Lp spectral multipliers on the free group N3,2

L^p SPECTRAL MULTIPLIERS ON THE FREE GROUP $N_{3,2}$

ALESSIO MARTINI AND DETLEF MÜLLER

Abstract. Let L be a homogeneous sublaplacian on the 6-dimensional free 2-step nilpotent Lie group $N_{3,2}$ on 3 generators. We prove a theorem of Mihlin-Hörmander type for the functional calculus of L, where the order of differentiability $s > 6/2$ is required on the multiplier.

1. Introduction

The free 2-step nilpotent Lie group $N_{3,2}$ on 3 generators is the simply connected, connected nilpotent Lie group defined by the relations

$$[X_1, X_2] = Y_3, \quad [X_2, X_3] = Y_1, \quad [X_3, X_1] = Y_2,$$

where $X_1, X_2, X_3, Y_1, Y_2, Y_3$ is a basis of its Lie algebra (that is, the Lie algebra of the left-invariant vector fields on $N_{3,2}$). In exponential coordinates, $N_{3,2}$ can be identified with $\mathbb{R}^3_x \times \mathbb{R}^3_y$, where the group law is given by

$$(x, y) \cdot (x', y') = (x + x', y + y' + x \wedge x'/2)$$

and $x \wedge x'$ denotes the usual vector product of $x, x' \in \mathbb{R}^3$. The family $(\delta_t)_{t>0}$ of automorphic dilations of $N_{3,2}$, defined by

$$\delta_t(x, y) = (tx, t^2 y),$$

turns $N_{3,2}$ into a stratified group of homogeneous dimension $Q = 9$.

Let L be a homogeneous sublaplacian on $N_{3,2}$; without loss of generality, we may assume that $L = -(X_1^2 + X_2^2 + X_3^2)$. L is a self-adjoint operator on $L^2(N_{3,2})$, hence a functional calculus for L is defined via spectral integration and, for all Borel functions $F : \mathbb{R} \to \mathbb{C}$, the operator $F(L)$ is bounded on $L^2(N_{3,2})$ whenever the "spectral multiplier" F is a bounded function. Here we are interested in giving a sufficient condition for the L^p-boundedness (for $p \neq 2$) of the operator $F(L)$, in terms of smoothness properties of the multiplier F.

Let $W_2^s(\mathbb{R})$ denote the L^2 Sobolev space of (fractional) order s. Then our main result reads as follows.

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Theorem 1.1. *Suppose that a function* $F : \mathbb{R} \to \mathbb{C}$ *satisfies*

$$\sup_{t>0} \|\eta\, F(t\cdot)\|_{W_2^s} < \infty$$

for some $s > 6/2$ *and some nonzero* $\eta \in C_c^\infty(]0,\infty[)$. *Then the operator* $F(L)$ *is of weak type* $(1,1)$ *and bounded on* $L^p(N_{3,2})$ *for all* $p \in]1,\infty[$.

Observe that the general multiplier theorem for homogeneous sublaplacians on stratified Lie groups by Christ [3] and Mauceri and Meda [16] requires the stronger regularity condition $s > Q/2 = 9/2$. To the best of our knowledge, in the case of $N_{3,2}$ none of the results and techniques known so far allowed one to go below the condition $s > Q/2$. Our result pushes the regularity assumption down to $s > d/2 = 6/2$, where $d = 6$ is the topological dimension of $N_{3,2}$. We conjecture that this condition is sharp.

The problem of L^p-boundedness for spectral multipliers on nilpotent Lie groups has a long history, and the theorem by Christ and Mauceri and Meda is itself an improvement of a series of previous results (see, e.g., [4, 8, 5]). Nevertheless it is still an open question, whether the homogeneous dimension in the smoothness condition may always be replaced by the topological dimension.

It has been known for a long time [10, 17] that such an improvement of the multiplier theorem holds true in the case of the Heisenberg and related groups (more precisely, for direct products of Métivier and abelian groups; see also [11, 14]). This class of groups, however, does not include $N_{3,2}$, nor any free 2-step nilpotent group $N_{n,2}$ on n generators (see [20, §3] for a definition), except for the smallest one, $N_{2,2}$, which is the 3-dimensional Heisenberg group. The free groups $N_{n,2}$ have in a sense the maximal structural complexity among 2-step groups, since every 2-step nilpotent Lie group is a quotient of a free one. Our result should then hopefully shed some new light and contribute to the understanding of the problem for general 2-step nilpotent Lie groups.

2. Strategy of the proof

The sublaplacian L is a left-invariant operator on $N_{3,2}$, hence any operator of the form $F(L)$ is left-invariant too. Let $\mathcal{K}_{F(L)}$ then denote the convolution kernel of $F(L)$. As shown, e.g., in [14, Theorem 4.6], the previous Theorem 1.1 is a consequence of the following L^1-estimate.

Proposition 2.1. *For all* $s > 6/2$, *for all compact sets* $K \subseteq]0,\infty[$, *and for all functions* $F : \mathbb{R} \to \mathbb{C}$ *such that* $\operatorname{supp} F \subseteq K$,

$$\| \mathcal{K}_{F(L)} \|_1 \leq C_{K,s} \|F\|_{W_2^s}. \tag{2.1}$$

Let $|\cdot|_\delta$ be any δ_t-homogeneous norm on $N_{3,2}$; take, e.g., $|(x,y)|_\delta = |x| + |y|^{1/2}$. The crucial estimate in the proof of [16] of the general theorem for stratified groups, that is,

$$\|(1+|\cdot|_\delta)^\alpha \mathcal{K}_{F(L)}\|_2 \leq C_{K,\alpha,\beta} \|F\|_{W_2^\beta} \tag{2.2}$$

for all $\alpha \geq 0$ and $\beta > \alpha$, implies (2.1) when $s > 9/2$, by Hölder's inequality. In order to push the condition down to $s > 6/2$, here we prove an enhanced version of (2.2), that is,

$$\|(1+|\cdot|_\delta)^\alpha w^r \mathcal{K}_{F(L)}\|_2 \leq C_{K,\alpha,\beta,r} \|F\|_{W_2^\beta}, \tag{2.3}$$

for some "extra weight" function w on $N_{3,2}$, and suitable constraints on the exponents α, β, r.

A similar approach is adopted in the mentioned works on the Heisenberg and related groups. However, in [17] the extra weight w is the full weight $1 + |\cdot|_\delta$, while [10] employs the weight $w(x,y) = 1 + |x|$. Here instead the weight $w(x,y) = 1 + |y|$ is used, and (2.3) is proved under the conditions $\alpha \geq 0$, $0 \leq r < 3/2$, $\beta > \alpha + r$ (see Proposition 4.6 below).

The proof of (2.3) when $\alpha = 0$ is based on a careful analysis exploiting identities for Laguerre polynomials, somehow in the spirit of [4, 17, 19], but with additional complexity due, inter alia, to the simultanous use of generalized Laguerre polynomials of different types. The estimate for arbitrary α is then recovered by interpolation with (2.2). An analogous strategy is followed in [15], where identities for Hermite polynomials are used in order to prove a sharp spectral multiplier theorem for Grushin operators.

3. A joint functional calculus

It is convenient for us to embed the functional calculus for the sublaplacian L in a larger functional calculus for a system of commuting left-invariant differential operators on $N_{3,2}$. Specifically, the operators

$$L, -iY_1, -iY_2, -iY_3 \tag{3.1}$$

are essentially self-adjoint and commute strongly, hence they admit a joint functional calculus (see, e.g., [13]).

If $\mathbf{Y}$ denotes the "vector of operators" $(-iY_1, -iY_2, -iY_3)$, then we can express the convolution kernel $\mathcal{K}_{G(L,\mathbf{Y})}$ of the operator $G(L,\mathbf{Y})$ in terms of Laguerre functions (cf. [7]). Namely, for all $n, k \in \mathbb{N}$, let

$$L_n^{(k)}(u) = \frac{u^{-k} e^u}{n!} \left(\frac{d}{du}\right)^n (u^{k+n} e^{-u})$$

be the n-th Laguerre polynomial of type k, and define

$$\mathcal{L}_n^{(k)}(t) = (-1)^n e^{-t} L_n^{(k)}(2t).$$

Further, for all $\eta \in \mathbb{R}^3 \setminus \{0\}$ and $\xi \in \mathbb{R}^3$, define $\xi^\eta_\parallel$ and $\xi^\eta_\perp$ by

$$\xi^\eta_\parallel = \langle \xi, \eta/|\eta| \rangle, \qquad \xi^\eta_\perp = \xi - \xi^\eta_\parallel \eta/|\eta|.$$

Proposition 3.1. *Let $G : \mathbb{R}^4 \to \mathbb{C}$ be in the Schwartz class, and set*

$$m(n, \mu, \eta) = G((2n+1)|\eta| + \mu^2, \eta), \tag{3.2}$$

for all $n \in \mathbb{N}$, $\mu \in \mathbb{R}$, $\eta \in \mathbb{R}^3$ with $\eta \neq 0$. Then

$$\mathcal{K}_{G(L,\mathbf{Y})}(x, y) = \frac{2}{(2\pi)^6} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \sum_{n \in \mathbb{N}} m(n, \xi^\eta_\parallel, \eta)\, \mathcal{L}_n^{(0)}(|\xi^\eta_\perp|^2/|\eta|)\, e^{i\langle \xi, x \rangle}\, e^{i\langle \eta, y \rangle}\, d\xi\, d\eta.$$

Proof. For all $\eta \in \mathbb{R}^3 \setminus \{0\}$, choose a unit vector $E_\eta \in \eta^\perp$, and set $\bar{E}_\eta = (\eta/|\eta|) \wedge E_\eta$; moreover, for all $x \in \mathbb{R}^3$, denote by x_1^η, x_2^η, $x_\parallel^\eta$ the components of x with respect to the positive orthonormal basis E^η, $\bar{E}^\eta$, $\eta/|\eta|$ of $\mathbb{R}^3$.

For all $\eta \in \mathbb{R}^3 \setminus \{0\}$ and all $\mu \in \mathbb{R}$, an irreducible unitary representation $\pi_{\eta,\mu}$ of $N_{3,2}$ on $L^2(\mathbb{R})$ is defined by

$$\pi_{\eta,\mu}(x, y)\phi(u) = e^{i\langle \eta, y \rangle} e^{i|\eta|(u + x_1^\eta/2)x_2^\eta} e^{i\mu x_\parallel^\eta} \phi(x_1^\eta + u)$$

for all $(x, y) \in N_{3,2}$, $u \in \mathbb{R}$, $\phi \in L^2(\mathbb{R})$. Following, e.g., [1, §2], one can see that these representations are sufficient to write the Plancherel formula for the group Fourier transform of $N_{3,2}$, and the corresponding Fourier inversion formula:

$$f(x, y) = (2\pi)^{-5} \int_{\mathbb{R}^3 \setminus \{0\}} \int_{\mathbb{R}} \operatorname{tr}(\pi_{\eta,\mu}(x, y)\, \pi_{\eta,\mu}(f))\, |\eta|\, d\mu\, d\eta \tag{3.3}$$

for all $f : N_{3,2} \to \mathbb{C}$ in the Schwartz class and all $(x, y) \in N_{3,2}$, where $\pi_{\eta,\mu}(f) = \int_{N_{3,2}} f(z)\, \pi_{\eta,\mu}(z^{-1})\, dz$.

Fix $\eta \in \mathbb{R}^3 \setminus \{0\}$ and $\mu \in \mathbb{R}$. The operators (3.1) are represented in $\pi_{\eta,\mu}$ as

$$d\pi_{\eta,\mu}(L) = -\partial_u^2 + |\eta|^2 u^2 + \mu^2, \qquad d\pi_{\eta,\mu}(-iY_j) = \eta_j. \tag{3.4}$$

If h_n is the n-th Hermite function, that is,

$$h_n(t) = (-1)^n (n!\, 2^n \sqrt{\pi})^{-1/2} e^{t^2/2} \left(\frac{d}{dt} \right)^n e^{-t^2},$$

and $\tilde{h}_{\eta,n}$ is defined by

$$\tilde{h}_{\eta,n}(u) = |\eta|^{1/4} h_n(|\eta|^{1/2} u),$$

then $\{\tilde h_{\eta,n}\}_{n\in\mathbb{N}}$ is a complete orthonormal system for $L^2(\mathbb{R})$, made of joint eigenfunctions of the operators (3.4); in fact,

$$\begin{aligned} d\pi_{\eta,\mu}(L)\tilde h_{\eta,n} &= (|\eta|(2n+1)+\mu^2)\tilde h_{\eta,n}, \\ d\pi_{\eta,\mu}(-iY_j)\tilde h_{\eta,n} &= \eta_j \tilde h_{\eta,n}. \end{aligned} \tag{3.5}$$

Moreover the corresponding diagonal matrix coefficients $\varphi_{\eta,\mu,n}$ of $\pi_{\eta,\mu}$ are given by

$$\begin{aligned} &\varphi_{\eta,\mu,n}(x,y) = \langle \pi_{\eta,\mu}(x,y)\tilde h_{\eta,n}, \tilde h_{\eta,n}\rangle \\ &= e^{i\langle\eta,y\rangle} e^{i\mu x_\parallel^\eta} |\eta|^{1/2} \int_{\mathbb{R}} e^{i|\eta| u x_2^\eta}\, h_n(|\eta|^{1/2}(u+x_1^\eta/2))\, h_n(|\eta|^{1/2}(u-x_1^\eta/2))\, du. \end{aligned}$$

The last integral is essentially the Fourier-Wigner transform of the pair (h_n, h_n), whose Fourier transform has a particularly simple expression (cf. [9, formula (1.90)]); the parity of the Hermite functions then yields

$$\begin{aligned} \varphi_{\eta,\mu,n}(x,y) = e^{i\langle\eta,y\rangle} e^{i\mu x_\parallel^\eta} \frac{(-1)^n}{\pi|\eta|} \int_{\mathbb{R}^2} e^{iv_2 x_2^\eta} e^{iv_1 x_1^\eta} \\ \times \int_{\mathbb{R}} e^{-it(2v_1/|\eta|^{1/2})}\, h_n(t+v_2/|\eta|^{1/2})\, h_n(t-v_2/|\eta|^{1/2})\, dt\, dv, \end{aligned}$$

that is,

$$\varphi_{\eta,\mu,n}(x,y) = \frac{1}{\pi|\eta|} e^{i\langle\eta,y\rangle} e^{i\mu x_\parallel^\eta} \int_{\mathbb{R}^2} e^{iv_1 x_1^\eta} e^{iv_2 x_2^\eta} \mathcal{L}_n^{(0)}(|v|^2/|\eta|)\, dv \tag{3.6}$$

(see [21, Theorem 1.3.4] or [9, Theorem 1.104]).

Note that $\mathcal{K}_{G(L,\mathbf{Y})} \in \mathcal{S}(N_{3,2})$ since $G \in \mathcal{S}(\mathbb{R}^4)$ (see [2, Theorem 5.2] or [12, §4.2]). Moreover

$$\pi_{\eta,\mu}(\mathcal{K}_{G(L,\mathbf{Y})})\tilde h_{\eta,n} = G(|\eta|(2n+1)+\mu^2, \eta)\tilde h_{\eta,n}$$

by (3.5) and [18, Proposition 1.1], hence

$$\langle \pi_{\eta,\mu}(x,y)\pi_{\eta,\mu}(\mathcal{K}_{G(L,\mathbf{Y})})\tilde h_{\eta,n}, \tilde h_{\eta,n}\rangle = m(n,\mu,\eta)\, \varphi_{\eta,\mu,n}(x,y).$$

Therefore, by (3.3) and (3.6),

$$\begin{aligned} &\mathcal{K}_{G(L,\mathbf{Y})}(x,y) \\ &= (2\pi)^{-5} \int_{\mathbb{R}^3\setminus\{0\}} \int_{\mathbb{R}} \sum_{n\in\mathbb{N}} m(n,\mu,\eta)\, \varphi_{\eta,\mu,n}(x,y)\, |\eta|\, d\mu\, d\eta \\ &= \frac{2}{(2\pi)^6} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \sum_{n\in\mathbb{N}} m(n,\xi_3,\eta)\, e^{i\langle\eta,y\rangle} e^{i\langle\xi,(x_1^\eta,x_2^\eta,x_\parallel^\eta)\rangle} \mathcal{L}_n^{(0)}((\xi_1^2+\xi_2^2)/|\eta|)\, d\xi\, d\eta. \end{aligned}$$

The conclusion follows by a change of variable in the inner integral. □

4. Weighted estimates

For convenience, set $\mathcal{L}_n^{(k)} = 0$ for all $n < 0$. The following identities are easily obtained from the properties of Laguerre polynomials (see, e.g., [6, §10.12]).

Lemma 4.1. *For all $k, n, n' \in \mathbb{N}$ and $t \in \mathbb{R}$,*

$$\mathcal{L}_n^{(k)}(t) = \mathcal{L}_{n-1}^{(k+1)}(t) + \mathcal{L}_n^{(k+1)}(t), \tag{4.1}$$

$$\frac{d}{dt}\mathcal{L}_n^{(k)}(t) = \mathcal{L}_{n-1}^{(k+1)}(t) - \mathcal{L}_n^{(k+1)}(t), \tag{4.2}$$

$$\int_0^\infty \mathcal{L}_n^{(k)}(t)\, \mathcal{L}_{n'}^{(k)}(t)\, t^k \, dt = \begin{cases} \frac{(n+k)!}{2^{k+1} n!} & \text{if } n = n', \\ 0 & \text{otherwise.} \end{cases} \tag{4.3}$$

We introduce some operators on functions $f : \mathbb{N} \times \mathbb{R} \times \mathbb{R}^3 \to \mathbb{C}$:

$$\begin{aligned} \tau f(n, \mu, \eta) &= f(n+1, \mu, \eta), \\ \delta f(n, \mu, \eta) &= f(n+1, \mu, \eta) - f(n, \mu, \eta), \\ \partial_\mu f(n, \mu, \eta) &= \frac{\partial}{\partial \mu} f(n, \mu, \eta), \\ \partial_\eta^\alpha f(n, \mu, \eta) &= \left(\frac{\partial}{\partial \eta}\right)^\alpha f(n, \mu, \eta), \end{aligned}$$

for all $\alpha \in \mathbb{N}^3$. For all multiindices $\alpha \in \mathbb{N}^3$, we denote by $|\alpha|$ its length $\alpha_1 + \alpha_2 + \alpha_3$. We set moreover $\langle t \rangle = 2|t| + 1$ for all $t \in \mathbb{R}$.

Note that, for all compactly supported $f : \mathbb{N} \times \mathbb{R} \times \mathbb{R}^3 \to \mathbb{C}$, $\tau^l f$ is null for all sufficiently large $l \in \mathbb{N}$; hence the operator $1 + \tau$, when restricted to the set of compactly supported functions, is invertible, with inverse given by

$$(1+\tau)^{-1} f = \sum_{l \in \mathbb{N}} (-1)^l \tau^l f,$$

and therefore the operator $(1+\tau)^q$ is well-defined for all $q \in \mathbb{Z}$.

Proposition 4.2. *Let $G : \mathbb{R}^4 \to \mathbb{C}$ be smooth and compactly supported in $\mathbb{R} \times (\mathbb{R}^3 \setminus \{0\})$, and let $m(n, \mu, \eta)$ be defined by (3.2). For all $\alpha \in \mathbb{N}^3$,*

$$\begin{aligned} &\int_{N_{3,2}} |y^\alpha \, \mathcal{K}_{G(L,\mathbf{Y})}(x,y)|^2 \, dx \, dy \\ &\quad \leq C_\alpha \sum_{\iota \in I_\alpha} \sum_{n \in \mathbb{N}} \int_{\mathbb{R}^3} \int_{\mathbb{R}} |\partial_\eta^{\gamma^\iota} \partial_\mu^{l_\iota} \delta^{k_\iota} (1+\tau)^{|\beta^\iota| - k_\iota} m(n, \mu, \eta)|^2 \\ &\qquad\qquad \times \mu^{2b_\iota} \, |\eta|^{2|\gamma^\iota| - 2|\alpha| - 2k_\iota + |\beta^\iota| + 1} \langle n \rangle^{|\beta^\iota|} \, d\mu \, d\eta, \end{aligned} \tag{4.4}$$

where I_α is a finite set and, for all $\iota \in I_\alpha$,

- *$\gamma^\iota \in \mathbb{N}^3$, $l_\iota, k_\iota \in \mathbb{N}$, $\gamma^\iota \leq \alpha$, $\min\{1, |\alpha|\} \leq |\gamma^\iota| + l_\iota + k_\iota \leq |\alpha|$,*

- $b_\iota \in \mathbb{N}$, $\beta^\iota \in \mathbb{N}^3$, $b_\iota + |\beta^\iota| = l_\iota + 2k_\iota$, $|\gamma^\iota| + l_\iota + b_\iota \leq |\alpha|$.

Proof. Proposition 3.1 and integration by parts allow us to write

$$\text{(4.5)}\quad y^\alpha \mathcal{K}_{G(L,\mathbf{Y})}(x,y) = \frac{2i^{|\alpha|}}{(2\pi)^6} \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \left[\left(\frac{\partial}{\partial\eta}\right)^\alpha \sum_{n\in\mathbb{N}} m(n,\xi^\eta_\parallel,\eta)\,\mathcal{L}^{(0)}_n(|\xi^\eta_\perp|^2/|\eta|)\right] e^{i\langle\xi,x\rangle}\, e^{i\langle\eta,y\rangle}\, d\xi\, d\eta.$$

From the definition of $\xi^\eta_\parallel$ and $\xi^\eta_\perp$, the following identities are not difficult to obtain:

$$\text{(4.6)}\quad \begin{gathered} \frac{\partial}{\partial\eta_j}\xi^\eta_\parallel = (\xi^\eta_\perp)_j \frac{1}{|\eta|}, \qquad \frac{\partial}{\partial\eta_j}(\xi^\eta_\perp)_k = -\xi^\eta_\parallel \frac{\partial}{\partial\eta_j}\frac{\eta_k}{|\eta|} - (\xi^\eta_\perp)_j \frac{\eta_k}{|\eta|^2}, \\ \frac{\partial}{\partial\eta_j}\frac{|\xi^\eta_\perp|^2}{|\eta|} = -\xi^\eta_\parallel (\xi^\eta_\perp)_j \frac{2}{|\eta|^2} - |\xi^\eta_\perp|^2 \frac{\eta_j}{|\eta|^3}. \end{gathered}$$

The multiindex notation will also be used as follows:

$$(\xi^\eta_\perp)^\beta = (\xi^\eta_\perp)_1^{\beta_1}(\xi^\eta_\perp)_2^{\beta_2}(\xi^\eta_\perp)_3^{\beta_3}$$

for all $\xi,\eta \in \mathbb{R}$, with $\eta \neq 0$, and all $\beta \in \mathbb{N}^3$; consequently

$$|\xi^\eta_\perp|^2 = (\xi^\eta_\perp)^{(2,0,0)} + (\xi^\eta_\perp)^{(0,2,0)} + (\xi^\eta_\perp)^{(0,0,2)}.$$

Via these identities, one can prove inductively that, for all $\alpha \in \mathbb{N}^3$,

$$\text{(4.7)}\quad \left(\frac{\partial}{\partial\eta}\right)^\alpha \sum_{n\in\mathbb{N}} m(n,\xi^\eta_\parallel,\eta)\,\mathcal{L}^{(0)}_n(|\xi^\eta_\perp|^2/|\eta|) = \sum_{\iota\in I_\alpha}\sum_{n\in\mathbb{N}} \partial_\eta^{\gamma^\iota}\partial_\mu^{l_\iota}\delta^{k_\iota} m(n,\xi^\eta_\parallel,\eta)\,(\xi^\eta_\parallel)^{b_\iota}\,(\xi^\eta_\perp)^{\beta^\iota}\,\Theta_\iota(\eta)\,\mathcal{L}^{(k_\iota)}_n(|\xi^\eta_\perp|^2/|\eta|),$$

where I_α, γ^ι, l_ι, k_ι, b_ι, β^ι are as in the statement above, while $\Theta_\iota : \mathbb{R}^3\setminus\{0\} \to \mathbb{R}$ is smooth and homogeneous of degree $|\gamma^\iota|-|\alpha|-k_\iota$. For the inductive step, one employs Leibniz' rule, and when a derivative hits a Laguerre function, the identity (4.2) together with summation by parts is used.

Note that, for all compactly supported $f : \mathbb{N}\times\mathbb{R}\times\mathbb{R}^3 \to \mathbb{C}$,

$$\sum_{n\in\mathbb{N}} f(n,\mu,\eta)\,\mathcal{L}^{(k)}_n(t) = \sum_{n\in\mathbb{N}} (1+\tau) f(n,\mu,\eta)\,\mathcal{L}^{(k+1)}_n(t),$$

by (4.1). Since $1+\tau$ is invertible, simple manipulations and iteration yield the more general identity

$$\sum_{n\in\mathbb{N}} f(n,\mu,\eta)\,\mathcal{L}^{(k)}_n(t) = \sum_{n\in\mathbb{N}} (1+\tau)^{k'-k} f(n,\mu,\eta)\,\mathcal{L}^{(k')}_n(t),$$

for all $k, k' \in \mathbb{N}$. This formula allows us to adjust in (4.7) the type of the Laguerre functions to the exponent of $\xi_\perp$, and to obtain that

$$\left(\frac{\partial}{\partial\eta}\right)^{\alpha} \sum_{n\in\mathbb{N}} m(n, \xi^\eta_\parallel, \eta)\, \mathcal{L}^{(0)}_n(|\xi^\eta_\perp|^2/|\eta|) = \sum_{\iota\in I_\alpha}\sum_{n\in\mathbb{N}} \partial^{\gamma^\iota}_\eta \partial^{l_\iota}_\mu \delta^{k_\iota} (1+\tau)^{|\beta^\iota|-k_\iota} m(n, \xi^\eta_\parallel, \eta) \times (\xi^\eta_\parallel)^{b_\iota}\, (\xi^\eta_\perp)^{\beta^\iota}\, \Theta_\iota(\eta)\, \mathcal{L}^{(|\beta^\iota|)}_n(|\xi^\eta_\perp|^2/|\eta|),$$

By plugging this identity into (4.5) and exploiting Plancherel's formula for the Fourier transform, the finiteness of I_α and the triangular inequality, we get that

$$\int_{N_{3,2}} |y^\alpha\, \mathcal{K}_{G(L,\mathbf{Y})}(x,y)|^2\, dx\, dy \leq C_\alpha \sum_{\iota\in I_\alpha} \int_{\mathbb{R}^3}\int_{\mathbb{R}}\int_{\mathbb{R}^2} \left| \sum_{n\in\mathbb{N}} \partial^{\gamma^\iota}_\eta \partial^{l_\iota}_\mu \delta^{k_\iota} (1+\tau)^{|\beta^\iota|-k_\iota} m(n,\mu,\eta)\, \mathcal{L}^{(|\beta^\iota|)}_n(|\zeta|^2/|\eta|) \right|^2 \times \mu^{2b_\iota}\, |\zeta|^{2|\beta^\iota|}\, |\eta|^{2|\gamma^\iota|-2|\alpha|-2k_\iota}\, d\zeta\, d\mu\, d\eta$$

A passage to polar coordinates in the ζ-integral and a rescaling then give that

$$\int_{N_{3,2}} |y^\alpha\, \mathcal{K}_{G(L,\mathbf{Y})}(x,y)|^2\, dx\, dy \leq C_\alpha \sum_{\iota\in I_\alpha} \int_{\mathbb{R}^3}\int_{\mathbb{R}}\int_0^\infty \left| \sum_{n\in\mathbb{N}} \partial^{\gamma^\iota}_\eta \partial^{l_\iota}_\mu \delta^{k_\iota} (1+\tau)^{|\beta^\iota|-k_\iota} m(n,\mu,\eta)\, \mathcal{L}^{(|\beta^\iota|)}_n(s) \right|^2 s^{|\beta^\iota|}\, ds \times \mu^{2b_\iota}\, |\eta|^{2|\gamma^\iota|-2|\alpha|-2k_\iota+|\beta^\iota|+1}\, d\mu\, d\eta,$$

and the conclusion follows by applying the orthogonality relations (4.3) for the Laguerre functions to the inner integral. □

Note that $\tau f(\cdot,\mu,\eta)$, $\delta f(\cdot,\mu,\eta)$ depend only on $f(\cdot,\mu,\eta)$; in other words, τ and δ can be considered as operators on functions $\mathbb{N} \to \mathbb{C}$. The next lemma will be useful in converting finite differences into continuous derivatives.

Lemma 4.3. *Let $f : \mathbb{N} \to \mathbb{C}$ have a smooth extension $\tilde{f} : [0,\infty[\to \mathbb{C}$, and let $k \in \mathbb{N}$. Then*

$$\delta^k f(n) = \int_{J_k} \tilde{f}^{(k)}(n+s)\, d\nu_k(s)$$

for all $n \in \mathbb{N}$, where $J_k = [0,k]$ and ν_k is a Borel probability measure on J_k. In particular

$$|\delta^k f(n)|^2 \leq \int_{J_k} |\tilde{f}^{(k)}(n+s)|^2\, d\nu_k(s)$$

for all $n \in \mathbb{N}$.

Proof. Iterated application of the fundamental theorem of integral calculus gives

$$\delta^k f(n) = \int_{[0,1]^k} \tilde{f}^{(k)}(n + s_1 + \cdots + s_k)\, ds.$$

The conclusion follows by taking as ν_k the push-forward of the uniform distribution on $[0,1]^k$ via the map $(s_1, \dots, s_k) \mapsto s_1 + \cdots + s_k$, and by Hölder's inequality. □

We give now a simplified version of the right-hand side of (4.4), in the case where we restrict to the functional calculus for the sublaplacian L alone. In order to avoid divergent series, however, it is convenient at first to truncate the multiplier along the spectrum of $\mathbf{Y}$.

Lemma 4.4. *Let $\chi \in C_c^\infty(\mathbb{R})$ be supported in $[1/2, 2]$, $K \subseteq]0, \infty[$ be compact and $M \in]0, \infty[$. If $F : \mathbb{R} \to \mathbb{C}$ is smooth and supported in K, and $F_M : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{C}$ is given by*

$$F_M(\lambda, \eta) = F(\lambda)\, \chi(|\eta|/M),$$

then, for all $r \in [0, \infty[$,

$$\int_{N_{3,2}} ||y|^r \, \mathcal{K}_{F_M(L,\mathbf{Y})}(x, y)|^2 \, dx\, dy \leq C_{K,\chi,r}\, M^{3-2r} \|F\|^2_{W_2^r}.$$

Proof. We may restrict to the case $r \in \mathbb{N}$, the other cases being recovered a posteriori by interpolation. Hence we need to prove that

$$\int_{N_{3,2}} |y^\alpha \, \mathcal{K}_{F_M(L,\mathbf{Y})}(x, y)|^2 \, dx\, dy \leq C_{K,\chi,\alpha}\, M^{3-2|\alpha|} \|F\|^2_{W_2^{|\alpha|}} \tag{4.8}$$

for all $\alpha \in \mathbb{N}^3$. On the other hand, if

$$m(n, \mu, \eta) = F(|\eta|\langle n\rangle + \mu^2)\, \chi(|\eta|/M),$$

then the left-hand side of (4.8) can be majorized by (4.4), and we are reduced to proving that

$$\begin{aligned} \sum_{n\in\mathbb{N}} \int_{\mathbb{R}^3} \int_{\mathbb{R}} |\partial_\eta^{\gamma^\iota} \partial_\mu^{l_\iota} \delta^{k_\iota} (1+\tau)^{|\beta^\iota| - k_\iota} m(n, \mu, \eta)|^2 \, \mu^{2b_\iota} \, |\eta|^{2|\gamma^\iota| - 2|\alpha| - 2k_\iota + |\beta^\iota| + 1} \\ \times \langle n\rangle^{|\beta^\iota|} \, d\mu\, d\eta \leq C_{K,\chi,\alpha}\, M^{3-2|\alpha|} \|F\|^2_{W_2^{|\alpha|}} \end{aligned} \tag{4.9}$$

for all $\iota \in I_\alpha$.

Consider first the case $|\beta^\iota| \geq k_\iota$. A smooth extension $\tilde{m} : \mathbb{R} \times \mathbb{R} \times \mathbb{R}^3 \to \mathbb{C}$ of m is defined by

$$\tilde{m}(t, \mu, \eta) = F(|\eta|(2t+1) + \mu^2)\, \chi(|\eta|/M).$$

Then, by Lemma 4.3,

$$\partial_\eta^{\gamma^\iota} \partial_\mu^{l_\iota} \delta^{k_\iota} (1+\tau)^{|\beta^\iota|-k_\iota} m(n,\mu,\eta) \\ = \sum_{j=0}^{|\beta^\iota|-k_\iota} \binom{|\beta^\iota|-k_\iota}{j} \int_{J_\iota} \partial_\eta^{\gamma^\iota} \partial_\mu^{l_\iota} \partial_t^{k_\iota} \tilde{m}(n+j+s,\mu,\eta)\, d\nu_\iota(s),$$

where $J_\iota = [0,k_\iota]$ and ν_ι is a suitable probability measure on J_ι; consequently (4.9) will be proved if we show that

$$\sum_{n\in\mathbb{N}} \int_{\mathbb{R}^3} \int_{\mathbb{R}} |\partial_\eta^{\gamma^\iota} \partial_\mu^{l_\iota} \partial_t^{k_\iota} \tilde{m}(n+s,\mu,\eta)|^2 \, \mu^{2b_\iota} \, |\eta|^{2|\gamma^\iota|-2|\alpha|-2k_\iota+|\beta^\iota|+1} \\ \times \langle n\rangle^{|\beta^\iota|} \, d\mu \, d\eta \le C_{K,\chi,\alpha} \, M^{3-2|\alpha|} \|F\|^2_{W_2^{|\alpha|}} \tag{4.10}$$

for all $s \in [0,|\beta^\iota|]$. On the other hand, it is easily proved inductively that

$$\partial_\eta^{\gamma^\iota} \partial_\mu^{l_\iota} \partial_t^{k_\iota} \tilde{m}(t,\mu,\eta) \\ = \sum_{r=\lceil l_\iota/2\rceil}^{l_\iota} \sum_{v=0}^{|\gamma^\iota|} \sum_{q=0}^{|\gamma^\iota|-v} \Psi_{\iota,v,q}(\eta) \, \langle t\rangle^v \mu^{2r-l_\iota} M^{-q} F^{(k_\iota+v+r)}(|\eta|\langle t\rangle + \mu^2) \, \chi^{(q)}(|\eta|/M)$$

for all $t \ge 0$, where $\Psi_{\iota,v,q} : \mathbb{R}^3 \setminus \{0\} \to \mathbb{R}$ is smooth and homogeneous of degree $k_\iota + v + q - |\gamma^\iota|$; hence

$$|\partial_\eta^{\gamma^\iota} \partial_\mu^{l_\iota} \partial_t^{k_\iota} \tilde{m}(t,\mu,\eta)|^2 \le C_{\chi,\alpha} \sum_{r=\lceil l_\iota/2\rceil}^{l_\iota} \sum_{v=0}^{|\gamma^\iota|} M^{2k_\iota+2v-2|\gamma^\iota|} \langle t\rangle^{2v} \mu^{4r-2l_\iota} \\ \times |F^{(k_\iota+v+r)}(|\eta|\langle t\rangle + \mu^2)|^2 \, \tilde{\chi}(|\eta|/M), \tag{4.11}$$

where $\tilde{\chi}$ is the characteristic function of $[1/2,2]$, and we are using the fact that $|\eta| \sim M$ in the region where $\tilde{\chi}(|\eta|/M) \neq 0$. Consequently the left-hand side of (4.10) is majorized by

$$\begin{aligned}
C_{\chi,\alpha} \sum_{r=\lceil l_\iota/2\rceil}^{l_\iota} \sum_{v=0}^{|\gamma^\iota|} M^{2v-2|\alpha|+|\beta^\iota|+1} \sum_{n\in\mathbb{N}} \langle n\rangle^{|\beta^\iota|} \langle n+s\rangle^{2v} \\
\times \int_{\mathbb{R}^3} \int_{\mathbb{R}} |F^{(k_\iota+v+r)}(|\eta|\langle n+s\rangle+\mu^2)|^2 \, \mu^{2b_\iota+4r-2l_\iota} \, \tilde\chi(|\eta|/M)\, d\mu\, d\eta \\
\le C_{\chi,\alpha} \sum_{r=\lceil l_\iota/2\rceil}^{l_\iota} \sum_{v=0}^{|\gamma^\iota|} M^{2v-2|\alpha|+|\beta^\iota|+3} \sum_{n\in\mathbb{N}} \langle n+s\rangle^{|\beta^\iota|+2v} \\
\times \int_0^\infty \int_0^\infty |F^{(k_\iota+v+r)}(\rho\langle n+s\rangle+\mu^2)|^2 \, \mu^{2b_\iota+4r-2l_\iota} \, \tilde\chi(\rho/M)\, d\mu \; d\rho \\
\le C_{\chi,\alpha} \sum_{r=\lceil l_\iota/2\rceil}^{l_\iota} \sum_{v=0}^{|\gamma^\iota|} M^{2v-2|\alpha|+|\beta^\iota|+3} \int_0^\infty \int_0^\infty |F^{(k_\iota+v+r)}(\rho+\mu^2)|^2 \\
\times \mu^{2b_\iota+4r-2l_\iota} \sum_{n\in\mathbb{N}} \langle n+s\rangle^{|\beta^\iota|+2v-1} \tilde\chi(\rho/(\langle n+s\rangle M))\, d\mu\, d\rho,
\end{aligned}$$

by passing to polar coordinates and rescaling. The last sum in n is easily controlled by $(\rho/M)^{|\beta^\iota|+2v}$, hence the left-hand side of (4.10) is majorized by

$$\begin{aligned}
C_{\chi,\alpha}\, M^{3-2|\alpha|} \sum_{r=\lceil l_\iota/2\rceil}^{l_\iota} \sum_{v=0}^{|\gamma^\iota|} \int_0^\infty \int_0^\infty |F^{(k_\iota+v+r)}(\rho+\mu^2)|^2 \mu^{2b_\iota+4r-2l_\iota} \, \rho^{|\beta^\iota|+2v} \, d\mu\, d\rho \\
\le C_{K,\chi,\alpha}\, M^{3-2|\alpha|} \sum_{r=\lceil l_\iota/2\rceil}^{l_\iota} \sum_{v=0}^{|\gamma^\iota|} \sup_{u\in[0,\max K]} \int_0^\infty |F^{(k_\iota+v+r)}(\rho+u)|^2\, d\rho,
\end{aligned}$$

because $2b_\iota + 4r - 2l_\iota \ge 0$ and $|\beta^\iota| + 2v \ge 0$ if r and v are in the range of summation, and $\operatorname{supp} F \subseteq K$. Since moreover $k_\iota + v + r \le k_\iota + |\gamma^\iota| + l_\iota \le |\alpha|$, the last integral is dominated by $\|F\|^2_{W_2^{|\alpha|}}$ uniformly in r, v, u, and (4.10) follows.

Consider now the case $|\beta^\iota| < k_\iota$. Via the identity

$$(1+\tau)^{-1} = (1-\tau)(1-\tau^2)^{-1} = -\delta(1-\tau^2)^{-1} = -\delta \sum_{j=0}^\infty \tau^{2j},$$

together with Lemma 4.3, we obtain that

$$\begin{aligned}
(4.12)\quad & \partial_\eta^{\gamma^\iota} \partial_\mu^{l_\iota} \delta^{k_\iota} (1+\tau)^{|\beta^\iota|-k_\iota} m(n,\mu,\eta) \\
&= (-1)^{k_\iota-|\beta^\iota|} \sum_{j=0}^\infty \binom{j+k_\iota-|\beta^\iota|-1}{k_\iota-|\beta^\iota|-1} \int_{J_\iota} \partial_\eta^{\gamma^\iota} \partial_\mu^{l_\iota} \partial_t^{2k_\iota-|\beta^\iota|} \tilde m(n+2j+s,\mu,\eta)\, d\nu_\iota(s),
\end{aligned}$$

where $J_\iota = [0, 2k_\iota - |\beta^\iota|]$ and ν_ι is a suitable probability measure on J_ι. Note that, because of the assumptions on the supports of F and χ, the sum on j

in the right-hand side of (4.12) is a finite sum, that is, the j-th summand is nonzero only if $\langle n+2j\rangle \leq 2M^{-1}\max K$; consequently, by applying the Cauchy-Schwarz inequality to the sum in j, and by (4.11),

$$
\begin{aligned}
&|\partial_\eta^{\gamma^\iota}\partial_\mu^{l_\iota}\delta^{k_\iota}(1+\tau)^{|\beta^\iota|-k_\iota}m(n,\mu,\eta)|^2 \\
&\quad \leq C_{K,\alpha}\, M^{1+2|\beta^\iota|-2k_\iota}\sum_{j=0}^\infty \int_{J_\iota} |\partial_\eta^{\gamma^\iota}\partial_\mu^{l_\iota}\partial_t^{2k_\iota-|\beta^\iota|}\tilde m(n+2j+s,\mu,\eta)|^2\,d\nu_\iota(s) \\
&\quad \leq C_{K,\chi,\alpha}\sum_{r=\lceil l_\iota/2\rceil}^{l_\iota}\sum_{v=0}^{|\gamma^\iota|} M^{1+2k_\iota+2v-2|\gamma^\iota|}\sum_{j=0}^\infty\int_{J_\iota}\langle n+2j+s\rangle^{2v}\mu^{4r-2l_\iota} \\
&\qquad\qquad \times |F^{(2k_\iota-|\beta^\iota|+v+r)}(|\eta|\langle n+2j+s\rangle+\mu^2)|^2\,\tilde\chi(|\eta|/M)\,d\nu_\iota(s).
\end{aligned}
$$

Remember that $|\eta|\sim M$ in the region where $\tilde\chi(|\eta|/M)\neq 0$. Hence the left-hand side of (4.9) is majorized by

$$
\begin{aligned}
&C_{K,\chi,\alpha}\sum_{r=\lceil l_\iota/2\rceil}^{l_\iota}\sum_{v=0}^{|\gamma^\iota|}\int_{J_\iota}\sum_{n\in\mathbb N}\sum_{j\in\mathbb N}\langle n+2j+s\rangle^{2v}\langle n\rangle^{|\beta^\iota|}\int_{\mathbb R^3}\int_{\mathbb R} M^{2+2v-2|\alpha|+|\beta^\iota|} \\
&\quad\times \mu^{2b_\iota+4r-2l_\iota}\,|F^{(2k_\iota-|\beta^\iota|+v+r)}(|\eta|\langle n+2j+s\rangle+\mu^2)|^2\,\tilde\chi(|\eta|/M)\,d\mu\,d\eta\,d\nu_\iota(s) \\
&\leq C_{K,\chi,\alpha}\sum_{r=\lceil l_\iota/2\rceil}^{l_\iota}\sum_{v=0}^{|\gamma^\iota|}\int_{J_\iota}\sum_{n\in\mathbb N}\sum_{j\in\mathbb N}\langle n+2j+s\rangle^{2v+|\beta^\iota|}\int_0^\infty\int_0^\infty M^{4+2v-2|\alpha|+|\beta^\iota|} \\
&\quad\times \mu^{2b_\iota+4r-2l_\iota}\,|F^{(2k_\iota-|\beta^\iota|+v+r)}(\rho\langle n+2j+s\rangle+\mu^2)|^2\,\tilde\chi(\rho/M)\,d\mu\,d\rho\,d\nu_\iota(s) \\
&\leq C_{K,\chi,\alpha}\sum_{r=\lceil l_\iota/2\rceil}^{l_\iota}\sum_{v=0}^{|\gamma^\iota|} M^{4+2v-2|\alpha|+|\beta^\iota|}\int_0^\infty\int_0^\infty |F^{(2k_\iota-|\beta^\iota|+v+r)}(\rho+\mu^2)|^2 \\
&\quad\times \mu^{2b_\iota+4r-2l_\iota}\int_{J_\iota}\sum_{(n,j)\in\mathbb N^2}\frac{\tilde\chi(\rho/(\langle n+2j+s\rangle M))}{\langle n+2j+s\rangle^{1-2v-|\beta^\iota|}}\,d\nu_\iota(s)\,d\mu\,d\rho,
\end{aligned}
$$

by passing to polar coordinates and rescaling. The sum in (n,j) is dominated by $(\rho/M)^{2v+|\beta^\iota|+1}$, uniformly in $s\in J_\iota$, and moreover $\operatorname{supp} F\subseteq K$. Therefore the left-hand side of (4.9) is majorized by

$$
C_{K,\chi,\alpha}\,M^{3-2|\alpha|}\sum_{r=\lceil l_\iota/2\rceil}^{l_\iota}\sum_{v=0}^{|\gamma^\iota|}\sup_{u\in[0,\max K]}\int_0^\infty |F^{(2k_\iota-|\beta^\iota|+v+r)}(\rho+u)|^2\,d\rho.
$$

On the other hand, $b_\iota+|\beta^\iota| = l_\iota+2k_\iota$, hence $2k_\iota-|\beta^\iota|+v+r\leq 2k_\iota-|\beta^\iota|+|\gamma^\iota|+l_\iota = b_\iota+|\gamma^\iota|\leq|\alpha|$ if r and v are in the range of summation, therefore the last integral is dominated by $\|F\|^2_{W_2^{|\alpha|}}$ uniformly in r,v,u, and (4.9) follows. □

Proposition 4.5. *Let $F : \mathbb{R} \to \mathbb{C}$ be smooth and such that* $\operatorname{supp} F \subseteq K$ *for some compact set $K \subseteq \,]0,\infty[$. For all $r \in [0, 3/2[$,*

$$\int_{N_{3,2}} \left|(1+|y|)^r \, \mathcal{K}_{F(L)}(x,y)\right|^2 \, dx \, dy \leq C_{K,r} \|F\|^2_{W_2^r}.$$

Proof. Take $\chi \in C_c^\infty(]0,\infty[)$ such that $\operatorname{supp} \chi \subseteq [1/2, 2]$ and $\sum_{k\in\mathbb{Z}} \chi(2^{-k}t) = 1$ for all $t \in \,]0,\infty[$. Note that, if (λ,η) belongs to the joint spectrum of $L, \mathbf{Y}$, then $|\eta| \leq \lambda$. Therefore, if $k_K \in \mathbb{Z}$ is sufficiently large so that $2^{k_K - 1} > \max K$, and if F_M is defined for all $M \in \,]0,\infty[$ as in Lemma 4.4, then

$$F(L) = \sum_{k\in\mathbb{Z},\, k\leq k_K} F_{2^k}(L, \mathbf{Y})$$

(with convergence in the strong sense). Hence an estimate for $\mathcal{K}_{F(L)}$ can be obtained, via Minkowski's inequality, by summing the corresponding estimates for $\mathcal{K}_{F_{2^k}(L,\mathbf{Y})}$ given by Lemma 4.4. If $r < 3/2$, then the series $\sum_{k\leq k_K} (2^k)^{3/2-r}$ converges, thus

$$\int_{N_{3,2}} \left||y|^r \, \mathcal{K}_{F(L)}(x,y)\right|^2 \, dx \, dy \leq C_{K,r} \|F\|^2_{W_2^r}.$$

The conclusion follows by combining the last inequality with the corresponding one for $r = 0$. □

Recall that $|\cdot|_\delta$ denotes a δ_t-homogeneous norm on $N_{3,2}$, thus $|(x,y)|_\delta \sim |x| + |y|^{1/2}$. Interpolation then allows us to improve the standard weighted estimate for a homogeneous sublaplacian on a stratified group.

Proposition 4.6. *Let $F : \mathbb{R} \to \mathbb{C}$ be smooth and such that* $\operatorname{supp} F \subseteq K$ *for some compact set $K \subseteq \,]0,\infty[$. For all $r \in [0, 3/2[$, $\alpha \geq 0$ and $\beta > \alpha + r$,*

$$\int_{N_{3,2}} \left|(1+|(x,y)|_\delta)^\alpha \, (1+|y|)^r \, \mathcal{K}_{F(L)}(x,y)\right|^2 \, dx \, dy \leq C_{K,\alpha,\beta,r} \|F\|^2_{W_2^\beta}. \tag{4.13}$$

Proof. Note that $1 + |y| \leq C(1 + |(x,y)|_\delta)^2$. Hence, in the case $\alpha \geq 0$, $\beta > \alpha + 2r$, the inequality (4.13) follows by the standard estimate [16, Lemma 1.2]. On the other hand, if $\alpha = 0$ and $\beta \geq r$, then (4.13) is given by Proposition 4.5. The full range of α and β is then obtained by interpolation (cf. the proof of [16, Lemma 1.2]). □

We can finally prove the fundamental L^1-estimate, and consequently Theorem 1.1.

Proof of Proposition 2.1. Take $r \in \,]9/2 - s, 3/2[$. Then $s - r > 3/2 + 3 - 2r$, hence we can find $\alpha_1 > 3/2$ and $\alpha_2 > 3 - 2r$ such that $s - r > \alpha_1 + \alpha_2$.

Therefore, by Proposition 4.6 and Hölder's inequality,

$$\| \mathcal{K}_{F(L)} \|_1^2 \leq C_{k,s} \|F\|_{W_2^s}^2 \int_{N_{3,2}} (1 + |(x,y)|_\delta)^{-2\alpha_1 - 2\alpha_2} (1 + |y|)^{-2r} \, dx \, dy.$$

The integral on the right-hand side is finite, because $2\alpha_1 > 3$, $\alpha_2 + 2r > 3$, and

$$(1 + |(x,y)|_\delta)^{-2\alpha_1 - 2\alpha_2} (1 + |y|)^{-2r} \leq C_s (1 + |x|)^{-2\alpha_1} (1 + |y|)^{-\alpha_2 - 2r},$$

and we are done. □

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Mathematisches Seminar, C.-A.-Universität zu Kiel, Ludewig-Meyn-Str. 4, D-24118 Kiel, Germany
E-mail address: martini@math.uni-kiel.de

Mathematisches Seminar, C.-A.-Universität zu Kiel, Ludewig-Meyn-Str. 4, D-24118 Kiel, Germany
E-mail address: mueller@math.uni-kiel.de