ACM vector bundles on projective surfaces of nonnegative Kodaira dimension

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ACM VECTOR BUNDLES ON PROJECTIVE SURFACES
OF NONNEGATIVE KODAIRA DIMENSION

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ABSTRACT. In this paper we contribute to the construction of families of arithmetically Cohen-Macaulay (aCM) indecomposable vector bundles on a wide range of polarized surfaces $(X, \mathcal{O}_X(1))$ for $\mathcal{O}_X(1)$ an ample line bundle. In many cases, we show that for every positive integer r there exists a family of indecomposable aCM vector bundles of rank r, depending roughly on r parameters, and in particular they are of *wild representation type*. We also introduce a general setting to study the complexity of a polarized variety $(X, \mathcal{O}_X(1))$ with respect to its category of aCM vector bundles. In many cases we construct indecomposable vector bundles on X which are aCM for all ample line bundles on X.

1. INTRODUCTION

In many areas of mathematics it plays a central role to understand the *complexity* of the objects one is interested in. This complexity can be measured in many different ways. For instance, in representation theory of quivers, Gabriel's theorem states that a connected quiver supports only finitely many irreducible representations, i.e. of indecomposable modules over the associated path algebra, if and only if it is of type *A, D, E*. The classification of *tame* quivers as *Euclidean graphs*, or *extended Dynkin diagrams*, of type $\tilde{A}, \tilde{D}, \tilde{E}$ was obtained right after. Remarkably, any other quivers support arbitrarily large families of indecomposable representations, i.e. they turn out to be of *wild representation type.*

Motivated by the results, similar questions were raised to understand the category of Cohen-Macaulay modules over an arbitrary **k**-algebra R. When $R := \mathbf{k}[x_0, \ldots, x_n]/I$ is a graded algebra finitely generated in degree one over a field **k**, Cohen-Macaulay modules correspond naturally to arithmetically Cohen-Macaulay sheaves over the closed subscheme $\mathrm{Proj}(R) \subset \mathbb{P}^n$; see [18].

Definition 1.1. A coherent sheaf $\mathcal{E}$ on a projective scheme $(X, \mathcal{O}_X(1))$ is called *arithmetically Cohen-Macaulay* (for short, aCM) if the following conditions hold:
  (i) $\mathcal{E}$ is locally Cohen-Macaulay, i.e. the stalk $\mathcal{E}_x$ has depth equal to $\dim \mathcal{O}_{X,x}$ for any point x on X;
 (ii) $H^i(\mathcal{E}(t)) = 0$ for all $t \in \mathbb{Z}$ and $i = 1, \ldots, \dim X - 1$.

The forementioned correspondence allowed to use a geometrical approach to this kind of questions. A milestone in this area was due to Horrocks, stating that the only indecomposable aCM sheaf on $\mathbb{P}^n$, up to twist, is $\mathcal{O}_{\mathbb{P}^n}$; see [15]. A similar classification was obtained for a smooth quadric hypersurface $Q \subset \mathbb{P}^n$: there exist, besides the structural sheaf $\mathcal{O}_Q$, only one (for n even)

or two (for n odd) irreducible aCM sheaves, the well-studied Spinor bundles; see [19]. The combined work of many mathematicians allowed to complete the list of projective schemes -of positive dimension- supporting a finite number of aCM sheaves, called the varieties of *finite aCM-representation type*: they are either a projective space $\mathbb{P}^n$, a smooth quadric hypersurface $X \subset \mathbb{P}^n$, a cubic scroll in $\mathbb{P}^4$, the Veronese surface in $\mathbb{P}^5$ or a rational normal curve; see [8].

The next degree of complexity is offered by the elliptic curves: in this case, vector bundles of a given rank and degree on an elliptic curve C are in bijection with the points of C; see [1]. They are called varieties of *tame aCM-representation type.* In [9] it was shown that smooth quartic surface scrolls in $\mathbb{P}^5$ are also tame. Notice that all the projective schemes $X \subset \mathbb{P}^n$ mentioned until now are arithmetically Cohen-Macaulay, namely the coordinate ring $R := \mathbf{k}[x_0,\dots,x_n]/I_X$ is a Cohen-Macaulay ring. Indeed, the represention type of the remaining aCM projective schemes $X \subset \mathbb{P}^n$ was set in [10]: they support arbitrarily large families of indecomposable non-isomorphic aCM sheaves. They are, therefore, of *wild aCM-representation type.*

On the other hand, up to our knowledge, a broader problem has been much less studied: which are the possible dimensions of families of aCM irreducible sheaves on polarized schemes $(X,\mathcal{O}_X(1))$, where the only requirement for the line bundle $\mathcal{O}_X(1)$ is to be ample. With this setting it is proved in [6] and [7] that polarized surfaces $(S,\mathcal{O}_S(1))$ such that $p_g = 0$, $q = 0$ or 1, and $\mathcal{O}_S(1)$ is very ample with $h^1(\mathcal{O}_S(1)) = 0$ are of wild representation type. Indeed, the aCM vector bundles witnessing wilderness own a special property: they have the maximal permitted number of global sections, namely they are the so-called *Ulrich vector bundles.* Again for $\mathcal{O}_X(1)$ very ample, it is proved in [22] that for polarized varieties $(X,\mathcal{O}_X(1))$ of dimension at least two, the embedding given by $\mathcal{O}_X(l)$ with $l \geq 3$ is of wild representation type under some mild assumptions on $\mathcal{O}_X(1)$.

The goal of the present paper is to contribute to this set of problems: we are constructing families of aCM vector bundles on a large range of polarized integral surfaces $(X,\mathcal{O}_X(1))$. In the following Theorem we summarize the results obtained:

Theorem 1.2. *Let X be an integral projective surface with a fixed ample line bundle $\mathcal{O}_X(1)$ listed below. Then for each integer $r \geq 2$ there exists an $b_X(r)$-dimensional irreducible family $\{\mathcal{E}_\alpha\}_{\alpha \in \Gamma}$ of indecomposable aCM vector bundles of rank r on X such that for each $\alpha \in \Gamma$ there are only finitely many $\beta \in \Gamma$ with $\mathcal{E}_\alpha \cong \mathcal{E}_\beta$.*

no.	X	$b_X(r)$
1	$\pi : X \to Y$ a birational morphism with $\omega_Y \cong \mathcal{O}_Y$ and $q(Y) = 0$ such that $\pi^{-1}(Y_{\mathrm{sing}}) \cong Y_{\mathrm{sing}}$	$2r$
2	$\omega_X \not\cong \mathcal{O}_X$ locally free with $h^0(\omega_X) = 0$ and $h^0(\omega_X^{\otimes 2}) = 1$, and $q(X) = 0$	$2\lceil \frac{r}{2} \rceil$
3	smooth and $q(X) = 1$ with $\omega_X^\vee \otimes \mathcal{O}_X(1)$ trivial or ample	1
4	$\pi : X \to Y$ a birational morphism with an abelian surface Y and $\omega_X^\vee \otimes \mathcal{O}_X(1)$ trivial or ample	$r+1$
5	$\pi : X \to Y$ a birational morphism with a hyperelliptic surface Y	1
6	$\omega_X \cong \mathcal{O}_X(1)$ with $h^1(\omega_X^{\otimes n}) = 0$ for all $n \in \mathbb{Z}$ and $p_g \geq 3$	r

Theorem 1.2 shows that the projective surfaces of Kodaira dimension zero, possibly with singularities, are of wild representation type, except the case of hyperelliptic surfaces. G. Casnati proved in [7] that hyperelliptic surfaces are of wild representation type with respect to a very ample polarization. Note that we do not assume in Theorem 1.2 that X is minimal or $\mathscr{O}_X(1)$ is very ample, while the result in [7] is more powerful in the sense that it gives wildness with respect to Ulrich vector bundles.

The strategy for Theorem 1.2 is two-fold. One is to consider zero-dimensional subschemes of length equal to the second Chern class of the aCM vector bundles in consideration, from which we construct aCM vector bundles of arbitrary rank by a series of extensions. The cases no. 1, 2 and 6 are handled by this method respectively in Theorem 2.4, Theorem 3.5 and Theorem 5.4; in case no. 6, for the construction of a family of aCM vector bundles of rank r even, it is enough to suppose that $p_g \geq 2$. The second strategy is to consider a family of aCM line bundles, parametrized by a non-empty open Zariski subset of $\mathrm{Pic}^0(X)$, from which we construct aCM vector bundles of arbitrary rank by iterated extensions. The cases no. 3, 4 and 5 are handled by this method respectively in Proposition 4.1, Theorem 1.3 and Proposition 4.5.

Based on the results in Theorem 1.2 we introduce a set-up to measure the complexity of a polarized variety $(X, \mathscr{O}_X(1))$. Define

$$a_{X,\mathscr{O}_X(1)}(r) := \sup_\Gamma \left\{ \dim \Gamma \,\middle|\, \begin{array}{l} \Gamma \text{ runs over the parameter spaces of indecomposable} \\ \text{aCM vector bundles of rank } r \text{ on } X \end{array} \right\}$$

with the convention that $a_{X,\mathscr{O}_X(1)}(r) = -\infty$ if there is no indecomposable aCM vector bundle of rank r. Then we have $a_{X,\mathscr{O}_X(1)}(r) \geq b_X(r)$ for the surfaces listed in Theorem 1.2. We also define

$$a_X(r) := \sup\left\{ a_{X,\mathscr{O}_X(1)}(r) \mid \mathscr{O}_X(1) \text{ ample} \right\}, \quad a'_X(r) := \inf\left\{ a_{X,\mathscr{O}_X(1)}(r) \mid \mathscr{O}_X(1) \text{ ample} \right\}.$$

In many construction of aCM vector bundles, the polarization is assumed to be very ample, in which case we give similar definitions for $a_X(r)$ and $a'_X(r)$, if we consider only very ample polarizations in their definitions. Then we may raise several questions.

- For a given X, what can be said about the following limits?

$$\limsup_{r \to \infty} a_X(r), \; \limsup_{r \to \infty} a'_X(r), \; \liminf_{r \to \infty} a_X(r) \text{ and } \liminf_{r \to \infty} a'_X(r)$$

- What can be said about following suprema

$$\sup_X \{a_X(r)\} \text{ and } \sup_X \{a'_X(r)\},$$

  where X runs over all smooth projective varieties, all varieties with a prescribed Kodaira dimension or all varieties in a prescribed interesting class, e.g. K3 surfaces?

In those questions concerning $(X, \mathscr{O}_X(1))$ polarized surfaces, we may allow singular surfaces, but locally CM, e.g. normal or with singularities of embedded dimension at most three, so that we may consider non-locally free aCM sheaves. We do not know if we may obtain bigger dimensional families of indecomposable aCM sheaves by considering non-locally free aCM sheaves.

For higher dimensional smooth varieties we prove the following result.

Theorem 1.3. *Let X be a smooth projective variety of dimension $n \geq 2$, birational to an abelian variety and fix an ample line bundle $\mathscr{O}_X(1)$ with $\omega_X^\vee \otimes \mathscr{O}_X(1)$ ample. Then X is wild with respect to $\mathscr{O}_X(1)$ and*

$$a_{X,\mathscr{O}_X(1)}(r) \geq (n-1)r + 1.$$

For the proof of Theorem 1.3 we use in an essential way a construction by S. Mukai of vector bundles on abelian varieties in [21], a generic vanishing for smooth varieties with maximal Albanese dimension in [12, 13] and results on the local Hilbert schemes in [5, 11].

Remark 1.4. In cases no. 1, 2 and 6 of Theorem 1.2 the indecomposable vector bundles that we construct are aCM for any ample line bundle on X. On the other hand, in cases no. 3, 4 and 5 of Theorem 1.2 and Theorem 1.3 the indecomposable vector bundles that we construct are aCM for every ample line bundle $\mathscr{O}_X(1)$ with $\omega_X^\vee \otimes \mathscr{O}_X(1)$ ample.

Recall from Theorem 1.2 that we obtain irreducible families of indecomposable aCM vector bundles of rank r on several projective surfaces, whose dimensions are at most linear polynomials in r. Nonetheless, we may not expect that $a_{X,\mathscr{O}_X(1)}(r)$ is linear in r for any projective surface. Indeed, Remark 1.5 shows that for X as in Theorem 1.3 with $n \geq 3$ we get a lower bound for $a_{X,\mathscr{O}_X(1)}(r)$ greater than linear, but less than quadratic, in r.

Remark 1.5. Let X be as in Theorem 1.3. Using the terminology from the proof of this theorem, we can consider the abelian variety Y birational to X and denote by $\widehat{Y} = \mathrm{Pic}^0(Y)$ the abelian variety dual to Y, by R be the completion of the local ring $\mathscr{O}_{\widehat{Y},0}$ and by $B_f[r]$ the set of all R-modules of finite length r. Then for $n \geq 3$ and $r \gg 0$, there are positive constants α_n and β_n such that
$$\alpha_n r^{2-2/n} \leq \dim B_f[r] \leq \beta_n r^{2-2/n}$$
by [5] and [11, page 6]. Since in the proof of Theorem 1.3 we are going to see that $\dim B_f[r] \leq a_{X,\mathscr{O}_X(1)}(r)$ we get
$$\liminf_{r \to \infty} \frac{a_r(X, \mathscr{O}_X(1))}{r^{2-2/n}} > 0.$$

On the other hand, in Section 6 we suggest examples of smooth surfaces of general type with at least a quadratic lower bound for $a_{X,\mathscr{O}_X(1)}(r)$.

We would like to thank C. Ciliberto for suggesting this problem.

2. K3-LIKE SURFACES

In this section we assume that X is integral with $\omega_X \cong \mathscr{O}_X$ and $q(X) = 0$. Let $\mathscr{O}_X(1)$ be an ample line bundle and set $\tilde{g} := h^0(\mathscr{O}_X(1))$; if X is a K3 surface, then we have $2\tilde{g} - 4 = d$ and $g := \tilde{g} - 1$ is called the genus of X. Notice that $h^1(\mathscr{O}_X(t)) = 0$ for all $t \in \mathbb{Z}$.

Proposition 2.1. *For each $r \in \mathbb{Z}$ with $2 \leq r \leq \tilde{g}$, there exists an indecomposable aCM vector bundle $\mathscr{E}$ of rank r on X with $\det(\mathscr{E}) \cong \mathscr{O}_X$ and $c_2(\mathscr{E}) = r$*

Proof. Take a general set of points $S \subset X_{\mathrm{reg}}$ with $|S| = r$. Let Ψ denote the set of all extensions of $\mathscr{I}_{S,X}$ by $\mathscr{O}_X^{\oplus(r-1)}$. Fix a general $\mathscr{E} \in \Psi$, i.e. let $\mathscr{E}$ be a general sheaf fitting into the following exact sequence

$$(1) \qquad\qquad\qquad 0 \to \mathscr{O}_X^{\oplus(r-1)} \xrightarrow{j} \mathscr{E} \to \mathscr{I}_{S,X} \to 0.$$

Note that $\mathrm{ext}_X^1(\mathscr{I}_{S,X}, \mathscr{O}_X) = h^1(\mathscr{I}_{S,X}) = r - 1$ and the sheaf $\mathrm{Im}(j)$ is the image of the evaluation map $H^0(\mathscr{E}) \otimes \mathscr{O}_X \to \mathscr{E}$. By generality of the extension (1) we may choose a basis $\{\varepsilon_1, \ldots, \varepsilon_{r-1}\}$ of $\mathrm{Ext}_X^1(\mathscr{I}_{S,X}, \mathscr{O}_X)$ inducing (1). In particular, $\mathscr{E}$ has no trivial factor. Let $\mathscr{F}$ be a general extension of $\mathscr{I}_{S,X}$ by $\mathscr{O}_X$. Since $\mathrm{Ext}_X^1(\mathscr{I}_{S',X}, \mathscr{O}_X) < \mathrm{Ext}_X^1(\mathscr{I}_{S,X}, \mathscr{O}_X)$ for all $S' \subset S$ such that $|S'| = r - 1$, the Cayley-Bacharach condition is satisfied and hence $\mathscr{F}$ is locally free. Since $\mathscr{O}_X^{\oplus(r-2)} \oplus \mathscr{F} \in \Psi$, $\mathscr{E}$ is general in Ψ and local freeness is an open condition, the sheaf $\mathscr{E}$ is locally free.

Assume $\mathscr{E} \cong \mathscr{F}_1 \oplus \mathscr{F}_2$ with $\mathrm{rank}(\mathscr{F}_1) = s$ and $0 < s < r$. For each $i \in \{1,2\}$, let $\mathscr{G}_i \subseteq \mathscr{F}_i$ be the image of the evaluation map $H^0(\mathscr{F}_i) \otimes \mathscr{O}_X \to \mathscr{F}_i$ with $s_i := \mathrm{rank}(\mathscr{G}_i)$. Then we get $\mathscr{G}_1 \oplus \mathscr{G}_2 \cong \mathscr{O}_X^{\oplus(r-1)}$. In particular, each $\mathscr{G}_i$ is trivial and $s_1 \in \{s, s-1\}$. Note that $(\mathscr{F}_1/\mathscr{G}_1) \oplus (\mathscr{F}_2/\mathscr{G}_2) \cong \mathscr{I}_{S,X}$ has no torsion. If $s_1 = s$, then we get $\mathscr{F}_1/\mathscr{G}_1 \cong 0$, i.e. $\mathscr{F}_1 \cong \mathscr{O}_X^{\oplus s}$, which is impossible since $\mathscr{E}$ has no trivial factor. If $s_1 = s - 1$, then we would get a contradiction similarly from $\mathscr{F}_2 \cong \mathscr{O}_X^{\oplus(r-s)}$. Thus $\mathscr{E}$ is indecomposable.

Then it remains to show that $\mathscr{E}$ is aCM. Since $h^0(\mathscr{O}_S) \le h^0(\mathscr{O}_X(1))$ and S is general, we have $h^1(\mathscr{I}_{S,X}(t)) = 0$ for all $t > 0$. Now $\{\varepsilon_1, \ldots, \varepsilon_{r-1}\}$ is a basis for $\mathrm{Ext}_X^1(\mathscr{I}_{S,X}, \mathscr{O}_X)$ and so it induces an isomorphism $H^1(\mathscr{I}_{S,X}) \to H^2(\mathscr{O}_X^{\oplus(r-1)})$. Thus we have $h^1(\mathscr{E}(t)) = 0$ for all $t \ge 0$. For any $\lambda \in \mathbf{k}$, let $\mathscr{E}_\lambda$ denote the middle term of the extension corresponding to $(\varepsilon_1, \lambda\varepsilon_2, \ldots, \lambda\varepsilon_{r-1})$; we have $\mathscr{E}_\lambda \cong \mathscr{E}$ for $\lambda \ne 0$ and $\mathscr{E}_0 \cong \mathscr{G} \oplus \mathscr{O}_X^{\oplus(r-2)}$ with $\mathscr{G}$ induced by the extension ε_1. As above we see that $h^1(\mathscr{G}(t)) = 0$ for all $t \ge 0$. Since $\mathscr{G}$ is locally free from the Cayley-Bacharach condition and generality of ε_1, we use Serre's duality to obtain $h^1(\mathscr{G}(t)) = h^1(\mathscr{G}(-t)) = 0$ for $t < 0$. Thus $\mathscr{E}_0$ is aCM. Now using the semicontinuity theorem for cohomology, we obtain $h^1(\mathscr{E}(t)) = 0$ because $\mathscr{E}_\lambda \cong \mathscr{E}$. $\qquad\square$

Remark 2.2. Consider the exact sequence (1) with $r = 2$. Since $\mathrm{ext}_X^1(\mathscr{I}_{S,X}, \mathscr{O}_X) = h^1(\mathscr{I}_{S,X}) = 1$, there exists a unique nontrivial extension of $\mathscr{I}_{S,X}$ by $\mathscr{O}_X$; denote its middle term by $\mathscr{G}_S$. Since the Cayley-Bacharach condition is satisfied, the sheaf $\mathscr{G}_S$ is an aCM vector bundle of rank two on X.

Theorem 2.3. *For each integer $2 \le r \le \tilde{g}$, there exists a $2r$-dimensional family $\{\mathscr{E}_\alpha\}_{\alpha \in \Gamma}$ of indecomposable aCM vector bundles of rank r on X with $\det(\mathscr{E}_\alpha) \cong \mathscr{O}_X$ and $c_2(\mathscr{E}_\alpha) = r$ such that for each $\alpha \in \Gamma$ there are only finitely many $\beta \in \Gamma$ with $\mathscr{E}_\beta \cong \mathscr{E}_\alpha$.*

Proof. For any subset $S \subset X_{\mathrm{reg}}$ with $|S| = r$, define $\mathbb{E}'(S)$ to be the subset of $\mathbb{E}(S) := \mathrm{Ext}_X^1(\mathscr{I}_{S,X}, \mathscr{O}_X^{\oplus(r-1)})$, consisting of all extensions whose corresponding middle terms are aCM and indecomposable vector bundles. By Proposition 2.1, $\mathbb{E}'(S)$ is a non-empty open subset of $\mathbb{E}(S)$ and each $[\mathscr{E}] \in \mathbb{E}'(S)$ has trivial determinant with $c_2(\mathscr{E}) = r$.

Letting $\mathbb{U} := \{S \subset X_{\mathrm{reg}} \mid |S| = r\}$, there is a vector bundle $\mathscr{V}$ of rank $(r-1)^2$ on $\mathbb{U}$ with $\mathbb{E}(S)$ as its fibre over $S \in \mathbb{U}$, since $\mathrm{ext}_X^1(\mathscr{I}_{S,X}, \mathscr{O}_X^{\oplus(r-1)}) = (r-1)^2$ for all $S \in \mathbb{U}$. Then there is a non-empty open subset $\mathscr{V}' \subset \mathscr{V}$ with $\mathscr{V}'_{|S} = \mathbb{E}'(S)$ for a general $S \in \mathbb{U}$. Thus there exists an irreducible variety $\Gamma \subset \mathscr{V}'$ such that the restriction of the map $\mathscr{V} \to \mathbb{U}$ to Γ is quasi-finite and dominant. In particular, we have $\dim \Gamma = \dim \mathbb{U} = 2r$.

For $[\mathscr{E}] \in \mathbb{E}'(S)$ we have $h^0(\mathscr{E}) = r - 1$ and the cokernel of the evaluation map $H^0(\mathscr{E}) \otimes \mathscr{O}_X \to \mathscr{E}$ is isomorphic to $\mathscr{I}_{S,X}$. Thus for $[\mathscr{E}] \in \mathbb{E}'(S)$ and $[\mathscr{F}] \in \mathbb{E}'(S')$ with $S \ne S' \in \mathbb{U}$, we have $\mathscr{E} \not\cong \mathscr{E}'$. Since the map $\Gamma \to \mathbb{U}$ is quasi-finite, the variety Γ satisfies the requirements for the assertion. $\qquad\square$

Theorem 2.4. *For each integer $r \ge 2$, there exists an $2r$-dimensional family $\{\mathscr{E}_\alpha\}_{\alpha \in \Gamma}$ of indecomposable aCM vector bundles of rank r on X with $\det(\mathscr{E}_\alpha) \cong \mathscr{O}_X$ and $c_2(\mathscr{E}_\alpha) = r$ such that for each $\alpha \in \Gamma$ there are only finitely many $\beta \in \Gamma$ with $\mathscr{E}_\beta \cong \mathscr{E}_\alpha$.*

For the proof of Theorem 2.4 we collect numerous technical results below. We fix subsets $S_0, \ldots, S_m \subset X_{\mathrm{reg}}$ with $|S_0| = 3$ and $|S_i| = 2$ for all $1 \le i \le m$ such that $S_i \cap S_j = \emptyset$ for any $i \ne j$.

Set $\mathbb{I}(S_1) := \{\mathscr{I}_{S_1,X}\}$ and define $\mathbb{I}(S_1, \ldots, S_i)$ for $i \ge 2$ inductively to be the set of all sheaves admitting an extension of $\mathscr{I}_{S_i,X}$ by an element in $\mathbb{I}(S_1, \ldots, S_{i-1})$. Thus for each $i \ge 2$ each sheaf $\mathscr{J} \in \mathbb{I}(S_1, \ldots, S_i)$ admits the following exact sequence for some $\mathscr{J}' \in \mathbb{I}(S_1, \ldots, S_{i-1})$

$$\text{(2)} \qquad\qquad 0 \to \mathscr{J}' \to \mathscr{J} \to \mathscr{I}_{S_i,X} \to 0.$$

For a subset $N = \{i_1, \ldots, i_k\} \subset \{1, \ldots, i\}$ with $i_1 < \ldots < i_k$, we denote $\mathbb{I}(S_{i_1}, \ldots, S_{i_k})$ by $\mathbb{I}(S_j; j \in N)$.

Set $\mathbb{J}(\emptyset; S_0) := \{\mathscr{I}_{S_0, X}\}$ and define $\mathbb{J}(S_1, \ldots, S_i; S_0)$ to be the set of all isomorphism classes of extensions of $\mathscr{I}_{S_0, X}$ by an element in $\mathbb{I}(S_1, \ldots, S_i)$. Similarly we define $\mathbb{J}(S_j; j \in N; S_0)$.

Lemma 2.5. *Each sheaf $\mathscr{J} \in \mathbb{I}(S_1, \ldots, S_i)$ admits an exact sequence*

$$(3) \qquad 0 \to \mathscr{J} \xrightarrow{\iota} \mathscr{J}^{\vee\vee} \cong \mathscr{O}_X^{\oplus i} \to \mathscr{O}_{S_1 \cup \cdots \cup S_i} \to 0,$$

where the map ι is the double dual map. In particular, we have $h^0(\mathscr{J}) = 0$ and $h^1(\mathscr{J}) = h^2(\mathscr{J}) = i$.

Proof. The assertion is clear for $i = 1$, i.e. $\mathscr{J} = \mathscr{I}_{S_1, X}$. Assume $i \geq 2$ and consider an exact sequence (2) with $\mathscr{J}' \in \mathbb{I}(S_1, \ldots, S_{i-1})$. By inductive hypothesis, the assertion holds for $\mathscr{J}'$ and $\mathscr{I}_{S_i, X}$ and we get the following commutative diagram:

$$
\begin{array}{ccccccccc}
& & 0 & & 0 & & 0 & & \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 \to & & \mathscr{J}' & \to & \mathscr{J} & \to & \mathscr{I}_{S_i, X} & \to & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 \to & & \mathscr{O}_X^{\oplus(i-1)} & \to & \mathscr{J}^{\vee\vee} & \to & \mathscr{O}_X & \to & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 \to & & \mathscr{O}_{S_1 \cup \cdots \cup S_{i-1}} & \to & \mathscr{J}^{\vee\vee}/\mathscr{J} & \to & \mathscr{O}_{S_i} & \to & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
& & 0 & & 0 & & 0 & &
\end{array}
$$

Since $\mathrm{ext}^1_X(\mathscr{O}_X, \mathscr{O}_X) = h^1(\mathscr{O}_X) = 0$, we get $\mathscr{J}^{\vee\vee} \cong \mathscr{O}_X^{\oplus i}$ from the second horizontal sequence. From the third horizontal sequence, we get $\mathscr{J}^{\vee\vee}/\mathscr{J} \cong \mathscr{O}_{S_1 \cup \cdots \cup S_i}$, because S_i's are disjoint to each other. Then we get the exact sequence (3). The vanishing $H^0(\mathscr{J}) = 0$ can be obtained by induction on i and $h^1(\mathscr{J}) = h^2(\mathscr{J}) = i$ can be obtained from (3). $\qquad\square$

Remark 2.6. By the same argument in the proof of Lemma 2.5, we have an exact sequence

$$0 \to \tilde{\mathscr{J}} \to \tilde{\mathscr{J}}^{\vee\vee} \cong \mathscr{O}_X^{\oplus(i+1)} \to \mathscr{O}_{S_0 \cup S_1 \cup \cdots \cup S_i} \to 0,$$

for $\tilde{\mathscr{J}} \in \mathbb{J}(S_1, \ldots, S_i; S_0)$. This gives $h^0(\tilde{\mathscr{J}}) = 0$, $h^1(\tilde{\mathscr{J}}) = i + 2$ and $h^2(\tilde{\mathscr{J}}) = i + 1$.

Lemma 2.7. *For a sheaf $\mathscr{J} \in \mathbb{I}(S_1, \ldots, S_i)$ and any finite subset $A \subset X$,*

   (i) *if $A \nsubseteq S_j$ for all $1 \leq j \leq i$, then we have $\mathrm{Hom}_X(\mathscr{J}, \mathscr{I}_{A,X}) = 0$;*

   (ii) *if $A \nsupseteq S_j$ for some $1 \leq j \leq i$, then we have $\mathrm{Hom}_X(\mathscr{I}_{A,X}, \mathscr{J}) = 0$.*

Proof. We only prove part (i), because part (ii) can be obtained similarly. Let us use induction on i; the case $i = 1$ is true, because $A \nsubseteq S_1$ is equivalent to $\mathrm{Hom}_X(\mathscr{I}_{S_1,X}, \mathscr{I}_{A,X}) = 0$. Now assume $i \geq 2$ and consider the sequence (2) with $\mathscr{J} \in \mathbb{I}(S_1, \ldots, S_{i-1})$. Since $\mathrm{Hom}_X(\mathscr{I}_{S_i,X}, \mathscr{I}_{A,X}) = 0$, any map $f \in \mathrm{Hom}_X(\mathscr{J}, \mathscr{I}_{A,X})$ is uniquely determined by $f' \in \mathrm{Hom}_X(\mathscr{J}', \mathscr{I}_{A,X})$. The inductive assumption gives $f' = 0$ and so we have $f = 0$. $\qquad\square$

Lemma 2.8. *We have $\mathrm{ext}^1_X(\mathscr{I}_{S_{i+1},X}, \mathscr{J}) = 2i$ for $\mathscr{J} \in \mathbb{I}(S_1, \ldots, S_i)$.*

Proof. Let $S := S_1 \cup \cdots \cup S_i$ and apply the functor $\mathrm{Hom}_X(\mathscr{I}_{S_{i+1},X}, -)$ to the sequence (3) to obtain

$$0 \to \mathrm{Hom}_X(\mathscr{I}_{S_{i+1},X}, \mathscr{J}) \to \mathrm{Hom}_X(\mathscr{I}_{S_{i+1},X}, \mathscr{O}_X^{\oplus i}) \to \mathrm{Hom}_X(\mathscr{I}_{S_{i+1},X}, \mathscr{O}_S)$$

$$\to \mathrm{Ext}^1_X(\mathscr{I}_{S_{i+1},X}, \mathscr{J}) \to \mathrm{Ext}^1_X(\mathscr{I}_{S_{i+1},X}, \mathscr{O}_X^{\oplus i}) \to \mathrm{Ext}^1_X(\mathscr{I}_{S_{i+1},X}, \mathscr{O}_S).$$

Here, we have $\hom_X(\mathscr{I}_{S_{i+1},X},\mathscr{O}_X^{\oplus i}) = i = \mathrm{ext}_X^1(\mathscr{I}_{S_{i+1},X},\mathscr{O}_X^{\oplus i})$. We also get $\hom_X(\mathscr{I}_{S_{i+1},X},\mathscr{O}_S) = 2i$, because S is disjoint from S_{i+1}. Now apply the functor $\mathrm{Hom}_X(-,\mathscr{O}_S)$ to the standard exact sequence for $S_{i+1} \subset X$ to obtain

$$\mathrm{Ext}_X^1(\mathscr{O}_X,\mathscr{O}_S) \to \mathrm{Ext}_X^1(\mathscr{I}_{S_{i+1},X},\mathscr{O}_S) \to \mathrm{Ext}_X^2(\mathscr{O}_{S_{i+1}},\mathscr{O}_S).$$

Here, we have $\mathrm{ext}_X^1(\mathscr{O}_X,\mathscr{O}_S) = h^1(\mathscr{O}_S) = 0$ and $\mathrm{ext}_X^2(\mathscr{O}_{S_{i+1}},\mathscr{O}_S) = 0$. In particular, we get $\mathrm{ext}_X^1(\mathscr{I}_{S_{i+1},X},\mathscr{O}_S) = 0$. Finally, apply the functor $\mathrm{Hom}_X(\mathscr{I}_{S_{i+1},X},-)$ to the sequence (2) to have

$$\mathrm{Hom}_X(\mathscr{I}_{S_{i+1},X},\mathscr{J}') \to \mathrm{Hom}_X(\mathscr{I}_{S_{i+1},X},\mathscr{J}) \to \mathrm{Hom}_X(\mathscr{I}_{S_{i+1},X},\mathscr{I}_{S_i,X}).$$

Since $S_i \cap S_{i+1} = \emptyset$, we get $\hom_X(\mathscr{I}_{S_{i+1},X},\mathscr{I}_{S_i,X}) = 0$. By inductive hypothesis, we get $\hom_X(\mathscr{I}_{S_{i+1},X},\mathscr{J}') = 0$. Thus we have $\hom_X(\mathscr{I}_{S_{i+1},X},\mathscr{J}) = 0$ and we get the assertion. □

Remark 2.9. Similarly as in the proof of Lemma 2.8, we see that $\mathrm{ext}_X^1(\mathscr{I}_{S_0,X},\mathscr{J}) = 3i$ for any $\mathscr{J} \in \mathbb{I}(S_1,\ldots,S_i)$. In particular, there exists a non-trivial extension

$$0 \to \mathscr{J} \to \tilde{\mathscr{J}} \to \mathscr{I}_{S_0,X} \to 0.$$

In this case, we have $\mathrm{ext}_X^1(\mathscr{I}_{S_0,X},\mathscr{O}_X^{\oplus i}) = 2i$ and the other numeric data in the proof of Lemma 2.8 are all same.

Lemma 2.10. *For each $i \geq 1$, there exists an indecomposable sheaf $\mathscr{J} \in \mathbb{I}(S_1,\ldots,S_i)$.*

Proof. Since $\mathscr{I}_{S_1,X}$ has rank one and X is an integral variety, $\mathscr{I}_{S_1,X}$ is indecomposable. Thus we may assume $i \geq 2$. Note that each $\mathscr{I}_{S_j,X}$ has the same Hilbert polynomial with respect to any polarization $\mathscr{O}_X(1)$. Thus any sheaf in $\mathbb{I}(S_1,\ldots,S_i)$ is strictly semistable with $\oplus_{j=1}^i \mathscr{I}_{S_j,X}$ as its Jordan-Hölder grading. Let $\mathscr{J}$ be a general sheaf fitting into an exact sequence

$$(4) \qquad 0 \to \oplus_{j=1}^{i-1} \mathscr{I}_{S_j,X} \xrightarrow{f} \mathscr{J} \xrightarrow{g} \mathscr{I}_{S_i,X} \to 0$$

and assume that $\mathscr{J}$ is decomposable, say $\mathscr{J} \cong \mathscr{A}_1 \oplus \cdots \oplus \mathscr{A}_h$ with $h \geq 2$ and each $\mathscr{A}_j$ indecomposable. Since $\mathscr{J}$ is strictly semistable with $gr(\mathscr{J}) \cong \oplus_{j=1}^i \mathscr{I}_{S_j,X}$, there is a subset $N_j \subset \{1,\ldots,i\}$ for each $j \in \{1,\ldots,h\}$ such that $gr(\mathscr{A}_j) \cong \oplus_{k \in N_j} \mathscr{I}_{S_k,X}$. Note that $\{N_j | 1 \leq j \leq h\}$ forms a partition of $\{1,\ldots,i\}$ with each N_j non-empty.

Assume first that $|N_j| = 1$ for all j. Then we have $\mathscr{J} \cong \oplus_{j=1}^i \mathscr{I}_{S_j,X}$. Since we have $\mathrm{Hom}_X(\mathscr{I}_{S_i,X},\mathscr{I}_{S_j,X}) = 0$ for all $j < i$ and $\mathrm{Hom}_X(\mathscr{I}_{S_i,X},\mathscr{I}_{S_i,X}) \cong \mathbf{k}$, we get that the sequence (4) splits, contradicting Lemma 2.8.

Now without loss of generality, assume $e := |N_1| \geq 2$. If $i \notin N_1$, then by permuting the first $i-1$ indices of S_j's we may assume $\mathscr{A}_1 \in \mathbb{I}(S_1,\ldots,S_e)$. Then by Lemma 2.7 we have $\hom_X(\mathscr{I}_{S_j,X},\mathscr{A}_1) = \hom_X(\mathscr{A}_1,\mathscr{I}_{S_j,X}) = 0$ for all $j \geq e+1$. Thus f induces an isomorphism $f' : \mathscr{A}_1 \to \oplus_{j=1}^e \mathscr{I}_{S_j,X}$, contradicting the assumption $e \geq 2$ and the indecomposability of $\mathscr{A}_1$. If $i \in N_1$, then by permuting the first $i-1$ indices of S_j's we may assume $\mathscr{A}_1 \in \mathbb{I}(S_{i-e+1},\ldots,S_i)$. From the case when $i \notin N_1$ we may also assume $|N_j| = 1$ for all $j > 1$, and this implies $\mathscr{J} \cong \mathscr{A}_1 \oplus (\oplus_{j=1}^{i-e} \mathscr{I}_{S_j,X})$. Then by Lemma 2.7 we have $\mathrm{Hom}_X(\mathscr{I}_{S_j,X},\mathscr{A}_1) = 0$ for all $j \leq i-e$. In particular, the extension class $\varepsilon = (\varepsilon_1,\ldots,\varepsilon_{i-1})$ corresponding to (4) with $\varepsilon_j \in \mathrm{Ext}_X^1(\mathscr{I}_{S_i,X},\mathscr{I}_{S_j,X})$ satisfies $\varepsilon_j = 0$ for all $j \leq i-e$, contradicting Lemma 2.8 and the generality of ε. □

Remark 2.11. As in the proof of Lemma 2.10, let us consider a general sheaf $\tilde{\mathscr{J}}$ fitting into an exact sequence

$$(5) \qquad 0 \to \oplus_{j=1}^i \mathscr{I}_{S_j,X} \to \tilde{\mathscr{J}} \to \mathscr{I}_{S_0,X} \to 0.$$

By Remark 2.9 the extension (5) is non-trivial. Here, $\tilde{\mathscr{J}} \in \mathbb{J}(S_1, \ldots, S_i; S_0)$ and the sequence (5) is the Harder-Narasimhan filtration of $\tilde{\mathscr{J}}$. Assume that $\tilde{\mathscr{J}}$ is decomposable, say $\tilde{\mathscr{J}} \cong \tilde{\mathscr{A}}_1 \oplus \cdots \oplus \tilde{\mathscr{A}}_h$. Note that the HN filtration of $\tilde{\mathscr{J}}$ is obtained from the ones of each $\tilde{\mathscr{A}}_i$. In particular, as in the proof of Lemma 2.10, we have a partition $\{N_j | 1 \le j \le h\}$ of $\{0, 1, \cdots, i\}$ such that $\tilde{\mathscr{A}}_j \in \mathbb{I}(S_k; k \in N_j)$ if $0 \notin N_j$, and $\tilde{\mathscr{A}}_j \in \mathbb{J}(S_k; k \in N_j \setminus \{0\}; S_0)$. Then by the same argument in the proof of Lemma 2.10, we get a contradiction. Thus we get an indecomposable sheaf in $\mathbb{J}(S_1, \ldots, S_i; S_0)$.

Lemma 2.12. *For each integer $i \ge 1$, the set $\mathbb{I}(S_1, \ldots, S_i)$ is parametrized by an affine space $T(S_1, \ldots, S_i)$, not necessarily finite-to-one, equipped with the universal sheaf, i.e. a sheaf $\mathscr{S}(S_1, \ldots, S_i)$ on $T(S_1, \ldots, S_i) \times X$ such that the fiber of $\mathscr{S}(S_1, \ldots, S_i)$ over $\{\mathscr{J}\} \times X$ with $\mathscr{J} \in \mathbb{I}(S_1, \ldots, S_i)$ is the sheaf $\mathscr{J}$ on X.*

Proof. For $i = 1$ we may take as $T(S_1)$ just a single point set, because $\mathbb{I}(S_1) = \{\mathscr{I}_{S_1, X}\}$. Assume that there exists an affine space $T(S_1, \ldots, S_{i-1})$ and a sheaf $\mathscr{S}(S_1, \ldots, S_{i-1})$ with prescribed property for $i \ge 2$. We set

$$T(S_1, \ldots, S_i) := \mathscr{E}xt^1_{p_1}(\mathscr{S}(S_1, \ldots, S_{i-1}), p_2^* \mathscr{I}_{S_i, X})$$

$$= R^1(p_{1*} \mathscr{H}om_{T(S_1, \ldots, S_{i-1}) \times X}(\mathscr{S}(S_1, \ldots, S_{i-1}), -))(p_2^* \mathscr{I}_{S_i, X})$$

to be the relative $\mathscr{E}xt^1_{p_1}$-sheaf, where p_j is the projection from $T(S_1, \ldots, S_{i-1}) \times X$ to its j-th factor; see [20, Proposition 3.1]. By Lemma 2.8 we have $\mathrm{ext}^1_X(\mathscr{J}', \mathscr{I}_{S_i, X}) = 2i - 2$ for each $\mathscr{J}' \in T(S_1, \ldots, S_{i-1})$. This implies that $T(S_1, \ldots, S_i)$ is a vector bundle of rank $2i - 2$ over $T(S_1, \ldots, S_{i-1})$ and so it is an affine space parametrizing $\mathbb{I}(S_1, \ldots, S_i)$ as required. We may also take as $\mathscr{S}(S_1, \ldots, S_i)$ the universal extension on $T(S_1, \ldots, S_i) \times X$ as in [20, Corollary 3.4]. $\qquad\square$

Remark 2.13. Following the same argument in the proof of Lemma 2.12, we can obtain an affine space $\tilde{T}(S_1, \ldots, S_i; S_0)$ parametrizing $\mathbb{J}(S_1, \ldots, S_i)$ equipped with the universal sheaf $\tilde{\mathscr{S}}(S_1, \ldots, S_i; S_0)$.

Proof of Theorem 2.4: Assume that r is even and set $m := r/2$. Fix subsets $S_1, \ldots, S_m \subset X_{\mathrm{reg}}$ such that $|S_i| = 2$ for all i and $S_i \cap S_j = \emptyset$ for all $i \ne j$. By Lemma 2.10 there exists an indecomposable sheaf $\mathscr{J} \in \mathbb{I}(S_1, \ldots, S_m)$, for which we consider a general sheaf $\mathscr{E}$ fitting into the following exact sequence:

$$(6) \qquad\qquad 0 \to \mathscr{O}_X^{\oplus m} \xrightarrow{f} \mathscr{E} \to \mathscr{J} \to 0.$$

Note that $\mathscr{E}$ has rank r with $\det(\mathscr{E}) \cong \mathscr{O}_X$ and $c_2(\mathscr{E}) = r$. Let $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_m) \in \mathrm{Ext}^1_X(\mathscr{J}, \mathscr{O}_X^{\oplus m})$ be the extension class corresponding to (6) with $\varepsilon_i \in \mathrm{Ext}^1_X(\mathscr{J}, \mathscr{O}_X)$. Note that $h^0(\mathscr{E}) = m$ and $f(\mathscr{O}_X^{\oplus m})$ is the image of the evaluation map $\rho_{\mathscr{E}} : H^0(\mathscr{E}) \otimes \mathscr{O}_X \to \mathscr{E}$ with $\mathscr{J} = \mathrm{coker}(\rho_{\mathscr{E}})$.

By Lemma 2.5 and Serre's duality, we have $\mathrm{ext}^1_X(\mathscr{J}, \mathscr{O}_X) = h^1(\mathscr{J}) = m$. From the generality of ε we see that the extensions $\varepsilon_1, \ldots, \varepsilon_m$ are linearly independent. In particular, we have $A \cdot \varepsilon \ne 0$ for all $A \in \mathrm{GL}(m)$, and so $\mathscr{E} \ncong \mathscr{O}_X \oplus \mathscr{G}$ with $\mathscr{G}$ an extension of $\mathscr{J}$ by $\mathscr{O}_X^{\oplus(m-1)}$. Since $f(\mathscr{O}_X^{\oplus m}) \subset \mathscr{E}$ is the image of $\rho_{\mathscr{E}}$, we get that $\mathscr{E} \ncong \mathscr{O}_X \oplus \mathscr{G}$ for any sheaf $\mathscr{G}$, i.e. $\mathscr{E}$ has no trivial factor.

Assume that $\mathscr{E}$ is decomposable, say $\mathscr{E} \cong \mathscr{E}_1 \oplus \mathscr{E}_2$ with each $\mathscr{E}_i \ne 0$. Since the global section functor $H^0(-)$ and the evaluation map commute with direct sums, we have $\mathscr{J} \cong \mathrm{coker}(\rho_{\mathscr{E}_1}) \oplus \mathrm{coker}(\rho_{\mathscr{E}_2})$. Since $\mathscr{J}$ is indecomposable, we get $\mathrm{coker}(\rho_{\mathscr{E}_i}) = 0$ for some $i \in \{1, 2\}$. This implies that $\mathscr{E}_i$ is trivial, which is impossible because $\mathscr{E}$ has no trivial factor.

To conclude the case r even we need to find a sheaf $\mathscr{E}$ that is locally free and aCM. Consider the variety $T(S_1, \ldots, S_m)$ together with the sheaf $\mathscr{S}(S_1, \ldots, S_m)$ in Lemma 2.12. Define

$$\mathscr{V}(S_1, \ldots, S_m) := \mathscr{E}xt^1_{p_2}(\mathscr{S}(S_1, \ldots, S_m), p_2^* \mathscr{O}_X^{\oplus m})$$

to be the relative $\mathscr{E}xt^1_{p_2}$-sheaf as in [20, Proposition 3.1]; the fibre of $\mathcal{V}(S_1,\ldots,S_m)$ over a point $\mathscr{J} \in T(S_1,\ldots,S_m)$ is the set of all extensions of $\mathscr{J}$ by $\mathscr{O}_X^{\oplus m}$. By Lemma 2.5 the sheaf $\mathcal{V}(S_1,\ldots,S_m)$ is a vector bundle of rank m^2 on $T(S_1,\ldots,S_m)$ and so it is an affine space. Pick an aCM and locally free sheaf $\mathscr{G}_{S_i}$ fitting into the sequence (6) with $r = 2$ for each S_i. Since $\mathscr{G}_{S_1} \oplus \cdots \oplus \mathscr{G}_{S_m}$ is locally free and aCM, the sheaf associated to a general point in $\mathcal{V}$ is also locally free and aCM. Define

$$\mathbb{U} := \left\{ (S_1,\ldots,S_m) \mid S_i \subset X_{\mathrm{reg}} \text{ with } |S_i| = 2 \text{ and } S_i \cap S_j = \varnothing \text{ for all } i \neq j \right\}$$

and consider a vector bundle $\mathcal{V}$ on $\mathbb{U}$, whose fibre over $(S_1,\ldots,S_m)$ is $\mathcal{V}(S_1,\ldots,S_m)$. Then there exists a non-empty open subset $\mathcal{V}' \subset \mathcal{V}$ such that the middle term of each extension in $\mathcal{V}'$ is aCM and locally free. As in the proof of Theorem 2.3 we can choose an irreducible subvariety $\Gamma \subset \mathcal{V}'$ such that the restriction of the map $\mathcal{V}' \to \mathbb{U}$ to Γ is quasi-finite and dominant. Hence we get the assertion for the case r even.

Now assume that r is odd, say $r = 2m+3$. The case $m = 0$ is true by Proposition 2.1 with $r = 3$, because we have $g = h^0(\mathscr{O}_X(1)) \geq 3$. Now assume $r \geq 5$, i.e. $m \geq 1$, and that Theorem 2.4 is true for all odd integers less than r. We fix subsets $S_0,\ldots,S_m \subset X_{\mathrm{reg}}$ with $|S_0| = 3$ and $|S_i| = 2$ for all $i \geq 1$ such that $S_i \cap S_j = \varnothing$ for all $i \neq j$. Define

$$\mathcal{W}(S_1,\ldots,S_m;S_0) := \mathscr{E}xt^1_{p_2}(\tilde{\mathscr{S}}(S_1,\ldots,S_m;S_0), p_2^*\mathscr{O}_X^{\oplus(m+2)}),$$

where $\tilde{\mathscr{S}}(S_1,\ldots,S_m;S_0)$ is the universal sheaf in Remark 2.13. Then it parametrizes all the extensions of some sheaf $\tilde{\mathscr{J}} \in \mathbb{J}(S_1,\ldots,S_m;S_0)$ by $\mathscr{O}_X^{\oplus(m+2)}$. Note that for each extension in $\mathcal{W}(S_1,\ldots,S_m;S_0)$ the corresponding middle term $\mathscr{E}$ is torsion-free and has rank $r = 2m+3$ with $\det(\mathscr{E}) \cong \mathscr{O}_X$ and $c_2(\mathscr{E}) = r$.

Let us denote by $\mathscr{G}_{S_0}$ an aCM and indecomposable vector bundle of rank three, admitting an extension of $\mathscr{I}_{S_0,X}$ by $\mathscr{O}_X^{\oplus 2}$ as in Proposition 2.1. Then $\oplus_{i=1}^m \mathscr{G}_{S_i}$ is the middle term of an extension in $\mathcal{W}(S_1,\ldots,S_m;S_0)$, which is locally free and aCM. So the general extension in $\mathcal{W}(S_1,\ldots,S_m;S_0)$ has an aCM and indecomposable middle term; the indecomposability can be seen by the exact same way as in the case of even r. Now fix an indecomposable sheaf $\tilde{\mathscr{J}} \in \mathbb{J}(S_1,\ldots,S_m;S_0)$ in Remark 2.11 and consider a general sheaf $\mathscr{E}$ fitting into the following exact sequence:

$$(7) \qquad\qquad 0 \to \mathscr{O}_X^{\oplus(m+2)} \xrightarrow{f} \mathscr{E} \xrightarrow{g} \tilde{\mathscr{J}} \to 0.$$

Assume that $\mathscr{E}$ is decomposable, say $\mathscr{E} \cong \mathscr{E}_1 \oplus \mathscr{E}_2$ with each $\mathscr{E}_i \not\cong 0$. As before, $f(\mathscr{O}_X^{\oplus(m+2)})$ is the image of the evaluation map $\rho_{\mathscr{E}} : H^0(\mathscr{E}) \otimes \mathscr{O}_X \to \mathscr{E}$ and $\mathrm{coker}(\rho_{\mathscr{E}}) = \tilde{\mathscr{J}}$. Since the global section functor $H^0(-)$ and the evaluation map commute with finite direct sums, we have $\tilde{\mathscr{J}} \cong \mathrm{coker}(\rho_{\mathscr{E}_1}) \oplus \mathrm{coker}(\rho_{\mathscr{E}_2})$. Since $\tilde{\mathscr{J}}$ is indecomposable, we get that $\mathscr{E}_i$ is trivial for some i, which contradicts to the generality of the extension (7), because we have $\mathrm{ext}^1_X(\tilde{\mathscr{J}},\mathscr{O}_X) = h^1(\tilde{\mathscr{J}}) = m+2$ by Remark 2.6. As in the case r even, we define

$$\tilde{\mathbb{U}} := \{(S_0, S_1,\ldots,S_m) \mid S_i \subset X_{\mathrm{reg}} \text{ with } |S_0| = 3,$$
$$|S_i| = 2 \text{ for all } 1 \leq i \leq m \text{ and } S_i \cap S_j = \varnothing \text{ for all } i \neq j\}.$$

We consider a vector bundle $\mathcal{W}$ on $\mathbb{U}$, whose fibre over $(S_0, S_1,\ldots,S_m)$ is $\mathcal{W}(S_1,\ldots,S_m;S_0)$. Then we get the assertion, following the same argument in the case r even. $\qquad\square$

Remark 2.14. Let $\pi : Y \to X$ be a birational morphism between integral projective surfaces with $\omega_X \cong \mathscr{O}_X$ and $q(X) = 0$ such that π induces an isomorphism $\pi^{-1}(X_{\mathrm{sing}}) \cong X_{\mathrm{sing}}$. In particular, we have $Y_{\mathrm{reg}} = \pi^{-1}(X_{\mathrm{reg}})$. This implies that $\pi_*\mathscr{O}_Y \cong \mathscr{O}_X$ and $R^1\pi_*\mathscr{O}_Y \cong 0$. Since each fiber of π has dimension at most one, we also have $R^2\pi_*\mathscr{F} \cong 0$ for any coherent sheaf $\mathscr{F}$ on X. Thus we have

$q(Y) = 0$ and $h^2(\mathcal{O}_Y) = 1$. Since π induces an isomorphism between $\pi^{-1}(X_{\text{sing}})$ and X_{sing}, the canonical sheaf ω_Y is locally free with $h^0(\omega_Y) = 1$ and so there is an effective divisor Δ such that $|\omega_Y| = \{\Delta\}$; we have $\Delta = \emptyset$ if and only if π is an isomorphism. By Serre's duality we have $\text{ext}_Y^1(\mathcal{I}_{S,Y}, \mathcal{O}_Y) = h^1(\mathcal{I}_{S,Y} \otimes \omega_Y)$. Since $|\omega_Y| = \{\Delta\}$ and $S \cap \Delta = \emptyset$, we may use the long exact sequence of cohomology of the following exact sequence

$$0 \to \mathcal{I}_{S,Y} \otimes \omega_Y \to \omega_Y \to \mathcal{O}_S \to 0$$

to obtain $\text{ext}_Y^1(\mathcal{I}_{S,Y}, \mathcal{O}_Y) = |S| - 1$ for any finite subset $S \subset Y_{\text{reg}} \setminus \Delta$. Then the same statement of Theorem 2.4 holds for Y, using the same argument in its proof with subsets $S_i \subset Y_{\text{reg}} \setminus \Delta$ for $i = 0, \cdots, m$.

3. ENRIQUES SURFACES

In this section we assume that X is an integral projective surface with $q(X) = 0$ and $\omega_X \not\cong \mathcal{O}_X$ locally free such that $h^0(\omega_X) = 0$ and $h^0(\omega_X^{\otimes 2}) = 1$. Let $\Delta \geq 0$ be the effective divisor such that $\omega_X^{\otimes 2} \cong \mathcal{O}_X(\Delta)$. When X is smooth, the minimal model of X is an Enriques surface. Note that $h^2(\mathcal{O}_X) = h^0(\omega_X) = 0$ and so $\chi(\mathcal{O}_X) = 1$. Set $X' := X_{\text{reg}} \cap (X \setminus \Delta)$.

Remark 3.1. We fix an ample line bundle $\mathcal{O}_X(1)$ on X such that $h^1(\mathcal{O}_X(t)) = 0$ for all $t \in \mathbb{Z}$; at least in characteristic zero Kodaira's vanishing theorem shows that we only need this assumption for $t \geq 0$. The case $t = 0$ is a general assumption of the surfaces considered in this article. Serre's duality gives $h^1(\omega_X(t)) = 0$ for all $t \in \mathbb{Z}$. Notice that using Riemann-Roch it is easy to see that under these hypothesis $h^0(\omega_X(1)) \neq 0$. In summary, we take a polarization $\mathcal{O}_X(1)$ such that $h^0(\omega_X(1)) \neq 0$ and $h^1(\mathcal{O}_X(t)) = h^1(\omega_X(t)) = 0$ for all $t \in \mathbb{Z}$. If $\Delta = \emptyset$, e.g. minimal Enriques surfaces, then we always have $h^1(\mathcal{O}_X(t)) = 0$ for $t > 0$, because $\omega_X(t)$ with $t > 0$ is ample; it is numerically equivalent to $\mathcal{O}_X(t)$ and so we can use Kodaira's vanishing theorem.

For any point $p \in X_{\text{reg}}$, we have $\text{ext}_X^1(\mathcal{I}_{p,X}, \mathcal{O}_X) = h^1(\mathcal{I}_{p,X} \otimes \omega_X) = 1$ by Serre's duality. Thus, up to isomorphisms, there is a unique sheaf $\mathcal{E}_p$ that fits into the following non-trivial extension:

$$(8) \qquad\qquad 0 \to \mathcal{O}_X \to \mathcal{E}_p \to \mathcal{I}_{p,X} \to 0.$$

Obviously $\mathcal{E}_p$ has rank two and it is locally free outside p with $\det(\mathcal{E}_p) \cong \mathcal{O}_X$. Since $p \in X_{\text{reg}}$ and $h^0(\omega_X) = 0$, the Cayley-Bacharach condition is satisfied. Thus $\mathcal{E}_p$ is locally free. Note that the point p is uniquely determined by the isomorphism class of $\mathcal{E}_p$, because we have $h^0(\mathcal{E}_p) = 1$ by the sequence (8) and any non-zero section of $\mathcal{E}_p$ vanishes only at p.

Lemma 3.2. *For a general $p \in X_{\text{reg}}$ the vector bundle $\mathcal{E}_p$ is aCM and indecomposable.*

Proof. The exact sequence (8) twisted by $\mathcal{O}_X(t)$ gives $h^1(\mathcal{E}_p(t)) = 0$ for all $t \geq 0$. From $\mathcal{E}_p^{\vee} \cong \mathcal{E}_p$ we see that $h^1(\mathcal{E}_p \otimes \omega_X) = h^1(\mathcal{E}_p) = 0$ by Serre's duality. Now fix an integer $t < 0$. The twist of the sequence (8) by $\omega_X(-t)$ gives

$$h^1(\mathcal{E}_p \otimes \omega_X(-t)) \leq h^1(\omega_X(-t)) + h^1(\mathcal{I}_{p,X} \otimes \omega_X(-t)) = h^1(\mathcal{I}_{p,X} \otimes \omega_X(-t)).$$

Here, we have $h^1(\omega_X(-t)) = 0$ by our assumptions on the polarization $\mathcal{O}_X$. We also have $h^0(\omega_X(-t)) > 0$ from the assumption that $h^0(\omega_X(1)) > 0$. Since p is general, we have $h^1(\mathcal{I}_{p,X} \otimes \omega_X(-t)) = 0$. By Serre's duality, this implies that $h^1(\mathcal{E}_p(t)) = h^1(\mathcal{E}_p \otimes \omega_X(-t)) = 0$. Thus $\mathcal{E}_p$ ia aCM.

Assume that $\mathcal{E}_p$ is decomposable; say $\mathcal{E}_p \cong \mathcal{A}_1 \oplus \mathcal{A}_2$ with each $\mathcal{A}_i$ a line bundle. Since $h^0(\mathcal{E}_p) = 1$, we may assume that $h^0(\mathcal{A}_1) = 1$ and $h^0(\mathcal{A}_2) = 0$. Since the evaluation map commutes with direct sums and $\mathcal{I}_{p,X}$ is isomorphic to the cokernel of the evaluation map $H^0(\mathcal{E}_p) \otimes \mathcal{O}_X \to \mathcal{E}_p$, we get $\mathcal{A}_2 \cong \mathcal{I}_{p,X}$, a contradiction. $\square$

Lemma 3.3. *For any two general points $p, q \in X_{\mathrm{reg}}$, we have $\mathrm{ext}^1_X(\mathscr{E}_p, \mathscr{E}_q) = 1$.*

Proof. Since $\det(\mathscr{E}_p) \cong \mathscr{O}_X$, we have $\mathscr{E}_p^\vee \cong \mathscr{E}_p$ and so $\mathrm{Ext}^1_X(\mathscr{E}_p, \mathscr{E}_q) \cong H^1(\mathscr{E}_p \otimes \mathscr{E}_q)$. Tensoring the exact sequence (8) with $\mathscr{E}_q$, we get the exact sequence

(9) $$0 \to \mathscr{E}_q \to \mathscr{E}_p \otimes \mathscr{E}_q \to \mathscr{I}_{p,X} \otimes \mathscr{E}_q \to 0.$$

Since $\mathscr{E}_q$ is aCM, we have $h^1(\mathscr{E}_q) = 0$. On the other hand, tensoring the sequence (8) for $\mathscr{E}_q$ with ω_X gives $h^0(\mathscr{E}_q \otimes \omega_X) = 0$, because $\omega_X \not\cong \mathscr{O}_X$. Thus by Serre's duality we get $h^2(\mathscr{E}_q) = h^0(\mathscr{E}_q \otimes \omega_X) = 0$ and therefore $H^1(\mathscr{E}_p \otimes \mathscr{E}_q) \cong H^1(\mathscr{I}_{p,X} \otimes \mathscr{E}_q)$. Then the assertion follows from the exact sequence

$$0 \to \mathscr{I}_{p,X} \otimes \mathscr{E}_q \to \mathscr{E}_q \to (\mathscr{E}_q)_{|\{p\}} \to 0$$

together with the fact that $\mathscr{E}_q$ is an aCM vector bundle of rank two and $H^0(\mathscr{E}_q)$ is one-dimensional whose nontrivial section vanishes only at q so that $h^0(\mathscr{I}_{p,X} \otimes \mathscr{E}_q) = 0$ and therefore $h^1(\mathscr{I}_{p,X} \otimes \mathscr{E}_q) = 1$. $\qquad\square$

Proposition 3.4. *Setting $\tilde{g} := h^0(\mathscr{O}_X(1))$, there exists an indecomposable aCM vector bundle $\mathscr{E}$ of rank r on X with $\det(\mathscr{E}) \cong \mathscr{O}_X$ and $c_2(\mathscr{E}) = r - 1$ for each integer $2 \le r \le \tilde{g} - 1$.*

Proof. As in the proof of Proposition 2.1, consider a general sheaf $\mathscr{E}$ fitting into the sequence (1) for a general $S \subset X_{\mathrm{reg}}$ with $|S| = r - 1$. Then we get $\mathrm{ext}^1_X(\mathscr{I}_{S,X}, \mathscr{O}_X) = r - 1$ and the proof of Proposition 2.1 works verbatim. $\qquad\square$

Theorem 3.5. *Let X be an integral projective surface with $q(X) = 0$ and $\omega_X \not\cong \mathscr{O}_X$ locally free such that $h^0(\omega_X) = 0$ and $h^0(\omega_X^{\otimes 2}) = 1$. Then for any $r \ge 2$ there exists a family $\{\mathscr{E}_\alpha\}_{\alpha \in \Gamma}$ of dimension $2\lceil \frac{r}{2} \rceil$ of indecomposable rank r aCM vector bundles with $c_1(\mathscr{E}_\alpha) \cong \mathscr{O}_X$ such that for each $\alpha \in \Gamma$ there are only finitely many $\beta \in \Gamma$ with $\mathscr{E}_\beta \cong \mathscr{E}_\alpha$.*

Proof. The proof follows exactly the same structure as in the case of Theorem 2.4. In the present setting, however, in the case of even rank $r = 2m$, the family Γ of indecomposable aCM vector bundles of rank r will be mapped by a quasi-finite dominant morphism to

$$\mathbb{U} := \big\{ (S_1, \ldots, S_m) \mid S_i \subset X_{\mathrm{reg}} \text{ with } |S_i| = 1 \text{ and } S_i \cap S_j = \emptyset \text{ for all } i \ne j \big\},$$

a variety of dimension r, while in the odd case $r = 2m + 3$ it will be mapped to

$$\widetilde{\mathbb{U}} := \{ (S_0, S_1, \ldots, S_m) \mid S_i \subset X_{\mathrm{reg}} \text{ with } |S_0| = 2,$$
$$|S_i| = 1 \text{ for all } 1 \le i \le m \text{ and } S_i \cap S_j = \emptyset \text{ for all } i \ne j \}.$$

a variety of dimension $2m + 4 = 2\lceil \frac{r}{2} \rceil$. $\qquad\square$

4. IRREGULAR SURFACES

In this section we deal with surfaces with $q(X) \ge 1$.

Proposition 4.1. *Let X be a smooth projective surface with $q(X) = 1$ and a fixed ample line bundle $\mathscr{O}_X(1)$, satisfying one of the following conditions:*

   (i) $\mathscr{O}_X(1) \cong \omega_X$;
   (ii) $\mathscr{O}_X(1) \otimes \omega_X^\vee$ *is ample.*

Then for each positive integer r there exists a one-dimensional family $\{\mathscr{E}_\alpha\}_{\alpha\in\Gamma}$ of indecomposable aCM vector bundles of rank r on X such that $\mathscr{E}_\alpha$ for each $\alpha \in \Gamma$ is strictly semistable with $\det(\mathscr{E}_\alpha) \in \mathrm{Pic}^0(X)$ and $c_2(\mathscr{E}_\alpha) = 0$ with respect to any polarization of X, and there are only finitely many $\beta \in \Gamma$ with $\mathscr{E}_\beta \cong \mathscr{E}_\alpha$.

Proof. Fix a general line bundle $\mathscr{L} \in \mathrm{Pic}^0(X)$. Then we have $h^1(\mathscr{L}) = 0$; see [3, Th. 0.1], [12, Theorem 1] or [13, Theorem 0.1]. We also have $h^1(\mathscr{L}(-t)) = 0$ for all $t > 0$ by Kodaira's vanishing. Note that Serre's duality gives $h^1(\mathscr{L}(t)) = h^1(\mathscr{L}^\vee \otimes \omega_X(-t))$. Then we have $h^1(\mathscr{L}^\vee \otimes \omega_X(-t)) = 0$ for all $t > 0$. Indeed, in case (i) we may apply Kodaira's vanishing for $t \geq 2$ and $h^1(\mathscr{L}^\vee) = 0$ for $t = 1$. In case (ii) $\omega_X^\vee(t)$ is ample and so we may apply Kodaira's vanishing. Thus $\mathscr{L}$ is aCM.

Let $\varphi : X \to C$ be the Albanese map of X onto an elliptic curve C. We have $\varphi_*\mathscr{O}_X \cong \mathscr{O}_C$ and $\mathrm{Pic}^0(X) = \varphi^*\mathrm{Pic}(C)$. By the classification of vector bundles on an elliptic curve in [1], there is an indecomposable vector bundle $\mathscr{F}$ of rank r on C, which is an iterated extension of $\mathscr{O}_C$. Define

$$\mathscr{E}_\mathscr{L} := \varphi^*\mathscr{F} \otimes \mathscr{L}.$$

Then $\mathscr{E}_\mathscr{L}$ is a vector bundle of rank r on X with $\det(\mathscr{E}_\mathscr{L}) \cong \mathscr{L}^{\otimes r} \in \mathrm{Pic}^0(X)$ and $c_2(\mathscr{E}_\mathscr{L}) = 0$, which is an iterated extension of $\mathscr{L}$. Since $\mathscr{L}$ is aCM, so is $\mathscr{E}_\mathscr{L}$. Moreover, $\mathscr{E}_\mathscr{L}$ is clearly strictly semistable with respect to any polarization.

Assume that $\mathscr{E}_\mathscr{L}$ is decomposable and this would imply that $\varphi^*\mathscr{F}$ is also decomposable, say $\varphi^*\mathscr{F} \cong \mathscr{F}_1 \oplus \mathscr{F}_2$ with each $\mathscr{F}_i$ an aCM vector bundle of rank r_i with $0 < r_i < r$. By the projection formula and $\varphi_*\mathscr{O}_X \cong \mathscr{O}_C$, we have $\mathscr{F} \cong \varphi_*\mathscr{F}_1 \oplus \varphi_*\mathscr{F}_2$. Now take a non-empty subset of C so that

- we have $\mathscr{F}_{|U} \cong \mathscr{O}_U^{\oplus r}$, and
- $\varphi^{-1}(q)$ is a smooth projective curve for each $q \in U$.

Since $(\varphi^*\mathscr{F})_{|\varphi^{-1}(q)}$ is the trivial vector bundle of rank r on the integral projective curve $\varphi^{-1}(q)$, we get $\mathscr{F}_{i|\varphi^{-1}(q)} \cong \mathscr{O}_{\varphi^{-1}(q)}^{\oplus r_i}$ for each i. In particular, we have $\varphi_*\mathscr{F}_i$ is not zero for each i, a contradiction to the indecomposability of $\mathscr{F}$. □

Remark 4.2. Let X be a smooth and connected projective variety of dimension $n \geq 2$ and $\varphi : X \to \mathrm{Alb}(X)$ its Albanese map. Assume that X has *maximal Albanese dimension*, i.e. $\dim\varphi(X) = n$. Note that this implies $q(X) = \dim\mathrm{Alb}(X) = n \geq 2$. In particular, an abelian variety has maximal Albanese dimension. Let $\mathscr{O}_X(1)$ be an ample line bundle on X such that $\omega_X^\vee\otimes\mathscr{O}_X(1)$ is ample; if X is an abelian variety, then $\mathscr{O}_X(1)$ can be arbitrary.

Now choose a general line bundle $\mathscr{L} \in \mathrm{Pic}^0(X)$. Since X has Albanese dimension n, we have $h^i(\mathscr{L}) = 0$ for all $1 \leq i \leq n-1$ by [12, Theorem 1] or [13, Theorem 0.1]. Fix a positive integer t. By Kleiman's numerical criterion of ampleness in [17], we get that $\mathscr{L}^\vee(t)$ and $\omega_X^\vee \otimes \mathscr{L}(t)$ are ample for $t > 0$. Then Kodaira's vanishing gives $h^i(\mathscr{L}(t)) = h^i(\omega_X \otimes \omega_X^\vee \otimes \mathscr{L}(t)) = 0$ for all $1 \leq i \leq n-1$. On the other hand, Serre's duality gives $h^i(\mathscr{L}(-t)) = h^{n-i}(\omega_X \otimes \mathscr{L}^\vee(t)) = 0$ for $1 \leq i \leq n-1$. This implies that $\mathscr{L}$ is aCM. Since $\dim\mathrm{Pic}^0(X) = q(X)$, there exists a n-dimensional family of pairwise non-isomorphic aCM lines bundles.

Now we work on the proof of Theorem 1.3 and the key tool is Mukai's study of vector bundles on abelian varieties; see [21].

Proof of Theorem 1.3: Since X is smooth and birational to an abelian variety, there are an n-dimensional abelian variety Y and a proper birational morphism $v : X \to Y$; see [23, Proposition 9.12]. In particular, we have $v_*\mathscr{O}_X \cong \mathscr{O}_Y$ by the Zariski Main Theorem in [14, Corollary III.11.4]). Let $\widehat{Y} = \mathrm{Pic}^0(Y)$ denote the abelian variety dual to Y. As in [21, Definitions 4.4, 4.5, 4.6] we

consider the following set

$$\mathbb{U}'_r := \{ \text{ the unipotent vector bundles of rank } r \text{ on } Y \},$$

i.e. the set of all vector bundles of rank r on Y, obtained by iterated extension; we have $\mathbb{U}'_1 = \{\mathscr{O}_Y\}$ and $\mathbb{U}'_r$ is the set of all vector bundles which admit extensions of $\mathscr{O}_Y$ by an element of $\mathbb{U}'_{r-1}$. If we let R be the completion of the local ring $\mathscr{O}_{\widehat{Y},0}$ and B_f the set of all R-modules with finite length, then by [21, Theorem 4.12] there is a bijection between $\mathbb{U}'_r$ and the set $B_f[r]$ of R-modules of length r. Note that this bijection preserves finite direct sums. Thus to an indecomposable vector bundle in $\mathbb{U}'_r$ it is enough to consider an indecomposable elements of $B_f[r]$. Define a subset

$$\mathbb{U}_r := \left\{ \mathscr{A} \in \mathbb{U}'_r \;\middle|\; \begin{array}{l} \mathscr{A} \text{ corresponds to an indecomposable elements of } B_f[r] \\ \text{of the form } R/I \text{ with } I \subset R \text{ an ideal of colength } r \end{array} \right\},$$

consisting of elements of the local Hilbert scheme of R corresponding to connected zero-dimensional subschemes of $\widehat{Y}$ of degree r with 0 as their support. Then we get an algebraic family $\mathbb{U}_r$ of indecomposable unipotent vector bundles of rank r. For the known results on the dimension of $\mathbb{U}_r$, refer to [11, page 6]. For $n = 2$ and arbitrary r, $\mathbb{U}_r$ is irreducible of dimension $r - 1$ by [4, 16], while it can be reducible for $n \geq 3$ by [11, 16]. In any case with $n \geq 2$, $\mathbb{U}_r$ has an irreducible family of dimension $(n-1)(r-1)$, whose general element is curvilinear, or collinear, by [11, pages 5–6].

For any line bundle $\mathscr{L} \in \mathrm{Pic}^0(X)$, set

$$\Theta_{\mathscr{L}} := \{ v^*(\mathscr{F}) \otimes \mathscr{L} \mid \mathscr{F} \in \mathbb{U}_r \}.$$

Each element of $\Theta_{\mathscr{L}}$ is a vector bundle of rank r on X, which is an iterated extension of $\mathscr{L}$. Thus each element of $\Theta_{\mathscr{L}}$ is strictly semistable with respect to any polarization on X and all its Chern classes are zero. Assume that $v^*(\mathscr{F}) \otimes \mathscr{L} \cong v^*(\mathscr{G}) \otimes \mathscr{L}$ for $\mathscr{F}, \mathscr{G} \in \mathbb{U}_r$. Then we get $v^*(\mathscr{F}) \cong v^*(\mathscr{G})$ and so $\mathscr{F} \cong \mathscr{G}$ by the projection formula and $v_* \mathscr{O}_X \cong \mathscr{O}_Y$. In particular, $\Theta_{\mathscr{L}}$ parametrizes one-to-one vector bundles of rank r on X and $\dim \Theta_{\mathscr{L}} = \dim \mathbb{U}_r$. Note that for each $\mathscr{A} \in \Theta_{\mathscr{L}}$ there are only finitely many $\mathscr{L}' \in \mathrm{Pic}^0(X)$ such that $\mathscr{A} \cong \mathscr{A}'$ for some $\mathscr{A}' \in \Theta_{\mathscr{L}'}$; indeed, we have at most $(2n)^r$ vector bundles $\mathscr{A}'$, because $\det(\mathscr{A}) \cong \mathscr{L}^{\otimes r}$ and so $\mathscr{L}' \otimes \mathscr{L}^{\vee}$ is an element of r-torsion of $\mathrm{Pic}^0(X)$. Now a general line bundle $\mathscr{L} \in \mathrm{Pic}^0(X)$ is aCM by Remark 4.2. Define a non-empty open subset

$$\mathbb{V} := \{ \mathscr{L} \in \mathrm{Pic}^0(X) \mid \mathscr{L} \text{ is aCM} \},$$

which is an algebraic variety of dimension $q(X) = n$. For each $\mathscr{L} \in \mathbb{V}$, every vector bundle $\mathscr{A} \in \Theta_{\mathscr{L}}$ is aCM, because it is an iterated extension of aCM vector bundles. Define a parameter space Γ over $\mathbb{V}$ whose fibre over $\mathscr{L}$ is $\Theta_{\mathscr{L}}$. Then it is a parameter space, finite-to-one, for indecomposable aCM vector bundles of rank r on X with $\dim \Gamma = n + \dim \mathbb{U}_r = (n-1)r + 1$. $\square$

Proposition 4.3. *Let X be a smooth projective surface with $q(X) \geq 2$ and a fixed ample line bundle $\mathscr{O}_X(1)$ satisfying one of the following conditions:*

  (i) *$\mathscr{O}_X(1) \cong \omega_X$;*
  (ii) *$\mathscr{O}_X(1) \otimes \omega_X^{\vee}$ is ample.*

Then for each integer r with $1 \leq r \leq q(X)$ there exists a $q(X)$-dimensional family $\{\mathscr{E}_\alpha\}_{\alpha \in \Gamma}$ of indecomposable aCM vector bundles of rank r on X such that $\mathscr{E}_\alpha$ for each $\alpha \in \Gamma$ is strictly semistable with $\det(\mathscr{E}_\alpha) \in \mathrm{Pic}^0(X)$ and $c_2(\mathscr{E}_\alpha) = 0$ with respect to any polarization of X, and there are only finitely many $\beta \in \Gamma$ with $\mathscr{E}_\beta \cong \mathscr{E}_\alpha$.

Proof. Fix a general line bundle $\mathscr{L} \in \mathrm{Pic}^0(X)$. Then as in Remark 4.2 we see that $\mathscr{L}$ is aCM. Set $\mathscr{G}_0 = 0$ the zero sheaf and $\mathscr{G}_1 := \mathscr{L}$. For an integer $r \geq 2$, we define $\mathscr{G}_r$ inductively as a general sheaf fitting into the following extension

$$(10) \qquad\qquad 0 \to \mathscr{G}_{r-1} \xrightarrow{u} \mathscr{G}_r \xrightarrow{v} \mathscr{L} \to 0.$$

Note that $\mathscr{G}_r$ is strictly semistable for any polarization and $\mathscr{G}_r \otimes \mathscr{L}^\vee$ is an iterated extension of $\mathscr{O}_X$ for each $r \geq 1$. Since $\mathscr{G}_{r-1} \otimes \mathscr{L}^\vee$ is an iterated extension of $\mathscr{O}_X$, we have $\det(\mathscr{G}_{r-1} \otimes \mathscr{L}^\vee) \cong \mathscr{O}_X$ and $c_2(\mathscr{G}_{r-1} \otimes \mathscr{L}^\vee) = 0$. Moreover, we may choose $\mathscr{G}_r$ admitting a non-trivial extension (10), because we have $\mathrm{ext}^1_X(\mathscr{L}, \mathscr{G}_{r-1}) > 0$; indeed, we have $h^1(\mathscr{G}_{r-1} \otimes \mathscr{L}^\vee) \geq q(X) - r + 2$, which is clearly true for $r = 2$. In general, we get the following exact sequence from (10)

$$H^0(\mathscr{O}_X) \to H^1(\mathscr{G}_{r-1} \otimes \mathscr{L}^\vee) \to H^1(\mathscr{G}_r \otimes \mathscr{L}^\vee).$$

Then we may apply the inductive hypothesis and $h^0(\mathscr{O}_X) = 1$.

Note that the coboundary map $H^0(\mathscr{O}_X) \to H^1(\mathscr{G}_{r-1} \otimes \mathscr{L}^\vee)$ is zero if and only if (10) is the trivial extension. Since we take a non-trivial extension at each step, we have $h^0(\mathscr{G}_r \otimes \mathscr{L}^\vee) = h^0(\mathscr{G}_{r-1} \otimes \mathscr{L}^\vee)$. By induction on r we get $h^0(\mathscr{G}_r \otimes \mathscr{L}^\vee) = 1$ for all $r \leq q(X)$. Assume now that $\mathscr{G}_r$ is decomposable, say $\mathscr{G}_r \cong \mathscr{F}_1 \oplus \mathscr{F}_2$ with each $\mathscr{F}_i$ nonzero. Then each $\mathscr{F}_i \otimes \mathscr{L}^\vee$ is a strictly semistable vector bundle with numerically trivial determinant. Since $gr(\mathscr{G}_{r-1} \otimes \mathscr{L}^\vee) = \mathscr{O}_X^{\oplus(r-1)}$, we get that $gr(\mathscr{F}_i \otimes \mathscr{L}^\vee)$ is trivial and so each $\mathscr{F}_i \otimes \mathscr{L}^\vee$ has a subsheaf isomorphic to $\mathscr{O}_X$. In particular, we have $h^0(\mathscr{G}_r \otimes \mathscr{L}^\vee) \geq 2$, a contradiction.

Note that $\det(\mathscr{G}_r) \cong \mathscr{L}^{\otimes r}$ and so there are only finitely many line bundles $\mathscr{L}' \in \mathrm{Pic}^0(X)$ such that $\mathscr{G}_r$ is also an iterated extension of $\mathscr{L}'$. Hence we get the assertion from $\dim \mathrm{Pic}^0(X) = q(X)$. $\qquad\square$

Remark 4.4. Let Y be a hyperelliptic surface, i.e. a smooth projective surface with $\omega_Y \not\cong \mathscr{O}_Y$, $q(Y) = 1$ and $\omega_Y^{\otimes 12} \cong \mathscr{O}_Y$. In particular, we have $h^2(\mathscr{O}_Y) = h^0(\omega_Y) = 0$ and so $\chi(\mathscr{O}_Y) = 0$. Let X be a smooth projective surface birational to Y. Then we have $h^i(\mathscr{O}_X) = h^i(\mathscr{O}_Y)$ for each i and $\omega_X \not\cong \mathscr{O}_X$ with $h^0(\omega_X^{\otimes 12}) = 1$. Fix an ample line bundle $\mathscr{O}_X(1)$ on X and take a line bundle $\mathscr{L} \in \mathrm{Pic}^0(X) \setminus \{\mathscr{O}_X, \omega_X^\vee\}$. Then we have $h^0(\mathscr{L}) = h^2(\mathscr{L}) = 0$. Since $\mathscr{L}$ is numerically equivalent to $\mathscr{O}_X$ and $\chi(\mathscr{O}_X) = 0$, we have $\chi(\mathscr{L}) = 0$ and so $h^1(\mathscr{L}) = 0$. Note that $\mathscr{L}(t)$ and $\mathscr{L}^\vee \otimes \omega_X(t)$ are ample for $t > 0$, because they are numerically equivalent to the ample line bundle $\mathscr{O}_X(t)$. So we get $h^1(\mathscr{L}(t)) = 0$ for all $t \neq 0$ by Kodaira's vanishing and Serre's duality. Thus $\mathscr{L}$ is aCM. Now we may construct indecomposable aCM vector bundles $\mathscr{G}_r$ of rank r as in the case of abelian surfaces. Indeed, we have $\mathrm{ext}^1_X(\mathscr{L}, \mathscr{L}) = h^1(\mathscr{O}_X) = 1$ and $\mathrm{ext}^1_X(\mathscr{L}, \mathscr{G}_{r-1}) > 0$. We have $\det(\mathscr{G}_r) \cong \mathscr{L}^{\otimes r}$. In particular, there are only finitely many line bundles $\mathscr{L}' \in \mathrm{Pic}^0(X)$ such that $\mathscr{G}_r$ is an iterated extension of $\mathscr{L}'$. We get the following result from $q(X) = 1$.

Proposition 4.5. *Let X be a smooth projective surface, birational to a hyperelliptic surface, with any polarization. For any positive integer r, there exists a one-dimensional family $\{\mathscr{E}_\alpha\}_{\alpha \in \Gamma}$ of indecomposable aCM vector bundles of rank r on X such that for each $\alpha \in \Gamma$ there are only finitely many $\beta \in \Gamma$ with $\mathscr{E}_\beta \cong \mathscr{E}_\alpha$.*

5. SURFACES OF GENERAL TYPE WITH AMPLE CANONICAL LINE BUNDLE

Let X be an integral projective surface, possibly singular, with ample ω_X satisfying the following conditions:

    (i) $h^1(\omega_X^{\otimes n}) = 0$ for all $n \in \mathbb{Z}$;

  (ii-ε) $p_g := h^0(\omega_X) \geq 2 + \varepsilon$ with $\varepsilon \in \{0, 1\}$.

We set $\mathcal{O}_X(1) := \omega_X$ with respect to which we consider aCM vector bundles on X.

Remark 5.1. Assume that X is smooth. The canonical line bundle ω_X is ample if and only if X is a minimal surface of general type without (-2)-curves, i.e. a smooth surface of general type without smooth rational curves $D \subset X$ with either $D^2 = -1$ or $D^2 = -2$; see [2]. There are surfaces X of general type with $p_g = h^0(\omega_X) \le 1$, but most surfaces have $p_g \ge 2$. The condition (i) for $n = 0$ is $h^1(\mathcal{O}_X) = 0$, i.e. the irregularity of X is $q(X) = 0$. This is a non-trivial requirement, but it is satisfied in many important cases. By Serre's duality this would imply that $h^1(\omega_X) = q(X) = 0$. In characteristic 0 the condition (i) for $n < 0$ comes from Kodaira's vanishing theorem by the ampleness of ω_X. Assume $h^1(\omega_X^{\otimes n}) = 0$ for all $n < 0$. By Serre's duality we have $h^1(\omega_X^{\otimes n}) = h^1(\omega_X^{\otimes(1-n)}) = 0$ for $n \ge 2$. Thus in characteristic 0 we have the condition (i) satisfied if and only if $h^1(\mathcal{O}_X) = 0$.

By the condition (ii-ε), the set

$$\Sigma := \mathrm{Sing}(X) \cap \{\text{the base locus of } |\omega_X|\}$$

is a proper closed subset of X. By the same argument in Remark 2.14 using Serre's duality we get the following lemma.

Lemma 5.2. *For a finite subset $S \subset X \setminus \Sigma$, we have $\mathrm{ext}^1_X(\mathcal{I}_{S,X}, \omega_X) = |S| - 1$ and a general extension of $\mathcal{I}_{S,X}$ by ω_X is locally free.*

Proof. For the first assertion, we may apply the same argument in Remark 2.14 using Serre's duality. The second assertion is clear, because the Cayley-Bacharach condition for S and the linear system $|\mathcal{O}_X|$ is satisfied. $\qquad\square$

Proposition 5.3. *For a fixed integer $2 \le r \le p_g$ and a general subset $S \subset X \setminus \Sigma$ with $|S| = r$, the general sheaf $\mathcal{E}$ fitting into an exact sequence*

$$\text{(11)} \qquad\qquad 0 \to \omega_X^{\oplus(r-1)} \to \mathcal{E} \to \mathcal{I}_{S,X} \to 0$$

is an indecomposable and aCM vector bundle of rank r.

Proof. Let Ψ denote the set of all extensions of $\mathcal{I}_{S,X}$ by $\omega_X^{\oplus(r-1)}$, and let $\mathcal{E}_0$ be a general extension of $\mathcal{I}_{S,X}$ by ω_X. Then by Lemma 5.2), the sheaf $\mathcal{E}_0$ is locally free. Then the vector bundle $\mathcal{E}_0 \oplus \omega_X^{\oplus(r-2)}$ is contained in the family Ψ. Since the local freeness is an open condition, the general sheaf $\mathcal{E}$ in the sequence (11) is locally free.

Now since we have $\mathrm{ext}^1_X(\mathcal{I}_{S,X}, \omega_X) = r - 1$ by Lemma 5.2, the extension (11) is induced by a choice of a basis $\{e_1, \ldots, e_{r-1}\}$ of $\mathrm{Ext}^1_X(\mathcal{I}_{S,X}, \omega_X)$. Thus the map $\varphi : H^1(\mathcal{I}_{S,X}) \to H^2(\omega_X^{\oplus(r-1)}) \cong \mathbf{k}^{\oplus(r-1)}$ is bijective, and in particular we have $h^1(\mathcal{E}) = 0$. Recall that we assume $\omega_X \cong \mathcal{O}_X(1)$. Then by the condition (i) we get $h^1(\omega_X(n)) = 0$ for all $n \in \mathbb{Z}$ and we get

$$0 \to H^1(\mathcal{E}(n)) \to H^1(\mathcal{I}_{S,X}(n)) \to H^2(\omega_X(n))^{\oplus(r-1)}.$$

Assume first that n is positive and this implies $h^2(\omega_X(n)) = h^0(\mathcal{O}_X(-n)) = 0$. Since S is general with $|S| = r \le h^0(\mathcal{O}_X(1)) \le h^0(\mathcal{O}_X(n))$, we get $h^1(\mathcal{I}_{S,X}(n)) = 0$. Thus we have $h^1(\mathcal{E}(n)) = 0$. It remains to show that $h^1(\mathcal{E}(-n)) = 0$ for $n \ge 1$. In fact, it is sufficient to prove the existence of an extension $\mathcal{F}$ of $\mathcal{I}_{S,X}$ by $\omega_X^{\oplus(r-1)}$ satisfying $h^1(\mathcal{F}(-n)) = 0$ for all $n \ge 1$. Take $\mathcal{F} \cong \mathcal{G} \oplus \omega_X^{\oplus(r-2)}$ with a general extension $\mathcal{G}$ of $\mathcal{I}_{S,X}$ by ω_X given by e_1. By the previous argument, we have $h^1(\mathcal{G}(n)) = 0$ for all $n \ge 1$. By Lemma 5.2, $\mathcal{G}$ is locally free with $\det(\mathcal{G}) \cong \omega_X$. Serre's duality gives $h^1(\mathcal{G}(-n)) = h^1(\mathcal{G}(n)) = 0$ for all $n \ge 1$. Thus we get that $\mathcal{E}$ is aCM. Note that if $r \ge 3$, then $\mathcal{G}$ is not aCM since we have $h^1(\mathcal{G}) = r - 2$.

For the indecomposability, we may use the same argument in the proof of Proposition 2.1 to $\mathcal{E} \otimes \omega_X^\vee$, because $\mathscr{I}_{S,X} \otimes \omega_X^\vee$ is indecomposable. $\qquad\square$

Now for the statement in Theorem 5.4, set $\varepsilon = r - 2\left\lfloor \frac{r}{2} \right\rfloor$ for which the condition (ii-ε) for X is assumed to be satisfied.

Theorem 5.4. *For each integer $r \geq 2$, there exists an r-dimensional family $\{\mathcal{E}_\alpha\}_{\alpha \in \Gamma}$ of indecomposable aCM vector bundles of rank r on X with $\det(\mathcal{E}_\alpha) \cong \omega_X^{\otimes \lceil r/2 \rceil}$ and $c_2(\mathcal{E}_\alpha) = r$ such that for each $\alpha \in \Gamma$ there are only finitely many $\beta \in \Gamma$ with $\mathcal{E}_\beta \cong \mathcal{E}_\alpha$.*

Proof. We use the same notations in the proof of Theorem 2.4 such as $\mathbb{I}(S_1,\ldots,S_i)$ and $\mathbb{J}(S_1,\ldots,S_i;S_0)$. Then we get the same assertions from Lemma 2.5 till Remark 2.13; the only difference occurs in Lemma 2.8 and Remark 2.9, where we have

$$\operatorname{ext}_X^1(\mathscr{I}_{S_{i+1},X}, \mathscr{J}) = \operatorname{ext}_X^1(\mathscr{I}_{S_0,X}, \mathscr{J}) = i$$

for $\mathscr{J} \in \mathbb{I}(S_1,\ldots,S_i)$ from $\operatorname{ext}_X^1(\mathscr{I}_{S_{i+1},X}, \mathcal{O}_X) = \operatorname{ext}_X^1(\mathscr{I}_{S_0,X}, \mathcal{O}_X) = 0$. Then we may consider the exact sequences (6) and (7) with $\mathcal{O}_X$ replaced by ω_X. $\qquad\square$

6. SURFACES MAPPED TO A CURVE OF GENUS ≥ 3 NOT AS THEIR ALBANESE IMAGE

Throughout this section, X is a smooth projective surface admitting a surjective map $\nu : X \to C$ with $g = g(C) \geq 3$ and $\mathcal{O}_X(1)$ is an ample line bundle positive enough to satisfy that $\omega_X^\vee \otimes \mathcal{O}_X(1)$ is ample as well. Assume that C is such a curve achieving maximum possible genus g and that $q(X) > g$. For example, we may take as X any smooth surface birational to $C \times D$, where D is a smooth curve with $1 \leq g(D) \leq g$; in this case we have $q(X) = g + g(D)$.

Proposition 6.1. *For each positive integer r there exists a family $\{\mathcal{E}_\alpha\}_{\alpha \in \Gamma}$ of indecomposable aCM vector bundles of rank r on X such that Γ is an integral variety with*

$$\dim \Gamma \geq q(X) + \frac{(r-1)(r-2)(g-1)}{2} - \frac{r(r-1)}{2}$$

and each $\mathcal{E}_\alpha$ is strictly semistable with $\det(\mathcal{E}_\alpha) \in \operatorname{Pic}^0(X)$ and $c_2(\mathcal{E}_\alpha) = 0$ with respect to any polarization of X such that there are only finitely many $\beta \in \Gamma$ with $\mathcal{E}_\beta \cong \mathcal{E}_\alpha$.

Set $\mathscr{A}_1 := \mathcal{O}_C$ and define inductively a vector bundle $\mathscr{A}_{i+1}$ of rank $i+1$ on C to be the middle term of the following extension:

$$(12) \qquad\qquad 0 \to \mathscr{A}_i \to \mathscr{A}_{i+1} \to \mathcal{O}_C \to 0,$$

where $\mathscr{A}_{i+1} = \mathscr{A}_{i+1}(e)$ corresponds to the extension class $e \in \operatorname{Ext}_C^1(\mathcal{O}_C, \mathscr{A}_i) \cong H^1(\mathscr{A}_i)$. Since we have $g \geq 3$ from the assumption, we get $h^1(\mathscr{A}_{i+1}) \neq 0$. In particular, we may assume that the extension (12) is non-trivial. The image of the coboundary map $H^0(\mathcal{O}_C) \to H^1(\mathscr{A}_i)$ corresponds to the extension (12), up to a sign, and therefore the coboundary map is injective. Thus, from the long exact sequence of cohomology groups associated to (12) we get $h^0(\mathscr{A}_{i+1}) = h^0(\mathscr{A}_i)$ and $h^1(\mathscr{A}_{i+1}) = h^1(\mathscr{A}_i) + g - 1$ for each i. By induction, we get

$$h^0(\mathscr{A}_i) = 1 \ \text{ and } \ h^1(\mathscr{A}_i) = i(g-1) + 1.$$

Note that each $\mathscr{A}_i$ is an iterated extension of $\mathcal{O}_C$, and in particular it is strictly semistable with $gr(\mathscr{A}_i) \cong \mathcal{O}_C^{\oplus i}$. Assume $\mathscr{A}_i \cong \mathscr{B}_1 \oplus \mathscr{B}_2$ with each $\mathscr{B}_i \neq 0$. Since each $\mathscr{B}_i$ has a HN-filtration with $\mathcal{O}_C$ as its first step, we have $h^0(\mathscr{B}_i) > 0$ and so $h^0(\mathscr{A}_i) \geq 2$, a contradiction. Thus each $\mathscr{A}_i$ is indecomposable.

Remark 6.2. Let $u : \mathscr{A} \to \mathscr{B}$ be a surjection of sheaves on C. Since $\dim C = 1$, we have $h^2(C, \ker(u)) = 0$. Thus the surjection u induces a surjective map $H^1(C, \mathscr{A}) \to H^1(C, \mathscr{B})$.

Lemma 6.3. *Let $\mathscr{M}, \mathscr{D}_1, \mathscr{D}_2$ be vector bundles on C fitting into exact sequences*

$$(13) \qquad 0 \to \mathscr{M} \xrightarrow{u_i} \mathscr{D}_i \to \mathscr{O}_C \to 0,$$

corresponding to an extension class $e_i \in \mathrm{Ext}^1_C(\mathscr{O}_C, \mathscr{M}) \cong H^1(\mathscr{M})$ for each i. If there exists an isomorphism $h : \mathscr{D}_2 \to \mathscr{D}_1$ such that $h(u_2(\mathscr{M})) = u_1(\mathscr{M})$, then e_1 and e_2 are in the same orbit of $H^1(\mathscr{M})$ for the action of the group $\mathrm{Aut}(\mathscr{M})$.

Proof. Note that $h^0(\mathscr{M}) \le h^0(\mathscr{D}_i) \le h^0(\mathscr{M}) + 1$, and $h^0(\mathscr{M}) = h^0(\mathscr{D}_i)$ if and only if $e_i \ne 0$. Since h is an isomorphism, $e_1 = 0$ if and only if $e_2 = 0$. Since the assertion is obvious when $e_1 = e_2 = 0$, we may assume $e_1 \ne 0$ and $e_2 \ne 0$. Since $h(u_2(\mathscr{M})) = u_1(\mathscr{M})$, h induces isomorphisms $h' : D_2/u_2(\mathscr{M}) \to \mathscr{D}_1/u_1(\mathscr{M})$ and $f : \mathscr{M} \to \mathscr{M}$. Since $\mathscr{D}_i/u_i(\mathscr{M}) \cong \mathscr{O}_C$, $i = 1, 2$, h' is induced by the multiplication by a constant, c. Note that e_i is determined by the image of 1 by the coboundary map $H^0(\mathscr{O}_C) \to H^1(\mathscr{M})$ in (13). Since $e_1 \ne 0$ and $e_2 \ne 0$, we have $c \ne 0$. Taking $\left(\frac{1}{c}\right) h$ instead of h we reduce to the case in which $h' : \mathscr{O}_C \to \mathscr{O}_C$ is the identity map. Thus we get a commutative diagram with exact rows:

$$
\begin{array}{ccccccccc}
0 \to & \mathscr{M} & \to & \mathscr{D}_2 & \to & \mathscr{O}_C & \to 0 \\
& \downarrow & & \downarrow & & \downarrow & \\
0 \to & \mathscr{M} & \to & \mathscr{D}_1 & \to & \mathscr{O}_C & \to 0,
\end{array}
$$

in which the three vertical arrows are respectively f, h and $\mathrm{Id}_{\mathscr{O}_C}$. By the definition of $\mathrm{Ext}^1_C(\mathscr{O}_C, \mathscr{M})$ as short exact sequences modulo an equivalence relation, we get $e_1 = f_*(e_2)$, i.e. $e_1 \in H^1(\mathscr{M})$ is contained in the orbit of e_2 for the action of the group $\mathrm{Aut}(\mathscr{M})$. $\qquad\square$

We set $\mathbf{T}_2 := H^1(\mathscr{O}_C) \setminus \{0\}$ and consider it as a parameter space, not finite-to-one, for non-trivial extensions of $\mathscr{O}_C$ by $\mathscr{O}_C$. Then we get a family $\{\mathscr{A}_2(e)\}_{e \in \mathbf{T}_2}$ of aCM vector bundles of rank two. Since we have $h^1(\mathscr{A}_2(e)) = 2g - 3$ for each $e \in \mathbf{T}_2$, there is a vector bundle $\pi_2 : \mathbf{T}'_3 \to \mathbf{T}_2$ of rank $2g - 3$ whose fibre over $\mathscr{A}_2(e)$ is $H^1(\mathscr{A}_2(e)) \cong \mathrm{Ext}^1_C(\mathscr{O}_C, \mathscr{A}_2(e))$. Then we get a family $\{\mathscr{A}_3(e)\}_{e \in \mathbf{T}'_3}$ of aCM vector bundles of rank three on C such that for each $e \in \mathbf{T}'_3$, $\mathscr{A}_3(e)$ is an extension of $\mathscr{O}_C$ by $\mathscr{A}_2(\pi(e))$. Let $\mathbf{T}_3$ be the non-empty Zariski open subset of $\mathbf{T}'_3$ parametrizing the non-trivial extensions of $\mathscr{O}_C$ by $\mathscr{A}_2(\pi(e))$. Thus we have a family $\{\mathscr{A}_3(e)\}_{e \in \mathbf{T}_3}$ of indecomposable aCM vector bundles of rank three, parametrized by $\mathbf{T}_3$.

Now we define a parameter space $\mathbf{T}_i$ inductively: fix an integer $i \ge 2$ and assume that $\mathbf{T}_i$ is defined, together with a family $\{\mathscr{A}_i(e)\}_{e \in \mathbf{T}_i}$ of indecomposable aCM vector bundles of rank i, parametrized by $\mathbf{T}_i$. Since we have $h^1(\mathscr{A}_i(e)) = i(g - 1) + 1$, there exists a vector bundle $\pi_i : \mathbf{T}'_{i+1} \to \mathbf{T}_i$ of rank $i(g-1)+1$ and a family $\{\mathscr{A}_{i+1}(e)\}_{e \in \mathbf{T}'_{i+1}}$ of aCM vector bundles of rank $i+1$ on C such that for each $e \in \mathbf{T}'_{i+1}$, $\mathscr{A}_{i+1}(e)$ is an extension of $\mathscr{O}_C$ by $\mathscr{A}_i(\pi(e))$. Let $\mathbf{T}_{i+1}$ be the non-empty Zariski open subset of $\mathbf{T}'_{i+1}$ parametrizing the non-trivial extensions of $\mathscr{O}_C$ by $\mathscr{A}_i(\pi(e))$.

If a vector bundle $\mathscr{A} = \mathscr{A}_r$ of rank r on C corresponding to $e \in \mathbf{T}_r$ is obtained as a successive extension of $\mathscr{O}_C$ by $\mathscr{A}_i(e_{i-1})$ corresponding to $e_i \in H^1(\mathscr{A}_i(e_{i-1})) \setminus \{0\}$ for each $i \le r$, then we simply denote it by $\mathscr{A}(e_1, \ldots, e_{r-1}) := \mathscr{A}$ and it has a filtration

$$0 \subset \mathscr{A}_1 = \mathscr{O}_C \subset \mathscr{A}_2 = \mathscr{A}(e_1) \subset \mathscr{A}_3 = \mathscr{A}(e_1, e_2) \subset \cdots \subset \mathscr{A}_r = \mathscr{A}(e_1, \ldots, e_{r-1}).$$

Fix a general $\mathscr{A} = \mathscr{A}(e_1, \ldots, e_{r-1})$ that is a non-trivial extension of $\mathscr{O}_C$ by $\mathscr{A}' := \mathscr{A}(e_1, \ldots, e_{r-2})$. Letting $u_{i,r} : \mathscr{A}_i \to \mathscr{A}$ with $1 \le i \le r - 1$ be the inclusion arising by the extensions reaching $\mathscr{A}$,

we have the following commutative diagram

$$
\begin{array}{ccccccc}
& 0 & & 0 & & & \\
& \downarrow & & \downarrow & & & \\
& \mathscr{A}_1 & = & \mathscr{A}_1 & & & \\
& \downarrow & & \downarrow & & & \\
0 \to & \mathscr{A}' & \to & \mathscr{A} & \to & \mathscr{O}_C & \to 0 \\
& \downarrow & & \downarrow & & \| & \\
0 \to & \mathscr{A}'/u_{1,r-1}(\mathscr{A}_1) & \to & \mathscr{A}/u_{1,r}(\mathscr{A}_1) & \to & \mathscr{O}_C & \to 0 \\
& \downarrow & & \downarrow & & & \\
& 0 & & 0 & & &
\end{array}
$$

so that $\mathscr{A}/u_{1,r}(\mathscr{A}_1)$ is an extension of $\mathscr{O}_C$ by $\mathscr{A}'/u_{1,r}(\mathscr{A}_1)$. Iterating the process, we see that $\mathscr{A}/u_{1,r}(\mathscr{A}_1)$ is an iterated extension of $\mathscr{O}_C$.

Lemma 6.4. *Fix a general $\mathscr{A}_r = \mathscr{A}(e_1,\ldots,e_{r-1}) \in \mathbf{T}_r$ with a filtration $\mathscr{A}_1 \subset \cdots \subset \mathscr{A}_{r-1} \subset \mathscr{A}_r$. Then we have*

(i) $h^0(\mathscr{A}_i/\mathscr{A}_j) = 1$ *for all $1 \le j < i \le r$;*

(ii) $f(\mathscr{A}_i) \subset \mathscr{A}_i$ *for any $f \in \mathrm{End}(\mathscr{A}_r)$ and each i;*

(iii) $\dim \mathrm{End}(\mathscr{A}_r) \le r$ *and* $\dim \mathrm{End}(\mathscr{A}_r) - \dim(\mathscr{A}_{r-1}) \le 1$.

(iv) $h(\mathscr{A}_i) = \mathscr{B}_i$ *for all i and any isomorphism $h : \mathscr{B}_r \to \mathscr{A}_r$, where $\mathscr{B}_r \in \mathbf{T}_r$ general with a filtration $\mathscr{B}_1 \subset \cdots \subset \mathscr{B}_{r-1} \subset \mathscr{B}_r$.*

Proof. For (i) consider the following sequence, obtained from (12):

(14) $$0 \to \mathscr{A}_i/\mathscr{A}_j \to \mathscr{A}_{i+1}/\mathscr{A}_j \to \mathscr{O}_C \to 0.$$

Since $e_i \in H^1(\mathscr{A}_i)$ is general by the generality of $\mathscr{A}_r$, we get that (14) is a general extension and $h^0(\mathscr{A}_{i+1}/\mathscr{A}_j) = h^0(\mathscr{A}_i/\mathscr{A}_j)$. Thus to prove the assertion for $j = 1$ it is enough to show it for the case $i = 2$, which is obvious from $\mathscr{A}_2/\mathscr{A}_1 \cong \mathscr{O}_C$. For $j \ge 2$ we use (14) starting from the case $i = j+1$, when we have $\mathscr{A}_{j+1}/\mathscr{A}_j \cong \mathscr{O}_C$.

For (ii) note first that $\mathscr{A}_1 = \mathscr{O}_C$ and $h^0(\mathscr{A}_r) = 1$. This implies that $\mathscr{A}_1$ is the image of the evaluation map $H^0(\mathscr{A}_r) \otimes \mathscr{O}_C \to \mathscr{A}_r$ and so $f(\mathscr{A}_1) \subseteq \mathscr{A}_1$, concluding the case $r = 2$. Now f induces a map $f' : \mathscr{A}_r/\mathscr{A}_1 \to \mathscr{A}_r/\mathscr{A}_1$. Since $h^0(\mathscr{A}_r/\mathscr{A}_1) = 1$ by (i) and $\mathscr{A}_2/\mathscr{A}_1 \cong \mathscr{O}_C$, we get $f'(\mathscr{A}_2/\mathscr{A}_1) \subseteq \mathscr{A}_2/\mathscr{A}_1$ and so $f(\mathscr{A}_2) \subseteq \mathscr{A}_2$. Thus we get the assertion by continuing this process together with (i).

For (iii) since the case $r = 1$ is trivial, we may assume $r \ge 2$ and use induction on r. For $f \in \mathrm{End}(\mathscr{A}_r)$, we have $\mathscr{A}_1 = \mathscr{O}_C$ and $f(\mathscr{A}_1) \subseteq \mathscr{A}_1$ by (ii). Thus there is $c \in \mathbf{k}$ such that $(f - c \cdot \mathrm{Id}_{\mathscr{A}_r})(\mathscr{A}_1) = 0$, and $f - c \cdot \mathrm{Id}_{\mathscr{A}_r}$ is uniquely determined by $f' \in \mathrm{End}(\mathscr{A}_r/\mathscr{A}_1)$. Since we may apply (i) and (ii) to $\mathscr{A}_r/\mathscr{A}_1$, we conclude by induction on r.

For (iv) note that $\mathscr{A}_1$ (resp. $\mathscr{B}_1$) is the image of the evaluation map of $\mathscr{A}_r$ (resp. $\mathscr{B}_r$) and h is an isomorphism. In particular, we have $h(\mathscr{A}_1) = \mathscr{B}_1$ and so h induces an isomorphism $h' : \mathscr{A}_r/\mathscr{A}_1 \to \mathscr{B}_r/\mathscr{B}_1$. Since $h^0(\mathscr{A}_i/\mathscr{A}_j) = h^0(\mathscr{B}_i/\mathscr{B}_j) = 1$ for all $i > j$ by (i), we iterate the previous argument. $\qquad\square$

Define a subset $\mathbf{J}_r$ to be

$$\mathbf{J}_r = \left\{ e \in \mathbf{T}_r \ \middle| \ \begin{array}{l} \mathscr{A}_r(e) \text{ admits a filtration } \mathscr{A}_1 \subset \cdots \subset \mathscr{A}_{r-1} \subset \mathscr{A}_r \\ \text{such that } h^0(\mathscr{A}_i/\mathscr{A}_j) = 1 \text{ for all } 1 \le j < i \le r \end{array} \right\},$$

i.e. the non-empty open subset of $\mathbf{T}_r$ parametrizing the vector bundles $\mathscr{A}_r$ satisfying (i) of Lemma 6.4; thus $\mathscr{A}_r$ satisfies (ii), (iii) and (iv) of Lemma 6.4.

Lemma 6.5. *For a general $\mathscr{A}_r \in \mathbf{J}_r$ there exists an algebraic subset of $\mathbf{J}_r$, parametrizing the vector bundles isomorphic to $\mathscr{A}_r$, with dimension at most $\frac{r(r-1)}{2}$.*

Proof. We use induction on r; the case $r = 1$ is trivial, because $\mathbf{J}_1 = \mathbf{T}_1 = \{\mathscr{O}_C\}$. We assume that $r \geq 2$ and fix $\mathscr{B}_r \in \mathbf{J}_r$, isomorphic to $\mathscr{A}_r$, with a filtration $\mathscr{B}_1 \subset \cdots \subset \mathscr{B}_r$. For any isomorphism $h : \mathscr{B}_r \to \mathscr{A}_r$, we have $h(\mathscr{B}_{r-1}) = \mathscr{A}_{r-1}$ by (iv) of Lemma 6.4. Since $\mathscr{A}_{r-1}$ is also general in $\mathbf{J}_{r-1}$, by inductive assumption there is an algebraic subset $\mathbf{J}'$ of $\mathbf{J}_{r-1}$ parametrizing the vector bundles isomorphic to $\mathscr{A}_{r-1}$. Fix $\mathscr{M} \in \mathbf{J}'$ and consider the subset $\mathbf{T}' \subset \mathbf{T}_r$ of all extensions of $\mathscr{O}_C$ by $\mathscr{M}$ which are isomorphic to $\mathscr{A}_r$. By Lemma 6.3 and (iii) of Lemma 6.4, we have $\dim \mathbf{T}' \leq r - 1$ and we get the assertion. $\qquad\square$

Proof of Proposition 6.1: Note that

$$g - 1 + \sum_{i=2}^{r-1} \left(i(g-1) - 1 \right) - \sum_{i=1}^{r-1} i = \frac{(r-1)(r-2)(g-1)}{2} - \frac{r(r-1)}{2}.$$

Set $\Delta := \{v^*(\mathscr{A}) \mid \mathscr{A} \in \mathbf{J}_r\}$ and then each element of Δ is indecomposable, because each $\mathscr{A} \in \mathbf{J}_r$ is indecomposable. Since we have $v_* v^* \mathscr{F} \cong \mathscr{F}$ for any vector bundle $\mathscr{F}$ on C by the projection formula and $v_* \mathscr{O}_X \cong \mathscr{O}_C$, we have $v^* \mathscr{A} \cong v^* \mathscr{B}$ if and only if $\mathscr{A} \cong \mathscr{B}$ for any $v^* \mathscr{A}, v^* \mathscr{B} \in \Delta$.

Fix a general $\mathscr{L} \in \mathrm{Pic}^0(X)$ and set $\Theta_{\mathscr{L}} := \{\mathscr{G} \otimes \mathscr{L} \mid \mathscr{G} \in \Delta\}$. Each element of $\Theta_{\mathscr{L}}$ is an indecomposable vector bundle of rank r on X and the isomorphism classes of elements in $\Theta_{\mathscr{L}}$ are also parametrized by $\mathbf{J}_r$. We have $h^1(\mathscr{L}) = 0$ by [3, Th. 0.1], because $q(X) > g$ and by our definition of g there is no non-constant morphism from X to a curve of genus $q(X)$. Then the same argument as in Remark 4.2 ensures that $\mathscr{L}$ is aCM.

Since each element of $\Theta_{\mathscr{L}}$ is an iterated extension of $\mathscr{L}$, each element of $\Theta_{\mathscr{L}}$ is also aCM. Note that each element of $\Theta_{\mathscr{L}}$ is strictly semistable with $gr(\mathscr{A}_r) \cong \mathscr{L}^{\oplus r}$ and so no element of $\Theta_{\mathscr{L}}$ is isomorphic to an element of $\Theta_{\mathscr{L}'}$ with $\mathscr{L} \not\cong \mathscr{L}'$. Now we may vary the general $\mathscr{L} \in \mathrm{Pic}^0(X)$ to obtain a family Γ whose fibre over $\mathscr{L}$ is $\Theta_{\mathscr{L}}$. Then we get the inequality in the assertion and all the requirements for Γ are satisfied. $\qquad\square$

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