

Analysis of joint spectral multipliers on Lie groups of polynomial growth

Original

Analysis of joint spectral multipliers on Lie groups of polynomial growth / Martini, A.. - In: ANNALES DE L'INSTITUT FOURIER. - ISSN 0373-0956. - STAMPA. - 62:4(2012), pp. 1215-1263. [10.5802/aif.2721]

Availability:

This version is available at: 11583/2949478 since: 2022-01-12T23:58:52Z

Publisher:

Annales de l'Institut Fourier

Published

DOI:10.5802/aif.2721

Terms of use:

This article is made available under terms and conditions as specified in the corresponding bibliographic description in the repository

Publisher copyright

(Article begins on next page)



ANNALES

DE

L'INSTITUT FOURIER

Alessio MARTINI

Analysis of joint spectral multipliers on Lie groups of polynomial growth

Tome 62, n° 4 (2012), p. 1215-1263.

http://aif.cedram.org/item?id=AIF_2012__62_4_1215_0

© Association des Annales de l'institut Fourier, 2012, tous droits réservés.

L'accès aux articles de la revue « Annales de l'institut Fourier » (<http://aif.cedram.org/>), implique l'accord avec les conditions générales d'utilisation (<http://aif.cedram.org/legal/>). Toute reproduction en tout ou partie de cet article sous quelque forme que ce soit pour tout usage autre que l'utilisation à fin strictement personnelle du copiste est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

cedram

*Article mis en ligne dans le cadre du
Centre de diffusion des revues académiques de mathématiques
<http://www.cedram.org/>*

ANALYSIS OF JOINT SPECTRAL MULTIPLIERS ON LIE GROUPS OF POLYNOMIAL GROWTH

by Alessio MARTINI

ABSTRACT. — We study the problem of L^p -boundedness ($1 < p < \infty$) of operators of the form $m(L_1, \dots, L_n)$ for a commuting system of self-adjoint left-invariant differential operators L_1, \dots, L_n on a Lie group G of polynomial growth, which generate an algebra containing a weighted subcoercive operator. In particular, when G is a homogeneous group and L_1, \dots, L_n are homogeneous, we prove analogues of the Mihlin-Hörmander and Marcinkiewicz multiplier theorems.

RÉSUMÉ. — On étudie la bornitude L^p ($1 < p < \infty$) des opérateurs de la forme $m(L_1, \dots, L_n)$ pour un système commutatif L_1, \dots, L_n d'opérateurs différentiels autoadjoints invariants à gauche sur un groupe de Lie G à croissance polynomiale, qui engendrent une algèbre contenant un opérateur sous-coercif pondéré. En particulier, quand G est un groupe homogène et L_1, \dots, L_n sont homogènes, on prouve des analogues des théorèmes de multiplicateurs de Mihlin-Hörmander et Marcinkiewicz.

1. Introduction

Let (X, μ) be a measure space, and let L_1, \dots, L_n be (possibly unbounded) self-adjoint operators on $L^2(X, \mu)$ which commute strongly, *i.e.*, which admit a joint spectral resolution E on \mathbb{R}^n . Then a joint functional calculus for L_1, \dots, L_n is defined via spectral integration and, for every Borel function $m : \mathbb{R}^n \rightarrow \mathbb{C}$, the operator

$$m(L) = m(L_1, \dots, L_n) = \int_{\mathbb{R}^n} m dE$$

is bounded on $L^2(X, \mu)$ if and only if the function m is (E -essentially) bounded. The characterization of the boundedness of $m(L)$ on other function spaces, such as $L^p(X, \mu)$ for $p \neq 2$, in terms of properties of the function m — which will be called (*spectral*) *multiplier* for L_1, \dots, L_n — is

Keywords: spectral multipliers, joint functional calculus, differential operators, Lie groups, polynomial growth, singular integral operators.

Math. classification: 43A22, 22E30, 42B15.

a much more difficult question, even in particular cases. Several problems and results of harmonic analysis fall into this frame, the classical examples being the Mihlin-Hörmander and Marcinkiewicz theorems for Fourier multipliers on \mathbb{R}^n , which give sufficient conditions for the L^p boundedness ($1 < p < \infty$) of joint functions $m(-i\partial_1, \dots, -i\partial_n)$ of the partial derivatives on \mathbb{R}^n , in terms of smoothness properties of the multiplier m .

Here we are interested in the case $X = G$ is a (connected) Lie group of polynomial volume growth (with a Haar measure μ) and L_1, \dots, L_n belong to the algebra $\mathfrak{D}(G)$ of left-invariant differential operators on G , with particular reference to homogeneous operators L_1, \dots, L_n on a homogeneous (nilpotent) Lie group G . For $n = 1$, i.e., for a single operator $L = L_1$, several results of Mihlin-Hörmander type are known (see, e.g., [28, 4, 17, 30, 1, 18, 20, 42, 8, 9] and references therein), dealing mainly with a sublaplacian L on G , or more generally with a positive operator L for which Gaussian-type heat kernel estimates hold. For $n > 1$, instead, most of the known results [27, 31, 32, 13, 52, 14, 15] refer to a specific class of groups G (namely, the H-type groups) and to specific choices of the operators L_1, \dots, L_n (i.e., sublaplacians and central derivatives); a recent work of Sikora [43], which has no such restrictions (in fact it applies to more general settings than Lie groups), is however limited to the case of direct products ($G = G_1 \times \dots \times G_n$ and each L_j operates on a different factor G_j of G), so that it does not cover the mentioned results on H-type groups.

In this paper, we propose a quite general setting where spectral multiplier theorems of Mihlin-Hörmander and Marcinkiewicz type for a system of operators can be obtained. Starting from the *weighted subcoercive operators*⁽¹⁾ of ter Elst and Robinson [10] — which are a large class of left-invariant differential operators on a Lie group (including positive elliptic operators, sublaplacians, and positive Rockland operators on homogeneous groups) for which Gaussian-type heat kernel estimates hold — we define a *weighted subcoercive system* to be a system L_1, \dots, L_n of pairwise commuting, formally self-adjoint, left-invariant differential operators on a connected Lie group G which generate a subalgebra of $\mathfrak{D}(G)$ containing a weighted subcoercive operator. An “abstract” study of weighted subcoercive systems, in relation with the algebraic structure and the representation

(1) The definition of weighted subcoercive operator given in [26], to which we refer, may appear more restrictive than the original in [10] (see [26, footnote 1] for details); however, a slight modification of the argument in [10, §11] (namely, a basis change may be needed before eliminating the “over-weight” directions) shows that the original definition is actually equivalent to the one in [26].

theory of the environment Lie group, is performed in [26], from which we get in particular that:

- the members of a weighted subcoercive system L_1, \dots, L_n on G are essentially self-adjoint and commute strongly (so that they admit a joint spectral resolution) in every unitary representation of G ;
- the operators of the form $m(L) = m(L_1, \dots, L_n)$ are convolution operators:

$$m(L)\phi = \phi * \mathcal{K}_L m = \phi * \check{m}$$

for some distribution $\check{m} = \mathcal{K}_L m$; in fact, if m is bounded and compactly supported, then $\mathcal{K}_L m \in L^2(G)$ together with all its left-invariant derivatives;

- a Plancherel formula holds:

$$\|\mathcal{K}_L m\|_{L^2(G)} = \|m\|_{L^2(\mathbb{R}^n, \sigma)}$$

for some regular Borel measure σ , which is called the *Plancherel measure* associated with L_1, \dots, L_n , and whose support is their joint L^2 spectrum;

- if G is a homogeneous group (with automorphic dilations δ_t), a commuting system L_1, \dots, L_n of *homogeneous*, formally self-adjoint left-invariant differential operators is a weighted subcoercive system if and only if L_1, \dots, L_n are *jointly injective* on the smooth vectors v of every non-trivial irreducible representation π of G :

$$d\pi(L_1)v = \dots = d\pi(L_n)v = 0 \implies v = 0$$

(this is a multi-variate analogue of the Rockland condition [35, 22]); in this case, we speak of a *homogeneous weighted subcoercive system* and, if L_1, \dots, L_n are homogeneous of degrees w_1, \dots, w_n respectively, then

$$(1.1) \quad \sigma(\epsilon_t(A)) = t^{Q_\delta} \sigma(A), \quad \mathcal{K}_L(m \circ \epsilon_t) = t^{-Q_\delta} (\mathcal{K}_L m) \circ \delta_{t^{-1}},$$

where Q_δ is the homogeneous dimension ($\det \delta_t = t^{Q_\delta}$) and

$$\epsilon_t(\lambda_1, \dots, \lambda_n) = (t^{w_1} \lambda_1, \dots, t^{w_n} \lambda_n)$$

are the dilations on \mathbb{R}^n associated with L_1, \dots, L_n .

In the following, under the hypothesis that L_1, \dots, L_n is a weighted subcoercive system on a Lie group G of polynomial growth of degree Q_G , we prove weighted L^1 estimates for the kernels $\mathcal{K}_L m$ corresponding to multipliers m with compact support, in terms of a Sobolev norm of m . If we suppose further that G is a homogeneous group and that L_1, \dots, L_n are homogeneous operators (with degrees w_1, \dots, w_n and associated dilations

ϵ_t), then a theorem of Mihlin-Hörmander type can be obtained: the operator $m(L)$ is of weak type $(1, 1)$ and bounded on $L^p(G)$ for $1 < p < \infty$ whenever the multiplier m satisfies an L^q Mihlin-Hörmander condition of order s , i.e.,

$$(1.2) \quad \|m\|_{M_\epsilon W_q^s} \stackrel{\text{def}}{=} \sup_{t>0} \|(m \circ \epsilon_t) \eta\|_{W_q^s} < \infty,$$

with $q \in [2, \infty]$ and

$$(1.3) \quad s > \frac{Q_G}{2} + \frac{n-1}{q},$$

where $W_q^s(\mathbb{R}^n)$ is the L^q Sobolev space of (fractional) order s and η is a non-negative smooth cut-off function on \mathbb{R}^n supported on an annulus centered at the origin. Notice that the condition (1.2) is independent of the choice of the cut-off η ; moreover, an L^∞ Mihlin-Hörmander condition of integral order s is essentially equivalent to the pointwise conditions

$$(1.4) \quad \sup_{\lambda \neq 0} |\lambda|_\epsilon^{\|\alpha\|_\epsilon} |\partial^\alpha m(\lambda)| < \infty$$

for $\alpha \in \mathbb{N}^n$ with $|\alpha| = \alpha_1 + \dots + \alpha_n \leq s$, where $|\cdot|_\epsilon$ is a ϵ_t -homogeneous norm on \mathbb{R}^n and $\|\alpha\|_\epsilon = w_1 \alpha_1 + \dots + w_n \alpha_n$.

In particular cases, the previous theorem can be improved by lowering the *regularity threshold*, i.e., the right-hand side of (1.3). For instance, by extending a technique due to Hebisch and Zienkiewicz [20], we prove that, if G is the direct product of Euclidean and Métivier groups, then the dimension at infinity Q_G can be replaced in (1.3) by the topological dimension $\dim G$; in fact, in § 3 we propose a systematic approach for exploiting such technique, by introducing the notion of *h-capacious groups*, for which Q_G can be replaced in (1.3) by $Q_G - h$. The term $(n-1)/q$ in (1.3) can be lowered too, by determining the volume growth rate with respect to the Plancherel measure σ of Euclidean balls with small radius: namely, if $\sigma(B(\lambda, r)) \leq Cr^d$ for $|\lambda| = 1$ and $r \leq 1$, then $n - 1$ can be replaced in (1.3) by $n - d$.

Finally, a sort of product theory can be developed, by considering several homogeneous Lie groups G_l , each of which endowed with a homogeneous weighted subcoercive system $L_{l,1}, \dots, L_{l,n_l}$, for $l = 1, \dots, \varrho$. Let $(\epsilon_{l,t})_{t>0}$ be the dilations on \mathbb{R}^{n_l} associated with the system $L_{l,1}, \dots, L_{l,n_l}$, and define the multi-parameter dilations

$$\mathfrak{J}_t = \epsilon_{1,t_1} \times \dots \times \epsilon_{\varrho,t_\varrho}$$

on $\mathbb{R}^{\vec{n}} = \mathbb{R}^{n_1} \times \dots \times \mathbb{R}^{n_\varrho}$; set moreover $X_{\vec{n}} = \{\lambda \in \mathbb{R}^{\vec{n}} : |\lambda_1| \dots |\lambda_\varrho| = 0\}$. If G is a connected Lie group, $\nu_l : G_l \rightarrow G$ are Lie group homomorphisms,

and $L_{l,j}^b = v_l'(L_{l,j})$, then we have a system

$$(1.5) \quad L_{1,1}^b, \dots, L_{1,n_1}^b, \dots, L_{\varrho,1}^b, \dots, L_{\varrho,n_\varrho}^b$$

of essentially self-adjoint, left-invariant differential operators on G . Under the sole hypothesis of (strong) commutativity of the operators (1.5) on $L^2(G)$, we prove a multi-variate analogue of the previous theorem: the operator $m(L^b)$ is bounded on $L^p(G)$ for $1 < p < \infty$ whenever the multiplier $m : \mathbb{R}^{\vec{n}} \rightarrow \mathbb{C}$ vanishes on $X_{\vec{n}}$ and satisfies an L^q Marcinkiewicz condition of order $\vec{s} = (s_1, \dots, s_\varrho)$, i.e.,

$$(1.6) \quad \|m\|_{M_2 S_q^{\vec{s}} W} \stackrel{\text{def}}{=} \sup_{t_1, \dots, t_\varrho > 0} \|(m \circ \mathfrak{I}_{\vec{t}}) \eta_1 \otimes \dots \otimes \eta_\varrho\|_{S_q^{\vec{s}} W} < \infty,$$

with $q \in [2, \infty]$ and

$$(1.7) \quad s_l > \frac{Q_{G_l}}{2} + \frac{n_l - 1}{q} \quad \text{for } l = 1, \dots, \varrho,$$

where $S_q^{\vec{s}} W(\mathbb{R}^{\vec{n}})$ is the L^q Sobolev space with dominating mixed smoothness of order \vec{s} (see, e.g., [40, 38]) and the $\eta_l : \mathbb{R}^{n_l} \rightarrow \mathbb{R}$ are cut-off functions as before. This result can also be improved in particular cases: in fact, each of the components (1.7) of the regularity threshold can be independently lowered, by the same techniques and amounts as for the threshold (1.3) in the previous result. The hypothesis $m|_{X_{\vec{n}}} = 0$, related to the possibility that the spectral measure of $X_{\vec{n}}$ is not null, can be relaxed too, by applying iteratively our Marcinkiewicz-type result to subsystems of (1.5).

Both our theorems can be applied to the direct-product setting of [43] (in the case of homogeneous groups), and also to the systems of operators considered in the above-mentioned works on H-type groups; in fact, the results of [32, 52] are sharper than ours, since they require a weaker condition on the multiplier. On the other hand, our theorems have a much wider range of applicability, with respect both to the groups and to the systems of differential operators under consideration. In particular, the environment group G in the second theorem need not be homogeneous. As a corollary, we obtain L^p multiplier theorems for distinguished sublaplacians on some non-nilpotent solvable Lie groups G of polynomial growth (such as the plane motion group, the oscillator groups, the diamond groups) with regularity threshold $(\dim G)/2$; to our knowledge, this threshold had been previously reached only for some homogeneous groups (i.e., Heisenberg and related groups [17, 30]) and for the compact group SU_2 [8].

Notation and preliminary remarks

For a topological space X , we denote by $C(X)$ the space of continuous (complex-valued) functions on X , whereas $C_0(X)$ is the subspace of continuous functions vanishing at infinity. If X is a smooth manifold, then $\mathcal{D}(X)$ is the space of compactly supported smooth functions on X .

If G is a Lie group, f is a function on G and $x, y \in G$, then we set

$$L_x f(y) = f(x^{-1}y), \quad R_x f(y) = f(yx).$$

$R : x \mapsto R_x$ is the (right) regular representation of G . For a fixed right Haar measure μ on G , R_x is an isometry of $L^p(G)$ for $1 \leq p \leq \infty$. With respect to such measure, convolution and involution take the form

$$f * g(x) = \int_G f(xy^{-1})g(y) dy, \quad f^*(x) = \Delta(x)\overline{f(x^{-1})}$$

(where Δ is the modular function) and we set, for every representation π ,

$$\pi(f) = \int_G f(x) \pi(x^{-1}) dx$$

(differently from the common usage), so that in particular

$$R(g)f = f * g, \quad \pi(f * g) = \pi(g)\pi(f), \quad \pi(Df) = d\pi(D)\pi(f)$$

for every $D \in \mathfrak{D}(G)$. We denote by D^+ the formal adjoint of a smooth differential operator D on G (with respect to the measure μ).

The above conditions (1.2), (1.6) on the multiplier m have been expressed in terms of Sobolev norms. In fact, there are several scales of spaces which can be used to express a differentiability condition of fractional order and with an L^q flavour; since the inequalities (1.3), (1.7) involving the order are strict, it does not really matter which of the various scales is used. In the following, we will use the scale of Besov spaces $B_{q,r}^s(\mathbb{R}^n)$ and their dominating-mixed-smoothness variants $S_{q,r}^{\vec{s}}B(\mathbb{R}^{\vec{n}})$, which are particularly convenient because of their embedding and interpolation properties (see, e.g., [2, 47, 49, 40, 37, 39, 38, 41] for a reference); accordingly, we will use the quantities $\|\cdot\|_{M_c B_{q,q}^s}$, $\|\cdot\|_{M_2 S_{q,q}^{\vec{s}}B}$ in place of the ones in (1.2), (1.6).

2. Weighted estimates

Let G be a Lie group of polynomial growth of degree Q_G , and set

$$|x|_G = d_G(x, e), \quad \langle x \rangle_G = 1 + |x|_G,$$

where d_G is a left-invariant connected distance on G [51, § III.4] and $e \in G$ is the identity element. Let L_1, \dots, L_n be a weighted subcoercive system on G , with associated Plancherel measure σ , and let \mathcal{O} be the subalgebra of $\mathfrak{D}(G)$ generated by them. The aim of this section is to obtain inequalities of the form

$$\| \langle \cdot \rangle_G^\alpha \mathcal{K}_L m \|_{L^p(G)} \leq C_{K,\alpha,\beta} \| m \|_{B_{q,q}^\beta(\mathbb{R}^n)}$$

for multipliers m with support contained in a fixed compact $K \subseteq \mathbb{R}^n$, and for suitable p, q, α, β . In order to do this, we will subsequently perform two “changes of variables” on the spectral side: the former corresponds to choosing a system of generators of \mathcal{O} made of positive weighted subcoercive operators, while the latter is the multi-variate analogue of an exponential change of variables which has often been used in the literature (see, e.g., [12, § 6.B]) and which allows, together with a Fourier-series decomposition, heat kernel estimates, and Hölder’s inequality, to obtain the required estimates. Properties of the Plancherel measure σ and interpolation will then be used (as in [28] and subsequent works) to improve the obtained inequalities.

As in [26, § 3.2], we can find a polynomial $p_* \geq 0$ on \mathbb{R}^n such that, if

$$p_0(\lambda) = p_*(\lambda) + \sum_{j=1}^n \lambda_j^2 + 1, \quad p_\nu(\lambda) = p_0(\lambda) + \lambda_\nu \quad \text{for } \nu = 1, \dots, n,$$

then $p_*(L), p_0(L), p_1(L), \dots, p_n(L)$ are all positive and weighted subcoercive, and moreover $p_0(L), \dots, p_n(L)$ generate \mathcal{O} . Let $p : \mathbb{R}^n \rightarrow \mathbb{R}^{1+n}$ be the map whose components are the polynomials p_0, \dots, p_n . For $l \in \mathbb{Z}^{1+n}$, set

$$E_l(\lambda) = e^{il \cdot e^{-p(\lambda)}} - 1 = e^{i(l_0 e^{-p_0(\lambda)} + l_1 e^{-p_1(\lambda)} + \dots + l_n e^{-p_n(\lambda)})} - 1.$$

Then $E_l \in C_0(\mathbb{R}^n)$, and in fact

$$E_l = \sum_{0 \neq k \in \mathbb{N}^{1+n}} \frac{(il_0)^{k_0} \dots (il_n)^{k_n}}{k_0! \dots k_n!} e^{-k_0 p_0} \dots e^{-k_n p_n},$$

with uniform convergence on \mathbb{R}^n . This means that, if $h_{\nu,t}$ is the heat kernel of $p_\nu(L)$ for $\nu = 0, \dots, n$ (with $h_{\nu,0}$ denoting the Dirac delta at the identity of G), then

$$(2.1) \quad \check{E}_l = \sum_{0 \neq k \in \mathbb{N}^{1+n}} \frac{(il_0)^{k_0} \dots (il_n)^{k_n}}{k_0! \dots k_n!} h_{0,k_0} * \dots * h_{n,k_n}$$

with convergence in the norm of the corresponding convolution operators on $L^2(G)$.

LEMMA 2.1. — *There exists $C > 0$ such that*

$$\| \check{E}_l \|_2 \leq C |l| \quad \text{for all } l \in \mathbb{Z}^{1+n}.$$

Proof. — We have

$$|E_l(\lambda)| \leq |l \cdot e^{-p(\lambda)}| \leq \sum_{\nu=0}^n |l_\nu| e^{-p_\nu(\lambda)} \leq (1+n)|l| e^{-p^*(\lambda)},$$

so that in particular, if $f = e^{p^*} E_l$, then

$$\|\check{E}_l\|_2 = \|f(L)\mathcal{K}_L(e^{-p^*})\|_2 \leq \|f\|_\infty \|\mathcal{K}_L(e^{-p^*})\|_2 \leq (1+n)\|\mathcal{K}_L(e^{-p^*})\|_2 |l|,$$

which is the conclusion. □

LEMMA 2.2. — *There exist $c, \omega > 0$ such that*

$$\|\check{E}_l\|_{L^2(G, e^{2|x|_G} dx)} \leq c e^{\omega|l|} \quad \text{for all } l \in \mathbb{Z}^{1+n}.$$

Proof. — Since all the connected left-invariant distances on G are equivalent in the large [51, Proposition III.4.2], by interpolating the inequalities (e) and (f) of [26, Theorem 2.3], we have that there exist $c \geq 1$ and $\omega > 0$ such that

$$\|h_{\nu,t} e^{|\cdot|_G}\|_q \leq c e^{\omega t} \quad \text{for } t \geq 1, \nu = 0, \dots, n \text{ and } q \in [1, \infty].$$

By Young’s inequality and submultiplicativity of $e^{|\cdot|_G}$, we then get

$$\|(h_{0,k_0} * \dots * h_{n,k_n}) e^{|\cdot|_G}\|_q \leq c^{1+n} e^{\omega(k_0 + \dots + k_n)}$$

for $k \in \mathbb{N}^{1+n} \setminus \{0\}$ and $q \in [1, \infty]$. This means in particular that the series in (2.1) converges absolutely in $L^2(G, e^{2|x|_G} dx)$, with

$$\sum_{0 \neq k \in \mathbb{N}^{1+n}} \left\| \frac{(il_0)^{k_0} \dots (il_n)^{k_n}}{k_0! \dots k_n!} h_{0,k_0} * \dots * h_{n,k_n} \right\|_{L^2(G, e^{2|x|_G} dx)} \leq c^{1+n} e^{\omega|l|},$$

and we are done. □

LEMMA 2.3. — *For all $\alpha \geq 0$, we have*

$$\|\check{E}_l\|_{L^2(G, \langle x \rangle_G^{2\alpha} dx)} \leq C_\alpha |l|^{\alpha+1} \quad \text{for } l \in \mathbb{Z}^{1+n}.$$

Proof. — By Lemma 2.2, it is sufficient to check the estimate for $|l|$ large, but then

$$\begin{aligned} \int_G |\check{E}_l(x)|^2 \langle x \rangle_G^{2\alpha} dx &\leq \int_{|x|_G \leq \omega|l|} + \int_{|x|_G > \omega|l|} \\ &\leq (1 + \omega|l|)^{2\alpha} \|\check{E}_l\|_2^2 + \sup_{r > \omega|l|} \frac{(1+r)^{2\alpha}}{e^{2r}} \|\check{E}_l\|_{L^2(G, e^{2|x|_G} dx)}^2 \leq C_\alpha |l|^{2(\alpha+1)} \end{aligned}$$

by Lemmata 2.1 and 2.2. □

LEMMA 2.4. — Let $K \subseteq \mathbb{R}^n$ be compact. For every $f \in \mathcal{D}(\mathbb{R}^n)$ supported in K , there exists $g \in \mathcal{D}(\mathbb{T}^{1+n})$, depending linearly on f , such that

$$(2.2) \quad f(\lambda) = g\left(e^{ie^{-p(\lambda)}}\right) = g\left(e^{ie^{-p_0(\lambda)}}, \dots, e^{ie^{-p_n(\lambda)}}\right),$$

$$(2.3) \quad g(1, \dots, 1) = 0,$$

$$(2.4) \quad \|g\|_{B_{2,2}^s(\mathbb{T}^{1+n})} \leq C_{K,s} \|f\|_{B_{2,2}^s(\mathbb{R}^n)} \quad \text{for all } s \geq 0.$$

In particular, if $g(e^{it}) = \sum_{l \in \mathbb{Z}^{1+n}} \hat{g}(l)e^{il \cdot t}$ is the Fourier-series development of g , then we have

$$(2.5) \quad f = \sum_{0 \neq l \in \mathbb{Z}^{1+n}} \hat{g}(l)E_l,$$

with uniform convergence on \mathbb{R}^n .

Proof. — Since $K \subseteq \mathbb{R}^n$ is compact and the polynomials p_0, \dots, p_n are strictly positive, $p(K)$ is a compact subset of $\Omega =]0, +\infty[^{1+n}$. Therefore we can choose $\psi_K \in \mathcal{D}(\Omega)$ such that $\psi_K|_{p(K)} \equiv 1$. If we put

$$\tilde{f}(y) = f(y_1 - y_0, \dots, y_n - y_0)\psi_K(y) \quad \text{for } y \in \mathbb{R}^{1+n},$$

we then have that $\tilde{f} \in \mathcal{D}(\Omega)$, $f = \tilde{f} \circ p$ and

$$\|\tilde{f}\|_{B_{2,2}^s(\mathbb{R}^{1+n})} \leq C_{K,s} \|f\|_{B_{2,2}^s(\mathbb{R}^n)} \quad \text{for all } s \geq 0,$$

since the change of variables has maximal rank.

Notice now that the map

$$\Phi : \Omega \ni y \mapsto e^{ie^{-y}} = (e^{ie^{-y_0}}, \dots, e^{ie^{-y_n}}) \in \mathbb{T}^{1+n}$$

is a smooth diffeomorphism with its image, which is an open subset of \mathbb{T}^{1+n} not containing $(1, \dots, 1)$. The function $g = \tilde{f} \circ \Phi^{-1} \in \mathcal{D}(\Phi(\Omega))$ can be then extended by zero to a smooth function on \mathbb{T}^{1+n} , and we have clearly

$$\|g\|_{B_{2,2}^s(\mathbb{T}^{1+n})} \leq C_{K,s} \|\tilde{f}\|_{B_{2,2}^s(\mathbb{R}^{1+n})} \quad \text{for all } s \geq 0.$$

The construction shows that g depends linearly on f and satisfies (2.2)-(2.4). In particular, we have $\sum_{l \in \mathbb{Z}^{1+n}} \hat{g}(l) = 0$, so that the Fourier decomposition of g can be rewritten as

$$g(e^{it}) = \sum_{0 \neq l \in \mathbb{Z}^{1+n}} \hat{g}(l)(e^{il \cdot t} - 1)$$

(with uniform convergence since g is smooth), which gives (2.5). □

PROPOSITION 2.5. — Let $K \subseteq \mathbb{R}^n$ be compact, $\alpha \geq 0, \beta > \alpha + (n+3)/2$. For all $f \in \mathcal{D}(\mathbb{R}^n)$ with $\text{supp } f \subseteq K$, we have

$$\|\tilde{f}\|_{L^2(G, \langle x \rangle^{2\alpha} dx)} \leq C_{K,\alpha,\beta} \|f\|_{B_{2,2}^\beta(\mathbb{R}^n)}.$$

Proof. — Let $g \in \mathcal{D}(\mathbb{T}^{1+n})$ be given by Lemma 2.4. Then

$$\check{f} = \sum_{0 \neq l \in \mathbb{Z}^{1+n}} \hat{g}(l) \check{E}_l$$

in the norm of convolution operators on $L^2(G)$. However, the series in the right-hand side converges absolutely in $L^2(G, \langle x \rangle_G^{2\alpha} dx)$, since

$$\begin{aligned} \sum_{0 \neq l \in \mathbb{Z}^{1+n}} |\hat{g}(l)| \|\check{E}_l\|_{L^2(G, \langle x \rangle_G^{2\alpha} dx)} &\leq C_\alpha \sum_{0 \neq l \in \mathbb{Z}^{1+n}} |\hat{g}(l)| |l|^{\alpha+1} \\ &\leq C_{\alpha, \beta} \|g\|_{B_{2,2}^\beta(\mathbb{T}^{1+n})} \leq C_{K, \alpha, \beta} \|f\|_{B_{2,2}^\beta(\mathbb{R}^n)} \end{aligned}$$

by Lemma 2.3 and Hölder’s inequality, and the conclusion follows. □

The previous proposition contains a “preliminary version” of the required inequalities, which we are now going to sharpen by interpolation with the Plancherel formula. In order to control the $L^2(\sigma)$ norm with a Besov norm, we will use a refined trace theorem due to Triebel.

Let τ be a (positive) regular Borel measure on \mathbb{R}^n , and let $0 \leq d \leq n$; we say that τ is *locally d -bounded* on an open $\Omega \subseteq \mathbb{R}^n$ if, for every compact $K \subseteq \Omega$ and for $0 \leq \gamma < d$, there exist $C, \bar{r} > 0$ such that

$$\tau(B(\lambda, r)) \leq Cr^\gamma \quad \text{for } \lambda \in K \text{ and } r \leq \bar{r}.$$

Notice that every regular Borel measure τ is locally 0-bounded on the whole \mathbb{R}^n ; moreover, if τ is homogeneous with respect to some system of dilations ϵ_t on \mathbb{R}^n (i.e., if $\tau(\epsilon_t(A)) = t^a \tau(A)$ for some $a \geq 0$ and every Borel $A \subseteq \mathbb{R}^n$), then τ is locally 1-bounded on $\mathbb{R}^n \setminus \{0\}$.

LEMMA 2.6. — *Let τ be a regular Borel measure which is locally d -bounded on an open $\Omega \subseteq \mathbb{R}^n$. If $s > (n - d)/2$ and $K \subseteq \Omega$ is compact, then*

$$\|f\|_{L^2(\tau)} \leq C_{K,s} \|f\|_{B_{2,2}^s(\mathbb{R}^n)}$$

for every $f \in \mathcal{D}(\mathbb{R}^n)$ with $\text{supp } f \subseteq K$.

Proof. — If $d = 0$, then $\|f\|_\infty \leq C_s \|f\|_{B_{2,2}^s(\mathbb{R}^n)}$ and the result is trivial. Suppose instead that $0 < d \leq n$, and let $K \subseteq \Omega$ be compact, $\varepsilon > 0$. Choose a compact neighborhood $K' \subseteq \Omega$ of K , and let $C, \bar{r} > 0$ such that

$$\tau(B(\lambda, r)) \leq Cr^{d-\varepsilon} \quad \text{for } \lambda \in K' \text{ and } 0 < r \leq \bar{r}.$$

Let moreover $\bar{r}' = \min\{\bar{r}, \text{dist}(\Omega \setminus K', K)\}$, $C' = \max\{C, \tau(K)/(\bar{r}')^{d-\varepsilon}\}$. The identity $\tau_K(E) = \tau(E \cap K)$ defines a positive regular Borel measure τ_K on \mathbb{R}^n , which coincides with τ on K , and with $\text{supp } \tau_K \subseteq K$. Moreover

$$\tau_K(B(\lambda, r)) \leq C' r^{d-\varepsilon} \quad \text{for every } r > 0 \text{ and } \lambda \in \mathbb{R}^n,$$

by construction. Therefore, since the Besov space $B_{2,2}^s(\mathbb{R}^n)$ coincides with the Triebel-Lizorkin space $F_{2,2}^s(\mathbb{R}^n)$, by [50, Corollary 9.8(ii)] we have

$$\|f\|_{L^2(\tau)} = \|f\|_{L^2(\tau_K)} \leq C_{K,s} \|f\|_{B_{2,2}^s(\mathbb{R}^n)}$$

for $s > (n - d)/2 + \varepsilon/2$ and $f \in \mathcal{D}(\mathbb{R}^n)$ with $\text{supp } f \subseteq K$. The conclusion follows from the arbitrariness of $\varepsilon > 0$. □

THEOREM 2.7. — *Suppose that, for some open $\Omega \subseteq \mathbb{R}^n$, the Plancherel measure σ is locally d -bounded on Ω . Let $K \subseteq \Omega$ be compact, $D \in \mathfrak{D}(G)$, $p, q \in [1, \infty]$, $\alpha \geq 0$,*

$$\beta > \alpha + Q_G \left(\frac{1}{\min\{2, p\}} - \frac{1}{2} \right) + \frac{n}{q} - \frac{d}{\max\{2, q\}}.$$

For all $m \in B_{q,q}^\beta(\mathbb{R}^n)$ with $\text{supp } m \subseteq K$, we have

$$\|\langle \cdot \rangle_G^\alpha D \check{m}\|_{L^p(G)} \leq C_{K,D,\alpha,\beta,p,q} \|m\|_{B_{q,q}^\beta(\mathbb{R}^n)}.$$

Proof. — Consider first the case $p = 2, D = 1$. Let $\xi \in \mathcal{D}(\mathbb{R}^n)$ be such that $\text{supp } \xi \subseteq \Omega, \xi|_K \equiv 1$, and let $K' \subseteq \Omega$ be a compact neighborhood of $\text{supp } \xi$. Proposition 2.5, together with the continuous inclusion $B_{q,2}^\beta(\mathbb{R}^n) \subseteq B_{2,2}^{\beta-n/2}(\mathbb{R}^n)$, then yields, for $m \in \mathcal{D}(\mathbb{R}^n)$ with $\text{supp } m \subseteq K'$, that

$$(2.6) \quad \|\check{m}\|_{L^2(G, \langle x \rangle_G^{2\alpha} dx)} \leq C_{K,\alpha,\beta,q} \|m\|_{B_{q,2}^\beta(\mathbb{R}^n)}$$

for $\beta > \alpha + (2n + 3)/2$. By the use of a suitable approximate identity, (2.6) can be easily extended to all $m \in B_{q,2}^\beta(\mathbb{R}^n)$ with $\text{supp } m \subseteq \text{supp } \xi$. Hence, if we consider the linear map $M : m \mapsto \mathcal{K}_L(m\xi)$, then we have that

$$(2.7) \quad M \text{ is bounded } B_{q,2}^\beta(\mathbb{R}^n) \rightarrow L^2(G, \langle x \rangle_G^{2\alpha} dx) \text{ for } \beta > \alpha + (2n + 3)/2.$$

On the other hand, for $\alpha = 0$, the Plancherel formula and Lemma 2.6 give

$$(2.8) \quad M \text{ is bounded } B_{q,2}^\beta(\mathbb{R}^n) \rightarrow L^2(G) \text{ for } \beta > \frac{n}{q} - \frac{d}{\max\{2, q\}}$$

(this is clear for $q = 2$ and $q = \infty$; for $1 \leq q < 2$, we exploit the continuous inclusion $B_{q,2}^\beta(\mathbb{R}^n) \subseteq B_{2,2}^{\beta-n/q+n/2}(\mathbb{R}^n)$; for $2 < q < \infty$, we interpolate). Therefore, by interpolating (2.7) and (2.8), we get

$$M \text{ is bounded } B_{q,2}^\beta(\mathbb{R}^n) \rightarrow L^2(G, \langle x \rangle_G^{2\alpha} dx) \text{ for } \beta > \alpha + \frac{n}{q} - \frac{d}{\max\{2, q\}}.$$

In order to conclude, it is sufficient to notice that, if $\beta > \alpha + n/q - d/\max\{2, q\}$, then for any $\beta' \in]\alpha + n/q - d/\max\{2, q\}, \beta[$ we have the continuous inclusion $B_{q,q}^\beta(\mathbb{R}^n) \subseteq B_{q,2}^{\beta'}(\mathbb{R}^n)$, and moreover $Mm = \check{m}$ for every m with $\text{supp } m \subseteq K$.

Take now an arbitrary $D \in \mathfrak{D}(G)$. For $m \in B_{\infty, \infty}^\beta(\mathbb{R}^n)$ with $\text{supp } m \subseteq K$, set $m_0 = me^{p*}$, $\xi = e^{-p*}$; then $\check{m} = \check{m}_0 * \check{\xi}$, so that, by Young’s inequality,

$$\|\langle \cdot \rangle_G^\alpha D\check{m}\|_2 \leq \|\langle \cdot \rangle_G^\alpha \check{m}_0\|_2 \|\langle \cdot \rangle_G^\alpha D\check{\xi}\|_1 \lesssim \|m_0\|_{B_{q,q}^\beta(\mathbb{R}^n)} \lesssim \|m\|_{B_{q,q}^\beta(\mathbb{R}^n)}$$

for $\beta > \alpha + n/q - d/\max\{2, q\}$. This concludes the proof for the case $p = 2$.

The case $1 \leq p < 2$ follows by Hölder’s inequality, since $\int_G \langle x \rangle_G^{-\alpha} dx < \infty$ for $\alpha > Q_G$.

Let now $p = \infty$. If $\zeta \in \mathcal{D}(\mathbb{R}^n)$ is nonnegative and $\zeta(e) > 0$, and if we set $w_\alpha = \langle \cdot \rangle_G^\alpha * \zeta$ for $\alpha \geq 0$, then w_α is smooth and nonnegative,

$$\langle x \rangle_G^\alpha \leq C_\alpha w_\alpha(x), \quad Dw_\alpha(x) \leq C_{D,\alpha} \langle x \rangle_G^\alpha$$

for all $D \in \mathfrak{D}(G)$. If \mathfrak{D}_* is a basis of the $A \in \mathfrak{D}(G)$ of order up to $\lfloor (\dim G)/2 \rfloor + 1$, then we have, for $\beta > \alpha + n/q - d/\max\{2, q\}$ and $D \in \mathfrak{D}(G)$,

$$\begin{aligned} \|\langle \cdot \rangle_G^\alpha D\check{m}\|_\infty &\lesssim \|w_\alpha D\check{m}\|_\infty \lesssim \sum_{A \in \mathfrak{D}_*} \|A(w_\alpha D\check{f})\|_2 \\ &\lesssim \sum_{A_1, A_2 \in \mathfrak{D}_*} \|(A_1 w_\alpha)(A_2 D\check{m})\|_2 \lesssim \sum_{A \in \mathfrak{D}_*} \|\langle \cdot \rangle_G^\alpha AD\check{m}\|_2 \lesssim \|m\|_{B_{q,q}^\beta} \end{aligned}$$

by Sobolev’s embedding, Leibniz’s rule and the case $p = 2$.

The remaining case $2 < p < \infty$ follows by interpolation. □

3. Improved weighted estimates

The weighted estimates given by Theorem 2.7 for $p = 1$ yield a “weak multiplier theorem” for a weighted subcoercive system L_1, \dots, L_n on a Lie group G of polynomial growth: the operator $m(L)$ is bounded on $L^p(G)$ for $1 \leq p \leq \infty$ if the multiplier m is compactly supported and sufficiently smooth; more precisely, by taking $q = \infty$, we require an order of smoothness $\gamma > Q_G/2$, where Q_G is the dimension at infinity of G . If $G = \mathbb{R}^n$, then Q_G coincides with the topological dimension $\dim G = n$; for non-abelian (simply connected) nilpotent groups, however, $Q_G > \dim G$. Nevertheless, for a particular class of 2-step nilpotent groups (namely, Heisenberg and related groups) multiplier theorems have been proved with $(\dim G)/2$ as the regularity threshold [17, 30, 20, 32, 52]. In this section, we extend to our context of weighted subcoercive systems the technique of Hebisch and Zienkiewicz [20], which allows in some cases to lower the threshold in the weighted L^1 estimates.

Let G be a nilpotent Lie group, with Lie algebra \mathfrak{g} . Let \mathfrak{z} be the center of \mathfrak{g} and set

$$(3.1) \quad \mathfrak{h} = \{v \in \mathfrak{g} : [v, \mathfrak{g}] \subseteq \mathfrak{z}\};$$

\mathfrak{h} is a characteristic ideal of \mathfrak{g} containing \mathfrak{z} (in fact, it is the term following \mathfrak{z} in the ascending central series of \mathfrak{g}). Let moreover $P : \mathfrak{g} \rightarrow \mathfrak{g}/\mathfrak{z}$ be the canonical projection. The bilinear map $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$ induces, by restriction, passage to the quotient and transposition, another bilinear map

$$J : \mathfrak{g}/\mathfrak{z} \times \mathfrak{z}^* \rightarrow \mathfrak{h}^*,$$

which we will call the *capacity map* of \mathfrak{g} , and is uniquely determined by

$$J(P(x), \tau)(y) = \tau([x, y])$$

for $x \in \mathfrak{g}$, $y \in \mathfrak{h}$, $\tau \in \mathfrak{z}^*$. The group G is said to be an *H-type group* if there exists an inner product on \mathfrak{g} such that, for every $\tau \in \mathfrak{z}^*$ of norm 1, the map $J(\cdot, \tau) : \mathfrak{g}/\mathfrak{z} \rightarrow \mathfrak{h}^*$ is an isometric embedding (this condition implies that $\mathfrak{g} = \mathfrak{h}$, so that G is 2-step). If G is an H-type group, then in particular

$$(3.2) \quad |J(\bar{x}, \tau)| \geq |\bar{x}| |\tau|$$

for suitable norms on $\mathfrak{g}/\mathfrak{z}$, \mathfrak{z}^* and \mathfrak{h}^* ; the validity of such an inequality defines the class of *Métivier groups*, which has been introduced in the study of analytic hypoellipticity of Rockland operators [29, 21]; this class is strictly larger than that of H-type groups (see [33] for an example), but is still contained in the class of 2-step groups.

In the following, we consider a more general inequality of the form

$$|J(\bar{x}, \tau)| \geq w(\bar{x}) \zeta(\tau)$$

for some non-negative functions $w : \mathfrak{g}/\mathfrak{z} \rightarrow \mathbb{R}$, $\zeta : \mathfrak{z}^* \rightarrow \mathbb{R}$, which may hold also on higher-step groups. Rewritten as

$$w(\bar{x})^\gamma \leq |J(\bar{x}, \tau)|^\gamma \zeta(\tau)^{-\gamma}$$

for some $\gamma > 0$, this inequality will be interpreted via the spectral theorem, in order to control a multiplication operator (corresponding to $w(\bar{x})^\gamma$) with a function of the central derivatives (corresponding to $\zeta(\tau)^{-\gamma}$); in this interpretation, it turns out that $|J(\bar{x}, \tau)|^2$ corresponds to a sum of products of left- and right-invariant differential operators on G , therefore the term $|J(\bar{x}, \tau)|^\gamma$ can be dominated by an *a priori* estimate for a weighted subcoercive operator on the direct product $G \times G$.

In order to fill in the details, it is convenient to introduce some notation. For every smooth differential operator D on G , the identity

$$(3.3) \quad (Df)^* = D^\circ f^*$$

defines another differential operator D° on G ; the map $D \mapsto D^\circ$ is a conjugate-linear involutive automorphism of the unital algebra of all smooth differential operators on G , which maps left-invariant operators to right-invariant ones and *vice versa*.

The Lie algebra $\tilde{\mathfrak{g}}$ of the direct product $\tilde{G} = G \times G$ is canonically isomorphic to $\mathfrak{g} \oplus \mathfrak{g}$; we define the correspondence $D \mapsto D^\bullet$ on $\mathfrak{D}(\tilde{G})$ as the unique conjugate-linear automorphism of the unital algebra $\mathfrak{D}(\tilde{G}) \cong U((\mathfrak{g} \oplus \mathfrak{g})_{\mathbb{C}})$ extending the Lie algebra automorphism $(X, Y) \mapsto (Y, X)$ of $\mathfrak{g} \oplus \mathfrak{g}$.

Let ξ be the unitary representation of \tilde{G} on $L^2(G)$ given by $\xi(x, y)f = R_x L_y f$. Then, for every $D \in \mathfrak{D}(\tilde{G})$, $d\xi(D)$ is a smooth differential operator on G , and

$$d\xi(D^\bullet) = d\xi(D)^\circ.$$

Finally, for $D \in \mathfrak{D}(G)$, let $\tilde{D} \in \mathfrak{D}(\tilde{G})$ be defined by $\tilde{D}(f \otimes g) = (Df) \otimes g$, so that in particular $d\xi(\tilde{D}) = D$.

LEMMA 3.1. — *Let $B = B^+ \in \mathfrak{D}(G)$ be weighted subcoercive, and set $\Delta = B^2$. Then $\tilde{\Delta} + \tilde{\Delta}^\bullet$ is positive weighted subcoercive on \tilde{G} .*

Proof. — For $D \in \mathfrak{D}(G)$, let $\bar{D} \in \mathfrak{D}(G)$ be the differential operator uniquely determined by the identity $\bar{D}f = \overline{Df}$. The map $D \mapsto \bar{D}$ defines a conjugate-linear involutive automorphism of the unital algebra $\mathfrak{D}(G)$, and it is easily proved that $\tilde{D}^\bullet(f \otimes g) = f \otimes (\bar{D}g)$. In particular, we have

$$(\tilde{\Delta} + \tilde{\Delta}^\bullet)(f \otimes g) = (B^2 f) \otimes g + f \otimes (\bar{B}^2 g).$$

In view of [26, Theorem 5.4], since B is self-adjoint and weighted subcoercive, in order to conclude it will be sufficient to show that \bar{B} is weighted subcoercive too.

As in [26, §2], fix a weighted structure of \mathfrak{g} and a weighted subcoercive form C such that $dR_G(C) = B$. If \bar{C} is the form defined by $\bar{C}(\alpha) = \overline{C(\alpha)}$, then it is easy to see that $\bar{B} = dR_G(\bar{C})$, and, on the other hand,

$$\Re\langle \phi, dR_G(\bar{C})\phi \rangle = \Re\langle \bar{\phi}, dR_G(C)\bar{\phi} \rangle,$$

thus \bar{C} is also weighted subcoercive by definition. □

Let $L_1, \dots, L_n \in \mathfrak{D}(G)$ be a weighted subcoercive system on the nilpotent Lie group G , and let $\Delta = p(L)^2$, where p is a real polynomial such that $p(L)$ is weighted subcoercive. We define

$$\tilde{A} = (\tilde{\Delta} + \tilde{\Delta}^\bullet)/2, \quad A = d\xi(\tilde{A}) = (\Delta + \Delta^\circ)/2.$$

By Lemma 3.1, \tilde{A} is a (left-invariant) positive weighted subcoercive operator on \tilde{G} , whereas A is a differential operator on G which in general is

neither left- nor right-invariant; since $\tilde{A}, \tilde{\Delta}, \tilde{\Delta}^\bullet$ form a weighted subcoercive system, the corresponding operators A, Δ, Δ° in the representation ξ admit a joint spectral resolution.

Let h_t ($t > 0$) be the convolution kernel of $e^{-t\Delta}$.

LEMMA 3.2. — Suppose that $u \in L^2(G)$ commutes with all the h_t ($t > 0$). For all Borel $m : \mathbb{R} \rightarrow \mathbb{C}$, u is in the domain of $m(\Delta)$ if and only if it is in the domain of $m(A)$, and in this case

$$(3.4) \quad m(A)u = m(\Delta)u.$$

Proof. — From (3.3) we easily deduce

$$e^{-t\Delta^\circ} f = (f^* * h_t)^* = h_t * f$$

and

$$e^{-tA} f = e^{-t\Delta/2} e^{-t\Delta^\circ/2} f = h_{t/2} * f * h_{t/2},$$

so that $e^{-tA}u = e^{-t\Delta}u$. If $\xi_t(\lambda) = e^{-t\lambda}$ and $\mathcal{J}_0 = \text{span}\{\xi_t : t > 0\}$, then we obtain (3.4) for $m \in \mathcal{J}_0$. It is not difficult to extend (3.4) to $m \in C_0(\mathbb{R})$ by the Stone-Weierstrass theorem, and then to all Borel $m : \mathbb{R} \rightarrow \mathbb{C}$ by the spectral theorem and dominated convergence. □

LEMMA 3.3. — Let $X \in \mathfrak{g}$. Then, for all $v \in \mathfrak{g}$,

$$(X + X^\circ)|_{\exp(v)} = d \exp_v([v, X]).$$

Proof. — The semigroup associated to $\tilde{X} + \tilde{X}^\bullet$ is $t \mapsto (\exp(tX), \exp(tX))$, so that, for all $f \in \mathcal{D}(G)$, $v \in \mathfrak{g}$,

$$(X + X^\circ)|_{\exp(v)} f = \left. \frac{d}{dt} \right|_{t=0} f(\exp(-tX) \exp(v) \exp(tX)).$$

Since $\exp(-tX) \exp(v) \exp(tX) = \exp(\text{Ad}(\exp(-tX))(v))$, we have

$$\left. \frac{d}{dt} \right|_{t=0} (\exp(-tX) \exp(v) \exp(tX)) = d \exp_v(\text{ad}(-X)(v)) = d \exp_v([v, X]),$$

which is the conclusion. □

In the following, we will identify G with \mathfrak{g} via the exponential map. Choose a basis ν_1, \dots, ν_r of $(\mathfrak{g}/\mathfrak{z})^*$ and a basis T_1, \dots, T_d of \mathfrak{z} , and set $P_j = \nu_j \circ P$. The functions $P_j : G \rightarrow \mathbb{R}$ can be thought of as multiplication operators on $L^2(G)$, and it is not difficult to show that the operators

$$P_1, \dots, P_r, -iT_1, \dots, -iT_d$$

are (essentially) self-adjoint on $L^2(G)$ and commute strongly pairwise, so that they admit a joint spectral resolution.

Through the chosen bases, J can be identified with a bilinear map $\mathbb{R}^r \times \mathbb{R}^d \rightarrow \mathfrak{h}^*$. Therefore, for every $Y \in \mathfrak{h}$, we have a bilinear form $J(\cdot, \cdot)(Y) : \mathbb{R}^r \times \mathbb{R}^d \rightarrow \mathbb{R}$, which in fact is a polynomial; we can then evaluate this particular polynomial in the operators $P_1, \dots, P_r, -iT_1, \dots, -iT_d$, and denote by $J(P, -iT)(Y)$ the resulting operator on $L^2(G)$. Finally, choose an inner product on \mathfrak{h} (which induces an inner product on \mathfrak{h}^*) and an orthonormal basis $\{Y_l\}_l$ of \mathfrak{h} ; then also the map $|J(\cdot, \cdot)|^2$ is a polynomial, thus as before we can consider the operator $|J(P, -iT)|^2$ on $L^2(G)$, and clearly $|J(P, -iT)|^2 = \sum_l (J(P, -iT)(Y_l))^2$.

LEMMA 3.4. — *For all $Y \in \mathfrak{h}$, $J(P, -iT)(Y)$ is a differential operator on G ; more precisely, $J(P, -iT)(Y) = -i(Y + Y^\circ)$. In particular*

$$|J(P, -iT)|^2 = - \sum_l (Y_l + Y_l^\circ)^2 = d\xi \left(- \sum_l (\tilde{Y}_l + \tilde{Y}_l^\bullet)^2 \right).$$

Proof. — Let $\hat{T}_1, \dots, \hat{T}_d \in \mathfrak{z}^*$ and $\hat{\nu}_1, \dots, \hat{\nu}_r \in \mathfrak{g}/\mathfrak{z}$ be the dual bases of T_1, \dots, T_d and ν_1, \dots, ν_r respectively. Then, by bilinearity, for every $Y \in \mathfrak{h}$,

$$J(P, -iT)(Y) = -i \sum_{j,k} J(\hat{\nu}_j, \hat{T}_k)(Y) P_j T_k.$$

This shows that $J(P, -iT)(Y)$ is a differential operator on G . In fact, for all $x \in G \in \mathfrak{g}$, we have $\sum_j P_j(x) \hat{\nu}_j = P(x)$, therefore

$$\begin{aligned} J(P, -iT)(Y)|_x &= -i \sum_k J(P(x), \hat{T}_k)(Y) T_k \\ &= -i \sum_k \hat{T}_k([x, Y]) T_k = -i[x, Y] = -i(Y + Y^\circ)|_x \end{aligned}$$

by Lemma 3.3 (notice that, since T_1, \dots, T_d are central, they are constant vector fields in exponential coordinates). □

Since T_1, \dots, T_d are central, the left-invariant differential operators

$$(3.5) \quad L_1, \dots, L_n, -iT_1, \dots, -iT_d$$

on G are a weighted subcoercive system. We can thus consider the Plancherel measure σ' on $\mathbb{R}^n \times \mathfrak{z}^*$ associated to this system, which can be shown not to depend on the choice of the basis of \mathfrak{z} .

The core of the technique under discussion is contained in the following

PROPOSITION 3.5. — *Suppose that, for some nonnegative Borel functions $w : \mathfrak{g}/\mathfrak{z} \rightarrow \mathbb{R}$ and $\zeta : \mathfrak{z}^* \rightarrow \mathbb{R}$, we have*

$$|J(\bar{x}, \tau)| \geq w(\bar{x}) \zeta(\tau) \quad \text{for all } \bar{x} \in \mathfrak{g}/\mathfrak{z}, \tau \in \mathfrak{z}^*.$$

If $K \subseteq \mathbb{R}^n$ is compact and $\gamma \geq 0$, then, for all $m \in \mathcal{D}(\mathbb{R}^n)$ with $\text{supp } m \subseteq K$,

$$\|w \circ P|\gamma \check{m}\|_2^2 \leq C_{K,\gamma} \int_{\mathbb{R}^n \times \mathfrak{z}^*} |m(\lambda)|^2 \zeta(\tau)^{-2\gamma} d\sigma'(\lambda, \tau).$$

Proof. — From the hypothesis we deduce, by the spectral theorem,

$$\|w \circ P|\gamma f\|_2 \leq C_\gamma \| |J(P, -iT)|^\gamma \zeta(-iT)^{-\gamma} f \|_2$$

for $f \in L^2(G)$. By Lemma 3.4, $|J(P, -iT)|^2 = d\xi(D)$ for some $D \in \mathfrak{D}(\tilde{G})$; since \tilde{A} is weighted subcoercive on \tilde{G} , by [26, Theorem 2.3(iii)], for some polynomial q_γ we have, in the representation ξ ,

$$\| |J(P, -iT)|^\gamma \psi \|_2 \leq C_\gamma \|q_\gamma(A)\psi\|_2,$$

therefore, by putting the two inequalities together, we get

$$\|w \circ P|\gamma f\|_2 \leq C_\gamma \| \zeta(-iT)^{-\gamma} q_\gamma(A) f \|_2$$

(since the T_j commute strongly with A). In particular, if we take $f = \check{m}$,

$$\|w \circ P|\gamma \check{m}\|_2 \leq C_\gamma \| \zeta(-iT)^{-\gamma} q_\gamma(A) \check{m} \|_2 = C_\gamma \| \zeta(-iT)^{-\gamma} q_\gamma(\Delta) \check{m} \|_2$$

by Lemma 3.2, since \check{m} commutes with all the h_t . On the other hand, by the Plancherel formula for the system (3.5),

$$\| \zeta(-iT)^{-\gamma} q_\gamma(\Delta) \check{m} \|_2^2 \leq C_{K,\gamma} \int_{\mathbb{R}^n \times \mathfrak{z}^*} |\check{m}(\lambda)|^2 \zeta(\tau)^{-2\gamma} d\sigma'(\lambda, \tau),$$

where $C_{K,\gamma} = \sup_{\lambda \in K} q_\gamma(p(\lambda)^2)^2$, and we are done. □

Simple manipulations give a slightly more general form of the previous estimate:

COROLLARY 3.6. — *Suppose that, for some nonnegative Borel functions $w_j : \mathfrak{g}/\mathfrak{z} \rightarrow \mathbb{R}$ and $\zeta_j : \mathfrak{z}^* \rightarrow \mathbb{R}$ ($j = 1, \dots, h$), we have*

$$|J(\bar{x}, \tau)| \geq w_j(\bar{x}) \zeta_j(\tau) \quad \text{for all } \bar{x} \in \mathfrak{g}/\mathfrak{z}, \tau \in \mathfrak{z}^*,$$

and set $\tilde{w}_j(x) = 1 + w_j(P(x))$. If $K \subseteq \mathbb{R}^n$ is compact, then for all $m \in \mathcal{D}(\mathbb{R}^n)$ with $\text{supp } m \subseteq K$ and for all $\vec{\gamma} = (\gamma_1, \dots, \gamma_h) \geq 0$ we have

$$\| \tilde{w}_1^{\gamma_1} \cdots \tilde{w}_h^{\gamma_h} \check{m} \|_{L^2(G)}^2 \leq C_{K,\vec{\gamma}} \int_{\mathbb{R}^n \times \mathfrak{z}^*} |m(\lambda)|^2 \prod_{j=1}^h (1 + \zeta_j(\tau)^{-2\gamma_j}) d\sigma'(\lambda, \tau).$$

Proof. — If we set, for $I \subseteq \{1, \dots, h\}$,

$$\gamma_I = \sum_{j \in I} \gamma_j, \quad w_{\vec{\gamma}, I}(\bar{x}) = \prod_{j \in I} w_j(\bar{x})^{\gamma_j / \gamma_I}, \quad \zeta_{\vec{\gamma}, I}(\tau) = \prod_{j \in I} \zeta_j(\tau)^{\gamma_j / \gamma_I},$$

then clearly

$$|J(\bar{x}, \tau)| \geq w_{\vec{\gamma}, I}(\bar{x}) \zeta_{\vec{\gamma}, I}(\tau) \quad \text{for all } \bar{x} \in \mathfrak{g}/\mathfrak{z}, \tau \in \mathfrak{z}^*,$$

and moreover

$$\prod_{j=1}^h \tilde{w}_j^{2\gamma_j} \leq C_{\tilde{\gamma}} \sum_{I \subseteq \{1, \dots, h\}} (w_{\tilde{\gamma}, I} \circ P)^{2\gamma_I}, \quad \prod_{j=1}^h (1 + \zeta_j^{-2\gamma_j}) = \sum_{I \subseteq \{1, \dots, h\}} \zeta_{\tilde{\gamma}, I}^{-2\gamma_I},$$

therefore the conclusion follows by repeated application of Proposition 3.5. □

Under some particular hypotheses, we may therefore control a weighted L^2 norm of \tilde{m} in terms of an $L^2(\sigma_{\tilde{\zeta}})$ norm of m , where $\sigma_{\tilde{\zeta}}$ is the push-forward of

$$(3.6) \quad \tilde{\zeta}(\tau) d\sigma'(\lambda, \tau)$$

on the first factor of $\mathbb{R}^n \times \mathfrak{z}^*$, for some nonnegative function $\tilde{\zeta} : \mathfrak{z}^* \rightarrow \mathbb{R}$.

LEMMA 3.7. — (i) Suppose that $\tilde{\zeta} \in L^1_{\text{loc}}(\mathfrak{z}^*)$ is nonnegative. Then $\sigma_{\tilde{\zeta}}$ is a regular Borel measure on \mathbb{R}^n .

(ii) Suppose moreover that G is a homogeneous group, with dilations δ_t and homogeneous dimension Q_δ , and that L_1, \dots, L_n is a homogeneous system, with associated dilations ϵ_t . If $\tilde{\zeta}$ is homogeneous of degree a , i.e., $\tilde{\zeta}(\tau \circ \delta_t) = t^a \tilde{\zeta}(\tau)$, then $\sigma_{\tilde{\zeta}}$ is homogeneous of degree $Q_\delta + a$, i.e., $\sigma_{\tilde{\zeta}}(\epsilon_t(A)) = t^{Q_\delta + a} \sigma_{\tilde{\zeta}}(A)$.

Proof. — (i) Let $K \subseteq \mathbb{R}^n$ be compact. By [26, Lemma 3.16], the canonical projection $\mathbb{R}^n \times \mathfrak{z}^* \rightarrow \mathbb{R}^n$ is a proper continuous map when restricted to $\text{supp } \sigma'$, hence there is a compact $K' \subseteq \mathfrak{z}^*$ such that $(K \times \mathfrak{z}^*) \cap \text{supp } \sigma' \subseteq K \times K'$, and consequently

$$\sigma_{\tilde{\zeta}}(K) \leq C_K \int_{K \times K'} e^{-2p(\lambda)^2} \tilde{\zeta}(\tau) d\sigma'(\lambda, \tau) = C_K \|(\tilde{\zeta}\chi_{K'})^{1/2}(-iT)h_1\|_2^2,$$

by the Plancherel formula. On the other hand, since h_1 is in the Schwartz class, the last quantity is easily seen to be finite by using the Euclidean Fourier transform and the fact that $(\tilde{\zeta}\chi_{K'})^{1/2} \in L^2(\mathfrak{z}^*)$. We have thus proved that $\sigma_{\tilde{\zeta}}$ is finite on compacta; by [36, Theorem 2.18], this means that $\sigma_{\tilde{\zeta}}$ is a regular Borel measure on \mathbb{R}^n .

(ii) Without loss of generality, we may take the basis T_1, \dots, T_d of \mathfrak{z} as composed by δ_t -homogeneous elements; thus (3.5) is a homogeneous system, and the associated dilations ϵ'_t on $\mathbb{R}^n \times \mathfrak{z}^*$ are given by $\epsilon'_t(\lambda, \tau) = (\epsilon_t(\lambda), \tau \circ \delta_t)$. By (1.1), σ' is ϵ'_t -homogeneous of degree Q_δ . Therefore, if $\tilde{\zeta}$ is homogeneous of degree a , then clearly the measure (3.6) is homogeneous of degree $Q_\delta + a$; since the canonical projection $\mathbb{R}^n \times \mathfrak{z}^* \rightarrow \mathbb{R}^n$ intertwines the two system of dilations, we infer that also $\sigma_{\tilde{\zeta}}$ is homogeneous of degree $Q_\delta + a$. □

Via interpolation, we then obtain an improvement of Theorem 2.7, where the role of the Plancherel measure σ is now played by some $\sigma_{\tilde{\zeta}}$.

PROPOSITION 3.8. — (i) Suppose that, for some nonnegative Borel functions $w_j : \mathfrak{g}/\mathfrak{z} \rightarrow \mathbb{R}$ and $\zeta_j : \mathfrak{z}^* \rightarrow \mathbb{R}$ ($j = 1, \dots, h$), we have

$$(3.7) \quad |J(\bar{x}, \tau)| \geq w_j(\bar{x}) \zeta_j(\tau) \quad \text{for all } \bar{x} \in \mathfrak{g}/\mathfrak{z}, \tau \in \mathfrak{z}^*,$$

and set $\tilde{w}_j(x) = 1 + w_j(P(x))$. Suppose moreover that, for some $\gamma_1, \dots, \gamma_h > 0$, if $\tilde{\zeta}_{\bar{\gamma}} = \prod_{j=1}^h (1 + \zeta_j^{-2\gamma_j})$, then the measure $\sigma_{\tilde{\zeta}_{\bar{\gamma}}}$ is locally d -bounded on some open $\Omega \subseteq \mathbb{R}^n$. If $K \subseteq \Omega$ is compact, $q \in [1, \infty]$, $\alpha \geq 0$,

$$\beta > \alpha + \frac{n}{q} - \frac{d}{\max\{2, q\}},$$

then, for all $m \in \mathcal{D}(\mathbb{R}^n)$ with $\text{supp } m \subseteq K$,

$$\|\langle \cdot \rangle_G^\alpha \tilde{w}_1^{\gamma_1} \cdots \tilde{w}_h^{\gamma_h} \check{m}\|_{L^2(G)} \leq C_{K, \alpha, \bar{\gamma}, \beta} \|m\|_{B_{q, q}^\beta(\mathbb{R}^n)}.$$

(ii) Suppose in addition that $\int_G \langle x \rangle_G^{-2\alpha} \prod_{j=1}^h \tilde{w}_j(x)^{-2\gamma_j} dx < \infty$ for $\alpha > \bar{\alpha}_{\bar{\gamma}}$. If $K \subseteq \Omega$ is compact, $q \in [1, \infty]$, $\alpha \geq 0$,

$$\beta > \alpha + \bar{\alpha}_{\bar{\gamma}} + \frac{n}{q} - \frac{d}{\max\{2, q\}},$$

then, for all $m \in \mathcal{D}(\mathbb{R}^n)$ with $\text{supp } m \subseteq K$,

$$\|\langle \cdot \rangle_G^\alpha \check{m}\|_{L^1(G)} \leq C_{K, \alpha, \bar{\gamma}, \beta} \|m\|_{B_{q, q}^\beta(\mathbb{R}^n)}.$$

Proof. — (i) Since $\sigma_{\tilde{\zeta}_{\bar{\gamma}}}$ is locally d -bounded, the function ζ_j cannot be everywhere null, therefore (3.7) and the bilinearity of J imply $w_j(P(x)) \leq C \langle x \rangle_G^\theta$ for some $C, \theta \geq 0$, thus also $\prod_{j=1}^h \tilde{w}_j(x)^{2\gamma_j} \leq C_{\bar{\gamma}} \langle x \rangle_G^{2\theta(\gamma_1 + \dots + \gamma_h)}$ for some $C_{\bar{\gamma}} \geq 0$.

Let $\psi \in \mathcal{D}(\mathbb{R}^n)$ such that $\psi|_K = 1$ and $K' = \text{supp } \psi \subseteq \Omega$. The operator $m \mapsto \mathcal{K}_L(m\psi)$ is then continuous

$$B_{q, q}^\beta(\mathbb{R}^n) \rightarrow L^2\left(G, \langle x \rangle_G^{2\alpha} \prod_{j=1}^h \tilde{w}_j(x)^{2\gamma_j} dx\right)$$

for $\alpha \geq 0$, $\beta > \alpha + \theta(\gamma_1 + \dots + \gamma_h) + n$ by Theorem 2.7, whereas it is continuous

$$B_{q, q}^\beta(\mathbb{R}^n) \rightarrow L^2\left(G, \prod_{j=1}^h \tilde{w}_j(x)^{2\gamma_j} dx\right)$$

for $\beta > n/q - d/\max\{2, q\}$ by Corollary 3.6 and Lemma 2.6 (cf. the proof of Theorem 2.7). The conclusion then follows by interpolation.

(ii) It follows from (i) by Hölder's inequality. □

The hypotheses of the previous proposition are quite involved, and it is not particularly clear which classes of groups and systems of operators satisfy them. Hebisch and Zienkiewicz [20] treat explicitly the case of direct products of H-type groups; however, as it is mentioned in a remark at the end of [20], there are further cases of homogeneous groups for which this technique gives an improvement of the weighted L^1 estimates. In order to attempt a systematic treatment of these various cases, we introduce the following definition: for $h \in \mathbb{N}$, we say that a homogeneous Lie group G is h -capacious if there exist linearly independent homogeneous elements $\omega_1, \dots, \omega_h \in (\mathfrak{g}/\mathfrak{z})^*$ and linearly independent homogeneous elements $z_1, \dots, z_h \in \mathfrak{z}$ such that, for $j = 1, \dots, h$,

$$(3.8) \quad |J(\bar{x}, \tau)| \geq |\omega_j(x)| |\tau(z_j)| \quad \text{for all } \bar{x} \in \mathfrak{g}/\mathfrak{z}, \tau \in \mathfrak{z}^*.$$

Clearly, every homogeneous group is 0-capacious. In the following proposition, we give some criteria which may be of some use in showing that a certain homogeneous group is h -capacious. Let us denote by

$$(3.9) \quad \mathfrak{g}_{[1]} = \mathfrak{g}, \quad \mathfrak{g}_{[r+1]} = [\mathfrak{g}, \mathfrak{g}_{[r]}]$$

the descending central series of a Lie algebra \mathfrak{g} .

PROPOSITION 3.9. — *Let G be a homogeneous group, with dilations δ_t .*

- (i) *If G is a Métivier group (with any family of automorphic dilations), then $\mathfrak{z} = [\mathfrak{g}, \mathfrak{g}]$ and G is $(\dim \mathfrak{z})$ -capacious.*
- (ii) *Suppose that, for some $r \geq 2$, $\dim \mathfrak{g}_{[r]} = 1$. Then G is 1-capacious.*
- (iii) *If \mathfrak{g} admits a \mathbb{C} -linear structure which is compatible with its homogeneous Lie algebra structure, and if moreover $\dim_{\mathbb{C}} \mathfrak{g}_{[r]} = 1$ for some $r \geq 2$, then \mathfrak{g} is 2-capacious.*
- (iv) *Suppose that $G = G_1 \times G_2$, where G_1 and G_2 are homogeneous Lie groups with dilations $\delta_{1,t}$ and $\delta_{2,t}$ respectively, so that $\delta_t = \delta_{1,t} \times \delta_{2,t}$. If G_1 is h_1 -capacious and G_2 is h_2 -capacious, then G is $(h_1 + h_2)$ -capacious.*

Proof. — (i) Notice that the capacity map J takes its values in the subspace of η^* corresponding to $(\eta/\mathfrak{z})^*$. The condition (3.2) implies that $J(\cdot, \tau) : \mathfrak{g}/\mathfrak{z} \rightarrow (\eta/\mathfrak{z})^*$ is injective for $\tau \neq 0$, and that $J(\bar{x}, \cdot) : \mathfrak{z}^* \rightarrow (\eta/\mathfrak{z})^*$ is injective for $\bar{x} \neq 0$. Therefore $\dim \mathfrak{g} \leq \dim \eta$, so that $\mathfrak{g} = \eta$ and $[\mathfrak{g}, \mathfrak{g}] \subseteq \mathfrak{z}$; on the other hand, $\dim(\mathfrak{g}/\mathfrak{z}) \geq \dim \mathfrak{z}$.

The δ_t are automorphisms, hence \mathfrak{z} is a homogeneous ideal. Thus, if $h = \dim \mathfrak{z}$, we can choose linearly independent homogeneous elements z_1, \dots, z_h of \mathfrak{z} , and also linearly independent homogeneous $\omega_1, \dots, \omega_h \in (\mathfrak{g}/\mathfrak{z})^*$, since

$h \leq \dim(\mathfrak{g}/\mathfrak{z})$. Modulo a suitable renormalization, from (3.2) we infer (3.8) for $j = 1, \dots, h$.

If $[\mathfrak{g}, \mathfrak{g}]$ were strictly contained in \mathfrak{z} , then we would find $\tau \in \mathfrak{z}^* \setminus \{0\}$ such that $\tau|_{[\mathfrak{g}, \mathfrak{g}]} = 0$, but then also $J(\cdot, \tau) = 0$, which contradicts (3.2); therefore $\mathfrak{z} = [\mathfrak{g}, \mathfrak{g}]$.

(ii) Since G is nilpotent, it must be r -step, so that $\mathfrak{g}_{[r]} \subseteq \mathfrak{z}$. Notice that the ideal $\mathfrak{g}_{[r-1]}$ is preserved by every automorphism of \mathfrak{g} , therefore it is generated by δ_t -homogeneous elements; since $[\mathfrak{g}, \mathfrak{g}_{[r-1]}] = \mathfrak{g}_{[r]} \neq 0$, then there must exist a δ_t -homogeneous element $y \in \mathfrak{g}_{[r-1]}$ such that, for some $x_0 \in \mathfrak{g}$, $[x_0, y] = z \neq 0$. In particular $y \neq 0$ and moreover, since the ideal $\mathfrak{g}_{[r]}$ is δ_t -homogeneous and 1-dimensional, necessarily z is δ_t -homogeneous.

Since $y \in \mathfrak{g}_{[r-1]}$, the linear map $[\cdot, y] : \mathfrak{g} \rightarrow \mathfrak{g}$ takes its values in $\mathfrak{g}_{[r]} = \mathbb{R}z$; therefore, there exists $\omega \in (\mathfrak{g}/\mathfrak{z})^*$ such that $[x, y] = \omega(P(x))z$ for all $x \in \mathfrak{g}$. Notice that $\omega(P(x_0)) = 1$, thus $\omega \neq 0$; moreover, since both y and z are homogeneous, also ω is homogeneous. Finally

$$(3.10) \quad J(\bar{x}, \tau)(y) = \omega(\bar{x})\tau(z) \quad \text{for all } \bar{x} \in \mathfrak{g}/\mathfrak{z}, \tau \in \mathfrak{z}^*,$$

which implies immediately that G is 1-capacious.

(iii) Arguing as in part (ii), but with a complex Lie algebra \mathfrak{g} , one finds an identity analogous to (3.10), where now ω is a \mathbb{C} -linear functional on $\mathfrak{g}/\mathfrak{z}$, and $z \in \mathfrak{z}$. The conclusion then follows by taking the \mathbb{R} -linearly independent \mathbb{R} -linear functionals $\Re\omega, \Im\omega$ on $\mathfrak{g}/\mathfrak{z}$, and the \mathbb{R} -linearly independent elements $z, iz \in \mathfrak{z}$.

(iv) Via the canonical identification $\mathfrak{g} = \mathfrak{g}_1 \times \mathfrak{g}_2$, we have (with the obvious meaning of the notation) $\mathfrak{z} = \mathfrak{z}_1 \times \mathfrak{z}_2, \eta = \eta_1 \times \eta_2$, thus also

$$\mathfrak{z}^* = \mathfrak{z}_1^* \times \mathfrak{z}_2^*, \quad \eta^* = \eta_1^* \times \eta_2^*, \quad \mathfrak{g}/\mathfrak{z} = (\mathfrak{g}_1/\mathfrak{z}_1) \times (\mathfrak{g}_2/\mathfrak{z}_2).$$

Moreover clearly $J((\bar{x}_1, \bar{x}_2), (\tau_1, \tau_2)) = (J_1(\bar{x}_1, \tau_1), J_2(\bar{x}_2, \tau_2))$, therefore

$$|J((\bar{x}_1, \bar{x}_2), (\tau_1, \tau_2))| \geq \max\{|J_1(\bar{x}_1, \tau_1)|, |J_2(\bar{x}_2, \tau_2)|\}$$

and the conclusion follows immediately. □

Notice that the previous proposition is not sufficient to exhaust all the cases of h -capacious groups; an example is shown in § 6.2.

LEMMA 3.10. — *Suppose that G is h -capacious, and let $\omega_1, \dots, \omega_h \in (\mathfrak{g}/\mathfrak{z})^*$ be as in the definition. Then the functionals $\omega_j \circ P$ are null on $[\mathfrak{g}, \mathfrak{g}]$. In particular*

$$h \leq \min\{\dim \mathfrak{z}, \dim \mathfrak{g} - \dim(\mathfrak{z} + [\mathfrak{g}, \mathfrak{g}])\}.$$

Moreover, we can find a homogeneous basis of \mathfrak{g} compatible with the descending central series such that the functionals $\omega_1 \circ P, \dots, \omega_h \circ P$ are part of the dual basis.

Proof. — Notice that $[[\mathfrak{g}, \mathfrak{g}], \mathfrak{v}] \subseteq [\mathfrak{g}, [\mathfrak{g}, \mathfrak{v}]] \subseteq [\mathfrak{g}, \mathfrak{z}] = 0$. Then from the definition of J it follows that, for every $x \in [\mathfrak{g}, \mathfrak{g}]$,

$$J(P(x), \tau) = 0 \quad \text{for all } \tau \in \mathfrak{z}^*.$$

Hence, by choosing in (3.8) a $\tau \in \mathfrak{z}^*$ such that $\tau(z_j) \neq 0$, we obtain that the functional $\omega_j \circ P$ is null on $[\mathfrak{g}, \mathfrak{g}]$. In particular, the $\omega_j \circ P$ correspond to linearly independent elements of $(\mathfrak{g}/([\mathfrak{g}, \mathfrak{g}] + \mathfrak{z}))^*$, and the inequality about h follows.

Let now $W = \ker(\omega_1 \circ P) \cap \dots \cap \ker(\omega_h \circ P)$. Then W is a homogeneous subspace of \mathfrak{g} containing $[\mathfrak{g}, \mathfrak{g}]$. Moreover, if $\tilde{\omega}_j$ is the element of $(\mathfrak{g}/W)^*$ corresponding to ω_j , then $\tilde{\omega}_1, \dots, \tilde{\omega}_h$ are a homogeneous basis of $(\mathfrak{g}/W)^*$. We can then choose homogeneous elements $v_1, \dots, v_h \in \mathfrak{g}$ such that the corresponding elements in the quotient \mathfrak{g}/W are the dual basis of $\tilde{\omega}_1, \dots, \tilde{\omega}_h$. Finally, we append to v_1, \dots, v_h a homogeneous basis of W compatible with the descending central series (which, apart from $\mathfrak{g}_{[1]}$, is contained in W , and is made of homogeneous ideals), and we are done. \square

Here is finally the improvement of Theorem 2.7 for h -capacious groups.

THEOREM 3.11. — *Suppose that G is h -capacious, and let Q_G be its degree of polynomial growth. Let moreover L_1, \dots, L_n be a homogeneous weighted subcoercive system on G . If $q \in [1, \infty]$, $\alpha \geq 0$ and*

$$\beta > \alpha + \frac{Q_G - h}{2} + \frac{n}{q} - \frac{1}{\max\{2, q\}},$$

then, for every $K \subseteq \mathbb{R}^n \setminus \{0\}$ compact,

$$\|\check{m}\|_{L^1(G, \langle x \rangle_\alpha^q dx)} \leq C_{K, \alpha, \beta} \|m\|_{B_{q, \alpha}^\beta}$$

for all $m \in \mathcal{D}(G)$ with $\text{supp } m \subseteq K$.

Proof. — Let $\omega_1, \dots, \omega_h \in (\mathfrak{g}/\mathfrak{z})^*$ and $z_1, \dots, z_h \in \mathfrak{z}$ be as in the definition of h -capacious, and set $w_j(\bar{x}) = |\omega_j(\bar{x})|$, $\zeta_j(\tau) = |\tau(z_j)|$. Notice now that, since the z_j are linearly independent, for every choice of $\gamma_1, \dots, \gamma_h \in]0, 1/2[$, the function $\tilde{\zeta}_\gamma = \prod_{j=1}^h (1 + \zeta_j^{-2\gamma_j})$ is in $L^1_{\text{loc}}(\mathfrak{z}^*)$, so that, by Lemma 3.7, $\sigma_{\tilde{\zeta}_\gamma}$ is a regular Borel measure; in fact, since the z_j are homogeneous, $\sigma_{\tilde{\zeta}_\gamma}$ is the sum of ϵ_t -homogeneous regular Borel measures on \mathbb{R}^n (with possibly different degrees of homogeneity), hence $\sigma_{\tilde{\zeta}_\gamma}$ is locally 1-bounded on $\mathbb{R}^n \setminus \{0\}$.

By Lemma 3.10, we can find a homogeneous basis v_1, \dots, v_k of \mathfrak{g} , compatible with the descending central series, such that, if $\hat{v}_1, \dots, \hat{v}_h$ is the dual basis, then $\hat{v}_j = \omega_j \circ P$ for $j = 1, \dots, h$; in particular we have

$\tilde{w}_j(x) = 1 + |\hat{v}_j(x)|$. If we set $\kappa_j = \max\{r : v_j \in \mathfrak{g}_{[r]}\}$, then $Q_G = \sum_{j=1}^k \kappa_j$ and $\langle x \rangle_G \sim 1 + \sum_{j=1}^k |\hat{v}_j(x)|^{1/\kappa_j}$ (cf. [26, Proposition 2.1]), thus

$$\langle x \rangle_G^{-2\alpha_j} \leq C_{\alpha_j} (1 + |\hat{v}_j(x)|)^{-2\alpha_j/\kappa_j}$$

for $j = 1, \dots, k$ and $\alpha_j \geq 0$. Moreover, since the $\omega_j \circ P$ are null on $[\mathfrak{g}, \mathfrak{g}]$, then $\kappa_j = 1$ for $j = 1, \dots, h$.

Notice now that, for fixed $\gamma_1, \dots, \gamma_h \in]0, 1/2[$, if $\alpha \geq 0$ satisfies

$$2\alpha > 2\alpha_{\bar{\gamma}} = \sum_{j=1}^h (1 - 2\gamma_j) + \sum_{j=h+1}^k \kappa_j,$$

then we may choose $\alpha_1, \dots, \alpha_k \geq 0$ such that

$$\alpha = \sum_{j=1}^k \alpha_j, \quad 2\alpha_j > \begin{cases} 1 - 2\gamma_j & \text{for } j = 1, \dots, h, \\ \kappa_j & \text{for } j = h + 1, \dots, k, \end{cases}$$

therefore

$$\langle x \rangle_G^{-2\alpha} \prod_{j=1}^h \tilde{w}_j(x)^{-2\gamma_j} \leq \prod_{j=1}^h (1 + |\hat{v}_j(x)|)^{-2(\alpha_j + \gamma_j)} \prod_{j=h+1}^k (1 + |\hat{v}_j(x)|)^{-2\alpha_j/\kappa_j},$$

and the right-hand side is clearly integrable over G . We can thus apply Proposition 3.8(ii), and the conclusion follows because, if the γ_j tend to $1/2$, then $\alpha_{\bar{\gamma}}$ tends to $\sum_{j=h+1}^k \kappa_j = Q_G - h$. \square

Notice that, if G is a Métivier group, by Proposition 3.9(i) we can take $h = \dim[\mathfrak{g}, \mathfrak{g}]$, so that $Q_G - h = \dim G$; in fact, by Proposition 3.9(iv), the same holds if G is a direct product of Métivier and Euclidean groups.

4. Mihlin-Hörmander multipliers

Let G be a homogeneous Lie group, with dimension at infinity Q_G , automorphic dilations δ_t and homogeneous dimension Q_δ ; as in [26, §2.1], we suppose that the homogeneity degrees of the elements of the Lie algebra \mathfrak{g} are not less than 1, so that $Q_\delta \geq Q_G$. Define $|\cdot|_G, \langle \cdot \rangle_G$ as in § 2, and denote by $|\cdot|_\delta$ a subadditive homogeneous norm on G (cf. [19]).

Let L_1, \dots, L_n be a homogeneous weighted subcoercive system on G , with associated dilations ϵ_t , and Plancherel measure σ . Denote moreover by $|\cdot|_\epsilon$ an ϵ_t -homogeneous norm on \mathbb{R}^n , smooth off the origin.

Our starting point is, for some $q \in [1, \infty]$ and $s \in \mathbb{R}$, the following

hypothesis $(I_{q,s})$: for some compact $K_0 \subseteq \mathbb{R}^n \setminus \{0\}$ such that

$$\bigcup_{t>0} \epsilon_t(\overset{\circ}{K}_0) = \mathbb{R}^n \setminus \{0\},$$

for all $\beta > s$ and for all $m \in \mathcal{D}(\mathbb{R}^n)$ with $\text{supp } m \subseteq K_0$, we have

$$\|\check{m}\|_{L^1(G)} \leq C_\beta \|m\|_{B_{q,q}^\beta(\mathbb{R}^n)}.$$

From Theorems 2.7 and 3.11 we deduce immediately

PROPOSITION 4.1. — *For every $q \in [1, \infty]$, the hypothesis $(I_{q,s})$ holds in each of the following cases:*

- $s = Q_G/2 + n/q - 1/\max\{2, q\}$;
- the Plancherel measure σ is locally d -bounded on $\mathbb{R}^n \setminus \{0\}$ and $s = Q_G/2 + n/q - d/\max\{2, q\}$;
- G is h -capacious and $s = (Q_G - h)/2 + n/q - 1/\max\{2, q\}$.

In the rest of this section, we forget how such hypothesis may be checked, and we focus on its consequences.

PROPOSITION 4.2. — *Suppose that $(I_{q,s})$ holds for some $q \in [1, \infty]$ and $s \in \mathbb{R}$. Then $s \geq n/q$. Moreover, for every compact $K \subseteq \mathbb{R}^n \setminus \{0\}$, for every $\alpha \geq 0$ and $\beta > \alpha + s$, for every $D \in \mathfrak{D}(G)$, for every $m \in B_{q,q}^\beta(\mathbb{R}^n)$ with $\text{supp } m \subseteq K$, we have*

$$(4.1) \quad \|\langle \cdot \rangle_G^\alpha D\check{m}\|_{L^1(G)} \leq C_{K,D,\alpha,\beta} \|m\|_{B_{q,q}^\beta(\mathbb{R}^n)}.$$

Proof. — Let $\lambda \in \overset{\circ}{K}_0 \cap \text{supp } \sigma$. For every $m \in \mathcal{D}(\mathbb{R}^n)$ with $\text{supp } m \subseteq \overset{\circ}{K}_0$, we have $|m(\lambda)| \leq \|m\|_{L^\infty(\sigma)} \leq \|\check{m}\|_1 \leq C_\beta \|m\|_{B_{q,q}^\beta}$ for all $\beta > s$. Such an inequality gives easily $\|f\|_\infty \leq C_\beta \|f\|_{B_{q,q}^\beta}$ for all f in the Schwartz class and $\beta > s$, which however can hold only if $s \geq n/q$ (cf. [48, § 2.6.2, Theorem 1]).

The hypothesis $(I_{q,s})$ gives (4.1) in the case $\alpha = 0$, $D = 1$, m smooth, $K = K_0$. The extension to a generic compact $K \subseteq \mathbb{R}^n \setminus \{0\}$ is performed by a partition-of-unity argument and exploiting homogeneity. The full generality is then reached by approximation and interpolation, as in the proof of Theorem 2.7. □

Notice that, by [26, Proposition 2.1], there are constants $a, C > 0$ such that

$$(4.2) \quad 1 + |x|_\delta \leq C \langle x \rangle_G^a.$$

COROLLARY 4.3. — Suppose that $(I_{q,s})$ holds. Let $K \subseteq \mathbb{R}^n \setminus \{0\}$ be compact, $\beta > s$. If $m \in B_{q,q}^\beta(\mathbb{R}^n)$ and $\text{supp } m \subseteq K$, then $\check{m} \in L^1(G)$. Moreover, for $0 \leq \alpha < (\beta - s)/a$,

$$(4.3) \quad \int_G (1 + |x|_\delta)^\alpha |\check{m}(x)| dx \leq C_{K,\alpha,\beta} \|m\|_{B_{q,q}^\beta}$$

and, for all $h \in G$,

$$(4.4) \quad \begin{aligned} \|R_h \check{m} - \check{m}\|_1 &\leq C_{K,\beta} \|m\|_{B_{q,q}^\beta} |h|_\delta, \\ \|L_h \check{m} - \check{m}\|_1 &\leq C_{K,\beta} \|m\|_{B_{q,q}^\beta} |h|_\delta. \end{aligned}$$

Proof. — Since $\beta > s + a\alpha$, by Proposition 4.2 and (4.2) we have

$$\int_G (1 + |x|_\delta)^\alpha |\check{m}(x)| dx \leq C_\alpha \int_G \langle x \rangle_G^{a\alpha} |\check{m}(x)| dx \leq C_{K,\alpha,\beta} \|m\|_{B_{q,q}^\beta},$$

and in particular $\check{m} \in L^1(G)$.

Starting from the inequality $\|R_{\exp(tX)} \check{m} - \check{m}\|_1 \leq \|X\check{m}\|_1 |t|$, true for all $X \in \mathfrak{g}$ and $t \in \mathbb{R}$, having chosen a basis X_1, \dots, X_k of \mathfrak{g} , with X_j homogeneous of degree d_j , we easily obtain $\|R_h \check{m} - \check{m}\|_1 \leq C \sum_{j=1}^k \|X_j \check{m}\|_1 |h|_\delta^{d_j}$, so that also

$$\|R_h \check{m} - \check{m}\|_1 \leq C \left(\|\check{m}\|_1 + \sum_{j=1}^k \|X_j \check{m}\|_1 \right) |h|_\delta,$$

since $d_j \geq 1$. However $\|\check{m}\|_1 + \sum_{j=1}^k \|X_j \check{m}\|_1 \leq C_{K,\beta} \|m\|_{B_{q,q}^\beta}$ by Proposition 4.2, thus we get the former of (4.4); the latter is obtained by replacing m with \bar{m} . \square

LEMMA 4.4. — Let m be a bounded Borel function on \mathbb{R}^n . Then we can find bounded Borel functions m_j on \mathbb{R}^n (for $j \in \mathbb{Z}$) such that

$$(4.5) \quad \text{supp } m_j \subseteq \{\lambda : 2^{-1} \leq |\lambda|_\epsilon \leq 2\}, \quad \|m_j\|_{B_{q,q}^\beta} \leq C_{q,\beta} \|m\|_{M_\epsilon B_{q,q}^\beta}$$

for all $q \in [1, \infty]$ and $\beta \geq 0$, and moreover

$$(4.6) \quad \check{m} = \sum_{j \in \mathbb{Z}} 2^{-Q\delta j} \check{m}_j \circ \delta_{2^{-j}},$$

in the sense of strong convergence of the corresponding convolution operators.

Proof. — Set $K = \{\lambda : 2^{-1} \leq |\lambda|_\epsilon \leq 2\}$. Choose a nonnegative $\eta \in \mathcal{D}(\mathbb{R}^n)$ supported in K and such that $\sum_{j \in \mathbb{Z}} \eta \circ \epsilon_{2^j} = 1$ off the origin, and let $m_j = (m \circ \epsilon_{2^{-j}}) \eta$. Then clearly (4.5) is satisfied, and moreover $m = \sum_{j \in \mathbb{Z}} m_j \circ \epsilon_{2^j}$ off the origin. In fact, this is locally a finite sum and the convergence is dominated by the constant $\|m\|_\infty$. Since $\sigma(\{0\}) = 0$, by

the spectral theorem and (1.1) we then have (4.6), in the sense of strong convergence of the corresponding convolution operators. \square

PROPOSITION 4.5. — *Suppose that $(I_{q,s})$ holds. Let $\beta > s$. If m is a bounded Borel function on \mathbb{R}^n such that $\|m\|_{M_\epsilon B_{q,q}^\beta} < \infty$, then $\check{m}|_{G \setminus \{e\}} \in L^1_{\text{loc}}(G \setminus \{e\})$, and moreover*

$$(4.7) \quad \int_{|x|_\delta \geq 2|h|_\delta} |\check{m}(xh) - \check{m}(x)| \, dx \leq C_\beta \|m\|_{M_\epsilon B_{q,q}^\beta},$$

$$(4.8) \quad \int_{|x|_\delta \geq 2|h|_\delta} |\check{m}(hx) - \check{m}(x)| \, dx \leq C_\beta \|m\|_{M_\epsilon B_{q,q}^\beta}$$

for all $h \in G \setminus \{e\}$.

Proof. — Let the m_j be given by Lemma 4.4 and set $u_j = 2^{-Q\delta j} \check{m}_j \circ \delta_{2^{-j}}$.

Firstly we prove that the convergence in (4.6) holds also in $L^1_{\text{loc}}(G \setminus \{e\})$. In fact, let $B_k = \{x \in G : 2^k \leq |x|_\delta \leq 2^{k+1}\}$; it is sufficient to prove the convergence in each $L^1(B_k)$. We have $\int_{B_k} |u_j| \, d\mu = \int_{B_{k-j}} |\check{m}_j| \, d\mu$ and, for $j \leq k$,

$$\int_{B_{k-j}} |\check{m}_j(x)| \, dx \leq 2^{\alpha(j-k)} \int_{B_{k-j}} |\check{m}_j(x)| |x|_\delta^\alpha \, dx \leq C 2^{\alpha(j-k)} \|m\|_{M_\epsilon B_{q,q}^\beta}$$

(where $\alpha > 0$ is as in (4.3)), whereas, for $j \geq k$,

$$\int_{B_{k-j}} |\check{m}_j(x)| \, dx \leq \|\check{m}_j\|_2 \sqrt{\mu(B_{k-j})} \leq \sqrt{\sigma(K)\mu(B_0)} \|m\|_\infty 2^{Q\delta(k-j)/2},$$

(here we use a uniform estimate on the L^2 -norms of the \check{m}_j) so that

$$\sum_j \int_{B_k} |u_j| \, d\mu \leq C' \sum_{j \leq k} 2^{\alpha(j-k)} + C'' \sum_{j \geq k} 2^{Q\delta(k-j)/2} < \infty.$$

This shows (by uniqueness of limits) that the restriction of the distribution \check{m} to $G \setminus \{e\}$ coincides with a function in $L^1_{\text{loc}}(G \setminus \{e\})$.

Since $\check{m} = \sum_{j \in \mathbb{Z}} u_j$ in $L^1_{\text{loc}}(G \setminus \{e\})$, then $R_h \check{m} - \check{m} = \sum_{j \in \mathbb{Z}} (R_h u_j - u_j)$ in $L^1_{\text{loc}}(G \setminus \{e, h^{-1}\})$, so that in particular

$$(4.9) \quad \int_{|x|_\delta \geq 2|h|_\delta} |\check{m}(xh) - \check{m}(x)| \, dx \leq \sum_{j \in \mathbb{Z}} \int_{|x|_\delta \geq 2|h|_\delta} |u_j(xh) - u_j(x)| \, dx.$$

Let $k \in \mathbb{Z}$. Then, for $j < k$, the j -th summand in the right-hand side of (4.9) is not greater than

$$2 \int_{|x|_\delta \geq |h|_\delta} |u_j(x)| \, dx \leq C_\beta \frac{2^{\alpha j}}{|h|_\delta^\alpha} \|m\|_{M_\epsilon B_{q,q}^\beta}$$

by (4.3), whereas, for $j \geq k$, it is not greater than

$$\int_G |\check{m}_j(y\delta_{2^{-j}}(h)) - \check{m}_j(y)| dy \leq C_\beta \frac{|h|_\delta}{2^j} \|m\|_{M_\epsilon B_{q,q}^\beta}$$

by (4.4). Putting all together, the left-hand side of (4.9) is majorized by

$$C_\beta \|m\|_{M_\epsilon B_{q,q}^\beta} \left(\frac{2^{k\alpha}}{|h|_\delta^\alpha} \sum_{j < 0} 2^{j\alpha} + \frac{|h|_\delta}{2^k} \sum_{j \geq 0} 2^{-j} \right)$$

and, in order to obtain an estimate independent of h , it is sufficient to choose a k such that $2^k \leq |h|_\delta < 2^{k+1}$. Hence we have proved (4.7); the inequality (4.8) is obtained analogously. \square

Here is finally the multiplier theorem.

THEOREM 4.6. — *Suppose that $(I_{q,s})$ holds. If m is a bounded Borel function on \mathbb{R}^n such that $\|m\|_{M_\epsilon B_{q,q}^\beta} < \infty$ for some $\beta > s$, then the operator $m(L)$ is of weak type $(1, 1)$ and bounded on $L^p(G)$ for $1 < p < \infty$, with*

$$\|m(L)\|_{p \rightarrow p} \leq C_{p,q,\beta} \|m\|_{M_\epsilon B_{q,q}^\beta}.$$

Proof. — Notice that $\|m\|_{L^\infty(\sigma)} \leq C_{q,\beta} \|m\|_{M_\epsilon B_{q,q}^\beta}$, since $\beta > n/p$ by Proposition 4.2. In view of Proposition 4.5, the conclusion then follows from the Calderón-Zygmund theory of singular integral operators [45, § I.5, Theorem 3 and § I.7.4(iii)]. \square

Notice that a compactly supported $m \in B_{q,q}^\beta(\mathbb{R}^n)$ does satisfy an L^q Mihlin-Hörmander condition of order β , at least for β sufficiently large. More precisely, let $\tilde{Q}_\epsilon = \sum_j w_j / \min_j w_j$ denote the *normalized homogeneous dimension* associated with the dilations $\epsilon_t(\lambda) = (t^{w_1} \lambda_1, \dots, t^{w_n} \lambda_n)$; then we have

PROPOSITION 4.7. — *If $K \subseteq \mathbb{R}^n$ is compact, $q \in [1, \infty]$, $\beta > \tilde{Q}_\epsilon/q$, then*

$$\|m\|_{M_\epsilon B_{q,q}^\beta} \leq C_{K,q,\beta} \|m\|_{B_{q,q}^\beta}$$

for all $m \in B_{q,q}^\beta(\mathbb{R}^n)$ with $\text{supp } m \subseteq K$.

Recall that $\|m\|_{M_\epsilon B_{q,q}^\beta} = \sup_{t>0} \|(m \circ \epsilon_t)\eta\|_{B_{q,q}^\beta}$ for a suitable cut-off function $\eta \in \mathcal{D}(\mathbb{R}^n)$ supported away from the origin. If $\text{supp } m \subseteq K$, then, for some $t_K > 0$ sufficiently large, we have $\|(m \circ \epsilon_t)\eta\|_{B_{q,q}^\beta} = 0$ for $t > t_K$; therefore Proposition 4.7 follows immediately from

LEMMA 4.8. — *If $p, q \in [1, \infty]$, $\beta > \tilde{Q}_\epsilon/p$ and $\eta \in \mathcal{D}(\mathbb{R}^n)$, then*

$$\sup_{0 < t \leq 1} \|(f \circ \epsilon_t)\eta\|_{B_{p,q}^\beta} \leq C_{\eta,p,q,\beta} \|f\|_{B_{p,q}^\beta}.$$

Proof. — Without loss of generality, we may suppose that β is not an integer (the missing values can be recovered *a posteriori* by interpolation), thus there exists $m \in \mathbb{N}$ such that $m - 1 < \beta < m$.

Define the k -th order difference operator Δ_y^k and the modulus of continuity ω_p^k as in [2, § 6.2]. In view of the characterization by differences of the Besov norm given in [2, Theorem 6.2.5], we have to estimate

$$(4.10) \quad \|(f \circ \epsilon_t)\eta\|_p + \left(\int_0^1 \left(\frac{\omega_p^m(r, (f \circ \epsilon_t)\eta)}{r^\beta} \right)^q \frac{dr}{r} \right)^{1/q}.$$

The former summand in (4.10) is immediately majorized by Hölder’s inequality and embeddings, since η is compactly supported and $\beta > n/p$:

$$\|(f \circ \epsilon_t)\eta\|_p \leq C_{\eta,p} \|f \circ \epsilon_t\|_\infty = C_{\eta,p} \|f\|_\infty \leq C_{\eta,p,q,\beta} \|f\|_{B_{p,q}^\beta}.$$

For the latter summand, notice first that

$$(4.11) \quad \|\Delta_y^k \psi\|_p \leq C_{k,p} \|\psi\|_{W_p^k} |y|^k;$$

this inequality, together with the Leibniz rule for finite differences, Hölder’s inequality and the fact that $\eta \in \mathcal{D}(\mathbb{R}^n)$, gives easily

$$\|\Delta_y^m((f \circ \epsilon_t)\eta)\|_p \leq C_{\eta,m,p,p_0,\dots,p_m} \sum_{k=0}^m |y|^{m-k} \|\Delta_y^k(f \circ \epsilon_t)\|_{p_k}$$

for any choice of $p_0, \dots, p_k \geq p$; since

$$\|\Delta_y^k(f \circ \epsilon_t)\|_{p_k} = t^{-Q_\epsilon/p_k} \|\Delta_{\epsilon_t(y)}^k f\|_{p_k} \quad \text{and} \quad |\epsilon_t(y)|_\infty \leq t^{w_*} |y|_\infty$$

for $t \leq 1$, where $w_* = \min\{w_1, \dots, w_n\}$, we then get also

$$\omega_p^m(r, (f \circ \epsilon_t)\eta) \leq C_{\eta,m,p,p_0,\dots,p_m} \sum_{k=0}^m r^{m-k} t^{-Q_\epsilon/p_k} \omega_{p_k}^k(t^{w_*} r, f).$$

Choose now $p_m = p$, and $p_k = p\beta/k$ for $k < m$. Then, for $k < m$, we have

$$p_k > p, \quad k - \frac{n}{p_k} = \frac{k}{\beta} \left(\beta - \frac{n}{p} \right) < \beta - \frac{n}{p}, \quad w_* k - \frac{Q_\epsilon}{p_k} = \frac{w_* k}{\beta} \left(\beta - \frac{\tilde{Q}_\epsilon}{p} \right) > 0,$$

so that, by (4.11) and the embeddings $B_{p,q}^\beta \subseteq B_{p_k,1}^k \subseteq W_{p_k}^k$,

$$\frac{r^{m-k} t^{-Q_\epsilon/p_k} \omega_{p_k}^k(t^{w_*} r, f)}{r^\beta} \leq C_{p,q,\beta} r^{m-\beta} \|f\|_{B_{p,q}^\beta}.$$

For $k = m$, instead,

$$\frac{t^{-Q_\epsilon/p} \omega_p^m(t^{w_*} r, f)}{r^\beta} = t^{w_* \beta - Q_\epsilon/p} \frac{\omega_p^m(t^{w_*} r, f)}{(t^{w_*} r)^\beta} \leq \frac{\omega_p^m(t^{w_*} r, f)}{(t^{w_*} r)^\beta}.$$

Putting all together, the latter summand in (4.10) is majorized by

$$C_{n,p,q,\beta} \left(\|f\|_{B_{p,q}^\beta} \left(\int_0^1 (r^{m-\beta})^q \frac{dr}{r} \right)^{1/q} + \left(\int_0^1 \left(\frac{\omega_p^m(r, f)}{r^\beta} \right)^q \frac{dr}{r} \right)^{1/q} \right),$$

and the conclusion follows again by [2, Theorem 6.2.5]. □

5. Marcinkiewicz multipliers

Let G be a homogeneous Lie group, with automorphic dilations δ_t and homogeneous dimension Q_δ . For $w \in L^1(G)$, we define the maximal operator $M_w \phi(x) = \sup_{t>0} |\phi * (t^{-Q_\delta} w \circ \delta_{t^{-1}})(x)|$. We say that the function w is M -admissible if M_w is bounded on $L^p(G)$ for $1 < p < \infty$.

In terms of maximal operators, we formulate the following hypothesis about the homogeneous group G and a chosen homogeneous weighted subcoercive system L_1, \dots, L_n on it:

hypothesis ($J_{s,d}$): for every $\beta > s$ there exist

- a Borel function u_β on G with $u_\beta = u_\beta^*$ and $u_\beta \geq c \langle \cdot \rangle_G^{-\theta}$ for some $c, \theta > 0$,
- a positive regular Borel measure σ_β on \mathbb{R}^n , which is locally d -bounded on $\mathbb{R}^n \setminus \{0\}$,
- a non-negative real number $\gamma_\beta < 2\beta$,

such that

- the function $\langle \cdot \rangle_G^{-\gamma_\beta} u_\beta$ is M -admissible, and
- for every compact $K \subseteq \mathbb{R}^n \setminus \{0\}$ and every $m \in \mathcal{D}(\mathbb{R}^n)$ with $\text{supp } m \subseteq K$, we have

$$(5.1) \quad \|\check{m}\|_{L^2(G, u_\beta^{-1}(x) dx)} \leq C_{K,\beta} \|m\|_{L^2(\sigma_\beta)}.$$

PROPOSITION 5.1. — *Let G be a homogeneous group, with degree of polynomial growth Q_G , and let L_1, \dots, L_n be a homogeneous weighted subcoercive system on G .*

- (i) *The hypothesis ($J_{Q_G/2,1}$) holds. More generally, if the Plancherel measure σ is locally d -bounded on $\mathbb{R}^n \setminus \{0\}$, then ($J_{Q_G/2,d}$) holds.*
- (ii) *If G is h -capacious, then ($J_{(Q_G-h)/2,1}$) holds.*

Proof. — (i) Let σ be the Plancherel measure associated with the system L_1, \dots, L_n . For $\beta > Q_G/2$, we choose $u_\beta \equiv 1$, $\sigma_\beta = \sigma$. By (1.1), σ is δ_t -homogeneous, so that it is locally 1-bounded on $\mathbb{R}^n \setminus \{0\}$. Therefore, by the Plancherel formula, in order to conclude, it is sufficient to show that, for $\gamma_\beta \in]Q_G, 2\beta[$, the function $w_\beta = \langle \cdot \rangle_G^{-\gamma_\beta}$ is M -admissible.

The ideals composing the descending central series (3.9) are characteristic and thus δ_t -invariant, hence we can find δ_t -invariant complements V_k of $\mathfrak{g}_{[k+1]}$ in $\mathfrak{g}_{[k]}$. The dilations $\tilde{\delta}_t$ of \mathfrak{g} defined by $\tilde{\delta}_t(x) = t^k x$ for $x \in V_k$ in general are not automorphic, but commute with the δ_t , and moreover, by [26, Proposition 2.1], if $|\cdot|_{\tilde{\delta}}$ is a $\tilde{\delta}_t$ -homogeneous norm, then $\langle \cdot \rangle_G \sim 1 + |\cdot|_{\tilde{\delta}}$. We then have

$$\begin{aligned} \left| \int_G \phi(x \delta_t(y)^{-1}) w_\beta(y) dy \right| &= \left| \int_{|y|_{\tilde{\delta}} \leq 1} + \sum_{h \geq 1} \int_{2^{h-1} < |y|_{\tilde{\delta}} \leq 2^h} \right| \\ &\leq C_{\gamma_\beta} \sum_{h \geq 0} 2^{-h(\gamma_\beta - Q_G)} \int_{|y|_{\tilde{\delta}} \leq 1} |\phi(x \delta_t(\tilde{\delta}_{2^h}(y))^{-1})| dy. \end{aligned}$$

Since $\gamma_\beta > Q_G$, if M_{strong} is the strong maximal function on G associated to a basis of simultaneous eigenvectors of the δ_t and the $\tilde{\delta}_t$ [5, § 2], we obtain

$$M_{w_\beta} \phi \leq C_{\gamma_\beta} M_{\text{strong}} \phi,$$

which gives the conclusion by [5, Theorem 2.1].

(ii) Let \mathfrak{z} be the center of \mathfrak{g} , and $P : \mathfrak{g} \rightarrow \mathfrak{g}/\mathfrak{z}$ be the canonical projection. Let $\omega_1, \dots, \omega_h \in (\mathfrak{g}/\mathfrak{z})^*$ and $z_1, \dots, z_h \in \mathfrak{z}$ be as in the definition of h -capacious. By Lemma 3.10, there exists a homogeneous basis v_1, \dots, v_k of \mathfrak{g} compatible with the descending central series such that, if $\hat{v}_1, \dots, \hat{v}_k$ is the dual basis, then $\omega_j \circ P = \hat{v}_j$ for $j = 1, \dots, h$. Moreover, if we set $\kappa_j = \max\{r : v_j \in \mathfrak{g}_{[r]}\}$, then $\kappa_j = 1$ for $j = 1, \dots, h$ and

$$(5.2) \quad Q_G = \sum_{j=1}^k \kappa_j, \quad \langle x \rangle_G \sim 1 + \sum_{j=1}^k |\hat{v}_j(x)|^{1/\kappa_j}$$

by [26, Proposition 2.1].

Let \mathcal{T}_t be the k -parameter family of dilations on \mathfrak{g} given by $\mathcal{T}_t(v_j) = t_j v_j$. Clearly the \mathcal{T}_t are in general not automorphisms, but the automorphic dilations δ_t can be obtained as a particular case: $\delta_t = \mathcal{T}_{(t^{b_1}, \dots, t^{b_k})}$, where b_j is the δ_t -homogeneous degree of v_j .

If $\beta > (Q_G - h)/2$, then $2\beta > Q_G - h = \sum_{j=h+1}^k \kappa_j$, so that we can find $\eta_{\beta,1}, \dots, \eta_{\beta,h} \in [0, 1[$ and $\gamma_{\beta,1}, \dots, \gamma_{\beta,k} > 0$ such that

$$2\beta > \gamma_\beta = \sum_{j=1}^k \gamma_{\beta,j}, \quad \gamma_{\beta,j} > \begin{cases} 1 - \eta_{\beta,j} & \text{for } j = 1, \dots, h, \\ \kappa_j & \text{for } j = h + 1, \dots, k. \end{cases}$$

Let now σ' be the Plancherel measure on $\mathbb{R}^n \times \mathfrak{z}^*$ associated to the system L_1, \dots, L_n extended with the central derivatives, as in § 3, and let σ_β be the push-forward of the measure $\prod_{j=1}^h (1 + |\tau(z_j)|^{-\eta_{\beta,j}}) d\sigma'(\lambda, \tau)$ via the

canonical projection on the first factor of $\mathbb{R}^n \times \mathfrak{J}^*$. By Lemma 3.7, since $\eta_{\beta,1}, \dots, \eta_{\beta,h} < 1$, the measure σ_β is a regular Borel measure on \mathbb{R}^n ; moreover, since the z_j are δ_t -homogeneous, σ_β is the sum of ϵ_t -homogeneous regular Borel measures of different degrees (where ϵ_t are the dilations associated with the system L_1, \dots, L_n), and consequently σ_β is locally 1-bounded on $\mathbb{R}^n \setminus \{0\}$. Finally, if we set $u_\beta(x) = \prod_{j=1}^h (1 + |\omega_j(P(x))|)^{-\eta_{\beta,j}} = \prod_{j=1}^h (1 + |\hat{v}_j(x)|)^{-\eta_{\beta,j}}$, then $u_\beta = u_\beta^*$ and, by (5.2), u_β^{-1} is dominated by some power of $\langle \cdot \rangle_G$; moreover, by Corollary 3.6, for every compact $K \subseteq \mathbb{R}^n \setminus \{0\}$ and every $m \in \mathcal{D}(\mathbb{R}^n)$ with $\text{supp } m \subseteq K$, we have (5.1).

In order to conclude, we must show that $w_\beta = \langle \cdot \rangle_G^{-\gamma_\beta} u_\beta$ is M-admissible. In fact, again by (5.2),

$$w_\beta(x) \leq C_\beta \prod_{j=1}^h (1 + |\hat{v}_j(x)|)^{-(\gamma_{\beta,j} + \eta_{\beta,j})} \prod_{j=h+1}^k (1 + |\hat{v}_j(x)|)^{-\gamma_{\beta,j}/\kappa_j},$$

and the exponents $\gamma_{\beta,j} + \eta_{\beta,j}$, $\gamma_{\beta,j}/\kappa_j$ are all greater than 1 by construction. The conclusion then follows as in part (i), but with a multi-variate decomposition, by [5, Theorem 2.1] applied to the multi-parameter dilations $\overline{\tau}_t$. □

Suppose now that, for $l = 1, \dots, \varrho$, G_l is a homogeneous Lie group, with dilations $(\delta_{l,t})_{t>0}$, and that $L_{l,1}, \dots, L_{l,n_l}$ is a homogeneous weighted subcoercive system on G_l . Set $G^\times = G_1 \times \dots \times G_\varrho$, and let $L_{l,j}^\times \in \mathfrak{D}(G^\times)$ be defined by

$$L_{l,j}^\times(f_1 \otimes \dots \otimes f_\varrho) = f_1 \otimes \dots \otimes (L_{l,j} f_j) \otimes \dots \otimes f_\varrho$$

for $l = 1, \dots, \varrho$, $j = 1, \dots, n_l$. By [26, Corollary 5.5], we know that

$$L_{1,1}^\times, \dots, L_{1,n_1}^\times, \dots, L_{\varrho,1}^\times, \dots, L_{\varrho,n_\varrho}^\times$$

is a homogeneous weighted subcoercive system on G^\times .

We then show how the hypotheses on the factor groups G_l can be put together in order to obtain weighted estimates on the product group G^\times . In the following, inequalities involving vectors are to be read componentwise.

PROPOSITION 5.2. — *Suppose that, for $l = 1, \dots, \varrho$, the homogeneous group G_l , with the system $L_{l,1}, \dots, L_{l,n_l}$, satisfies (J_{s_l, d_l}) . For $q \in [1, \infty]$, if*

$$\vec{\beta} > \vec{s} + \frac{\vec{n}}{q} - \frac{\vec{d}}{\max\{2, q\}},$$

where $\vec{s} = (s_1, \dots, s_\varrho)$, $\vec{d} = (d_1, \dots, d_\varrho)$, then there exists

$$w_{\vec{\beta}} = w_{\vec{\beta},1} \otimes \dots \otimes w_{\vec{\beta},\varrho} \in L^1(G^\times),$$

with $w_{\vec{\beta}} > 0$, $w_{\vec{\beta}}^* = w_{\vec{\beta}}$, such that $w_{\vec{\beta},l}$ is M -admissible on G_l for $l = 1, \dots, \varrho$, and moreover, for every compact $K = \prod_{l=1}^{\varrho} K_l \subseteq \prod_{l=1}^{\varrho} (\mathbb{R}^{n_l} \setminus \{0\})$, and for every $m \in S_{q,q}^{\vec{\beta}} B(\mathbb{R}^{\vec{n}})$ with $\text{supp } m \subseteq K$, we have

$$\|\mathcal{K}_{L \times m}\|_{L^2(G^\times, w_{\vec{\beta}}^{-1}(x) dx)} \leq C_{K, \vec{\beta}, p, q} \|m\|_{S_{q,q}^{\vec{\beta}} B(\mathbb{R}^{\vec{n}})}.$$

Proof. — Take $\vec{\alpha}$ such that $\vec{\alpha} > \vec{s}$, $\vec{\beta} > \vec{\alpha} + \vec{n}/q - \vec{d}/\max\{2, q\}$. For $l = 1, \dots, \varrho$, since $\alpha_l > s_l$, by (J_{s_l, d_l}) we can find a function $u_{\vec{\alpha}, l} = u_{\vec{\alpha}, l}^* > 0$ on G_l such that $u_{\vec{\alpha}, l} \geq c_l \langle \cdot \rangle_{G_l}^{-\theta_l}$ for some $c_l, \theta_l > 0$, a positive regular Borel measure $\sigma_{\vec{\alpha}, l}$ on \mathbb{R}^{n_l} locally d_l -bounded on $\mathbb{R}^{n_l} \setminus \{0\}$, and a positive real number $\gamma_{\vec{\alpha}, l} < 2\alpha_l$ such that the function $w_{\vec{\beta}, l} = \langle \cdot \rangle_{G_l}^{-\gamma_{\vec{\alpha}, l}} u_{\vec{\alpha}, l}$ is M -admissible on G_l and

$$(5.3) \quad \|\mathcal{K}_{L_l} m_l\|_{L^2(G_l, u_{\vec{\alpha}, l}^{-1}(x_l) dx_l)} \leq C_{K_l, \alpha_l} \|m_l\|_{L^2(\sigma_{\vec{\alpha}, l})}$$

for every compact $K_l \subseteq \mathbb{R}^{n_l} \setminus \{0\}$ and every $m_l \in \mathcal{D}(\mathbb{R}^{n_l})$ with $\text{supp } m_l \subseteq K_l$.

Set $u_{\vec{\alpha}} = u_{\vec{\alpha}, 1} \otimes \dots \otimes u_{\vec{\alpha}, \varrho}$, $\sigma_{\vec{\alpha}} = \sigma_{\vec{\alpha}, 1} \times \dots \times \sigma_{\vec{\alpha}, \varrho}$. By “taking the Hilbert tensor product” of the inequalities (5.3), from [26, Corollary 5.5] we deduce that

$$\|\mathcal{K}_L \times m\|_{L^2(G^\times, u_{\vec{\alpha}}^{-1}(x) dx)} \leq C_{K, \vec{\alpha}} \|m\|_{L^2(\sigma_{\vec{\alpha}})}$$

for every compact $K = \prod_{l=1}^{\varrho} K_l \subseteq \prod_{l=1}^{\varrho} (\mathbb{R}^{n_l} \setminus \{0\})$ and every $m \in \mathcal{D}(\mathbb{R}^{\vec{n}})$ with $\text{supp } m \subseteq K$.

Notice now that, again by taking tensor products, Lemma 2.6 gives

$$\|m\|_{L^2(\sigma_{\vec{\alpha}})} \leq C_{K, \vec{\alpha}, \vec{\eta}} \|m\|_{S_{2,2}^{\vec{\eta}} B(\mathbb{R}^{\vec{n}})}$$

for $\vec{\eta} > (\vec{n} - \vec{d})/2$, whereas trivially

$$\|m\|_{L^2(\sigma_{\vec{\alpha}})} \leq C_{K, \vec{\alpha}} \|m\|_{\infty} \leq C_{K, \vec{\alpha}} \|m\|_{S_{\infty,1}^0 B(\mathbb{R}^{\vec{n}})},$$

so that, by embeddings and interpolation (cf. the proof of Theorem 2.7),

$$\|m\|_{L^2(\sigma_{\vec{\alpha}})} \leq C_{K, \vec{\alpha}, \vec{\eta}, q} \|m\|_{S_{q,q}^{\vec{\eta}} B(\mathbb{R}^{\vec{n}})}$$

for $\vec{\eta} > \vec{n}/q - \vec{d}/\max\{2, q\}$.

Putting all together, we have

$$\|\mathcal{K}_{L \times m}\|_{L^2(G^\times, u_{\vec{\alpha}}^{-1}(x) dx)} \leq C_{K, \vec{\alpha}, \vec{\eta}, q} \|m\|_{S_{q,q}^{\vec{\eta}} B(\mathbb{R}^{\vec{n}})}$$

for $\vec{\eta} > \vec{n}/q - \vec{d}/\max\{2, q\}$. On the other hand, by Theorem 2.7,

$$\|\mathcal{K}_{L_l} m_l\|_{L^2(G_l, \langle \cdot \rangle_{G_l}^{\gamma_l} u_{\vec{\alpha}, l}^{-1}(x_l) dx_l)} \leq C_{K_l, \vec{\alpha}, \gamma_l, \eta_l} \|m_l\|_{B_{2,2}^{\eta_l}(\mathbb{R}^{n_l})}$$

for $\eta_l > \gamma_l/2 + \theta_l/2 + n_l/2$, so that, by tensor products and embeddings,

$$\|\mathcal{K}_{L \times m}\|_{L^2(G^\times, \langle x_1 \rangle_{G_1}^{\gamma_1} \cdots \langle x_\varrho \rangle_{G_\varrho}^{\gamma_\varrho} u_{\vec{\alpha}}^{-1}(x) dx)} \leq C_{K, \vec{\alpha}, \vec{\gamma}, \vec{\eta}, q} \|m\|_{S_{q, q}^{\vec{\eta}} B(\mathbb{R}^{\vec{n}})}$$

for $\vec{\eta} > \vec{\gamma}/2 + \vec{\theta}/2 + \vec{n}$. By interpolation, we obtain that

$$\|\mathcal{K}_{L \times m}\|_{L^2(G^\times, \langle x_1 \rangle_{G_1}^{\gamma_1} \cdots \langle x_\varrho \rangle_{G_\varrho}^{\gamma_\varrho} u_{\vec{\alpha}}^{-1}(x) dx)} \leq C_{K, \vec{\alpha}, \vec{\gamma}, \vec{\eta}, q} \|m\|_{S_{q, q}^{\vec{\eta}} B(\mathbb{R}^{\vec{n}})}$$

for $\vec{\eta} > \vec{\gamma}/2 + \vec{n}/q - \vec{d}/\max\{2, q\}$.

In particular, if we take $\vec{\gamma} = (\gamma_{\vec{\alpha}, 1}, \dots, \gamma_{\vec{\alpha}, \varrho})$, $\vec{\eta} = \vec{\alpha} + \vec{n}/q - \vec{d}/\max\{2, q\}$ and set $w_{\vec{\beta}} = w_{\vec{\beta}, 1} \otimes \cdots \otimes w_{\vec{\beta}, \varrho}$, we get

$$\|\mathcal{K}_{L \times m}\|_{L^2(G^\times, w_{\vec{\beta}}^{-1}(x) dx)} \leq C_{K, \vec{\beta}, q} \|m\|_{S_{q, q}^{\vec{\eta}} B(\mathbb{R}^{\vec{n}})},$$

for every compact $K = \prod_{l=1}^{\varrho} K_l \subseteq \prod_{l=1}^{\varrho} (\mathbb{R}^{n_l} \setminus \{0\})$ and every $m \in \mathcal{D}(\mathbb{R}^{\vec{n}})$ with $\text{supp } m \subseteq K$. The conclusion then follows by approximation. \square

Notice that, in the particular case $\varrho = 1$, the previous proposition, together with Hölder’s inequality and Proposition 4.2, gives the following

COROLLARY 5.3. — *Suppose that a homogeneous weighted subcoercive system L_1, \dots, L_n on a homogeneous Lie group G satisfies $(J_{s, d})$. Then, for $q \in [1, \infty]$, it satisfies also $(I_{q, s+n/q-d/\max\{2, q\}})$. In particular, $s \geq d/2$.*

The weighted estimate on G^\times given by Proposition 5.2 are the starting point for the following multi-variate multiplier results. In fact, we are going to consider a setting which is more general than the product group G^\times .

Let G be a connected Lie group, endowed with Lie group homomorphisms

$$v_l : G_l \rightarrow G \quad \text{for } l = 1, \dots, \varrho.$$

Then, for $l = 1, \dots, \varrho$, the operators $L_{l, 1}, \dots, L_{l, n_l}$ correspond (via the derivative v_l^b of the homomorphism) to operators $L_{l, 1}^b, \dots, L_{l, n_l}^b \in \mathfrak{D}(G)$, which are essentially self-adjoint. Since we want to give a meaning to joint functions of these operators on G , we suppose in the following that $L_{1, 1}^b, \dots, L_{1, n_1}^b, \dots, L_{\varrho, 1}^b, \dots, L_{\varrho, n_\varrho}^b$ commute strongly, i.e., they admit a joint spectral resolution E^b on $L^2(G)$.

In order to obtain multiplier results on G , we would like to “transfer” to G the estimates obtained on the product group G^\times . However, we cannot apply directly the classical transference results (cf. [6, 3, 7]), since the map

$$v^\times : G^\times \ni (x_1, \dots, x_n) \mapsto v_1(x_1) \cdots v_\varrho(x_\varrho) \in G$$

in general is not a group homomorphism and consequently it does not yield an action of G^\times on $L^p(G)$ by translations. Nevertheless, under the sole assumption of (strong) commutativity of the differential operators $L_{l, j}^b$ on

G , we are able to express the operator $m(L^b)$ on G by a sort of convolution with the kernel $\mathcal{K}_{L^\times} m$ of the operator $m(L^\times)$ on G^\times .

PROPOSITION 5.4. — (i) For every $m \in \mathcal{D}(\mathbb{R}^{\vec{n}})$ and $\phi \in L^2 \cap C_0(G)$,

$$m(L^b)\phi(x) = \int_{G^\times} \phi(x v^\times(y)^{-1}) \mathcal{K}_{L^\times} m(y) dy.$$

(ii) Under the hypotheses of Proposition 5.2, the previous identity holds for every $m \in S_{q,q}^{\vec{\beta}} B(\mathbb{R}^{\vec{n}})$ with compact support $\text{supp } m \subseteq \prod_{l=1}^{\varrho} (\mathbb{R}^{n_l} \setminus \{0\})$.

Proof. — (i) If $m \in \mathcal{D}(\mathbb{R}^{\vec{n}})$, then we can decompose $m = \sum_{k \in \mathbb{N}} g_{k,1} \otimes \cdots \otimes g_{k,\varrho}$, where $g_{k,l} \in \mathcal{D}(\mathbb{R}^{n_l})$ for $k \in \mathbb{N}, l = 1, \dots, \varrho$, and the convergence is in $\mathcal{D}(\mathbb{R}^{\vec{n}})$. In particular, by applying [26, Corollary 5.5] and Theorem 2.7 to the group G^\times , we obtain that

$$\mathcal{K}_{L^\times} m = \sum_{k \in \mathbb{N}} \mathcal{K}_{L_1} g_{k,1} \otimes \cdots \otimes \mathcal{K}_{L_\varrho} g_{k,\varrho}$$

in $L^1(G^\times)$. On the other hand, for all $\phi \in L^2 \cap C_0(G)$, we have

$$g_{k,l}(L_l^b)\phi(x) = \int_{G_l} \phi(x v_l(y_l)^{-1}) \mathcal{K}_{L_l} g_{k,l}(y_l) dy_l$$

by [26, Proposition 3.7], and in particular (being $\mathcal{K}_{L_l} g_{k,l} \in L^1(G_l)$) also $g_{k,l}(L_l^b)\phi \in L^2 \cap C_0(G)$, so that, by iterating,

$$(g_{k,1} \otimes \cdots \otimes g_{k,\varrho})(L^b)\phi(x) = \int_{G^\times} \phi(x v^\times(y)^{-1}) \prod_{l=1}^{\varrho} \mathcal{K}_{L_l} g_{k,l}(y_l) dy.$$

Summing over $k \in \mathbb{N}$, the left-hand side converges in $L^2(G)$ to $m(L^b)\phi$, whereas (since $y \mapsto \phi(x v^\times(y)^{-1})$ is bounded) the right-hand side converges pointwise to $\int_{G^\times} \phi(x v^\times(y)^{-1}) \mathcal{K}_{L^\times} m(y) dy$, and the conclusion follows.

(ii) Choose $\vec{\beta}'$ such that $\vec{\beta} > \vec{\beta}' > \vec{s} + \vec{n}/q - \vec{d}/\max\{2, q\}$. Take a compact $K = \prod_{l=1}^{\varrho} K_l \subseteq \prod_{l=1}^{\varrho} (\mathbb{R}^{n_l} \setminus \{0\})$ and a sequence $m_k \in \mathcal{D}(\mathbb{R}^{\vec{n}})$ with $\text{supp } m_k \subseteq K$ such that $m_k \rightarrow m$ in $S_{q,q}^{\vec{\beta}'}(\mathbb{R}^{\vec{n}})$. By Proposition 5.2 and Hölder's inequality, we then have $\mathcal{K}_{L^\times} m_k \rightarrow \mathcal{K}_{L^\times} m$ in $L^1(G^\times)$; moreover, by Corollary 5.3, $\beta'_l > n_l/q$ for $l = 1, \dots, \varrho$, so that $m_k \rightarrow m$ uniformly. Therefore the conclusion follows by applying (i) to the functions m_k and passing to the limit. \square

We are now going to exploit the Littlewood-Paley theory. An important tool will be the following result, which summarizes a well-known argument for proving properties of square functions.

LEMMA 5.5. — Let (X, μ) be a σ -finite measure space, $1 \leq p < \infty$, $T_{\vec{k}}$ ($\vec{k} \in \mathbb{N}^{\varrho}$) bounded linear operators on $L^p(X, \mu)$. Let $A > 0$ be such that,

for all choices of $\varepsilon_k^i \in \{-1, 1\}$ ($1 \leq i \leq \varrho, k \in \mathbb{N}$) and of a finite subset $I \subseteq \mathbb{N}^\varrho$, we have

$$(5.4) \quad \left\| \sum_{\vec{k} \in I} \varepsilon_{k_1}^1 \cdots \varepsilon_{k_\varrho}^\varrho T_{\vec{k}} \right\|_{p \rightarrow p} \leq A.$$

Then, for all $f \in L^p(X, \mu)$,

$$(5.5) \quad \left\| \left(\sum_{\vec{k} \in \mathbb{N}^\varrho} |T_{\vec{k}} f|^2 \right)^{1/2} \right\|_p \leq C_{\varrho,p} A \|f\|_p.$$

Moreover, if $p > 1$, for all $\{f_{\vec{k}}\}_{\vec{k} \in \mathbb{N}^\varrho} \subseteq L^p(X, \mu)$, if $(\sum_{\vec{k}} |f_{\vec{k}}|^2)^{1/2} \in L^p(X, \mu)$, then

$$\left\| \sum_{\vec{k} \in \mathbb{N}^\varrho} T_{\vec{k}} f_{\vec{k}} \right\|_p \leq C_{\varrho,p'} A \left\| \left(\sum_{\vec{k} \in \mathbb{N}^\varrho} |f_{\vec{k}}|^2 \right)^{1/2} \right\|_p$$

where the series on the left-hand side converges unconditionally in L^p .

Proof. — For $n \in \mathbb{N}$, let $r_n : [0, 1] \rightarrow \mathbb{R}$ be the n -th Rademacher function, $r_n(t) = (-1)^{\lfloor 2^n t \rfloor}$, and set $r_{\vec{k}} = r_{k_1} \otimes \cdots \otimes r_{k_\varrho}$ for $\vec{k} \in \mathbb{N}^\varrho$. Then $(r_{\vec{k}})_{\vec{k}}$ is an (incomplete) orthonormal system in $L^2([0, 1]^\varrho)$, and Khinchin’s inequalities hold: for $1 \leq p < \infty$, there exist $c_{\varrho,p}, C_{\varrho,p} > 0$ such that

$$c_{\varrho,p}^{-1} \|f\|_p \leq \|f\|_2 \leq C_{\varrho,p} \|f\|_p \quad \text{for all } f \in \text{span}\{r_{\vec{k}} : \vec{k} \in \mathbb{N}^\varrho\}.$$

(see [44, Appendix D] or [16, Appendix C]).

Consequently, for all finite $I \subseteq \mathbb{N}^\varrho$ and $f \in L^p(X, \mu)$, we have

$$\begin{aligned} \left\| \left(\sum_{\vec{k} \in I} |T_{\vec{k}} f|^2 \right)^{1/2} \right\|_p^p &= \int_X \left(\sum_{\vec{k} \in I} |T_{\vec{k}} f(x)|^2 \right)^{p/2} d\mu(x) \\ &\leq C_{\varrho,p}^p \int_X \int_{[0,1]^\varrho} \left| \sum_{\vec{k} \in I} T_{\vec{k}} f(x) r_{\vec{k}}(t) \right|^p dt d\mu(x) \\ &= C_{\varrho,p}^p \int_{[0,1]^\varrho} \left\| \left(\sum_{\vec{k} \in I} r_{\vec{k}}(t) T_{\vec{k}} \right) f \right\|_p^p dt \leq C_{\varrho,p}^p A^p \|f\|_p^p. \end{aligned}$$

Since $I \subseteq \mathbb{N}^\varrho$ was arbitrary, (5.5) follows by monotone convergence.

Notice now that the vector-valued Lebesgue space $V_p = L^p(X, \mu; l^2(\mathbb{N}^\varrho))$ can be thought of as a space of sequences of $L^p(X, \mu)$ -functions:

$$V_p = \left\{ (f_{\vec{k}})_{\vec{k} \in \mathbb{N}^\varrho} \in L^p(X, \mu)^{\mathbb{N}^\varrho} : \left(\sum_{\vec{k}} |f_{\vec{k}}|^2 \right)^{1/2} \in L^p(X, \mu) \right\},$$

with norm $\|(f_{\vec{k}})_{\vec{k} \in \mathbb{N}^{\varrho}}\|_{V_p} = \|(\sum_{\vec{k}} |f_{\vec{k}}|^2)^{1/2}\|_p$. The inequality (5.5) therefore means that the operator $f \mapsto (T_{\vec{k}}^* f)_{\vec{k} \in \mathbb{N}^{\varrho}}$ is bounded $L^p(X, \mu) \rightarrow V_p$, with norm not greater than $C_{\varrho,p} A$.

If $p > 1$, the hypothesis (5.4) is equivalent to

$$\left\| \sum_{\vec{k} \in I} \varepsilon_{k_1}^1 \cdots \varepsilon_{k_n}^n T_{\vec{k}}^* \right\|_{p' \rightarrow p'} \leq A;$$

consequently we have that $S : f \mapsto (T_{\vec{k}}^* f)_{\vec{k} \in \mathbb{N}^{\varrho}}$ is bounded $L^{p'}(X, \mu) \rightarrow V_{p'}$, with norm not greater than $C_{\varrho,p'} A$. This means that the transpose operator $S^* : V_p \rightarrow L^p(X, \mu)$ is bounded too, with the same norm; since

$$S^* ((f_{\vec{k}})_{\vec{k}}) = \sum_{\vec{k}} T_{\vec{k}} f_{\vec{k}},$$

where the series on the right-hand side converges unconditionally in L^p , the remaining part of the conclusion follows. \square

For $l = 1, \dots, \varrho$, let $\varepsilon_{l,t}$ be the dilations on \mathbb{R}^{n_l} associated to the weighted subcoercive system $L_{l,1}, \dots, L_{l,n_l}$, and fix an ε_l -homogeneous norm $|\cdot|_{\varepsilon_l}$ on \mathbb{R}^{n_l} , smooth off the origin. Choose a non-negative $\xi \in \mathcal{D}(\mathbb{R})$ with $\text{supp } \xi \subseteq [1/2, 2]$ and such that, if $\xi_k(t) = \xi(2^{-k}t)$, then

$$(5.6) \quad \sum_{k \in \mathbb{Z}} \xi_k^2(t) = 1 \quad \text{for } t > 0,$$

and set, for $l = 1, \dots, \varrho$ and $k \in \mathbb{Z}$, $\chi_{l,k}(\lambda) = \xi(|\varepsilon_{l,2^{-k}}(\lambda)|_{\varepsilon_l}) = \xi_k(|\lambda|_{\varepsilon_l})$ for $\lambda \in \mathbb{R}^{n_l}$. Moreover, for $\vec{k} = (k_1, \dots, k_{\varrho}) \in \mathbb{Z}^{\varrho}$, let $\chi_{\vec{k}} = \chi_{1,k_1} \otimes \cdots \otimes \chi_{\varrho,k_{\varrho}}$, $T_{\vec{k}} = \chi_{\vec{k}}(L^b)$, and define the square function

$$g(\phi) = \left(\sum_{\vec{k} \in \mathbb{Z}^{\varrho}} |T_{\vec{k}} \phi|^2 \right)^{1/2}.$$

Finally, set $X_{\vec{n}} = \{\lambda \in \mathbb{R}^{\vec{n}} : |\lambda_1|_{\varepsilon_1} \cdots |\lambda_{\varrho}|_{\varepsilon_{\varrho}} = 0\}$.

LEMMA 5.6. — For $1 < p < \infty$ and for all $\phi \in L^2 \cap L^p(G)$,

$$c_p \|E^b(\mathbb{R}^{\vec{n}} \setminus X_{\vec{n}})\phi\|_p \leq \|g(\phi)\|_p \leq C_p \|\phi\|_p.$$

Proof. — Using the characterization (1.4) of L^∞ Mihlin-Hörmander conditions, it is not difficult to prove, for $l = 1, \dots, \varrho$, $s \in \mathbb{N}$, $(\varepsilon_k^l)_{k \in \mathbb{Z}} \in \{-1, 0, 1\}^{\mathbb{Z}}$, $N \in \mathbb{N}$, that

$$\left\| \sum_{|k| \leq N} \varepsilon_k^l \chi_{l,k} \right\|_{M_{\varepsilon_l} B_{\infty, \infty}^s} \leq C_{l,s},$$

where $C_{l,s} > 0$ does not depend on $(\varepsilon_k^l)_k$ or N .

By Theorem 4.6 applied to the group G_l , and by transference to the group G (see [3, Theorem 2.7] and [26, Proposition 3.7]), we then have

$$\left\| \sum_{|k| \leq N} \varepsilon_k^l \chi_{l,k}(L_l^b) \right\|_{p \rightarrow p} \leq \left\| \sum_{|k| \leq N} \varepsilon_k^l \chi_{l,k}(L_l) \right\|_{p \rightarrow p} \leq C_{l,p}$$

for $1 < p < \infty$, $l = 1, \dots, \varrho$, where $C_{l,p} > 0$ does not depend on $(\varepsilon_k^l)_k$ or N , and consequently also

$$\left\| \sum_{|k_1|, \dots, |k_\varrho| \leq N} \varepsilon_{k_1}^1 \cdots \varepsilon_{k_\varrho}^\varrho T_{\vec{k}} \right\|_{p \rightarrow p} \leq C_{1,p} \cdots C_{\varrho,p}.$$

Moreover, by (5.6) and the properties of the spectral integral, $\sum_{\vec{k} \in \mathbb{Z}^\varrho} T_{\vec{k}}^2$ converges strongly to $E^b(\mathbb{R}^{\vec{n}} \setminus X_{\vec{n}})$. The conclusion follows then immediately by Lemma 5.5. \square

In the following, we will consider Marcinkiewicz conditions on $\mathbb{R}^{\vec{n}}$ adapted to the system $\mathfrak{J}_{\vec{t}} = \varepsilon_{1,t_1} \times \cdots \times \varepsilon_{\varrho,t_\varrho}$ of multi-variate dilations.

THEOREM 5.7. — *Suppose that, for $l = 1, \dots, \varrho$, the homogeneous group G_l , with the system $L_{l,1}, \dots, L_{l,n_l}$, satisfies (J_{s_l, d_l}) . If $q \in [1, \infty]$ and*

$$\vec{\beta} > \vec{s} + \frac{\vec{n}}{q} - \frac{\vec{d}}{\max\{2, q\}},$$

then, for every Borel $m : \mathbb{R}^{\vec{n}} \rightarrow \mathbb{C}$ with $m|_{X_{\vec{n}}} = 0$ and $\|m\|_{M_{\mathfrak{J}_{\vec{t}} S_{q,q}^{\vec{\beta}} B}} < \infty$, the operator $m(L^b)$ is bounded on $L^p(G)$ for $1 < p < \infty$ and

$$\|m(L^b)\|_{p \rightarrow p} \leq C_{\vec{\beta}, p, q} \|m\|_{M_{\mathfrak{J}_{\vec{t}} S_{q,q}^{\vec{\beta}} B}}.$$

Proof. — Choose a non-negative $\zeta \in \mathcal{D}(\mathbb{R})$ with $\text{supp } \zeta \subseteq [1/4, 4]$ and such that $\zeta \equiv 1$ on $[1/2, 2]$. For $l = 1, \dots, \varrho$, set $\eta_l(\lambda) = \zeta(|\lambda|_{\varepsilon_l})$ and $\eta = \eta_1 \otimes \cdots \otimes \eta_\varrho$. If we set $m_{\vec{k}} = (m \circ \mathfrak{J}_{(2^{k_1}, \dots, 2^{k_\varrho})})\eta$, $f_{\vec{k}} = m_{\vec{k}} \circ \mathfrak{J}_{(2^{-k_1}, \dots, 2^{-k_\varrho})}$ for $\vec{k} \in \mathbb{Z}^\varrho$, then we have $\chi_{\vec{k}} m = f_{\vec{k}} \chi_{\vec{k}}$, so that $T_{\vec{k}} m(L^b) = f_{\vec{k}}(L^b) T_{\vec{k}}$.

Let $w_{\vec{\beta}} = w_{\vec{\beta},1} \otimes \cdots \otimes w_{\vec{\beta},\varrho} \in L^1(G^\times)$ be given by Proposition 5.2. Set $w_{\vec{\beta},l,k} = 2^{kQ_{\delta_l}} w_{\vec{\beta},l} \circ \delta_{l,2^k}$ for $k \in \mathbb{Z}$, $l = 1, \dots, \varrho$, and let $w_{\vec{\beta},\vec{k}} = w_{\vec{\beta},1,k_1} \otimes \cdots \otimes w_{\vec{\beta},\varrho,k_\varrho}$ for $\vec{k} \in \mathbb{Z}^\varrho$. For $l = 1, \dots, \varrho$, if π_l denotes the unitary representation of G_l on $L^2(G)$ induced by the homomorphism ν_l , since $w_{\vec{\beta},l}$ is M-admissible on G_l , then the maximal function $M_{\vec{\beta},l}$ on G defined by $M_{\vec{\beta},l} \phi(x) = \sup_{k \in \mathbb{Z}} |\pi_l(w_{\vec{\beta},l,k}) \phi(x)|$ is bounded on $L^p(G)$ for $1 < p < \infty$, by transference [3, Theorem 2.11].

If $\phi \in L^2 \cap C_0(G)$, then we have, by Proposition 5.4(ii) and Hölder's inequality,

$$\begin{aligned} |f_{\vec{k}}(L^b)T_{\vec{k}}\phi(x)|^2 &\leq \left(\int_{G^\times} |T_{\vec{k}}\phi(xv^\times(y)^{-1})| |\mathcal{K}_{L^\times} f_{\vec{k}}(y)| dy \right)^2 \\ &\leq \int_{G^\times} |T_{\vec{k}}\phi(xv^\times(y)^{-1})|^2 w_{\vec{\beta},\vec{k}}(y) dy \int_{G^\times} |\mathcal{K}_{L^\times} m_{\vec{k}}(y)|^2 w_{\vec{\beta}}^{-1}(y) dy \\ &\leq C_{\vec{\beta},q} \|m_{\vec{k}}\|_{S_{q,q}^{\vec{\beta}}(\mathbb{R}^{\vec{n}})}^2 \pi_1(w_{\vec{\beta},1,k_1}) \cdots \pi_\ell(w_{\vec{\beta},\ell,k_\ell}) (|T_{\vec{k}}\phi|^2) \end{aligned}$$

thus

$$\begin{aligned} \|g(m(L^b)\phi)\|_p &\leq C_{\vec{\beta},q} \|m\|_{M_2 S_{q,q}^{\vec{\beta}} B} \left\| \sum_{\vec{k} \in \mathbb{Z}^e} \pi_1(w_{\vec{\beta},1,k_1}) \cdots \pi_\ell(w_{\vec{\beta},\ell,k_\ell}) (|T_{\vec{k}}\phi|^2) \right\|_{p/2}^{1/2} \end{aligned}$$

for $2 \leq p < \infty$.

On the other hand, since $w_{\vec{\beta}} = w_{\vec{\beta}}^*$, for every $\psi \in L^{(p/2)'}(G)$ we have

$$\begin{aligned} &\left| \int_G \left(\sum_{\vec{k} \in \mathbb{Z}^e} \pi_1(w_{\vec{\beta},1,k_1}) \cdots \pi_\ell(w_{\vec{\beta},\ell,k_\ell}) (|T_{\vec{k}}\phi|^2) \right) \psi d\mu_G \right| \\ &\leq \sum_{\vec{k} \in \mathbb{Z}^e} \int_G \left(\pi_1(w_{\vec{\beta},1,k_1}) \cdots \pi_\ell(w_{\vec{\beta},\ell,k_\ell}) (|T_{\vec{k}}\phi|^2) \right) |\psi| d\mu_G \\ &\leq \int_G \left(\sum_{\vec{k} \in \mathbb{Z}^e} |T_{\vec{k}}\phi|^2 \right) M_{\vec{\beta},\ell} \cdots M_{\vec{\beta},1} (|\psi|) d\mu_G \\ &\leq C_{\vec{\beta},p} \left\| \sum_{\vec{k} \in \mathbb{Z}^e} |T_{\vec{k}}\phi|^2 \right\|_{p/2} \|\psi\|_{(p/2)'}, \end{aligned}$$

that is,

$$\left\| \sum_{\vec{k} \in \mathbb{Z}^e} \pi_1(w_{\vec{\beta},1,k_1}) \cdots \pi_\ell(w_{\vec{\beta},\ell,k_\ell}) (|T_{\vec{k}}\phi|^2) \right\|_{p/2} \leq C_{\vec{\beta},p} \|g(\phi)\|_p^2.$$

Putting all together, and applying Lemma 5.6, we get the conclusion for $2 \leq p < \infty$ (notice that $E^b(\mathbb{R}^{\vec{n}} \setminus X_{\vec{n}})m(L^b) = m(L^b)$ because $m|_{X_{\vec{n}}} = 0$). Thus we are done when $m(L^b)$ is self-adjoint, i.e., when m is real-valued; in the general case, one can decompose m in its real and imaginary parts and then apply the previous result to each part. \square

The hypothesis $m|_{X_{\vec{n}}} = 0$ in Theorem 5.7 does not have an analogue in Theorem 4.6, because the spectral measure of the origin for a homogeneous

weighted subcoercive system on a homogeneous group is zero. On the other hand, if E_l^b is the joint spectral resolution of $L_{l,1}^b, \dots, L_{l,n_l}^b$, then $E_l^b(\{0\})$ need not be zero. However we have the following

PROPOSITION 5.8. — $E_l^b(\{0\})$ is bounded on $L^p(G)$ for $1 \leq p \leq \infty$. If moreover $\overline{v_l(G_l)}$ is not compact in G , then $E_l^b(\{0\}) = 0$.

Proof. — Choose $\psi \in \mathcal{D}(\mathbb{R}^{n_l})$ with $\psi(0) = 1$, so that $\psi_t = \psi \circ \epsilon_{l,t} \rightarrow \chi_{\{0\}}$ pointwise for $t \rightarrow +\infty$, and then $\psi_t(L_l^b) \rightarrow E_l^b(\{0\})$ strongly as operators on $L^2(G)$. By [26, Proposition 3.7] we have $\psi_t(L_l^b) = \pi_l(\mathcal{K}_{L_l} \psi_t)$, thus

$$\|\psi_t(L_l^b)\|_{L^p(G) \rightarrow L^p(G)} \leq \|\mathcal{K}_{L_l} \psi_t\|_{L^1(G_l)} = \|\mathcal{K}_{L_l} \psi\|_{L^1(G_l)} < \infty$$

by (1.1) and Theorem 2.7. For every $f \in L^2 \cap L^p(G)$ and $g \in L^2 \cap L^{p'}(G)$, we then have

$$|\langle E_l^b(\{0\})f, g \rangle| = \lim_{t \rightarrow +\infty} |\langle \psi_t(L_l^b)f, g \rangle| \leq \|\mathcal{K}_{L_l} \psi\|_{L^1(G_l)} \|f\|_{L^p(G)} \|g\|_{L^{p'}(G)},$$

which gives the required boundedness of $E_l^b(\{0\})$.

Suppose now that $\overline{v_l(G_l)}$ is not compact and that $E_l^b(\{0\})f = f$ for some $f \in L^2(G)$. This means that $d\pi_l(L_{l,1})f = \dots = d\pi_l(L_{l,n_l})f = 0$, and proceeding analogously as in the proof of [26, Theorem 5.2] one gets that $\pi_l(y)f = f$ for every $y \in G_l$. If $f \neq 0$, we can find a compact $K \subseteq G$ such that $\int_K |f(x)|^2 dx \neq 0$; on the other hand, since $\overline{v_l(G_l)}$ is not compact, it is easy to construct inductively a sequence $\{y_k\}_{k \in \mathbb{N}}$ in G_l such that the sets $Kv_l(y_k)$ for $k \in \mathbb{N}$ are pairwise disjoint, but then

$$\int_G |f(x)|^2 dx \geq \sum_{k \in \mathbb{N}} \int_{Kv_l(y_k)} |f(x)|^2 dx = \sum_{k \in \mathbb{N}} \int_K |f(x)|^2 dx = \infty,$$

contradiction. Hence $f = 0$, and then $E_l^b(\{0\}) = 0$ by arbitrariness of f . \square

Proposition 5.8 allows to relax the hypothesis $m|_{X_{\vec{n}}} = 0$ in Theorem 5.7. Namely, for $I \subseteq \{1, \dots, \varrho\}$, let $\vec{n}_I = (n_l)_{l \in I}$, so that $\mathbb{R}^{\vec{n}_I} = \prod_{l \in I} \mathbb{R}^{n_l}$; let moreover $\iota_I : \mathbb{R}^{\vec{n}_I} \rightarrow \mathbb{R}^{\vec{n}}$ be the canonical linear embedding, and define on $\mathbb{R}^{\vec{n}_I}$ the system of multi-variate dilations $\mathfrak{J}_{I,(t_l)_{l \in I}} = \prod_{l \in I} \epsilon_{l,t_l}$. Then the decomposition

$$m(L^b) = \sum_{I \subseteq \{1, \dots, \varrho\}} \prod_{l \notin I} E_l^b(\{0\}) \prod_{l \in I} E_l^b(\mathbb{R}^{n_l} \setminus \{0\})(m \circ \iota_I)(L_I^b),$$

where $L_I^b = (L_{l,j}^b)_{l \in I, 1 \leq j \leq n_l}$, shows that the L^p -boundedness of $m(L^b)$ can be obtained by applying Theorem 5.7 to the subsystems L_I^b of L^b :

COROLLARY 5.9. — Suppose that, for $l = 1, \dots, \varrho$, the homogeneous group G_l , with the system $L_{l,1}, \dots, L_{l,n_l}$, satisfies (J_{s_l, d_l}) . If $q \in [1, \infty]$ and

$$\vec{\beta} > \vec{s} + \frac{\vec{n}}{q} - \frac{\vec{d}}{\max\{2, q\}},$$

then, for every Borel $m : \mathbb{R}^{\vec{n}} \rightarrow \mathbb{C}$ we have

$$(5.7) \quad \|m(L^b)\|_{p \rightarrow p} \leq C_{\vec{\beta}, p, q} \sum_{I \in \mathcal{I}} \|m \circ \iota_I\|_{M_{\mathbb{Z}_I} S_{q, q}^{\vec{\beta}_I} B},$$

where \mathcal{I} is the set of the $I \subseteq \{1, \dots, \varrho\}$ such that $\prod_{l \notin I} E_l^b(\{0\}) \neq 0$, and where $\|m \circ \iota_I\|_{M_{\mathbb{Z}_I} S_{q, q}^{\vec{\beta}_I} B} = |m(0)|$ for $I = \emptyset$.

In particular, if all the $\overline{v_l(G_l)}$ are not compact, then the hypothesis $m|_{X_{\vec{n}}} = 0$ in Theorem 5.7 can be dropped.

We conclude the section with a comparison of the Mihlin-Hörmander and Marcinkiewicz conditions, which constitute the hypotheses of Theorems 4.6 and 5.7 respectively: we obtain that, under suitable hypotheses on the orders of smoothness, a Marcinkiewicz condition is weaker than the corresponding Mihlin-Hörmander condition.

PROPOSITION 5.10. — If $q \in [1, \infty]$ and $\beta_l > \tilde{Q}_{\epsilon_l}/q$ for $l = 1, \dots, \varrho$, then

$$\|m\|_{M_{\mathbb{Z}} S_{q, q}^{\vec{\beta}} B} \leq C_{q, \vec{\beta}} \|m\|_{M_{\epsilon} B_{q, q}^{\beta_1 + \dots + \beta_{\varrho}}},$$

where $\epsilon_t = \mathfrak{J}_{(t, \dots, t)} = \epsilon_{1, t} \times \dots \times \epsilon_{\varrho, t}$.

Proof. — For $q < \infty$, in view of the characterization of $S_{q, q}^{\vec{\beta}} B(\mathbb{R}^{\vec{n}})$ as a tensor product of the $B_{q, q}^{\beta_l}(\mathbb{R}^{n_l})$ (cf. [41, Theorem 2.2]), from Lemma 4.8 we immediately get

$$(5.8) \quad \sup_{\vec{t} > 0, |\vec{t}|_{\infty} \leq 1} \|(f \circ \mathfrak{J}_{\vec{t}}) \eta\|_{S_{q, q}^{\vec{\beta}} B} \leq C_{\eta, q, \vec{\beta}} \|f\|_{S_{q, q}^{\vec{\beta}} B}$$

for $\eta = \eta_1 \otimes \dots \otimes \eta_{\varrho} \in \mathcal{D}(\mathbb{R}^{\vec{n}})$; the same holds also for $q = \infty$, as it is easily proved via the characterization by differences of the $S_{\infty, \infty}^{\vec{\beta}} B$ -norm (cf. [40, § 2.3.4]).

Suppose now that $\text{supp } \eta_l \subseteq \{\lambda_l \in \mathbb{R}^{n_l} : a \leq |\lambda_l|_{\infty} \leq b\}$ for some $b > a > 0$ and $l = 1, \dots, \varrho$, and take $\tilde{\eta} \in \mathcal{D}(\mathbb{R}^{\vec{n}})$ such that $\tilde{\eta}|_{\{\lambda : a \leq |\lambda|_{\infty} \leq b\}} \equiv 1$. If $\vec{t} > 0$ and $|\vec{t}|_{\infty} = 1$, then $(\tilde{\eta} \circ \mathfrak{J}_{\vec{t}}) \eta = \eta$, thus from (5.8) we get

$$(5.9) \quad \sup_{\vec{t} > 0, |\vec{t}|_{\infty} = 1} \|(f \circ \mathfrak{J}_{\vec{t}}) \eta\|_{S_{q, q}^{\vec{\beta}} B} \leq C_{\eta, q, \vec{\beta}} \|f \tilde{\eta}\|_{S_{q, q}^{\vec{\beta}} B}.$$

For an arbitrary $\vec{t} > 0$, set $r = |\vec{t}|_\infty$, so that $|r^{-1}\vec{t}|_\infty = 1$; then we have $m \circ \mathfrak{J}_{\vec{t}} = (m \circ \epsilon_r) \circ \mathfrak{J}_{r^{-1}\vec{t}}$, so that, by (5.9) applied to $f = m \circ \epsilon_r$,

$$\|(m \circ \mathfrak{J}_{\vec{t}})\eta\|_{S_{q,q}^{\vec{\beta}}B} \leq C_{\eta,q,\vec{\beta}}\|(m \circ \epsilon_r)\tilde{\eta}\|_{S_{q,q}^{\vec{\beta}}B} \leq C_{\eta,q,\vec{\beta}}\|(m \circ \epsilon_r)\tilde{\eta}\|_{B_{q,q}^{\beta_1+\dots+\beta_\varrho}},$$

and the conclusion follows by a suitable choice of η and $\tilde{\eta}$. □

Notice that a Mihlin-Hörmander condition on $m : \mathbb{R}^{\vec{n}} \rightarrow \mathbb{C}$ gives some control also on the restriction of m to $X_{\vec{n}} \setminus \{0\}$, so that it can be used to satisfy the more involved hypothesis of Corollary 5.9. Namely, by the trace theorem for Besov spaces, under the hypothesis on $\beta_1, \dots, \beta_\varrho$ of Proposition 5.10, if m satisfies an L^q Mihlin-Hörmander condition of order $\beta_1 + \dots + \beta_\varrho$, then $m \circ \iota_I$ satisfies an L^q Mihlin-Hörmander condition of order $\sum_{l \in I} \beta_l$ for $\emptyset \neq I \subseteq \{1, \dots, \varrho\}$; therefore, by Proposition 5.10, all the summands in the right-hand side of (5.7), except possibly for $|m(0)|$, are majorized by $\|m\|_{M_\epsilon B_{q,q}^{\beta_1+\dots+\beta_\varrho}}$.

6. Examples and applications

6.1. Multipliers for a single operator

Although the present work focuses on L^p multipliers for systems of multiple operators, some results can be deduced also for single operators.

In view of the characterization stated in § 1, a homogeneous weighted subcoercive system made of a single operator L is simply a self-adjoint Rockland operator. Hence from Theorem 4.6 and Proposition 4.1 we get

COROLLARY 6.1. — *Let L be a self-adjoint Rockland operator on a homogeneous Lie group G . Suppose that G is h -capacious, and let Q_G be its dimension at infinity. If $m : \mathbb{R} \rightarrow \mathbb{C}$ satisfies an L^2 Mihlin-Hörmander condition of order $s > (Q_G - h)/2$, then $m(L)$ is of weak type $(1, 1)$ and bounded on $L^p(G)$ for $1 < p < \infty$.*

This corollary summarizes several results present in the literature. For a general (positive) Rockland operator, this result is stated in the unpublished paper [18] with regularity threshold $Q_G/2$ (see also [23, 46]); the improvement on the threshold is proved in [20] for products of Euclidean and H-type groups. By restricting to the case of a homogeneous sublaplacian L on a stratified Lie group G , we recover the result of [28, 4], where the threshold is half the homogeneous dimension of G .

Notice that, when a homogeneous Lie group G is stratified (i.e., when the elements of degree 1 generate the whole Lie algebra), then the homogeneous

dimension coincides with Q_G . However, on a nilpotent Lie group G there may be multiple homogeneous structures, and the homogeneous dimension Q_δ depends on the chosen automorphic dilations δ_t (for instance, if X, Y, T is a basis of the Lie algebra of the Heisenberg group with $[X, Y] = T$, then we can set $\delta_t(X) = tX$, $\delta_t(Y) = t^a Y$, $\delta_t(T) = t^{1+a} T$ for all $a \geq 1$, and we have $Q_\delta = 2 + 2a$), whereas Q_G is intrinsic of the Lie group structure of G . In fact (under the hypothesis that all the homogeneity degrees are not less than 1) we always have $Q_\delta \geq Q_G$, with equality if and only if G is stratified [26, Proposition 2.2].

The existence of a homogeneous sublaplacian forces G to be stratified (modulo rescaling the homogeneity degrees). On the other hand, on non-stratified homogeneous groups G there might exist higher-order self-adjoint Rockland operators L (for instance, in the previous example of the Heisenberg group, one can take $L = (-iX)^{2a} - Y^2$ when $a \in \mathbb{N}$), to which Corollary 6.1 applies, with threshold (at most) $Q_G/2$. Therefore our result is also an improvement of [9, Corollary 7.1], where the required threshold is $Q_\delta/2$.

Multiplier results for a single operator can be deduced from Theorem 5.7 too, through a sort of “spectral mapping”; some examples in the context of a non-nilpotent Lie group are presented in the following § 6.3.

6.2. Plancherel measure and capacity map

In order to obtain the sharpest results from the previous multiplier theorems, properties of the Plancherel measure associated with a weighted subcoercive system and of the capacity map of a group must be investigated.

If L_1, \dots, L_n is a weighted subcoercive system on a nilpotent group G , then the associated Plancherel measure σ is related to the group Plancherel measure, defined on the set \widehat{G} of (equivalence classes of) irreducible (unitary) representations of G . In fact, for every irreducible representation π of G on a Hilbert space \mathcal{H}_π , we can find⁽²⁾ a complete orthonormal system $\{v_{\pi,\alpha}\}_\alpha$ of \mathcal{H}_π made of joint eigenvectors of $d\pi(L_1), \dots, d\pi(L_n)$. If $\lambda_{\pi,\alpha} \in \mathbb{R}^n$ denotes the eigenvalue of $d\pi(L_1), \dots, d\pi(L_n)$ corresponding to

⁽²⁾ If $\Delta = p(L_1, \dots, L_n)$ is a positive weighted subcoercive operator, then $\pi(\mathcal{K}_L(e^{-p}))$ is compact (since G is CCR), thus $d\pi(\Delta)$ has discrete spectrum and finite-dimensional eigenspaces, and moreover it commutes with $d\pi(L_1), \dots, d\pi(L_n)$.

the eigenvector $v_{\pi,\alpha}$, then, for every $m \in \mathcal{D}(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} |m(\lambda)|^2 d\sigma(\lambda) = \|\check{m}\|_{L^2(G)}^2 = \int_{\widehat{G}} \|\pi(\check{m})\|_{\text{HS}}^2 d\pi = \int_{\widehat{G}} \sum_{\alpha} |m(\lambda_{\pi,\alpha})|^2 d\pi$$

(cf. [26, Proposition 3.7]). If one is able to determine both the group Plancherel measure and the eigenvectors $v_{\pi,\alpha}$ in such a way that the function $(\pi, \alpha) \mapsto \lambda_{\pi,\alpha}$ is sufficiently regular, then the measure σ on \mathbb{R}^n is determined by the previous identity as the push-forward of the product of the group Plancherel measure times a counting measure.

This route can be followed, e.g., for the free 2-step nilpotent group on 3 generators $N_{3,2}$, which is determined by the relations

$$[X_1, X_2] = T_3, \quad [X_2, X_3] = T_1, \quad [X_3, X_1] = T_2,$$

where $X_1, X_2, X_3, T_1, T_2, T_3$ is a basis of its Lie algebra $\mathfrak{n}_{3,2}$; the dilations δ_t given by $\delta_t(X_j) = tX_j, \delta_t(T_j) = t^2T_j$ define a stratification of $N_{3,2}$, so that $Q_{N_{3,2}} = Q_{\delta} = 9$.

If $L = -(X_1^2 + X_2^2 + X_3^2)$ is the sublaplacian, $\Delta = -(T_1^2 + T_2^2 + T_3^2)$ is the central Laplacian and $D = -(X_1T_1 + X_2T_2 + X_3T_3)$, then L, Δ, D is a homogeneous weighted subcoercive system, with Plancherel measure σ given by

$$\int_{\mathbb{R}^3} f d\sigma = \sum_{\alpha \in 2\mathbb{N}+1} \int_0^{\infty} \int_{-1}^1 f \left(\lambda, \lambda^{3/2} \frac{\theta(1-\theta^2)}{\alpha}, \lambda^2 \frac{(1-\theta^2)^2}{\alpha} \right) \frac{(1-\theta^2)^3}{8\pi^4\alpha^4} d\theta \lambda^{7/2} d\lambda$$

(cf. [11]). This measure is supported on a countable family of surfaces accumulating on the axis $\mathbb{R} \times \{0\} \times \{0\}$, and it is not difficult to show that σ is locally 2-bounded on $\mathbb{R}^3 \setminus \{0\}$, so that the system L, Δ, D satisfies the hypothesis $(J_{9/2,2})$.

On the group $N_{3,2}$ we can also consider the system $L, -iT_1, -iT_2, -iT_3$; in this case the Plancherel measure σ is given by

$$\int_{\mathbb{R}^4} f d\sigma = \sum_{\alpha \in 2\mathbb{N}+1} \int_0^{\infty} \int_{S^2} \int_0^1 f \left(\lambda, \lambda \frac{\nu\omega}{\alpha} \right) \frac{\nu^3}{32\pi^5\alpha^4\sqrt{1-\nu}} d\nu d\omega \lambda^{7/2} d\lambda.$$

One can show that σ is locally $\frac{7}{2}$ -bounded on $\mathbb{R}^4 \setminus \{0\}$, so that the system $L, -iT_1, -iT_2, -iT_3$ satisfies $(J_{9/2,7/2})$.

As an example of computation of the capacity map J , we consider instead the 3-step group $G_{6,23}$ of [34], which is defined by the relations

$$\begin{aligned} [X_6, X_5] &= X_4, & [X_6, X_4] &= X_2, & [X_6, X_3] &= -X_1, \\ [X_5, X_4] &= X_1, & [X_5, X_3] &= X_2, \end{aligned}$$

where $X_6, X_5, X_4, X_3, X_2, X_1$ is a basis of its Lie algebra $\mathfrak{g} = \mathfrak{g}_{6,23}$. It can be shown that the unique automorphic dilations δ_t (modulo automorphisms and rescaling) are given by $\delta_t(X_j) = t^{w_j} X_j$ with $w_6 = w_5 = 1, w_4 = w_3 = 2, w_2 = w_1 = 3$, so that this is an example of a non-stratifiable group, with $12 = Q_\delta > Q_{G_{3,2}} = 11$.

With the notation of § 3, we have

$$\mathfrak{z} = \text{span}\{X_2, X_1\}, \quad \mathfrak{h} = \text{span}\{X_4, X_3, X_2, X_1\}.$$

If we denote by $\bar{X}_6, \bar{X}_5, \bar{X}_4, \bar{X}_3$ and X_2^*, X_1^* the bases induced by X_6, \dots, X_1 on $\mathfrak{g}/\mathfrak{z}$ and \mathfrak{z}^* respectively, then we have

$$|J(x_6\bar{X}_6 + x_5\bar{X}_5 + x_4\bar{X}_4 + x_3\bar{X}_3, t_2X_2^* + t_1X_1^*)|^2 = (x_6^2 + x_5^2)(t_2^2 + t_1^2)$$

with respect to a suitable norm on \mathfrak{h}^* , therefore the dual elements $\bar{X}_6^*, \bar{X}_5^* \in (\mathfrak{g}/\mathfrak{z})^*$ and $X_2, X_1 \in \mathfrak{z}$ attest that $G_{6,23}$ is 2-capacious (despite the fact that Proposition 3.9 does not apply to this group). Consequently, every homogeneous weighted subcoercive system on $G_{6,23}$, such as

$$(-iX_6)^{4k} + (-iX_5)^{4k} + (-iX_3)^{2k}, \quad -iX_2, \quad -iX_1$$

for $k \in \mathbb{N} \setminus \{0\}$, satisfies the hypothesis $(J_{9/2,1})$.

Further examples and details may be found in [25].

6.3. Non-nilpotent groups

Theorem 5.7 allows one to obtain multiplier theorems also on groups which are not homogeneous, even not nilpotent. An interesting class of examples comes by considering an action of a torus $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ on a homogeneous group N by automorphisms which commute with dilations, and the corresponding semidirect product $N \rtimes \mathbb{T}^d$ (or alternatively its universal covering group $N \rtimes \mathbb{R}^d$).

Take for instance a diamond group $G = H_n \rtimes \mathbb{T}^d$ (see [24]). If L is a \mathbb{T}^d -invariant homogeneous sublaplacian on H_n and U_1, \dots, U_d are the partial derivatives on the torus \mathbb{T}^d , then $L, -iU_1, \dots, -iU_d$ is a weighted subcoercive system on G , since these operators commute and they generate an algebra containing the sublaplacian $\Delta = L + (-iU_1)^2 + \dots + (-iU_d)^2$. In

fact, each of the operators $L, -iU_1, \dots, -iU_d$ can be considered as a homogeneous weighted subcoercive system in itself: L is Rockland on H_n , and therefore satisfies $(J_{(\dim H_n)/2,1})$, whereas $-iU_j$ comes from the corresponding derivative on the j -th factor of \mathbb{R}^d , which satisfies $(J_{1/2,1})$. By applying Theorem 5.7, we then obtain that, if $m : \mathbb{R}^{1+d} \rightarrow \mathbb{C}$ vanishes on the coordinate hyperplanes and

$$\|m\|_{M_{\mathbb{1}}S_{2,2}^{\vec{s}}B(\mathbb{R}^{1+d})} < \infty \quad \text{for } \vec{s} > \left(\frac{\dim H_n}{2}, \frac{1}{2}, \dots, \frac{1}{2} \right),$$

then $m(L, -iU_1, \dots, -iU_d)$ is bounded on $L^p(G)$ for $1 < p < \infty$.

Thanks to Corollary 5.9 and Proposition 5.10, this result in turn yields a multiplier theorem for the sublaplacian Δ : if $m : \mathbb{R} \rightarrow \mathbb{C}$ satisfies

$$\|m\|_{M_c B_{2,2}^s(\mathbb{R})} < \infty \quad \text{for } s > \frac{\dim H_n + d}{2} = \frac{\dim G}{2},$$

then $m(\Delta)$ is bounded on $L^p(G)$ for $1 < p < \infty$. We remark that:

- this condition is sharper than the one following by the general result of Alexopoulos [1], which instead requires an L^∞ condition of order $s > \frac{\dim G + 1}{2}$;
- this is an example of a group in which the regularity threshold in a multiplier theorem can be lowered to half the topological dimension, which is neither a Heisenberg or related group, nor SU_2 ;
- the sublaplacian Δ can be replaced by any operator of the form

$$L^{k_0} + -(iU_1)^{2k_1} + \dots + (-iU_d)^{2k_d} \quad \text{or} \quad L^{k_0}(-iU_1)^{k_1} \dots (-iU_d)^{k_d}$$

for some $k_0, k_1, \dots, k_d \in \mathbb{N} \setminus \{0\}$, obtaining an analogous multiplier result with identical smoothness requirement.

Spectral multipliers for operators such as the complete Laplacian

$$\Delta_c = L + (-iT)^2 + (-iU_1)^2 + \dots + (-iU_d)^2,$$

where T is the central derivative on H_n , can also be studied. By considering $L, -iT$ together as a homogeneous system on H_n , and each of the $-iU_j$ separately as before, one obtains a multiplier theorem for Δ_c , with an L^∞ condition of order $s > \frac{\dim G}{2}$.

Analogous considerations hold if one replaces H_n by any M etivier group, and also if one takes the universal covering group $H_n \rtimes \mathbb{R}^d$; this last case comprises, for $d = 1$, the oscillator groups. Notice that the previous result about the Laplacian Δ_c , when stated on the universal covering group, is sharper than [1], since the degree of growth of the group is greater than its topological dimension.

Further examples include the plane motion group $\mathbb{R}^2 \rtimes \mathbb{T}$, and the semidirect product $N_{2,3} \rtimes \mathbb{T}$ determined by the action of SO_2 on the free 3-step nilpotent group $N_{2,3}$ considered, e.g., in [26, § 5.3]. In these last cases, for some distinguished sublaplacians, we still get a sharpening of the result by Alexopoulos: although the required order of smoothness is the same, our condition is expressed in terms of an L^2 instead of an L^∞ Besov norm.

Acknowledgements. I thank Fulvio Ricci for drawing my attention to the subject of this work, and for his continuous encouragement and support.

BIBLIOGRAPHY

- [1] G. ALEXOPOULOS, “Spectral multipliers on Lie groups of polynomial growth”, *Proc. Amer. Math. Soc.* **120** (1994), no. 3, p. 973-979.
- [2] J. BERGH & J. LÖFSTRÖM, *Interpolation spaces. An introduction*, Springer-Verlag, Berlin, 1976, Grundlehren der Mathematischen Wissenschaften, No. 223, x+207 pages.
- [3] E. BERKSON, M. PALUSZYŃSKI & G. WEISS, “Transference couples and their applications to convolution operators and maximal operators”, in *Interaction between functional analysis, harmonic analysis, and probability (Columbia, MO, 1994)*, Lecture Notes in Pure and Appl. Math., vol. 175, Dekker, New York, 1996, p. 69-84.
- [4] M. CHRIST, “ L^p bounds for spectral multipliers on nilpotent groups”, *Trans. Amer. Math. Soc.* **328** (1991), no. 1, p. 73-81.
- [5] ———, “The strong maximal function on a nilpotent group”, *Trans. Amer. Math. Soc.* **331** (1992), no. 1, p. 1-13.
- [6] R. R. COIFMAN & G. WEISS, *Transference methods in analysis*, American Mathematical Society, Providence, R.I., 1976, Conference Board of the Mathematical Sciences Regional Conference Series in Mathematics, No. 31, ii+59 pages.
- [7] M. COWLING, “Herz’s “principe de majoration” and the Kunze-Stein phenomenon”, in *Harmonic analysis and number theory (Montreal, PQ, 1996)*, CMS Conf. Proc., vol. 21, Amer. Math. Soc., Providence, RI, 1997, p. 73-88.
- [8] M. COWLING & A. SIKORA, “A spectral multiplier theorem for a sublaplacian on $SU(2)$ ”, *Math. Z.* **238** (2001), no. 1, p. 1-36.
- [9] X. T. DUONG, E. M. OUHABAZ & A. SIKORA, “Plancherel-type estimates and sharp spectral multipliers”, *J. Funct. Anal.* **196** (2002), no. 2, p. 443-485.
- [10] A. F. M. TER ELST & D. W. ROBINSON, “Weighted subcoercive operators on Lie groups”, *J. Funct. Anal.* **157** (1998), no. 1, p. 88-163.
- [11] V. FISCHER & F. RICCI, “Gelfand transforms of $SO(3)$ -invariant Schwartz functions on the free group $N_{3,2}$ ”, *Ann. Inst. Fourier (Grenoble)* **59** (2009), no. 6, p. 2143-2168.

- [12] G. B. FOLLAND & E. M. STEIN, *Hardy spaces on homogeneous groups*, Mathematical Notes, vol. 28, Princeton University Press, Princeton, N.J., 1982, xii+285 pages.
- [13] A. J. FRASER, “Marcinkiewicz multipliers on the Heisenberg group”, PhD Thesis, Princeton University, 1997.
- [14] ———, “Convolution kernels of $(n + 1)$ -fold Marcinkiewicz multipliers on the Heisenberg group”, *Bull. Austral. Math. Soc.* **64** (2001), no. 3, p. 353-376.
- [15] ———, “An $(n + 1)$ -fold Marcinkiewicz multiplier theorem on the Heisenberg group”, *Bull. Austral. Math. Soc.* **63** (2001), no. 1, p. 35-58.
- [16] L. GRAFAKOS, *Classical Fourier analysis*, second ed., Graduate Texts in Mathematics, vol. 249, Springer, New York, 2008, xvi+489 pages.
- [17] W. HEBISCH, “Multiplier theorem on generalized Heisenberg groups”, *Colloq. Math.* **65** (1993), no. 2, p. 231-239.
- [18] ———, “Functional calculus for slowly decaying kernels”, 1995, Preprint. Available on the web <http://www.math.uni.wroc.pl/~hebisch/>.
- [19] W. HEBISCH & A. SIKORA, “A smooth subadditive homogeneous norm on a homogeneous group”, *Studia Math.* **96** (1990), no. 3, p. 231-236.
- [20] W. HEBISCH & J. ZIENKIEWICZ, “Multiplier theorem on generalized Heisenberg groups. II”, *Colloq. Math.* **69** (1995), no. 1, p. 29-36.
- [21] B. HELFFER, “Conditions nécessaires d’hypoanalyticité pour des opérateurs invariants à gauche homogènes sur un groupe nilpotent gradué”, *J. Differential Equations* **44** (1982), no. 3, p. 460-481.
- [22] B. HELFFER & J. NOURRIGAT, “Caractérisation des opérateurs hypoelliptiques homogènes invariants à gauche sur un groupe de Lie nilpotent gradué”, *Comm. Partial Differential Equations* **4** (1979), no. 8, p. 899-958.
- [23] A. HULANICKI, “A functional calculus for Rockland operators on nilpotent Lie groups”, *Studia Math.* **78** (1984), no. 3, p. 253-266.
- [24] J. LUDWIG, “Dual topology of diamond groups”, *J. Reine Angew. Math.* **467** (1995), p. 67-87.
- [25] A. MARTINI, “Algebras of differential operators on Lie groups and spectral multipliers”, PhD Thesis, Scuola Normale Superiore, Pisa, 2010, [arXiv:1007.1119](https://arxiv.org/abs/1007.1119).
- [26] ———, “Spectral theory for commutative algebras of differential operators on Lie groups”, *J. Funct. Anal.* **260** (2011), no. 9, p. 2767-2814.
- [27] G. MAUCERI, “Zonal multipliers on the Heisenberg group”, *Pacific J. Math.* **95** (1981), no. 1, p. 143-159.
- [28] G. MAUCERI & S. MEDA, “Vector-valued multipliers on stratified groups”, *Rev. Mat. Iberoamericana* **6** (1990), no. 3-4, p. 141-154.
- [29] G. MÉTIVIER, “Hypoellipticité analytique sur des groupes nilpotents de rang 2”, *Duke Math. J.* **47** (1980), no. 1, p. 195-221.

- [30] D. MÜLLER & E. M. STEIN, “On spectral multipliers for Heisenberg and related groups”, *J. Math. Pures Appl. (9)* **73** (1994), no. 4, p. 413-440.
- [31] D. MÜLLER, F. RICCI & E. M. STEIN, “Marcinkiewicz multipliers and multi-parameter structure on Heisenberg (-type) groups. I”, *Invent. Math.* **119** (1995), no. 2, p. 199-233.
- [32] ———, “Marcinkiewicz multipliers and multi-parameter structure on Heisenberg (-type) groups. II”, *Math. Z.* **221** (1996), no. 2, p. 267-291.
- [33] D. MÜLLER & A. SEEGER, “Singular spherical maximal operators on a class of two step nilpotent Lie groups”, *Israel J. Math.* **141** (2004), p. 315-340.
- [34] O. A. NIELSEN, *Unitary representations and coadjoint orbits of low-dimensional nilpotent Lie groups*, Queen’s Papers in Pure and Applied Mathematics, vol. 63, Queen’s University, Kingston, ON, 1983, xiii+117 pages.
- [35] C. ROCKLAND, “Hypoellipticity on the Heisenberg group-representation-theoretic criteria”, *Trans. Amer. Math. Soc.* **240** (1978), p. 1-52.
- [36] W. RUDIN, *Real and complex analysis*, second ed., McGraw-Hill Book Co., New York, 1974, McGraw-Hill Series in Higher Mathematics, xii+452 pages.
- [37] T. RUNST & W. SICKEL, *Sobolev spaces of fractional order, Nemytskij operators, and nonlinear partial differential equations*, de Gruyter Series in Nonlinear Analysis and Applications, vol. 3, Walter de Gruyter & Co., Berlin, 1996, x+547 pages.
- [38] H.-J. SCHMEISSER, “Recent developments in the theory of function spaces with dominating mixed smoothness”, in *Nonlinear Analysis, Function Spaces and Applications. Proceedings of the Spring School held in Prague, May 30-June 6, 2006*, vol. 8, Czech Academy of Sciences, Mathematical Institute, Praha, 2007, p. 145-204.
- [39] H.-J. SCHMEISSER & W. SICKEL, “Spaces of functions of mixed smoothness and approximation from hyperbolic crosses”, *J. Approx. Theory* **128** (2004), no. 2, p. 115-150.
- [40] H.-J. SCHMEISSER & H. TRIEBEL, *Topics in Fourier analysis and function spaces*, Mathematik und ihre Anwendungen in Physik und Technik [Mathematics and its Applications in Physics and Technology], vol. 42, Akademische Verlagsgesellschaft Geest & Portig K.-G., Leipzig, 1987, 300 pages.
- [41] W. SICKEL & T. ULLRICH, “Tensor products of Sobolev-Besov spaces and applications to approximation from the hyperbolic cross”, *J. Approx. Theory* **161** (2009), no. 2, p. 748-786.
- [42] A. SIKORA, “On the $L^2 \rightarrow L^\infty$ norms of spectral multipliers of “quasi-homogeneous” operators on homogeneous groups”, *Trans. Amer. Math. Soc.* **351** (1999), no. 9, p. 3743-3755.
- [43] ———, “Multivariable spectral multipliers and analysis of quasielliptic operators on fractals”, *Indiana Univ. Math. J.* **58** (2009), no. 1, p. 317-334.
- [44] E. M. STEIN, *Singular integrals and differentiability properties of functions*, Princeton Mathematical Series, No. 30, Princeton University Press, Princeton, N.J., 1970, xiv+290 pages.

- [45] ———, *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*, Princeton Mathematical Series, vol. 43, Princeton University Press, Princeton, NJ, 1993, With the assistance of Timothy S. Murphy. Monographs in Harmonic Analysis, III, xiv+695 pages.
- [46] K. STEMPAK, “A weighted multiplier theorem for Rockland operators”, *Colloq. Math.* **51** (1987), p. 335-344.
- [47] H. TRIEBEL, *Interpolation theory, function spaces, differential operators*, North-Holland Mathematical Library, vol. 18, North-Holland Publishing Co., Amsterdam, 1978, 528 pages.
- [48] ———, *Spaces of Besov-Hardy-Sobolev type*, BSB B. G. Teubner Verlagsgesellschaft, Leipzig, 1978, Teubner-Texte zur Mathematik, With German, French and Russian summaries, 207 pages.
- [49] ———, *Theory of function spaces*, Monographs in Mathematics, vol. 78, Birkhäuser Verlag, Basel, 1983, 284 pages.
- [50] ———, *The structure of functions*, Monographs in Mathematics, vol. 97, Birkhäuser Verlag, Basel, 2001, xii+425 pages.
- [51] N. T. VAROPOULOS, L. SALOFF-COSTE & T. COULHON, *Analysis and geometry on groups*, Cambridge Tracts in Mathematics, vol. 100, Cambridge University Press, Cambridge, 1992, xii+156 pages.
- [52] A. VENERUSO, “Marcinkiewicz multipliers on the Heisenberg group”, *Bull. Austral. Math. Soc.* **61** (2000), no. 1, p. 53-68.

Manuscrit reçu le 6 octobre 2010,
accepté le 15 mars 2011.

Alessio MARTINI
Scuola Normale Superiore
Piazza dei Cavalieri, 7
56126 Pisa (Italy)
Current address:
School of Mathematics and Statistics
University of New South Wales
UNSW Sydney NSW 2052 (Australia)
a.martini@unsw.edu.au