$$

where $\xi \in \mathbb{R}^{\mathcal{V}_s}$ can be any vector satisfying the constraints

$$B^\top \xi = \xi, \quad 0 \leq \xi_i \leq \bar{p}_i \, \forall i \in \mathcal{V}_s, \quad B \doteq (a_{ij})_{i,j \in \mathcal{V}_s}. \tag{11}$$

Notice that although the subvector ξ in (10) is defined non-uniquely, some its elements are in fact uniquely determined due to Lemma 2. As we know, $\xi_i = p_i^* = 0$ whenever i does not belong to a sink component and is not reachable from C^+. Combining Theorem 1 with the result of Lemma 2, we can establish an alternative uniqueness criterion, which *does not* require knowledge of the vector p^*.

Theorem 2: [20, Theorem 2] The following two conditions are equivalent:

(i) the clearing vector is unique (and equals p^*);
(ii) each non-trivial sink component of $\mathcal{G}$ either has a node from C^+ or is reachable from C^+.

D. Theorems 1 and 2 vs. previously known results

In this subsection, we briefly summarize differences between our results and previously known ones.

1) Criteria from [1] and [6]: The uniqueness criterion from [1] states that the clearing vector is unique if the set C^+ is globally reachable, being thus a special case of Lemma 3. The direct proof of this criterion (found in [1] and simplified in [7]) can be extended, with some variations, to more general models [7], [18]. The criterion from [6] can be considered as another special case of Lemma 3. It guarantees uniqueness in the situation where all nodes are connected to the external sector by chains of liability. Adding a fictitious sink node standing for the external sector, this node is thus reachable from all other nodes (that is, S is globally reachable). It should be noted that the works [1], [6], [7] did not present parameterizations of all clearing vectors.

2) An extension of the Eisenberg-Noe model and the uniqueness criterion from [17]: The original Eisenberg-Noe model assumes that, in case of default, the available assets of the bank are distributed pro rata between *all* creditors, including the external ones. At the same time, one can suppose that some external payments (e.g., operational costs) cannot be reduced in spite of the dropping outside assets. In this situation, vector $c \geq 0$ is replaced by vector $e \in \mathbb{R}^{\mathcal{V}}$ whose elements may be negative. Such an extension leads to a modified definition [18] of the clearing vector

$$p = \min(\bar{p}, \max(A^\top p + e, 0)). \tag{12}$$

Obviously, in the case where $e = c \geq 0$, the definition of the clearing vector (12) is equivalent to (6). However, the criteria developed for studying uniqueness of the generalized clearing vectors (12) appear to be superfluous and inconvenient in the classical model with $e \geq 0$. The most general of such criteria, proposed in [17], requires to know not only the irreducible decomposition of the matrix A (equivalently, the structure of the network's strongly connected components), but also to find the left and right Perron-Frobenius eigenvector for each irreducible block. Theorems 1 and 2 (restricted to the case $e \geq 0$) do not require such an information. The result from [17] extends the previously known result from [24] that was confined to *strongly connected* financial networks.

V. RELAXING THE PRO-RATA CONDITION

We next examine the case of clearing payments without the pro-rata rule. As soon as the pro-rata constraint is removed, the clearing matrix is no longer unique. One matrix corresponds to the pro-rata rule (as we have seen, this matrix always exists and, under some natural assumption, is unique), however there are other clearing matrices that can be found, e.g., by solving optimization problems similar to (8).

There are several reasons to consider clearing matrices different from the pro-rata matrix. One reason is that, as discussed in [6], [15], the creditors of different standings may have different priorities, and the principles of proportionality and priority are equally important. Another reason is that the relaxation of the pro-rata constraint can visibly reduce the systemic loss as will be shown in the next section.

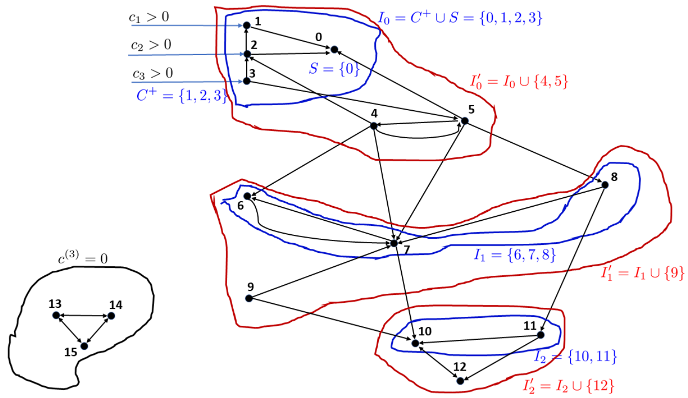

Fig. 1. The sets I_s (encircled by blue lines) and I_s' (encircled by red lines) for a special financial network with $n = 15$ nodes with $p_i^* > 0\,\forall i \neq 0$.

Recalling that $[0, \bar{P}] = \{P \in \mathbb{R}^{n\times n} : 0 \le p_{ij} \le \bar{p}_{ij}\,\forall i, j\}$, we consider the convex polyhedron in the space of matrices

$$\mathcal{D}_{n\times n} = [0, \bar{P}] \cap \{P \in \mathbb{R}^{n\times n} : c + P^\top \mathbf{1} \ge P\mathbf{1}\}.$$

We call a function $F : [0, \bar{P}] \to \mathbb{R}$ *decreasing* if $F(P^1) > F(P^2)$ whenever $P^1 \lneq P^2$. For any such function, consider the optimization problem

$$\min F(P) \quad \text{subject to} \quad P \in \mathcal{D}_{n\times n}. \tag{13}$$

Lemma 4: [20, Lemma 5] For a decreasing function $F : [0, \bar{P}] \to \mathbb{R}$, each local minimizer in (13) is a clearing matrix.

Notice that Lemma 4 does not require the function to be continuous. For a continuous function, the global minimum always exists due to compactness of $\mathcal{D}_{n\times n}$. Two examples of functions continuous and decreasing on $[0, \bar{P}]$ are

$$\|\bar{\phi}^{\text{in}} - \phi^{\text{in}}\|_2^2 = \|(\bar{c} - c) + (\bar{P} - P)^\top \mathbf{1}\|_2^2, \tag{14}$$

$$\sum_{i=1}^{n} (\bar{\phi}_i^{\text{in}} - \phi_i^{\text{in}}) = \mathbf{1}^\top (\bar{c} - c) + \mathbf{1}^\top (\bar{P} - P)^\top \mathbf{1} \tag{15}$$

Both these functions provide a global measure of the impact of individual defaults on financial network as a whole.

As it follows from Lemma 1, the set of matrices obeying *pro-rata* constraint 5 always has a maximal element, corresponding to the dominant clearing vector p^*. Also, this set is closed with respect to the operation $\max$, because the maximum of two clearing vectors, as can be easily seen, is also a clearing vector. In some situations, clearing matrices automatically satisfy the pro-rata rule (for instance, if each node of the graph has only one outgoing arc). However, in general the set of all clearing matrices has a non-trivial structure and is not a complete lattice [20, Example 2].

VI. Numerical Experiments

Imposing a pro-rata rule on the payments has a systemic impact on the network, which we can evaluate by comparing it with the minimum level achievable by an optimal clearing matrix. Indeed, using a global loss function such as, e.g., function (14) or (15), we can compute the minimal system loss when the pro-rata rule is respected ($p_{ij} = a_{ij} p_i^*$, where p^* is the optimal clearing vector) and when the pro-rata rule is relaxed to (3). As a testbench for the numerical experiments, we used synthetic random networks similar to ones proposed in [19] as described in the Subsect. VI-A.

A. A model of random network

The random graphs used for simulations are constructed using a technique inspired by [19]. The topology of the graph is given by the standard Erdös-Renyi $G(n, p)$ graph. The interbank liabilities $\bar{p}_{ij}$ for every edge (i, j) of the random graph are then found by sampling from a uniform distribution $\bar{p}_{ij} \sim \mathcal{U}(0, P_{max})$, where P_{max} is the maximum possible value of a single interbank payment. In the experiments we set $P_{max} = 100$. Unlike [19], the values $\bar{p}_{ij}$ can thus be heterogeneous. Also, we do not consider payments to external sector (deposits etc.): as it has been discussed, we can always get rid of them by introducing a fictitious node.

Following [19], we define the total amount of the external assets $E = \frac{\beta}{1-\beta} I$, where $I = \sum_{i,j=1}^{n} \bar{p}_{ij}$ is the total amount of the interbank liabilities and $\beta = E/(E+I)$ is a parameter representing the percentage of external assets in total assets at the system level; in our experiments $\beta = 0.05$. The nominal asset vector $\bar{c}$ is then computed in two steps: 1) each bank is given the minimal value of external assets under which its balance sheet equals zero; 2) the remainder of the aggregated external assets is evenly distributed among all banks.

The financial shock is modeled by randomly choosing one bank of the system and nullifying its external financial assets.

B. The price of pro-rata rule: a numerical study

To evaluate the "price" of imposing the pro-rata rule, we consider the affine function of *system loss* proposed in [6], that is $l \doteq \sum_i (\bar{p}_i - p_i)$, where $p_i = \sum_{j=1}^{n} p_{ij}$ (this function is equivalent to (15)) . Obviously, this function is strictly monotone. Hence, its minimal value over all matrices obeying the pro-rata constraint (5) is $l_{pr} = \sum_i (\bar{p}_i - p_i^*)$, where p^* is the dominating clearing vector from Lemma 1, which is found by solving problem (8) with $F(p) = \|\bar{\phi}^{\text{in}} - \phi^{\text{in}}\|_2^2$. Relaxing the pro-rata constraint, we find the globally optimal clearing matrix P^*, resulting in the system loss $l_{nopr} = \mathbf{1}^\top (\bar{P} - P^*)\mathbf{1}$. The price, or global effect, of the pro-rata rule is estimated by

$$G = \frac{l_{pr} - l_{nopr}}{l_{pr}} \in [0, 1].$$

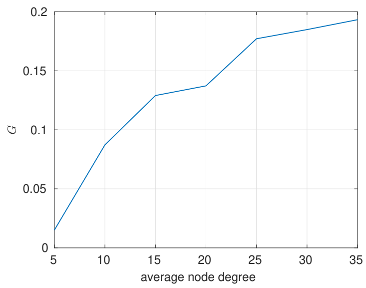

Fig. 2. Gain obtained by relaxing the pro-rata rule.

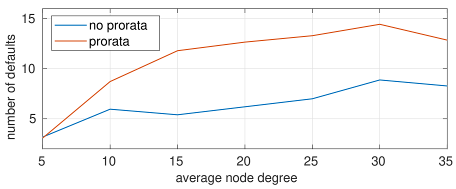

Fig. 3. Number of defaulted nodes with or without the pro-rata rule.

If $G = 0$ (as e.g. in Example 2 from [20]), where all clearing matrices are optimal), the imposition of pro-rata constraint is "gratuitous" in the sense that it does not increase the aggregate system loss: $l_{pr} = l_{nopr}$. The larger value G we obtain, the more "costful" is the pro-rata restriction.

It seems natural that G is growing as the graph is becoming more dense, since in this situation the pro-rata rule visibly reduces the number of free variables in the optimization problem. We have tested this conjecture using the random model described above. The random graph contained $n = 50$ nodes, whereas the average node degree $d = np$ varied from 0 to 35. The resulting gain G was averaged over 50 runs. The resulting dependence between G and d is shown in Fig. 2. The relaxation of the pro-rata rule can give gain up to 19%.

To evaluate the price of the pro-rata rule, we also considered a metric which measures the number of defaulted nodes. This metric evaluates the dimension of the failure cascade caused by the initial shock. Figure 3 compares the number of defaulted nodes with or without the pro-rata rule. We can observe that relaxing the pro-rata rule can significantly reduce the failure cascade.

VII. Conclusions

Based on the financial networks model of [1], we explored in this paper the concept of a clearing vector of payments, and we developed new necessary and sufficient conditions for its uniqueness, together with a characterization of the set of all clearing vectors, see Theorem 1 and Theorem 2. Further, we examined matrices of clearing payments that naturally arise if one relaxes the pro-rata rule. Optimal clearing matrices can be computed efficiently by solving a convex optimization problem. Using numerical experiments with randomly generated synthetic networks, we showed that relaxation of the pro-rata rule allows to reduce significantly the overall systemic loss and the number of defaulted nodes.

Many aspects remain to be explored. First and foremost, the biggest gap from theory to practice is that, in practice, the overall structure of the financial network is not known precisely, let alone the inter-bank liability amounts. For effective practical implementation, therefore, one should develop a system-level (i.e., *global*) approach whose iterations are based only on *local* exchange of information between nodes. The development of such distributed and decentralized optimal clearing payment algorithm is the subject of ongoing research. Another interesting aspect arises if one allows the liabilities to be spread over some interval of time, rather than cleared instantly, as it is supposed in the mainstream model of [1]. This would lead to *dynamic* clearing payments, and to the ensuing optimal dynamic optimization problems.

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