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The Cohen-Macaulay representation type of projective arithmetically Cohen-Macaulay varieties

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Abstract. We show that any reduced non-degenerate closed subscheme $X \subset \mathbb{P}^n$ of dimension $m \geq 1$ whose graded coordinate ring is Cohen-Macaulay is of wild Cohen-Macaulay type, except for a few cases which we completely classify.

Keywords. ACM vector sheaves and bundles; Ulrich sheaves; MCM modules; Graded Cohen-Macaulay rings; Representation type

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[Français]

Le type de représentation de Cohen-Macaulay des variétés projectives arithmétiquement Cohen-Macaulay

Résumé. Nous montrons que tout sous-schéma fermé réduit et non dégénéré $X \subset \mathbb{P}^n$ de dimension $m \geq 1$ dont l'anneau de coordonnées homogènes est Cohen-Macaulay est de type de Cohen-Macaulay sauvage, excepté dans quelques cas que nous classifions complètement.

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Contents

1. Introduction

A classical result in representation theory of quivers is Gabriel's theorem, stating that a finite connected quiver supports only finitely many irreducible representations (that is, indecomposable modules over the associated path algebra) if and only if it is of type A, D, E. The classification of tame quivers as *Euclidean graphs*, or *extended Dynkin diagrams*, of type $\tilde{A}$, $\tilde{D}$, $\tilde{E}$ came shortly afterwards. Remarkably, any other finite connected quiver supports arbitrarily large families of indecomposable representations, which is to say, it is of *wild representation type*.

In algebraic geometry and commutative algebra, the relevant problem in terms of representation theory of algebras concerns the complexity of the category of maximal Cohen-Macaulay modules over the coordinate ring $\boldsymbol{k}[X]$ of a closed m-dimensional subvariety $X \subset \mathbb{P}^n$ over a field $\boldsymbol{k}$. For $m > 0$, assuming $\boldsymbol{k}[X]$ to be Cohen-Macaulay (so X is said to be arithmetically Cohen-Macaulay, briefly ACM), these correspond to ACM sheaves, namely coherent sheaves $\mathcal{E}$ on X without intermediate cohomology, that is, satisfying $\mathrm{H}^i(X, \mathcal{E}(t)) = 0$ for all $t \in \mathbb{Z}$ and $0 < i < m$. For hypersurfaces [Eis80] these modules correspond to matrix factorizations, which in turn are related to mirror symmetry, see [Orl04].

In this sense, reduced projective ACM varieties of finite CM-type are classified, see [EH88], see also [BGS87, AR87, Knö87, Her78]. Their list (for positive dimension) consists of rational normal curves, projective spaces, smooth quadrics, the Veronese surface in $\mathbb{P}^5$ and the cubic scroll in $\mathbb{P}^4$. Of course this result is connected with Horrocks' and Grothendieck's classical splitting theorems for vector bundles over $\mathbb{P}^n$, which in turn relates to ideas going back to Segre, [Seg84].

The next class consists of CM-tame varieties. These include CM-countable varieties (for example quadrics of corank one) and varieties where all indecomposable ACM sheaves are parametrized by a curve. Besides smooth elliptic curves (by seminal work of Atiyah, [Ati57], also related to classical work of Segre, *cf.* [Seg86]), trees and cycles of rational curves (see [DG93, DG01], see also [BD04]), two sporadic examples were given in [FM17], consisting of smooth quartic surface scrolls in $\mathbb{P}^5$.

The main goal of this paper is to prove that, besides theses cases, all reduced closed ACM subschemes $X \subset \mathbb{P}^n$ of positive dimension are CM-wild. Without loss of generality, we may assume that X is non-degenerate, namely X is contained in no hyperplane.

Theorem. *Let $X \subset \mathbb{P}^n$ be a reduced non-degenerate closed ACM subscheme of dimension $m \geq 1$. Then X is of wild CM-type unless X is one of the following:*

(i) *a linear space;*
(ii) *a quadric hypersurface of corank at most one;*
(iii) *a tree of rational curves;*
(iv) *a smooth elliptic curve or a cycle of rational curves;*
(v) *a smooth rational scroll of dimension 2 and degree $d = 3$ or $d = 4$;*
(vi) *the Veronese surface in $\mathbb{P}^5$.*

As for the terminology used here, a *rational scroll* is a variety obtained as the image in $\mathbb{P}^n$ of the projective bundle $Y = \mathbb{P}(\oplus_{i=1}^m \mathcal{O}_{\mathbb{P}^1}(a_i))$, for some integers $0 \leq a_1 \leq \cdots \leq a_m \neq 0$, by the relatively ample line bundle $\mathcal{O}_Y(1)$. A rational scroll is smooth if and only if $a_1 > 0$ or $a_{m-1} < a_m = 1$ (in which case $Y = \mathbb{P}^m$), otherwise it is a cone, see §7.1. As we will recall in a minute, rational scrolls and quadrics form the class of varieties of *minimal degree*, that play a rather special role in representation theory of algebras.

A *tree of rational curves* is the union of distinct smooth rational curves $X_1, \ldots, X_s$, namely each X_i is isomorphic to $\mathbb{P}^1$, such that $X_i \cap X_j$ is a single point if $j \in \{i-1, i+1\}$ and empty otherwise. A *cycle of rational curves* can be either the same thing, but using cyclic notation on the indices (so $X_1 \cap X_s \neq \emptyset$) or an irreducible rational curve with a single ordinary double point. This means that the only singularities of the whole scheme X are ordinary double points, so the intersections points $X_{i-1} \cap X_i$ and $X_{i+1} \cap X_i$ are distinct.

A word on the base field $\boldsymbol{k}$ is in order. The result holds for an algebraically closed field of arbitrary characteristic except 2. Actually all the results that we prove in this paper are valid also in characteristic 2. The only point where $\mathrm{char}(\boldsymbol{k}) \neq 2$ is needed is when we recall the fact, due to Knörrer and Buchweitz-Greuel-Schreyer, that quadric cones of corank 1 are CM-countable. We refer to Remark 7.6 for a discussion of this issue.

As a consequence of our main result, we get a strong version of the finite-tame-wild trichotomy, namely that any reduced ACM closed subscheme $X \subset \mathbb{P}^n$ of dimension $m > 0$ falls in exactly one of the following classes:

Finite: there are only finitely many indecomposable ACM sheaves on X up to isomorphism and degree shift. This happens in cases (i), (vi), (v) for $d = 3$ and the smooth cases of (ii), (iii).

Tame: in turn also classically divided into *tame discrete*: the parameter space of indecomposable non-isomorphic ACM sheaves is a countable set of points (in the singular cases of (ii), (iii)); or *properly tame*: for any given rank r, the parameter space of indecomposable non-isomorphic ACM sheaves of rank r is a finite union of curves (in cases (v) for $d = 4$ and (iv)).

Wild: the category of modules of any finite-dimensional algebra admits a representation embedding into the category of MCM $\boldsymbol{k}[X]$-modules; in particular X supports families of arbitrarily large dimension of indecomposable non-isomorphic ACM sheaves.

The result was known for some specific cases, such as smooth cubic surfaces (see [CH11]), all linearly embedded Segre varieties besides the CM finite ones (see [CMRPL12]), smooth del Pezzo surfaces (see [MRPL12, CKM13]), positive-dimensional hypersurfaces of degree at least 4 and some complete intersections (see [DT14]), some Fano varieties (see [MRPL14]), the triple Veronese embedding of any variety (see [MR15]).

One should expect that non-projective varieties may behave differently (see [LW12] for a detailed picture, see also [Sto14]). More varieties of tame type appear from germs of elliptic singularities, see [Kah89] or from non-isolated affine surface singularities, see also [BD17].

Let us indicate the strategy of our proof. The first step is to isolate some datum in order to build large families of indecomposable non-isomorphic MCM modules on $\boldsymbol{k}[X]$. This will be accomplished by Theorem A, where we establish that this datum should be a pair of ACM sheaves $\mathcal{A}$ and $\mathcal{B}$ on X, whose only endomorphisms are homotheties (that is, $\mathcal{A}$ and $\mathcal{B}$ are *simple*), satisfying some semistability condition or a mutual orthogonality condition $\mathrm{Hom}_X(\mathcal{B},\mathcal{A}) = \mathrm{Hom}_X(\mathcal{A},\mathcal{B}) = 0$, and such that $\mathrm{Ext}^1_X(\mathcal{B},\mathcal{A})$ is sufficiently large, namely of dimension $w \geq 3$. From this datum we construct a representation embedding from the category $\mathbf{Rep}_\Upsilon$ of finite-dimensional representations of the Kronecker quiver $\Upsilon = \Upsilon_w$ with w arrows to the category $\mathbf{MCM}_{\boldsymbol{k}[X]}$ of MCM modules over $\boldsymbol{k}[X]$. This quiver should be seen as parametrizing ACM sheaves appearing as extension of copies of $\mathcal{A}$ and $\mathcal{B}$. In turn, by a standard argument, the existence of this embedding suffices to prove CM-wildness of X. This procedure is not completely new, see [DG01], but we endow it with a quite more general flavour.

The next step is to actually construct the sheaves $\mathcal{A}$ and $\mathcal{B}$. This turns out to be quite complicated to achieve by working directly on X in general. For this we need our next result, Theorem B, which shows how to deduce CM-wildness of X from CM-wildness of Y when Y is a linear section of X of codimension $c > 0$, except when X has *minimal degree* that is $\deg(X) = n - m + 1$, or equivalently when the sectional genus p_X of X is 0.

In order to do this, we need to further assume that $\mathcal{A}$ and $\mathcal{B}$ are Ulrich sheaves, namely their modules of global sections have the maximal number of generators. We see this as a further indication of the importance of these sheaves, see [ESW03, ES09].

The idea of Theorem B is that taking the c^{th} syzygy $\boldsymbol{\Omega}^c_{\boldsymbol{k}[X]}$ of the $\boldsymbol{k}[X]$-module of an MCM module L over $\boldsymbol{k}[Y]$ one obtains an MCM module over $\boldsymbol{k}[X]$ and that this entails no essential loss of information if L is Ulrich and $p_X > 0$. In fact, for $\boldsymbol{\Omega}^c_{\boldsymbol{k}[X]}$ to be a functor we need to pass to the stable category $\underline{\mathbf{MCM}}_{\boldsymbol{k}[X]}$ where we quotient out by morphisms factoring through a free module. The point is that the stable syzygy functor $\underline{\boldsymbol{\Omega}}^c$ is fully faithful on Ulrich modules when $p_X > 0$. The proof uses cohomology vanishing of Ulrich sheaves combined with duality.

The next result, Theorem C, shows how to put these two ingredients together. Indeed, by resolving over $\boldsymbol{k}[X]$ the module of global sections of the universal extension of the sheaves $\mathcal{B}$ by $\mathcal{A}$ over Y needed for Theorem A and taking its c^{th} syzygy, we get a functor $\mathbf{Rep}_\Upsilon \to \mathbf{MCM}_X$ whose stabilization is fully faithful by Theorem B. Then, although the functor itself is not quite fully faithful, nevertheless it is a *representation embedding*, that is, it sends non-isomorphic (respectively, irreducible) representations to non-isomorphic (respectively, indecomposable) modules, and this suffices to show CM-wildness of X.

In view of these results, in order to complete the proof it remains to treat directly the case $p_X = 0$, and to construct the Ulrich sheaves $\mathcal{A}$ and $\mathcal{B}$ as above over a linear section Y of X, which we take to be of dimension 1 when $p_X \geq 2$, or of dimension 2 for $p_X = 1$.

The case $p_X \geq 2$ is rather easily seen to provide only CM-wild varieties, as $\mathcal{A}$ and $\mathcal{B}$ can be taken to be sufficiently general bundles of rank 2 over Y of slope $\deg(Y) + g - 1$, where g is the geometric genus of Y. This is treated in §9.

For $p_X = 1$ our proof of the existence of $\mathcal{A}$ and $\mathcal{B}$ is based on a study of locally free Ulrich sheaves of rank 2 on surfaces of sectional genus 1, also called of *almost minimal degree*. Special care has to be taken to allow Y to be singular and even non-normal (yet neither reducible nor a cone, see the next paragraphs); nevertheless these varieties are completely classified and sufficiently detailed information is available, in particular on their divisor class group, to construct the required sheaves and to control their deformations. Theorem 8.1 gathers the results of §8, devoted to this case.

A different method is needed for reducible or non-reduced subschemes, since our basic technique to construct the sheaves $\mathcal{A}$ and $\mathcal{B}$ may fail for various reasons. The two major ones are the following: first, the sheaves $\mathcal{A}$ and $\mathcal{B}$ may degenerate to non-simple ones when more components appear. Second, we partially rely on the classification of varieties of minimal and almost minimal degree, and for reducible subschemes this is a far more complicated question than for irreducible ones. We deal with this in Theorems 6.2 and 6.3.

To summarize again, it remains to work out the case of minimal degree, which is equivalent to $p_X = 0$. This is easy if X is smooth, but needs some care if X is singular, or equivalently if X is a cone, which is to say, the generators of the homogeneous ideal of X do not involve a given set of variables (see §7.1). This case is not quite straightforward, mainly because some smooth finite CM-type varieties degenerate to singular ones that turn out to be CM-wild. In §7 we describe a method to deal with cones and varieties of minimal degree in a uniform manner. This proves the stated CM-wildness of all cones except for a single exceptional variety, namely the cone over a rational normal cubic. In turn, in Theorem 7.5 we treat this intriguing case with an ad-hoc method based on representations of a certain quiver with three vertices.

If one carefully goes through the constructions carried out in this paper, it can be observed that many CM-wild varieties actually support unbounded families of Ulrich sheaves. A strong conjecture in this sense would be the following.

Conjecture 1. *Let $X \subset \mathbb{P}^n$ be a closed non-degenerate integral subscheme of dimension $m \geq 2$, not of minimal degree. Then X is strictly Ulrich wild.*

Our main theorem offers an affirmative answer in case X is ACM, after replacing "strictly Ulrich wild" by "CM-wild". The conjecture is known to hold for several classes of varieties, most notably of surfaces, like del Pezzo surfaces or K3 surfaces (see Theorem 8.1 for surfaces of almost minimal degree, which coincide with del Pezzo surfaces in the smooth case). It is also true that curves of arithmetic genus greater or equal than two (see Section 9) and smooth varieties of minimal degree of dimension $m \geq 2$ are strictly Ulrich wild except in cases (i), (ii), (v), (vi) of our main theorem. However this fails in general for singular varieties of minimal degree. For example, consider a quadric cone $X \subset \mathbb{P}^n$ over a vertex Λ of dimension at least 1. Then X is CM-wild. On the other hand, an Ulrich sheaf on X is the sheafification of $E^0 \otimes \boldsymbol{k}[\Lambda]$, where E^0 is the module associated with a direct sum of spinor bundles on a smooth quadric X^0, the base of the cone. These sheaves are rigid, so X is not Ulrich-wild.

In a sense, the statement of the previous conjecture admits no converse, as it turns out that the Segre variety $\mathbb{P}^1 \times \mathbb{P}^2 \subset \mathbb{P}^5$ is a CM-wild ACM variety whose only infinite family of non-isomorphic indecomposable ACM sheaves (up a degree shift) consists of Ulrich bundles. We refer to [FMS19] for this and related issues.

One may also observe that many singular CM-wild varieties admit unbounded families of non-isomorphic ACM sheaves of fixed rank, while this does seem not to happen for smooth varieties. This motivates the following question.

Problem 1. *Let $X \subset \mathbb{P}^n$ be a smooth projective variety of positive dimension. Given $r \geq 1$, is the family of isomorphism classes of indecomposable ACM initialized sheaves of rank r parametrized by a finite union of irreducible quasi-projective schemes?*

The problem of classifying the representation type of integral subschemes $X \subset \mathbb{P}^n$ of dimension $m \geq 2$ which are not ACM seems interesting. Some cases are known, such as abelian and Enriques surfaces, (see [Cas17] and [Bea16]) but the general problem remains wide open even for smooth surfaces. For reducible subschemes, already the representation type of 2-regular subschemes seems to be unknown in general.

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2. CM-wild varieties

Let $\boldsymbol{k}$ be a field and set $S = \boldsymbol{k}[x_0,\dots,x_n]$ for the symmetric graded $\boldsymbol{k}$-algebra with $n+1$ indeterminates, seen also as the coordinate ring of the projective n-space $\mathbb{P}^n = \mathrm{Proj}(S)$ of 1-dimensional linear quotients of the vector space $V = \boldsymbol{k}^{n+1}$.

2.1. ACM varieties and modules

We first recall some basic terminology for various Cohen-Macaulay properties of varieties, sheaves and modules.

2.1.1. ACM subschemes.— Let $X \subset \mathbb{P}^n$ be a closed subscheme of dimension $m > 0$. Write I_X for the saturated homogeneous ideal of X and $R = \boldsymbol{k}[X] = S/I_X$ for its coordinate ring.

Definition 2.1. The subscheme $X \subset \mathbb{P}^n$ is arithmetically Cohen-Macaulay (ACM) if $R = \boldsymbol{k}[X]$ is a graded Cohen-Macaulay ring, namely if R has a graded S-free resolution of length $n-m$.

2.1.2. Terminology on coherent sheaves.— Let $X \subset \mathbb{P}^n$ be a closed subscheme, $m = \dim(X)$. We write $\mathbf{Coh}_X$ for the category of coherent sheaves on X. We denote by $\mathcal{O}_X(1)$ the restriction to X of $\mathcal{O}_{\mathbb{P}^n}(1)$ and we employ the usual notation $\mathcal{E}(t) = \mathcal{E} \otimes \mathcal{O}_X(1)^{\otimes t}$. The ideal sheaf of a subscheme $Z \subset X$ will be denoted by $\mathcal{I}_{Z|X}$. Given $\mathcal{E}, \mathcal{F}$ in $\mathbf{Coh}_X$ and $i \in \mathbb{Z}$, we consider the Ext modules:

$$\mathrm{Ext}^i_X(\mathcal{E},\mathcal{F})_* = \bigoplus_{t\in\mathbb{Z}} \mathrm{Ext}^i_X(\mathcal{E},\mathcal{F}(t))$$

as R-modules. For $i \in \mathbb{N}$, we write $\mathrm{H}^i_*(\mathcal{E}) = \mathrm{Ext}^i_X(\mathcal{O}_X,\mathcal{E})_*$ for the i^{th} cohomology module of $\mathcal{E}$. One may also replace $t \in \mathbb{Z}$ by any truncation $t \geq t_0$. The module $\mathrm{H}^0_*(\mathcal{E})$ is also denoted by $\Gamma_*(\mathcal{E})$. It is finitely generated if $\mathcal{E}$ has no zero-dimensional subsheaf.

We say that a coherent sheaf $\mathcal{E}$ on X is ***simple*** if its only endomorphisms are homotheties, that is, if $\mathrm{Hom}_X(\mathcal{E},\mathcal{E}) = \boldsymbol{k}\,\mathrm{id}_{\mathcal{E}}$. We write $\chi(\mathcal{E},\mathcal{F})$ for the Euler characteristic of two coherent sheaves $\mathcal{E}$ and $\mathcal{F}$ over X, namely

$$\chi(\mathcal{E},\mathcal{F}) = \sum_{i\in\mathbb{Z}} (-1)^i \dim_{\boldsymbol{k}} \mathrm{Ext}^i_X(\mathcal{E},\mathcal{F}),$$

provided this is a finite sum. This is the case for instance when X is smooth or when $\mathcal{E}$ or $\mathcal{F}$ are locally free. We abbreviate $\chi(\mathcal{F}) = \chi(\mathcal{O}_X,\mathcal{F})$.

We write H_X for the very ample divisor class on X associated with $\mathcal{O}_X(1)$. The ***Hilbert polynomial*** of a coherent sheaf $\mathcal{E}$ is defined as $P(\mathcal{E},t) := \chi(\mathcal{E}(t))$. The degree $d = \deg(X)$ is defined in terms of the polynomial $P(\mathcal{O}_X,t)$, namely by the condition that the leading term of $P(\mathcal{O}_X,t)$ be $d/m!$. Similarly, for $\mathcal{E}$ in $\mathbf{Coh}_X$, the rank $r = \mathrm{rk}(\mathcal{E}) \in \mathbb{Q}$ is defined by the condition that the leading term of $P(\mathcal{E},t)$ be $rd/m!$. We write $\mathrm{p}(\mathcal{E},t) := P(\mathcal{E},t)/r$ for the ***reduced Hilbert polynomial.***

We write $p \geq q$ (resp. $p > q$) for polynomials $p,q \in \mathbb{Q}[t]$ if $p(t) \geq q(t)$ (resp. $p(t) > q(t)$) for $t \gg 0$. A coherent sheaf $\mathcal{E}$ is called ***pure*** if all of its subsheaves are supported in dimension m. A pure sheaf is H_X-***semistable*** in the sense of Gieseker-Maruyama if, for any coherent subsheaf $\mathcal{F} \subsetneq \mathcal{E}$, one has $\mathrm{p}(\mathcal{E},t) \geq \mathrm{p}(\mathcal{F},t)$. The sheaf is called H_X-stable if for all $\mathcal{F}$ as above $\mathrm{p}(\mathcal{E},t) > \mathrm{p}(\mathcal{F},t)$. We will often suppress H_X from the notation.

2.1.3. Cohen-Macaulay and Ulrich conditions.— Again $X \subset \mathbb{P}^n$ is a closed subscheme of dimension $m \geq 1$ with coordinate ring $R = \boldsymbol{k}[X]$. Given a graded R-module M and $t \in \mathbb{Z}$, we denote by M_t its degree-t piece and $M_{\geq t} = \oplus_{i \geq t} M_i$. Analogously $M_{<t} = \oplus_{i<t} M_i$.

Definition 2.2. A coherent sheaf $\mathcal{E}$ on an m-dimensional closed subscheme $X \subset \mathbb{P}^n$ is called *arithmetically Cohen-Macaulay (ACM)* if $\mathcal{E}$ is locally Cohen-Macaulay (that is, $\mathcal{E}_x$ is a Cohen-Macaulay $\mathcal{O}_{X,x}$-module for all $x \in X$) and $\mathrm{H}^i_*(\mathcal{E}) = 0$ for $i = 1, \dots, m-1$.

This is equivalent to asking that $E = \Gamma_*(\mathcal{E})$ is a maximal graded Cohen-Macaulay module over R. In turn, this amounts to requiring that E has a graded free S-resolution of length $n - m$.

Let $d = \deg(X)$. Given an MCM module E of rank r over $R = \boldsymbol{k}[X]$, the number of independent minimal generators of E is at most dr. Analogously, for an ACM sheaf $\mathcal{E}$, assuming that $\mathcal{E}$ is *initialized* (*i.e.*, $\mathrm{H}^0(X, \mathcal{E}) > \mathrm{H}^0(X, \mathcal{E}(-1)) = 0$), we have:

$$\dim_{\boldsymbol{k}} \mathrm{H}^0(X, \mathcal{E}) \leq dr. \tag{2.1}$$

An ACM coherent sheaf $\mathcal{E}$ on X is called an *Ulrich sheaf* on X (and $E = \Gamma_*(\mathcal{E})$ is called an Ulrich module over R) if $\mathrm{H}^0(X, \mathcal{E}(-1)) = 0$ and equality is attained in (2.1). The reader can consult [ESW03] for an account on Ulrich sheaves. Let us just gather here the main properties that will be used throughout this paper:

a) Any Ulrich sheaf $\mathcal{E}$ of rank r on an m-dimensional closed subscheme $X \subset \mathbb{P}^n$ of degree d has a linear $\mathcal{O}_{\mathbb{P}^n}$-resolution of the form
$$0 \leftarrow \mathcal{E} \leftarrow \mathcal{O}_{\mathbb{P}^n}^{dr} \xleftarrow{d_1} \mathcal{O}_{\mathbb{P}^n}(-1)^{a_1} \leftarrow \cdots \xleftarrow{d_{n-m}} \mathcal{O}_{\mathbb{P}^n}(m-n)^{a_{n-m}} \leftarrow 0.$$
The length of the resolution is $n - m$ and the maps $(d_i \mid i \in \{1, \dots, n-m\})$ are given by matrices whose entries are linear forms of S. Also one has $a_i = \binom{n-m}{i} dr$ for all $i \in \{1, \dots, n-m\}$ This follows from [ESW03, Proposition 2.1], see also the comments after this proposition.

b) Any Ulrich sheaf $\mathcal{E}$ is globally generated. Its Hilbert polynomial is $P(\mathcal{E}, t) = dr\binom{t+m}{m}$. This is a consequence of the previous point.

c) For any linear projection $\pi : X \to \mathbb{P}^m$, the direct image $\pi_* \mathcal{E}$ is isomorphic to $\mathcal{O}_{\mathbb{P}^m}^{dr}$. Again [ESW03, Proposition 2.1].

d) Any $\mathcal{E}$ Ulrich sheaf on a subscheme $X \subset \mathbb{P}^n$ is semistable and any destabilizing subsheaf of $\mathcal{E}$ is also Ulrich. This has been proved in a number of papers, with variable hypothesis; see for instance [CHGS12, Theorem 2.9] for smooth varieties. We refer to Lemma 8.3 for a statement on an arbitrary closed subscheme.

Let us denote by $\mathbf{ACM}_X$ (resp. $\mathbf{Ulr}_X$) the full subcategory of $\mathbf{Coh}_X$ consisting of ACM sheaves (resp. of Ulrich sheaves). We denote by $\mathbf{MCM}_{R,0}$ (resp. $\mathbf{Ulr}_{R,0}$) the subcategory of the category $\mathbf{Mod}_R$ of finitely generated R-modules whose objects are MCM modules (resp. Ulrich modules) and whose morphisms are degree-0 morphisms of R-modules. There is a basic equivalence between these notions as in the next lemma (see [KL71, Proposition 2.2.4]).

Lemma 2.3. *The functor* $\Gamma_* : \mathbf{ACM}_X \to \mathbf{MCM}_{R,0}$ *is an equivalence, whose inverse is the sheafification functor* $M \mapsto \tilde{M}$. *The equivalence carries* $\mathbf{Ulr}_X$ *to* $\mathbf{Ulr}_{R,0}$.

Let $\mathcal{E}$ and $\mathcal{F}$ be coherent sheaves on X whose associated modules $E = \Gamma_*(\mathcal{E})$ and $F = \Gamma_*(\mathcal{F})$ are finitely generated as R-modules. In spite of the previous lemma, $\mathrm{Ext}^i_X(\mathcal{E}, \mathcal{F}(t))$ and $\mathrm{Ext}^i_R(E, F)_t$ may differ. The following lemma will be useful to compare them.

Lemma 2.4. *Assume $\mathcal{F}$ is ACM. Then, there is $t \in \mathbb{Z}$ depending on the minimal graded free resolution of F as S-module, such that there is an isomorphism:*
$$\mathrm{Ext}^i_R(E_{\geq t}, F)_{\geq 0} \simeq \bigoplus_{q \geq 0} \mathrm{Ext}^i_X(\mathcal{E}, \mathcal{F}(q)), \qquad \textit{as graded } R\textit{-modules.}$$

i) If $i \leq m-1$, in the above isomorphism we may replace $E_{\geq t}$ by E.

ii) If F is Ulrich, we may take $t = 1 - i$.

iii) If F is linearly presented and E is generated in degree 0 *over S, then* $\mathrm{Ext}^1_R(E, F)_{<-1} = 0$.

Proof. Since the sheaf $\mathcal{F}$ is ACM of positive dimension, the associated R-module F is finitely generated. The first statement follows from [Smi00, Theorem 1]. If F (or $\mathcal{F}$) is Ulrich, the minimal graded free resolution of F as S-module is linear, so the same result allows $t = 1 - i$, so we get ii).

Next, F is a graded Cohen-Macaulay R-module, so $\mathrm{Ext}^i_R(\boldsymbol{k}, F) = 0$ for $i \leq m$, where $\boldsymbol{k}$ is the residue field seen as a R-module. Then, for any $j \in \mathbb{Z}$, since the module $E_{<j} := E/E_{\geq j}$ is Artinian, by induction on the length of the composition series of $E_{<j}$ we get $\mathrm{Ext}^i_R(E_{<j}, F) = 0$ for $i \leq m$. The isomorphism needed for i) follows from the exact sequence:

$$\mathrm{Ext}^i_R(E_{<j}, F) \to \mathrm{Ext}^i_R(E, F) \to \mathrm{Ext}^i_R(E_{\geq j}, F) \to \mathrm{Ext}^{i+1}_R(E_{<j}, F).$$

It remains to prove iii). A minimal graded presentation of F as S-module is an exact sequence of the form $S^{\beta_1}(-1) \to S^{\beta_0} \twoheadrightarrow F$, while E admits a surjection $S^{\alpha_0} \twoheadrightarrow E$, for some positive integers α_0, β_0 and β_1. The surjection $S^{\beta_0} \to F$, whose kernel is an S-module that we denote by N, descends to a surjection of R-modules $R^{\beta_0} \to F$, whose kernel we call M. Using $\mathrm{Hom}_R(-, E)$ we get an exact sequence of R-modules:

$$\mathrm{Hom}_R(R^{\beta_0}, E) \to \mathrm{Hom}_R(M, E) \to \mathrm{Ext}^1_R(F, E) \to 0.$$

Now, M is the quotient of $N \otimes_S R$ by $\mathrm{Tor}^S_1(R, F)$, so we have surjections $R(-1)^{\beta_1} \twoheadrightarrow N \otimes_S R \twoheadrightarrow M$. We deduce that $\mathrm{Hom}_R(M, E)$ is a submodule of $\mathrm{Hom}_R(R(-1)^{\beta_1}, E)$. From the surjection $S^{\alpha_0} \twoheadrightarrow E$ we get $R^{\alpha_0} \twoheadrightarrow E$ and thus a surjection:

$$\mathrm{Hom}_R(R(-1)^{\beta_1}, R^{\alpha_0}) \twoheadrightarrow \mathrm{Hom}_R(R(-1)^{\beta_1}, E).$$

This proves that $\mathrm{Hom}_R(R(-1)^{\beta_1}, E)$ vanishes in degree strictly below -1 so the same happens to $\mathrm{Hom}_R(M, E)$ and thus to $\mathrm{Ext}^1_R(F, E)$. □

2.2. CM-wildness

We will consider a couple of related notions of CM-wildness for a closed scheme $X \subset \mathbb{P}^n$. Algebraically this means that, for any finitely generated associative $\boldsymbol{k}$-algebra Σ, the category of MCM modules over $R = \boldsymbol{k}[X]$ contains, in some sense, the category $\mathbf{Mod}_\Sigma$ of finitely generated left Σ-modules. We spell this out in more detail in the next paragraph. We adopt [SS07, Chapter XIX] as general reference for this part.

Definition 2.5. Let $X \subset \mathbb{P}^n$ be a closed subscheme and set $R = \boldsymbol{k}[X]$. For any finitely generated associative $\boldsymbol{k}$-algebra Σ, and any finitely generated R-graded (R, Σ)-bimodule $\boldsymbol{M}$, flat over Σ, define the functor:

$$\Phi_{\boldsymbol{M}} : \mathbf{Mod}_\Sigma \to \mathbf{Mod}_R, \qquad N \mapsto \boldsymbol{M} \otimes_\Sigma N.$$

The variety X is of *wild CM-type* if, for any Σ as above, there is $\boldsymbol{M}$ such that $\Phi_{\boldsymbol{M}}$ takes values in $\mathbf{MCM}_R$ and is a *representation embedding* in $\mathbf{MCM}_R$, which is to say:

a) the module N is decomposable whenever $\Phi_{\boldsymbol{M}}(N)$ is;
b) for any pair (N, N') of modules in $\mathbf{Mod}_\Sigma$, we have:

$$N \simeq N' \Leftrightarrow \Phi_{\boldsymbol{M}}(N) \simeq \Phi_{\boldsymbol{M}}(N').$$

The variety X is of *wild Ulrich type* if moreover:

c) for any N in $\mathbf{Mod}_\Sigma$, $\Phi_{\boldsymbol{M}}(N)$ is Ulrich.

The variety X is said to be *strictly CM-wild* if for any Σ as above there is an $\boldsymbol{M}$ such that $\Phi_{\boldsymbol{M}}$ is fully faithful into $\mathbf{MCM}_{R,0}$, that is

$$\mathrm{Hom}_\Sigma(N, N') \simeq \mathrm{Hom}_R(\Phi(N), \Phi(N'))_0,$$

If moreover $\Phi_{\boldsymbol{M}}(N)$ is Ulrich for all N, then X is *strictly Ulrich wild.*

Remark 2.6. The following facts are well-known, or quickly proved in the next lines.

i) To check that X is CM-wild, it is enough to construct a representation embedding from $\mathbf{FMod}_\Sigma$ into $\mathbf{MCM}_R$, where Σ is a wild algebra of finite dimension over $\boldsymbol{k}$ and $\mathbf{FMod}_\Sigma$ is the category of Σ-modules of finite dimension over $\boldsymbol{k}$. Thanks to Lemma 2.3, we may work interchangeably with $\mathbf{ACM}_X$ or $\mathbf{MCM}_{R,0}$.

ii) If all non-zero graded R-modules $\Phi(M)$ are generated in the same degree and X is strictly CM-wild, then it is CM-wild, and if X is strictly Ulrich wild then it is of wild Ulrich type. Indeed, given $M, N \in \mathbf{Mod}_\Sigma$, since $\Phi(M)$ and $\Phi(N)$ are generated in the same degree, an isomorphism $\Phi(M) \to \Phi(N)$ must be of degree 0 and therefore must come from an isomorphism $M \to N$ by full faithfulness. Also any idempotent of $\Phi(M)$ must have degree 0 and is therefore induced by an idempotent of M.

iii) If X is of wild CM-type, then for any $r \in \mathbb{N}$ there are families of dimension at least r consisting of indecomposable ACM sheaves on X, all non-isomorphic to one another. In other words, X is of wild CM-type in the geometric sense. If X is of wild Ulrich type, these families can be taken to consist of Ulrich sheaves.

iv) Any exact functor $\Phi : \mathbf{Mod}_\Sigma \to \mathbf{Mod}_R$ is of the form $\Phi_{\boldsymbol{M}}$ for some finitely generated Σ-flat (R,Σ)-bimodule $\boldsymbol{M}$.

Let $w \geq 1$ be an integer and consider the Kronecker quiver $\Upsilon = \Upsilon_w$ with two vertices and w arrows from the first vertex to the second. Write $\mathbf{Rep}_\Upsilon$ for the abelian category of finite-dimensional $\boldsymbol{k}$-representations of Υ.

v) To check that X is strictly CM-wild (resp., of wild CM-type), it suffices to construct a fully faithful exact functor (resp., a representation embedding):

$$\Phi : \mathbf{Rep}_\Upsilon \to \mathbf{ACM}_X$$

where $\Upsilon = \Upsilon_w$ is the Kronecker quiver with $w \geq 3$. If moreover $\Phi(\mathcal{R})$ is Ulrich for any $\mathcal{R}$ in $\mathbf{Rep}_\Upsilon$, then X is strictly Ulrich wild (resp., of wild Ulrich type). The same argument works if we replace $\mathbf{Rep}_\Upsilon$ with $\mathbf{FMod}_\Sigma$ where $\Sigma = \boldsymbol{k}[x_1, x_2]$.

3. CM-wildness from extensions

Let $X \subset \mathbb{P}^n$ be a closed $\boldsymbol{k}$-subscheme, and let $\mathcal{A}$ and $\mathcal{B}$ be coherent sheaves on X such that:

$$\mathrm{Ext}^1_X(\mathcal{B}, \mathcal{A}) \neq 0.$$

We describe how extensions of $\mathcal{B}$ by $\mathcal{A}$ are parametrized by representations of the *Kronecker quiver* Υ_w having two vertices and as many arrows as $w = \dim_{\boldsymbol{k}} \mathrm{Ext}^1_X(\mathcal{B}, \mathcal{A})$, pointing in the same direction.

3.1. The functor from the Kronecker quiver to $\mathbf{Coh}_X$

Set $W = \mathrm{Ext}^1_X(\mathcal{B}, \mathcal{A})$ and consider the projective space $\mathbb{P}(W^*)$ of lines through the origin in W. Then, over $X \times \mathbb{P}(W^*)$, there is a universal extension:

$$0 \to \mathcal{A} \boxtimes \mathcal{O}_{\mathbb{P}(W^*)} \to \mathcal{U} \to \mathcal{B} \boxtimes \mathcal{O}_{\mathbb{P}(W^*)}(-1) \to 0,$$

where we write p and q for the projections from $X \times \mathbb{P}(W^*)$ to X and $\mathbb{P}(W^*)$, and for $\mathcal{E} \in \mathbf{Coh}_X$ and $\mathcal{F} \in \mathbf{Coh}_{\mathbb{P}(W^*)}$, we set $\mathcal{E} \boxtimes \mathcal{F} = p^*(\mathcal{E}) \otimes q^*(\mathcal{F})$. Then we consider:

$$\Phi_{\mathcal{U}} = \mathbf{R}p_*(q^*(-) \otimes \mathcal{U}) : \mathbf{D}^b(\mathbf{Coh}_{\mathbb{P}(W^*)}) \to \mathbf{D}^b(\mathbf{Coh}_X).$$

It is clear that:

$$\Phi_{\mathcal{U}}(\mathcal{O}_{\mathbb{P}(W^*)}) \simeq \mathcal{A}, \qquad \Phi_{\mathcal{U}}(\Omega_{\mathbb{P}(W^*)}(1)) \simeq \mathcal{B}[-1].$$

Set $w = \dim_{\boldsymbol{k}} \mathrm{Ext}^1_X(\mathcal{B},\mathcal{A})$ and consider the Kronecker quiver $\Upsilon = \Upsilon_w$. Then, the natural isomorphism $W \simeq \mathrm{Hom}_{\mathbb{P}(W^*)}(\Omega_{\mathbb{P}(W^*)}(1),\mathcal{O}_{\mathbb{P}(W^*)})$ provides an equivalence:

$$\Xi : \mathbf{D}^b(\mathbf{Rep}_\Upsilon) \simeq \langle \Omega_{\mathbb{P}(W^*)}(1),\mathcal{O}_{\mathbb{P}(W^*)}\rangle. \tag{3.1}$$

We compose this equivalence with the inclusion of $\langle \Omega_{\mathbb{P}(W^*)}(1),\mathcal{O}_{\mathbb{P}(W^*)}\rangle$ into $\mathbf{D}^b(\mathbf{Coh}_{\mathbb{P}(W^*)})$. Explicitly, this is described as follows. Choose a basis $(e_1,\dots,e_w)$ of $W = \mathrm{Ext}^1_X(\mathcal{B},\mathcal{A})$. Let $\mathcal{R}$ be a representation of Υ having dimension vector (a,b). Then $\mathcal{R}$ corresponds to the choice of w linear maps $m_1,\dots,m_w : \boldsymbol{k}^a \to \boldsymbol{k}^b$. Take the element:

$$\xi = \sum_{i=1}^{w} m_i \otimes e_i \in \mathrm{Hom}_{\boldsymbol{k}}(\boldsymbol{k}^b,\boldsymbol{k}^a)\otimes W. \tag{3.2}$$

Then, under the identification $W \cong \mathrm{Hom}_{\mathbb{P}(W^*)}(\Omega_{\mathbb{P}(W^*)}(1),\mathcal{O}_{\mathbb{P}(W^*)})$, we obtain from ξ a morphism:

$$M : \Omega_{\mathbb{P}(W^*)}(1)^b \to \mathcal{O}^a_{\mathbb{P}(W^*)}. \tag{3.3}$$

The cone of M is the element of $\mathbf{D}^b(\mathbf{Coh}_{\mathbb{P}(W^*)})$ associated with $\mathcal{R}$ via Ξ. This is directly extended to morphisms.

We consider a functor Φ which can be thought of as the restriction of $\Phi_{\mathcal{U}} \circ \Xi$ to $\mathbf{Rep}_\Upsilon$:

$$\Phi : \mathbf{Rep}_\Upsilon \to \mathbf{Coh}_X. \tag{3.4}$$

Let us first give an explicit description of Φ. At the level of objects, given a representation $\mathcal{R}$ of the quiver $\mathbf{Rep}_\Upsilon$ with dimension vector (a,b) let $(m_1,\dots,m_w)$ be the w linear maps associated with $\mathcal{R}$ and let ξ be as in (3.2). Then $\Phi(\mathcal{R})$ fits as middle term of a representative of the extension class corresponding to ξ:

$$0 \to \mathcal{A}^a \to \Phi(\mathcal{R}) \to \mathcal{B}^b \to 0.$$

Let us check now that this is well-defined on morphisms. Let $\mathcal{S}$ be another representation of Υ, of dimension vector (c,d), corresponding to the linear maps $(n_1,\dots,n_w)$. A morphism $\lambda : \mathcal{R} \to \mathcal{S}$ of representations is given by linear maps $\alpha : \boldsymbol{k}^a \to \boldsymbol{k}^c$ and $\beta : \boldsymbol{k}^b \to \boldsymbol{k}^d$ such that:

$$n_i\alpha = \beta m_i, \qquad \text{for all } i = 1,\dots,w. \tag{3.5}$$

Consider the map of coherent sheaves $\beta_{\mathcal{A}} = \beta \otimes \mathrm{id}_{\mathcal{A}} : \mathcal{A}^a \to \mathcal{A}^c$. Then, $\beta_{\mathcal{A}}$ defines a morphism of extensions:

$$\begin{array}{ccccccccc} 0 & \to & \mathcal{A}^a & \to & \Phi(\mathcal{R}) & \to & \mathcal{B}^b & \to & 0 \\ & & \downarrow{\scriptstyle \beta_{\mathcal{A}}} & & \downarrow{\scriptstyle \phi} & & \| & & \\ 0 & \to & \mathcal{A}^c & \xrightarrow{i} & \mathcal{D} & \xrightarrow{p} & \mathcal{B}^b & \to & 0 \end{array} \tag{3.6}$$

for a certain sheaf $\mathcal{D}$ representing the element:

$$\sum_{i=1}^{w} \beta m_i \otimes e_i \in \mathrm{Ext}^1_X(\mathcal{B}^b,\mathcal{A}^c).$$

Analogously $\alpha_{\mathcal{B}} = \alpha \otimes \mathrm{id}_{\mathcal{B}} : \mathcal{B}^b \to \mathcal{B}^d$ defines:

$$\begin{array}{ccccccccc} 0 & \to & \mathcal{A}^c & \xrightarrow{i'} & \mathcal{D}' & \xrightarrow{p'} & \mathcal{B}^b & \to & 0 \\ & & \| & & \downarrow{\scriptstyle \phi'} & & \downarrow{\scriptstyle \alpha_{\mathcal{B}}} & & \\ 0 & \to & \mathcal{A}^c & \to & \Phi(\mathcal{S}) & \to & \mathcal{B}^d & \to & 0 \end{array} \tag{3.7}$$

with the upper row representing an extension class in:

$$\sum_{i=1}^{w} n_i\alpha \otimes e_i \in \mathrm{Ext}^1_X(\mathcal{B}^b,\mathcal{A}^c).$$

Because of (3.5), the lower extension of (3.6) is the same as the upper one in (3.7). Then the morphisms ϕ' and ϕ of extensions compose to give a map from $\Phi(\mathcal{R})$ to $\Phi(\mathcal{S})$.

This construction agrees with the principle of considering Φ as restriction of $\Phi_{\mathcal{U}} \circ \Xi$ to $\mathbf{Rep}_{\Upsilon}$. Indeed, the representation $\mathcal{R}$ is mapped by Ξ to the cone of the matrix M of (3.3), which is sent by $\Phi_{\mathcal{U}}$ to the cone of:

$$\Phi_{\mathcal{U}}(M) : \mathcal{B}^b[-1] \to \mathcal{A}^a.$$

By construction this cone is represented by the extension class $\Phi(\mathcal{R})$.

3.2. Representation embeddings of the Kronecker quiver

Now we state a basic result on representation embeddings and fully faithful embeddings of the Kronecker quiver via extensions. Some forms of this result have been used already by several authors, however the following rather general statement seems to be new.

Theorem A. *Let $\mathcal{A}$ and $\mathcal{B}$ be simple coherent sheaves on a closed subscheme $X \subset \mathbb{P}^N$.*

i) Let $\mathcal{A}$ and $\mathcal{B}$ be semistable with $\mathrm{p}(\mathcal{B}) \le \mathrm{p}(\mathcal{A})$ and suppose that any non-zero morphism $\mathcal{A} \to \mathcal{B}$ is an isomorphism. Then the functor Φ from (3.4) is a representation embedding.

ii) Assume that $\mathrm{Hom}_X(\mathcal{A},\mathcal{B}) = \mathrm{Hom}_X(\mathcal{B},\mathcal{A}) = 0$. Then the functor Φ is fully faithful.

Proof. To check i) we first prove that, given an irreducible representation $\mathcal{R}$ of Υ, the associated sheaf $\mathcal{F} = \Phi(\mathcal{R})$ is indecomposable. Let $\mathcal{R}$ have dimension vector (b,a) so we have:

$$0 \to \mathcal{A}^a \xrightarrow{i} \mathcal{F} \xrightarrow{p} \mathcal{B}^b \to 0. \tag{3.8}$$

Assume first $\mathrm{p}(\mathcal{B}) \prec \mathrm{p}(\mathcal{A})$. Then, (3.8) is the Harder-Narasimhan filtration of $\mathcal{F}$, so the graded object $\mathrm{gr}(\mathcal{F})$ associated with $\mathcal{F}$ is just $\mathcal{A}^a \oplus \mathcal{B}^b$. Assume $\mathcal{F} \simeq \mathcal{F}' \oplus \mathcal{F}''$, with $\mathcal{F}' \neq 0 \neq \mathcal{F}''$. By the uniqueness of $\mathrm{gr}(\mathcal{F})$, we have $\mathrm{gr}(\mathcal{F}') \simeq \mathcal{A}^{a'} \oplus \mathcal{B}^{b'}$ and $\mathrm{gr}(\mathcal{F}'') \simeq \mathcal{A}^{a''} \oplus \mathcal{B}^{b''}$ for some (b',a') and (b'',a'') with $a'+a''=a$ and $b'+b''=b$. It follows that $\mathcal{A}^{a'}$ is the maximal destabilizing subsheaf of $\mathcal{F}'$ with quotient $\mathcal{B}^{b'}$, that is, $\mathcal{F}'$ is an extension of the form:

$$0 \to \mathcal{A}^{a'} \to \mathcal{F}' \to \mathcal{B}^{b'} \to 0,$$

associated with some $\xi' \in W \otimes \boldsymbol{k}^{a'} \otimes \boldsymbol{k}^{b'}$. Similarly, there is $\xi'' \in W \otimes \boldsymbol{k}^{a''} \otimes \boldsymbol{k}^{b''}$ corresponding to $\mathcal{F}''$. Moreover, composing the embedding $\mathcal{A}^{a'} \hookrightarrow \mathcal{F}'$ with $\mathcal{F}' \hookrightarrow \mathcal{F}$ we get a map $\mathcal{A}^{a'} \hookrightarrow \mathcal{F}$, that composes to zero with p, for $\mathcal{A}$ and $\mathcal{B}$ are semistable with $\mathrm{p}(\mathcal{B}) \prec \mathrm{p}(\mathcal{A})$.

We obtain thus a map $\mathcal{A}^{a'} \hookrightarrow \mathcal{A}^a$ which must be of the form $\alpha \otimes \mathrm{id}_{\mathcal{A}}$ for some monomorphism $\alpha : \boldsymbol{k}^{a'} \to \boldsymbol{k}^a$, because $\mathcal{A}$ is simple. This induces a map $\mathcal{B}^{b'} \to \mathcal{B}^b$ which is likewise of the form $\beta \otimes \mathrm{id}_{\mathcal{B}}$ for some $\beta : \boldsymbol{k}^{b'} \to \boldsymbol{k}^b$. This defines a representation $\mathcal{R}'$ of dimension vector (b',a') corresponding to ξ' which is a subrepresentation of $\mathcal{R}$, the embedding being given by (β,α). The quotient $\mathcal{R}'' = \mathcal{R}/\mathcal{R}'$ corresponds then to ξ''. The embedding $\mathcal{F}'' \hookrightarrow \mathcal{F}$ provides a splitting $\mathcal{R}'' \to \mathcal{R}$, so $\mathcal{R} \simeq \mathcal{R}' \oplus \mathcal{R}''$, with $\mathcal{R}' \neq 0 \neq \mathcal{R}''$ which is what we wanted.

Now assume $\mathrm{p}(\mathcal{B}) = \mathrm{p}(\mathcal{A})$ and take $\mathcal{R}$ indecomposable such that $\mathcal{F} = \mathcal{F}' \oplus \mathcal{F}''$ with $\mathcal{F}' \neq 0 \neq \mathcal{F}''$. Then (3.8) is a Jordan-Hölder filtration of $\mathcal{F}$, so the graded object $\mathrm{gr}(\mathcal{F})$ associated with $\mathcal{F}$ is again $\mathcal{A}^a \oplus \mathcal{B}^b$, hence $\mathrm{gr}(\mathcal{F}')$ and $\mathrm{gr}(\mathcal{F}'')$ take the same forms as above, in particular $\mathcal{F}'$ and $\mathcal{F}''$ are semistable with $\mathrm{p}(\mathcal{F}') = \mathrm{p}(\mathcal{A}) = \mathrm{p}(\mathcal{F}'')$.

Next, we compose i with the projection $\mathcal{F} \to \mathcal{F}''$ to get a map $q : \mathcal{A}^a \to \mathcal{F}''$. The sheaf $\mathrm{Im}(q)$ is semistable with $\mathrm{p}(\mathrm{Im}(q)) = \mathrm{p}(\mathcal{A})$. Composing with the projection to $\mathcal{B}^{b''}$, since any non-zero map $\mathcal{A} \to \mathcal{B}$ is an isomorphism, we get as image a direct sum of copies of $\mathcal{A}$. So $\mathrm{Im}(q)$ projects onto copies of $\mathcal{A}$ and thus, since $\mathcal{A}$ is simple, we actually have $\mathrm{Im}(q) \simeq \mathcal{A}^{a''}$ for some integer a'', which also gives $\ker(q) \simeq \mathcal{A}^{a'}$ with $a' = a - a''$. By the same argument, composing the injection $\mathcal{F}' \hookrightarrow \mathcal{F}$ with p gives a map j whose image is $\mathcal{B}^{b'}$, for some integer b', and whose cokernel is then $\mathcal{B}^{b''}$ with $b'' = b - b'$. Using that $\mathcal{A}$ and $\mathcal{B}$ are simple, we finally get a commutative exact diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{A}^{a'} & \longrightarrow & \mathcal{F}' & \xrightarrow{p} & \mathcal{B}^{b'} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \alpha'\otimes \mathrm{id}_{\mathcal{A}}} & & \downarrow & & \downarrow{\scriptstyle \beta'\otimes \mathrm{id}_{\mathcal{B}}} & & \\
0 & \longrightarrow & \mathcal{A}^{a} & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{B}^{b} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \alpha''\otimes \mathrm{id}_{\mathcal{A}}} & & \downarrow & & \downarrow{\scriptstyle \beta''\otimes \mathrm{id}_{\mathcal{B}}} & & \\
0 & \longrightarrow & \mathcal{A}^{a''} & \longrightarrow & \mathcal{F}'' & \xrightarrow{p} & \mathcal{B}^{b''} & \longrightarrow & 0
\end{array}$$

for some maps $\alpha' : \boldsymbol{k}^{\alpha'} \to \boldsymbol{k}^{\alpha}$, $\alpha'' : \boldsymbol{k}^{\alpha} \to \boldsymbol{k}^{\alpha''}$, and similarly for β. This says that there are representations $\mathcal{R}' \neq 0 \neq \mathcal{R}''$ of Υ with dimension vectors (b', a') and (b'', a'') such that $\mathcal{F}' \simeq \Phi(\mathcal{R}')$ and $\mathcal{F}'' \simeq \Phi(\mathcal{R}'')$. Also, the maps α', α'', β', β'' provide a exact sequence:

$$0 \to \mathcal{R}' \xrightarrow{(\beta',\alpha')} \mathcal{R} \xrightarrow{(\beta'',\alpha'')} \mathcal{R}'' \to 0.$$

Using the splitting map $\mathcal{F} \to \mathcal{F}'$ we see that $\mathcal{R} = \mathcal{R}' \oplus \mathcal{R}''$, which is what we needed.

Finally, we would like to show that two representations $\mathcal{R}$ and $\mathcal{S}$ of Υ are isomorphic if and only if their images via Φ are. Let $\mathcal{R}$ and $\mathcal{S}$ have dimension vectors (b, a) and (d, c). We may suppose that $\mathcal{R}$ and $\mathcal{S}$ are irreducible, so that $\mathcal{F} = \Phi(\mathcal{R})$ and $\mathcal{G} = \Phi(\mathcal{S})$ are indecomposable, by the first part of the proof. Take an isomorphism $\phi : \mathcal{F} \to \mathcal{G}$. Composing ϕ on the left with the injection $\mathcal{A}^a \to \mathcal{F}$ and on the right with the projection $\mathcal{G} \to \mathcal{B}^b$, we get a map ϕ_0. We distinguish two cases according to whether ϕ_0 is zero or not.

In the latter case, there is a summand $\mathcal{A}$ of $\mathcal{A}^a$ that maps non-trivially, hence isomorphically, to a summand $\mathcal{B}$ of $\mathcal{B}^d$. We deduce that $\mathcal{B}$ is a direct summand of $\mathcal{G}$. By the assumption on irreducibility of $\mathcal{S}$, $\mathcal{G} \simeq \mathcal{B}$, which gives the conclusion.

In the former case, we get a map $\alpha_{\mathcal{A}} : \mathcal{A}^a \to \mathcal{A}^c$ inducing an exact commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{A}^{a} & \xrightarrow{i} & \mathcal{F} & \xrightarrow{p} & \mathcal{B}^{b} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \alpha_{\mathcal{A}}} & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \beta_{\mathcal{B}}} & & \\
0 & \longrightarrow & \mathcal{A}^{c} & \longrightarrow & \mathcal{G} & \longrightarrow & \mathcal{B}^{d} & \longrightarrow & 0
\end{array}$$

Hence $\mathrm{coker}(\alpha_{\mathcal{A}}) \simeq \mathcal{A}^{c-a} \simeq \mathcal{B}^{b-d} \simeq \ker(\beta_{\mathcal{B}})$. Then $a \leq c$ and $d \leq b$. But using ϕ^{-1} we get the opposite inequalities, which implies that $\alpha_{\mathcal{A}}$ and $\beta_{\mathcal{B}}$ are isomorphisms. Again $\alpha_{\mathcal{A}} = \alpha \otimes \mathrm{id}_{\mathcal{A}}$ and $\beta_{\mathcal{B}} = \beta \otimes \mathrm{id}_{\mathcal{B}}$. It follows that (β, α) induces an isomorphism $\mathcal{R} \to \mathcal{S}$.

It remains to prove ii). To check this, consider a commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{A} & \xrightarrow{i} & \mathcal{D} & \xrightarrow{p} & \mathcal{B} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle \lambda} & & \downarrow{\scriptstyle 0} & & \\
0 & \longrightarrow & \mathcal{A} & \xrightarrow{i'} & \mathcal{D}' & \xrightarrow{p'} & \mathcal{B} & \longrightarrow & 0
\end{array}$$

Since $p' \circ \lambda = 0$, we have $\mathrm{Im}\,\lambda \subseteq \mathcal{A}$. But $\lambda i = 0$ implies that λ factors as:

$$\begin{array}{ccc}
\mathcal{D} & \xrightarrow{\quad\lambda\quad} & \mathcal{A} \\
 & \searrow \qquad \nearrow{\scriptstyle \bar{\lambda}} & \\
 & \mathcal{B} = \mathcal{D}/\mathcal{A} &
\end{array}$$

If $\lambda \neq 0$, this would give a nonzero map $\bar{\lambda} : \mathcal{B} \to \mathcal{A}$, contradicting $\mathrm{Hom}_X(\mathcal{B}, \mathcal{A}) = 0$. With this in mind, we deduce the injectivity of the natural map:

$$\mathrm{Hom}_{\Upsilon}(\mathcal{R}, \mathcal{S}) \to \mathrm{Hom}_X(\Phi(\mathcal{R}), \Phi(\mathcal{S})). \tag{3.9}$$

As for surjectivity, given a morphism $\mu : \Phi(\mathcal{R}) \to \Phi(\mathcal{S})$, we compose μ on one side with the projection $\Phi(\mathcal{S}) \to \mathcal{B}^d$, and with the injection $\mathcal{A}^a \to \Phi(\mathcal{R})$ on the other side. We obtain thus a map $\mathcal{A}^a \to \mathcal{B}^d$, which must vanish since $\mathrm{Hom}_X(\mathcal{A},\mathcal{B}) = 0$. We deduce that μ defines maps $\mathcal{A}^a \to \mathcal{A}^b$ and $\mathcal{B}^c \to \mathcal{B}^d$, which must be of the form $\beta \otimes \mathrm{id}_\mathcal{A}$ and $\alpha \otimes \mathrm{id}_\mathcal{B}$ by the assumption that $\mathcal{A}$ and $\mathcal{B}$ are simple. The pair (β, α) defines a morphism $\mathcal{R} \to \mathcal{S}$ whose image via (3.9) is μ. □

We deduce a criterion for an ACM variety being strictly CM-wild.

Corollary 3.1. *In the hypothesis of Theorem A, case i), resp. case ii), we have:*

i) if $w \geq 3$ and $\mathcal{A}$ and $\mathcal{B}$ are ACM, then X is CM-wild, resp. strictly CM-wild;
ii) if moreover $\mathcal{A}$ and $\mathcal{B}$ are Ulrich, then X is Ulrich wild, resp. strictly Ulrich wild.

Proof. By construction of the functor Φ of Theorem A, the sheaf $\Phi(\mathcal{R})$ associated with a representation $\mathcal{R}$ of Υ is ACM (respectively, Ulrich) if $\mathcal{A}$ and $\mathcal{B}$ are ACM (respectively, Ulrich).

The composition of Φ with the equivalence $\mathbf{ACM}_X \simeq \mathbf{MCM}_{R,0}$ gives the statement in case ii). For case i), we further compose Φ with the inclusion $\mathbf{MCM}_{R,0} \to \mathbf{MCM}_R$. The resulting functor is a representation embedding by Remark 2.6. To see this, we denote by F and F' the R-modules associated with $\Phi(\mathcal{R})$ and $\Phi(\mathcal{R}')$. We claim that an isomorphism $F \to F'$ of graded R-modules must have degree 0.

Indeed, given an injective morphism $F \to F'$ of degree t, that is, an injective map $F \to F'(t)$, assuming $t < 0$ and letting A and B be the R-modules associated with $\mathcal{A}$ and $\mathcal{B}$, we see that any submodule A of E maps to zero in $A(t)$ by semistability of $\mathcal{A}$. Also, A maps to zero in $B(t)$ because choosing an injection $B(t) \hookrightarrow B$ we would get a map $A \to B$ which is not an isomorphism. So the map $F \to F'(t)$ cannot be injective, namely $t \geq 0$. Using the inverse $F' \to F$ we see that $t \geq 0$, so finally $t = 0$.

The same argument applies to idempotents of F and shows that any non-trivial splitting of F into R-modules takes place in $\mathbf{MCM}_{R,0}$. This shows that the functor $\mathbf{Rep}_\Upsilon \to \mathbf{MCM}_R$ is a representation embedding. □

Remark 3.2. The hypothesis $\mathrm{Hom}_X(\mathcal{B},\mathcal{A}) = 0$ in Theorem A, case ii), is necessary. If $\mathrm{Hom}_X(\mathcal{B},\mathcal{A}) \neq 0$ we could indeed consider the map:

$$\phi : \mathcal{D} \xrightarrow{p} \mathcal{B} \to \mathcal{A} \xrightarrow{i} \mathcal{D}$$

This map ϕ is not zero, and makes the following diagram commute:

$$\begin{array}{ccccccccc} 0 & \to & \mathcal{A} & \xrightarrow{i} & \mathcal{D} & \xrightarrow{p} & \mathcal{B} & \to & 0 \\ & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle 0} & & \\ 0 & \to & \mathcal{A} & \xrightarrow{i'} & \mathcal{D} & \xrightarrow{p'} & \mathcal{B} & \to & 0 \end{array}$$

Notice, however that any ϕ fitting in such a diagram will be nilpotent.

As an explicit example, let $X \subset \mathbb{P}^{m+1}$ be a hypersurface of degree d and $Z \subset X$ be an arithmetically Gorenstein (which is to say, $\boldsymbol{k}[Z]$ is a graded Gorenstein ring) subscheme of codimension two and index i_Z, where:

$$i_Z = \max\{s \in \mathbb{Z} \mid \mathrm{H}^{m-1}(X, \mathcal{I}_{Z|X}(s)) \neq 0\}.$$

Define $e = i_Z + m + 2 - d$ and assume that $e < 0$ so that $\mathrm{Hom}_X(\mathcal{I}_{Z|X}(e), \mathcal{O}_X) \neq 0$. Let $\mathcal{D}$ be the sheaf fitting as the middle term of the non-trivial extension of $\mathcal{I}_{Z|X}(e)$ by $\mathcal{O}_X$. (One can show that this extension exists and is unique up to a nonzero scalar, by the definition of e and by Serre duality.)

Whenever Z is not a complete intersection inside X, $\mathcal{D}$ is indecomposable. Anyway $\mathcal{D}$ is an ACM sheaf of rank 2 over X which is never simple, as it always admits a nonzero nilpotent endomorphism. The conclusion of Theorem A fails in this case.

4. Stable syzygies of Ulrich modules

Let $X \subset \mathbb{P}^n$ be a closed ACM subscheme of dimension $m \geq 1$, let Y be a general linear section of X of codimension $c < m$. Set $T = \boldsymbol{k}[Y]$, $R = \boldsymbol{k}[X]$ and write ω_Y for the dualizing sheaf of Y. The ideal $I_{Y|X}$ of

Y in X is generated by a regular sequence of linear forms of R of length c. Looking at a finitely generated graded module E over T as a graded module over R, we take its minimal graded R-free resolution:

$$0 \leftarrow E \leftarrow F_0 \xleftarrow{d_1} F_1 \leftarrow \cdots \leftarrow F_{\ell-1} \xleftarrow{d_\ell} F_\ell \leftarrow \cdots \tag{4.1}$$

Write $\mathbf{\Omega}_R^\ell(E)$ for the ℓ^{th} syzygy of E over R, by which we mean $\mathbf{\Omega}_R^\ell(E) = \mathrm{Im}(d_\ell)$. It is well-known that, if E is MCM over T, then $\mathbf{\Omega}_R^\ell(E)$ is MCM over R for $\ell \geq c$.

Let $\underline{\mathbf{MCM}}_R$ be the stable category of graded maximal Cohen-Macaulay (MCM) modules over R. Given E, E' in $\mathbf{MCM}_R$, we write $\underline{\mathrm{Hom}}_R(E,E')$ for the morphisms in this category, namely the morphisms from E to E', modulo the ideal of morphisms that factor through a free R-module. We write $\underline{\mathrm{Hom}}_R(E,E')_t$ for the graded piece of degree t of $\underline{\mathrm{Hom}}_R(E,E')$. We will also use the notation $\underline{\mathbf{MCM}}_{R,0}$, the stable category where we take $\underline{\mathrm{Hom}}_R(E,E')_0$ as set of morphisms from E to E'. We write Π for the stabilization functor:

$$\Pi : \mathbf{MCM}_R \to \underline{\mathbf{MCM}}_R.$$

For $\ell \geq c$, we have also the ℓ^{th} syzygy stable functor:

$$\begin{aligned} \underline{\mathbf{\Omega}}^\ell : \mathbf{MCM}_T &\to \underline{\mathbf{MCM}}_R, \\ E &\mapsto \Pi \circ \mathbf{\Omega}_R^\ell(E). \end{aligned}$$

The following theorem is the center of this section and will play a major role throughout the rest of the paper.

Theorem B. *Let $X \subset \mathbb{P}^n$ be a closed non-degenerate ACM subscheme of dimension $m \geq 1$. Assume that X is not of minimal degree, namely,* $\deg(X) > n - m + 1$. *Then the restriction to $\mathbf{Ulr}_{T,0}$ of the c^{th} stable syzygy functor provides a fully faithful embedding:*

$$\underline{\mathbf{\Omega}}^c : \mathbf{Ulr}_{T,0} \to \underline{\mathbf{MCM}}_{R,0}.$$

We start with a lemma that characterizes varieties of minimal degree by a negativity condition on the canonical sheaf.

Lemma 4.1. *Let $X \subset \mathbb{P}^n$ be a closed ACM subscheme of dimension $m \geq 1$. Then X has minimal degree* $\deg(X) = n - m + 1$ *if and only if* $\mathrm{H}^0(X, \omega_X(m-1)) = 0$.

Proof. Without loss of generality, we may assume that $\boldsymbol{k}$ is algebraically closed. Let us work by induction on m. For $m = 1$, the statement holds as an ACM subscheme $X \subset \mathbb{P}^n$ has minimal degree if and only if the sectional genus of X is zero, see the discussion at §7. For $m \geq 2$, if Y is a hyperplane section of X, the adjunction formula gives an exact sequence

$$0 \to \omega_X(m-2) \to \omega_X(m-1) \to \omega_Y(\dim(Y)-1) \to 0$$

Because X is ACM and $m \geq 2$ we get $\mathrm{H}^1_*(X, \omega_X) = 0$. So, since by the induction hypothesis the statement holds for Y, taking global sections of the above sequence we see that it also holds for X.

For a proof in the language of modules, note that the dual of the minimal S-resolution of $\boldsymbol{k}[X]$ provides a resolution of the canonical module K_X (see [Mig98, Remark 1.2.4]) and therefore, by [EGHP06, Theorem 0.4], X is of minimal degree if and only if it is 2-regular if and only if $\mathrm{H}^0(X, \omega_X(m-1)) = (K_X)_{m-1} = 0$. □

The following lemma will be one of the keystones of our analysis. Given a finitely generated graded R-module M, we write $\langle M_{\leq d} \rangle$ for the graded submodule of M generated by the elements of degree at most d of M. We also write M^* for the dual $\mathrm{Hom}_R(M,R)$.

Lemma 4.2. *Fix the hypothesis as in Theorem B, let L be an Ulrich module over T, and set $M = \mathbf{\Omega}_R^c(L)$.*

Then we have a functorial exact sequence:

$$0 \to \langle M^*_{\leq 1-c} \rangle \to M^* \to \mathrm{Hom}_T(L, T(c)) \to 0.$$

Proof. Recall that L is generated in a single degree, and that the number of minimal generators of L equals $\alpha_0 = \deg(X)\mathrm{rk}(L)$. In other words, we can assume $F_0 \simeq R^{\alpha_0}$. By the minimality of the resolution, we have, for $i \geq 1$:

$$F_\ell \simeq \bigoplus_{j \geq \ell} R(-j)^{\alpha_{\ell,j}}, \qquad \text{for some integers } \alpha_{\ell,j}.$$

Also, X is not a linear space, so neither is Y, so that L has no free summands.

We are going to apply the functor $\mathrm{Hom}_R(-,R)$ to (4.1). We have:

- $\mathrm{Ext}^i_R(L,R) = 0$ for $i = 0,\dots,c-1$. This is due to the fact that the smallest integer l such that $\mathrm{Ext}^l_R(L,R) \neq 0$ is the grade (see [BH98, Definition 1.2.11 and ff.]), and

$$\mathrm{grade}(M) := \mathrm{grade}(\mathrm{Ann}_R(L),R) = \mathrm{grade}(I_{Y|X},R) = c.$$

- $\mathrm{Ext}^c_R(L,R) \simeq \mathrm{Hom}_T(L,T(c))$, which follows from noticing that $I_{Y|X}$ is generated by a regular sequence of linear forms of length c and applying inductively the graded version of Rees' Theorem (see [Rot09, Theorem 8.34]).

From the previous remarks, we obtain the long exact sequence:

$$0 \to F_0^* \to \cdots \to F_{c-2}^* \xrightarrow{d_{c-1}^*} F_{c-1}^* \xrightarrow{\pi} M^* \to \mathrm{Hom}_T(L,T(c)) \to 0. \tag{4.2}$$

Now comes the main point, namely that $\mathrm{Hom}_T(L,T)_1 = 0$. To see this, recall the isomorphism $\mathrm{Hom}_T(L,T)_1 \simeq \mathrm{Hom}_Y(\mathcal{L},\mathcal{O}_Y(1))$ and that $\mathcal{L}$ is Ulrich on Y as well as $\mathcal{H}om_Y(\mathcal{L},\omega_Y)$. Then, by Serre duality and [ESW03, Proposition 2.1] we get:

$$\mathrm{Hom}_Y(\mathcal{L},\omega_Y(m-c)) \simeq \mathrm{H}^{m-c}(Y,\mathcal{L}(c-m))^* = 0.$$

Note that, by the assumption and Lemma 4.1, $\mathrm{H}^0(Y,\omega_Y(m-c-1)) \neq 0$ and therefore we get an embedding:

$$\mathcal{O}_Y(1) \hookrightarrow \omega_Y(m-c).$$

Therefore, $\mathrm{Hom}_Y(\mathcal{L},\omega_Y(m-c)) = 0$ implies $\mathrm{Hom}_Y(\mathcal{L},\mathcal{O}_Y(1)) = 0$.

We have thus established that $\mathrm{Hom}_T(L,T(c))$ contains no element of degree $\leq 1-c$. Also, we may write:

$$F_{c-1}^* = R(c-1)^{\alpha_{c-1,c-1}} \oplus R(c)^{\alpha_{c-1,c}} \oplus \cdots$$

Then, (4.2) says that F_{c-1}^* generates all the elements of M^* of degree at most $1-c$, that is, the image of π is the submodule $\langle M^*_{\leq 1-c}\rangle$ of M^*. This is clearly functorial, and the lemma is proved. □

Proof of Theorem B. Let L and N be two Ulrich modules over T. Our goal will be to describe two mutually inverse maps:

$$\mathrm{Hom}_T(L,N)_0 \leftrightarrows \underline{\mathrm{Hom}}_R(\mathbf{\Omega}^c_R(L),\mathbf{\Omega}^c_R(N))_0.$$

Set $M = \mathbf{\Omega}^c_R(L)$ and $P = \mathbf{\Omega}^c_R(N)$. First, let $\varphi : L \to N$ belong to $\mathrm{Hom}_T(L,N)_0$. Consider the minimal graded free resolutions of L and N over R and choose a lifting of φ to these resolutions:

$$\begin{array}{ccccccccccc} 0 & \leftarrow & L & \leftarrow & F_0 & \leftarrow & \cdots & \leftarrow & F_{c-1} & \leftarrow & M \\ & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \varphi_0} & & & & \downarrow{\scriptstyle \varphi_{c-1}} & & \vdots{\scriptstyle \tilde{\varphi}} \\ 0 & \leftarrow & N & \leftarrow & G_0 & \leftarrow & \cdots & \leftarrow & G_{c-1} & \leftarrow & P \end{array} \tag{4.3}$$

The morphism $\tilde{\varphi}$ induced on the c^{th} syzygy modules gives the class $\bar{\varphi}$ in $\underline{\mathrm{Hom}}_R(M,P)_0$. This does not depend on the choice of the lifting φ_i, as any other choice would provide a map $\tilde{\varphi}'$ such that $\tilde{\varphi}-\tilde{\varphi}'$ factors through a free module.

Conversely, given $\bar{\psi} \in \underline{\mathrm{Hom}}_R(M,P)_0$, we choose a representative $\psi : M \to P$ with dual $\psi^* : P^* \to M^*$. Since ψ^* is homogeneous of degree 0, it maps elements of degree at most $1-c$ in P^* to elements of degree at most $1-c$ in M^*. By Lemma 4.2 we obtain a diagram:

$$\begin{array}{ccccccccc} 0 & \to & \langle M^*_{\le 1-c}\rangle & \to & M^* & \to & \mathrm{Hom}_T(L,T(c)) & \to & 0 \\ & & \uparrow \psi^* & & \uparrow \psi^* & & \vdots\uparrow \hat{\psi} & & \\ 0 & \to & \langle P^*_{\le 1-c}\rangle & \to & P^* & \to & \mathrm{Hom}_T(N,T(c)) & \to & 0 \end{array} \tag{4.4}$$

We wish to associate with $\bar{\psi}$ the morphism $\hat{\psi}^* : L \to N$. To do this, we have to check that $\hat{\psi}$ does not depend on the choice of the representative ψ of $\bar{\psi}$. By definition any other representative differs from ψ by a map $\zeta : M \to P$ that factors through a free module, which we call F, which means $\zeta = \zeta_2\zeta_1$ with $\zeta_1 : M \to F$ and $\zeta_2 : F \to P$. Therefore ζ^* factors through F^* and again by Lemma 4.2 we get the commutative diagram:

$$\begin{array}{ccccccccc} 0 & \to & \langle M^*_{\le 1-c}\rangle & \to & M^* & \to & \mathrm{Hom}_T(L,T(c)) & \to & 0 \\ & & \uparrow \zeta_1^* & & \uparrow \zeta_1^* & & \vdots\uparrow & & \\ 0 & \to & \langle F^*_{\le 1-c}\rangle & \to & F^* & & \hat{\zeta}\ \vdots & & \\ & & \uparrow \zeta_2^* & & \uparrow \zeta_2^* & & \vdots & & \\ 0 & \to & \langle P^*_{\le 1-c}\rangle & \to & P^* & \to & \mathrm{Hom}_T(N,T(c)) & \to & 0 \end{array}$$

Call G the quotient $F^*/\langle F^*_{\le 1-c}\rangle$. The diagram says that $\hat{\zeta}$ factors through G.

Now observe that G is a free R-module. Indeed, any direct summand of F takes the form $R(a)$ for some $a \in \mathbb{Z}$, and:

$$\langle R(a)_{\le 1-c}\rangle = \begin{cases} R(a), & \text{if } a \ge c-1, \\ 0, & \text{if } a < c-1. \end{cases} \tag{4.5}$$

Therefore G is the direct sum of all summands $R(a)$ of F^* with $a < c-1$, hence G is a graded free R-module. But $\mathrm{Hom}_T(N,T(c))$ is a torsion R-module. So it admits no non-trivial morphism with target in G, and therefore $\hat{\zeta} = 0$.

Let us check now that these maps are mutually inverse. Given $\varphi \in \mathrm{Hom}_T(L,N)_0$, we consider the representative $\psi = \tilde{\varphi}$ of the class $\bar{\varphi}$. Dualizing (4.3) we obtain a commutative diagram:

$$\begin{array}{ccccccccc} \cdots & \to & F^*_{c-1} & \to & M^* & \to & \mathrm{Hom}_T(L,T(c)) & \to & 0 \\ & & \uparrow \varphi^*_{c-1} & & \uparrow \psi^* & & \vdots\uparrow \varphi^* & & \\ \cdots & \to & G^*_{c-1} & \to & P^* & \to & \mathrm{Hom}_T(N,T(c)) & \to & 0 \end{array}$$

This diagram is the extension of (4.4) to a minimal resolution of $\langle P^*_{\le 1-c}\rangle$ and $\langle M^*_{\le 1-c}\rangle$. This says that $\hat{\psi} = \varphi^*$, so $\hat{\psi}^* = \varphi$.

Conversely, let ψ be a representative of $\bar{\psi} \in \underline{\mathrm{Hom}}_R(M,P)_0$ and set $\varphi = \hat{\psi}^*$. Let us lift the map $\langle P^*_{\le 1-c}\rangle \to \langle M^*_{\le 1-c}\rangle$ induced by ψ to the minimal graded free resolutions of these modules and dualize to obtain:

$$\begin{array}{ccccccccccccc} 0 & \leftarrow & L & \leftarrow & F_0 & \leftarrow & \cdots & \leftarrow & F_{c-1} & \leftarrow & M & \leftarrow & 0 \\ & & \downarrow \varphi & & \downarrow \psi_0 & & & & \downarrow \psi_{c-1} & & \downarrow \psi & & \\ 0 & \leftarrow & N & \leftarrow & G_0 & \leftarrow & \cdots & \leftarrow & G_{c-1} & \leftarrow & P & \leftarrow & 0 \end{array}$$

Then ψ is induced by a lifting of $\varphi : L \to N$ to the minimal resolutions of L and N. Our proof is thus complete. □

We isolate the following consequence of Lemma 4.2.

Lemma 4.3. *If L is an Ulrich module L over T, then $\mathbf{\Omega}^c_R(L)$ has no free summands.*

Proof. Suppose $\mathbf{\Omega}^c_R(L) = M \oplus F$, with F a nonzero direct summand. Lemma 4.2 gives:

$$0 \to \langle M_{\le 1-c}\rangle \oplus \langle F_{\le 1-c}\rangle \xrightarrow{\pi} M \oplus F \to \mathrm{Hom}_T(L, T(c)) \to 0,$$

with π block-diagonal. Then, (4.5) says that the restriction of π is an isomorphism between $\langle F_{\le 1-c}\rangle$ and F, as L has no free summand. Therefore, looking at (4.2) we see that d_c is surjective onto F, which contradicts minimality of the resolution (4.1). □

Example 4.4. Let $X \subset \mathbb{P}^{m+1}$ be a hypersurface of degree d and Y be a linear section of codimension c of X. Based on the theory of matrix factorizations developed in [Eis80], the resolution of an Ulrich module L of rank r on $T = \boldsymbol{k}[Y]$ reads:

$$0 \leftarrow L \leftarrow T^{rd} \leftarrow T(-1)^{rd} \leftarrow T(-d)^{rd} \leftarrow \cdots \tag{4.6}$$

Since $L(-d) \simeq \ker(T(-1)^{rd} \to T^{rd})$, this yields a resolution:

$$0 \leftarrow \mathrm{Hom}_T(L, T(d-1)) \leftarrow T^{rd} \leftarrow T(-1)^{rd} \leftarrow T(-d)^{rd} \leftarrow \cdots \tag{4.7}$$

Combining (4.6) with the Koszul resolution of Y in X we get a resolution over $R = \boldsymbol{k}[X]$:

$$0 \leftarrow L \leftarrow R^{rd} \leftarrow R(-1)^{rd(c+1)} \leftarrow R(-2)^{rd(c+\binom{c}{2})} \oplus R(-d)^{rd} \leftarrow \cdots \tag{4.8}$$

The k^{th} term F_k of this resolution looks as follows (here $\varepsilon \in \{0,1\}$):

$$F_k = \bigoplus_{2h+\varepsilon+j=k} R(-(j+hd+\varepsilon))^{\binom{c}{j}rd}.$$

Let $M = \mathbf{\Omega}^c_R(L)$. The resolution of the dualized syzygy M^* starts with:

$$\cdots \to R(c-d)^{rd(c+1)} \oplus F^*_{c-2} \to R(c-d+1)^{rd} \oplus F^*_{c-1} \to M^* \to 0.$$

Now we may remove from this resolution the dual of the truncation at $M = \mathbf{\Omega}^c_R(L)$ of (4.8), which is to say, by Lemma 4.2, the resolution of $\langle M^*_{1-c}\rangle$. The residual strand recovers precisely (4.7), twisted by $R(c-d+1)$. The two strands of the resolution do not mix if $d > 2$.

Remark 4.5. Theorem B is sharp, in the sense that it fails in general for ACM closed schemes $X \subset \mathbb{P}^n$ of minimal degree. Take for instance $\mathrm{char}(\boldsymbol{k}) \neq 2$, choose a positive integer k and let X be a smooth quadric hypersurface in $\mathbb{P}^{2k+1}$. Let Y be a smooth hyperplane section of X. It is well-known that, over $T = \boldsymbol{k}[Y]$ there is a unique indecomposable ACM (and Ulrich) module L with no free summands, namely the module associated with the spinor bundle. This module has rank 2^{k-1}. On the other hand $R = \boldsymbol{k}[X]$ supports exactly two non-isomorphic ACM (and Ulrich) modules F' and F'', which have both rank 2^{k-1}. There is a short exact sequence

$$0 \leftarrow L \leftarrow R^{2^k} \leftarrow F' \oplus F'' \leftarrow 0, \qquad \text{hence } \Omega^1_R(L) \simeq F' \oplus F''.$$

Therefore, the functor $\underline{\mathbf{\Omega}}^1$ is not even a representation embedding as it sends indecomposable modules to decomposable ones. The condition of preserving non-isomorphy of modules also fails in general, as choosing $n = 2k+2$ we get L' and L'' non-isomorphic spinor modules on T, but both $\Omega^1_R(L')$ and $\Omega^1_R(L'')$ are isomorphic to the single spinor module on X.

For ACM schemes of minimal degree, even though Theorem B cannot be applied, the following proposition shows that the syzygy functor enjoys a somehow opposite kind of nice feature, namely it preserves the property of being Ulrich.

Proposition 4.6. *Let $X \subset \mathbb{P}^n$ be a variety of minimal degree $d = \deg(X) = n - m + 1$ and dimension $m \ge 1$. Let $Y \subset X$ be a general linear section of codimension $c < m$. Then, for any Ulrich module L over $T = \boldsymbol{k}[Y]$, the c^{th} syzygy module $\Omega^c_R(L)$ is an Ulrich module over $R = \boldsymbol{k}[X]$.*

Proof. By induction, we can suppose that $c = 1$. Therefore, given an Ulrich module L over T, since we already know that $\Omega^1_R(L)$ is MCM over R, we just need to show that it is minimally generated by $d\,\mathrm{rk}(\Omega^1_R(L))$ elements. From [ESW03, Prop. 2.1], we see that the beginning of the minimal resolution of L over T has the following form

$$0 \leftarrow L \leftarrow T^{rd} \leftarrow T(-1)^{rd(n-m)} \leftarrow \cdots$$

Therefore, merging it with the minimal resolution of T over R we obtain that the minimal resolution of L over R starts

$$0 \leftarrow L \leftarrow R^{rd} \overset{d_1}{\leftarrow} R(-1)^{rd^2} \leftarrow \cdots$$

Namely, $\Omega^1_R(L) = \mathrm{Im}(d_1)$ is a MCM module over R of rank rd generated by rd^2 elements of the same degree. In other words, $\Omega^1_R(L)$ is Ulrich. □

5. CM-wildness from syzygies of Ulrich extensions

Let us fix the setup for this section. Let $X \subset \mathbb{P}^n$ be an ACM subscheme of dimension $m \geq 1$, put $R = \boldsymbol{k}[X]$ and let Y be a linear section of X of codimension $c < m$. Let $\mathcal{A}$ and $\mathcal{B}$ be two simple Ulrich sheaves on Y. Set $W = \mathrm{Ext}^1_Y(\mathcal{B}, \mathcal{A})$, $w = \dim_{\boldsymbol{k}} W$. Here we want to prove the following fundamental result.

Theorem C. *Assume $w \geq 3$, $X \subset \mathbb{P}^n$ is not of minimal degree and suppose that $\mathcal{A}$ and $\mathcal{B}$ satisfy hypothesis i) or ii) of Theorem A. Then X is of wild CM-type.*

Let us assume for the time being that $w \neq 0$ and write $\Upsilon = \Upsilon_w$. Over $Y \times \mathbb{P}(W^*)$, there is a universal extension:

$$0 \to \mathcal{A} \boxtimes \mathcal{O}_{\mathbb{P}(W^*)} \to \mathcal{U} \to \mathcal{B} \boxtimes \mathcal{O}_{\mathbb{P}(W^*)}(-1) \to 0.$$

Take the sheafified minimal graded free resolutions of $\mathcal{A}$ and $\mathcal{B}$ as $\mathcal{O}_X$-modules, pull-back via p to $X \times \mathbb{P}(W^*)$, and use the mapping cone construction to build a minimal graded free resolution of $\mathcal{U}$ over $X \times \mathbb{P}(W^*)$:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathcal{A} \boxtimes \mathcal{O}_{\mathbb{P}(W^*)} & \longrightarrow & \mathcal{U} & \longrightarrow & \mathcal{B} \boxtimes \mathcal{O}_{\mathbb{P}(W^*)}(-1) & \longrightarrow & 0 \\
 & & \uparrow & & {\scriptstyle d_0}\uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathcal{F}_0 \boxtimes \mathcal{O}_{\mathbb{P}(W^*)} & \longrightarrow & \mathcal{H}_0 & \longrightarrow & \mathcal{G}_0 \boxtimes \mathcal{O}_{\mathbb{P}(W^*)}(-1) & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & \vdots & & \vdots & & \vdots & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathcal{F}_{c-1} \boxtimes \mathcal{O}_{\mathbb{P}(W^*)} & \longrightarrow & \mathcal{H}_{c-1} & \longrightarrow & \mathcal{G}_{c-1} \boxtimes \mathcal{O}_{\mathbb{P}(W^*)}(-1) & \longrightarrow & 0 \\
 & & \uparrow & & {\scriptstyle d_c}\uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathcal{F}_c \boxtimes \mathcal{O}_{\mathbb{P}(W^*)} & \longrightarrow & \mathcal{H}_c & \longrightarrow & \mathcal{G}_c \boxtimes \mathcal{O}_{\mathbb{P}(W^*)}(-1) & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

Here, $\mathcal{H}_i = \mathcal{F}_i \boxtimes \mathcal{O}_{\mathbb{P}(W^*)} \oplus \mathcal{G}_i \boxtimes \mathcal{O}_{\mathbb{P}(W^*)}(-1)$. Set:

$$\mathcal{V} = \mathrm{Im}(d_c).$$

Then we consider:

$$\Phi_{\mathcal{V}} = \mathbf{R}p_*(q^*(-) \otimes \mathcal{V}) : \mathbf{D}^b(\mathbf{Coh}_{\mathbb{P}(W^*)}) \to \mathbf{D}^b(\mathbf{Coh}_X).$$

Lemma 5.1. *Let $T = \boldsymbol{k}[Y]$, and set L and N for the modules of global sections of $\mathcal{A}$ and $\mathcal{B}$. Let $\mathcal{M}$ and $\mathcal{N}$ be the sheafifications of $\boldsymbol{\Omega}^c_R(L)$ and $\boldsymbol{\Omega}^c_R(N)$. Then:*

$$\Phi_{\mathcal{V}}(\mathcal{O}_{\mathbb{P}(W^*)}) \simeq \mathcal{M}, \qquad \Phi_{\mathcal{V}}(\Omega_{\mathbb{P}(W^*)}(1)) \simeq \mathcal{N}[-1].$$

Proof. By the diagram we have an exact sequence:

$$0 \to \mathcal{M} \boxtimes \mathcal{O}_{\mathbb{P}(W^*)} \to \mathcal{V} \to \mathcal{N} \boxtimes \mathcal{O}_{\mathbb{P}(W^*)}(-1) \to 0,$$

We get the conclusion by using Künneth formula, see [Huy06, §3.3], and the vanishing of cohomology of $\mathcal{O}_{\mathbb{P}(W^*)}$ and $\Omega_{\mathbb{P}(W^*)}$ except in degree 0 and 1, respectively. □

Now consider the equivalence Ξ of (3.1). Then the restriction of $\Phi_{\mathcal{V}} \circ \Xi$ to $\mathbf{Rep}_\Upsilon$, composed with the global sections functor, gives an exact functor:

$$\Psi_0 : \mathbf{Rep}_\Upsilon \to \mathbf{MCM}_{R,0}.$$

We denote by Ψ the induced functor $\mathbf{Rep}_\Upsilon \to \mathbf{MCM}_R$. Theorem C amounts to the next result.

Theorem 5.2. *If $X \subset \mathbb{P}^n$ is not of minimal degree and $\mathcal{A}$ and $\mathcal{B}$ are Ulrich, then $\Psi : \mathbf{Rep}_\Upsilon \to \mathbf{MCM}_R$ is a representation embedding. So if $\mathcal{A}$ and $\mathcal{B}$ satisfy hypothesis i) or ii) of Theorem A and $w \geq 3$, X is of wild CM representation type.*

Proof. By construction we have the commutative diagram of functors:

$$\begin{array}{ccc} \mathbf{Rep}_\Upsilon & \xrightarrow{\Psi_0} & \mathbf{MCM}_{R,0} \\ \downarrow{\scriptstyle \Phi} & & \downarrow{\scriptstyle \Pi} \\ \mathbf{Ulr}_{T,0} & \xrightarrow{\underline{\boldsymbol{\Omega}}^c_R} & \underline{\mathbf{MCM}}_{R,0} \end{array}$$

We proved in Theorem B that $\underline{\boldsymbol{\Omega}}^c_R$ is fully faithful, and in Theorem A that Φ is also fully faithful. So the same happens to $\underline{\boldsymbol{\Omega}}^c_R \circ \Phi$ and hence to $\Pi \circ \Psi_0$.

Therefore, if $\mathcal{R}$ and $\mathcal{S}$ are two representations of Υ such that $\Psi_0(\mathcal{R}) \simeq \Psi_0(\mathcal{S})$, we still have an isomorphism $\Pi(\Psi_0(\mathcal{R})) \simeq \Pi(\Psi_0(\mathcal{S}))$ and thus $\mathcal{R} \simeq \mathcal{S}$ by full faithfulness.

Moreover, if $\Psi_0(\mathcal{R})$ is decomposable, then $\mathrm{Hom}_R(\Psi_0(\mathcal{R}), \Psi_0(\mathcal{R}))_0$ contains a non-trivial idempotent ψ. The class $\bar{\psi}$ is also an idempotent, which is trivial if and only if the summand of $\Psi_0(\mathcal{R})$ associated with ψ is free. But this cannot happen by Lemma 4.3. Also, again by full faithfulness of $\Pi \circ \Psi_0$, $\bar{\psi}$ corresponds to a non-trivial idempotent of $\mathcal{R}$, so $\mathcal{R}$ is also decomposable. This finishes the proof that Ψ_0 is a representation embedding.

The consequence that Ψ is also a representation embedding follows from the argument of Corollary 3.1. Therefore, X of wild CM-type. □

Example 5.3. The first class of varieties where it is a priori unknown how to construct large families of ACM bundles is given by general cubic hypersurfaces of dimension $m \geq 4$. We can do this with Theorem C. Indeed, start with a cubic hypersurface X in $\mathbb{P}^{m+1}$, sufficiently general to admit a smooth surface section Y. Then we may take $\mathcal{A} = \mathcal{O}_Y(A)$ and $\mathcal{B} = \mathcal{O}_Y(B)$, where A and B are twisted cubics in Y meeting at 5 points, see [Fae08] : these will satisfy the assumptions of Theorem C. Indeed, $\mathrm{H}^0(\omega_Y(m-c-1)) = \mathrm{H}^0(\mathcal{O}_Y) = \boldsymbol{k}$ and, by Riemann-Roch:

$$\dim_{\boldsymbol{k}} \mathrm{Ext}^1(\mathcal{B}, \mathcal{A}) = -\chi(\mathcal{O}_Y(A-B)) = 3$$

In the next section we will see how to deal in a similar fashion with any variety besides the non-wild varieties listed in the main result.

6. Wildness of non-integral projective schemes

In this short section we pay attention to the case of non-integral projective schemes. We first focus on the case of reducible subschemes. For them, the next lemma is our starting point. In order to use it one starts with an ACM subscheme X, recalls that X is thus equidimensional, takes X_0 to be a union of components of X, and studies the representation type of X in terms of that of X_0. Let us put $R = \boldsymbol{k}[X]$ and $R_0 = \boldsymbol{k}[X_0]$.

Lemma 6.1. *Let $X_0 \subset X \subset \mathbb{P}^n$ be closed subschemes of the same dimension m and suppose that X_0 is CM-wild. Then X is CM-wild.*

Proof. The inclusion $X_0 \subset X$ gives a surjective morphism of rings $R \to R_0$ that bestows a structure of R-module to any R_0-module. Because X_0 and X have the same dimension, any MCM R_0-module is also an MCM R-module. Non-isomorphic R_0-modules remain non-isomorphic R-modules. Also, an indecomposable R_0-module is indecomposable as R-module. In other words, we have a representation embedding $\mathbf{MCM}_{R_0} \to \mathbf{MCM}_R$, so the lemma is proved. □

As a consequence of the previous lemma, in order to classify reducible projective schemes, it only remains to take care of reducible varieties having no CM-wild component. This is the content of the following result.

Theorem 6.2. *Let $X_1, X_2 \subset \mathbb{P}^n$ be m-dimensional closed integral varieties with $m \geq 2$. Assume that $X_1 \cap X_2$ is a Weil divisor in X_1, that X_2 is ACM and that X_1 carries an ACM sheaf. Then $X_1 \cup X_2$ is CM-wild.*

Proof. Put $X = X_1 \cup X_2$ and $Y = X_1 \cap X_2$. The surjection $\mathcal{O}_X \to \mathcal{O}_Y$ factors as $\mathcal{O}_X \to \mathcal{O}_{X_1} \to \mathcal{O}_Y$ and $\mathcal{O}_X \to \mathcal{O}_{X_2} \to \mathcal{O}_Y$ and, since $X = X_1 \cup X_2$, this induces an isomorphism of $\mathcal{O}_X$-sheaves $\mathcal{I}_{X_2|X} \simeq \mathcal{I}_{Y|X_1}$. In other words, we have an exact sequence:

$$0 \to \mathcal{I}_{Y|X_1} \to \mathcal{O}_X \to \mathcal{O}_{X_2} \to 0.$$

Let $\mathcal{F}_1$ be an ACM sheaf on X_1. Note that $\mathcal{H}om_X(\mathcal{O}_{X_2}, \mathcal{F}_1) = \mathcal{H}om_X(\mathcal{F}_1, \mathcal{O}_{X_2}) = 0$, because $\mathcal{F}_1$ and $\mathcal{O}_{X_2}$ are respectively supported on X_1 and X_2, and these varieties have no common component. So:

$$\mathrm{Hom}_X(\mathcal{O}_{X_2}, \mathcal{F}_1(q)) = \mathrm{Hom}_X(\mathcal{F}_1(q), \mathcal{O}_{X_2}) = 0, \qquad \text{for all } q \in \mathbb{Z}.$$

Also, $\mathcal{H}om_X(\mathcal{O}_X, \mathcal{F}_1) \simeq \mathcal{F}_1$. So, applying $\mathcal{H}om_X(-, \mathcal{F}_1)$ to the previous exact sequence, we get:

$$0 \to \mathcal{F}_1 \to \mathcal{H}om_X(\mathcal{I}_{Y|X_1}, \mathcal{F}_1) \to \mathcal{E}xt^1_X(\mathcal{O}_{X_2}, \mathcal{F}_1). \tag{6.1}$$

We look at $\mathcal{I}_{Y|X_1}$ as the kernel of $\mathcal{O}_{X_1} \to \mathcal{O}_Y$. Applying $\mathcal{H}om_{X_1}(-, \mathcal{F}_1)$ gives:

$$0 \to \mathcal{F}_1 \to \mathcal{H}om_{X_1}(\mathcal{I}_{Y|X_1}, \mathcal{F}_1) \to \mathcal{E}xt^1_{X_1}(\mathcal{I}_{Y|X_1}, \mathcal{F}_1) \to 0, \tag{6.2}$$

because $\mathcal{F}_1$ is locally Cohen-Macaulay. On the other hand, applying $\mathcal{H}om_{X_1}(-, \mathcal{O}_{X_1})$ gives:

$$0 \to \mathcal{O}_{X_1} \to \mathcal{H}om_{X_1}(\mathcal{I}_{Y|X_1}, \mathcal{O}_{X_1}) \to \mathcal{N}_{Y|X_1} \to 0,$$

where we used the standard identification of the sheaf $\mathcal{E}xt^1_{X_1}(\mathcal{I}_{Y|X_1}, \mathcal{O}_{X_1})$ with the normal sheaf $\mathcal{N}_{Y|X_1}$ of Y in X_1. Tensoring the previous exact sequence with $\mathcal{F}_1$ gives:

$$\cdots \to \mathcal{F}_1 \to \mathcal{H}om_{X_1}(\mathcal{I}_{Y|X_1}, \mathcal{O}_{X_1}) \otimes \mathcal{F}_1 \to \mathcal{N}_{Y|X_1} \otimes \mathcal{F}_1 \to 0. \tag{6.3}$$

Because $\mathcal{H}om_X(\mathcal{I}_{Y|X_1}, \mathcal{F}_1) \simeq \mathcal{H}om_{X_1}(\mathcal{I}_{Y|X_1}, \mathcal{F}_1)$, putting together (6.1) and (6.2) yields an inclusion $\mathcal{E}xt^1_{X_1}(\mathcal{I}_{Y|X_1}, \mathcal{F}_1) \hookrightarrow \mathcal{E}xt^1_X(\mathcal{O}_{X_2}, \mathcal{F}_1)$. Therefore, for $q \in \mathbb{N}$, we have a linear inclusion:

$$\mathrm{H}^0(X_1, \mathcal{E}xt^1_{X_1}(\mathcal{I}_{Y|X_1}, \mathcal{F}_1(q))) \hookrightarrow \mathrm{H}^0(X, \mathcal{E}xt^1_X(\mathcal{O}_{X_2}, \mathcal{F}_1(q)) \simeq \mathrm{Ext}^1_X(\mathcal{O}_{X_2}, \mathcal{F}_1(q)). \tag{6.4}$$

where the isomorphism follows from the (degenerate) local-to-global spectral sequence.

There is a dense open subset of the reduced structure over Y where Y is Cartier and $\mathcal{F}_1$ is locally free. Over such open set, the exact sequences (6.2) and (6.3) become the same and $\mathcal{N}_{Y|X_1} \otimes \mathcal{F}_1$ is locally free of positive rank. Therefore, since $\dim(Y) = m - 1 \geq 1$, the dimension of $\mathrm{H}^0(Y, \mathcal{N}_{Y|X_1} \otimes \mathcal{F}_1(q))$ grows at least linearly when $q \gg 0$. Hence the same happens to $\mathrm{H}^0(X_1, \mathcal{E}xt^1_{X_1}(\mathcal{I}_{Y|X_1}, \mathcal{F}_1(q)))$. Therefore, in view of (6.4), $\mathrm{Ext}^1_X(\mathcal{O}_{X_2}, \mathcal{F}_1(q))$ has unbounded dimension for growing q. The result now follows by applying item ii) of Theorem A with $\mathcal{B} = \mathcal{O}_{X_2}$ and $\mathcal{A} = \mathcal{F}_1(q)$ for $q \gg 0$. □

We finish this section with a foray into non-reduced schemes.

Theorem 6.3. *Let $X \subset \mathbb{P}^n$ be an m-dimensional closed subscheme containing a double structure over an integral ACM subscheme X_0 of dimension $m \geq 1$. Then X is CM-wild.*

Proof. This follows the same path as the previous lemma. We have an exact sequence:

$$0 \to \mathcal{I}_{X_0|X} \to \mathcal{O}_X \to \mathcal{O}_{X_0} \to 0.$$

Since X contains a double structure over X_0, the sheaf $\mathcal{I}_{X_0|X}$ has rank at least one as a sheaf over X_0. Applying $\mathcal{H}om_X(-, \mathcal{O}_{X_0})$ to this sequence, we get an exact sequence:

$$0 \to \mathcal{O}_{X_0} \xrightarrow{\simeq} \mathcal{O}_{X_0} \to \mathcal{H}om_X(\mathcal{I}_{X_0|X}, \mathcal{O}_{X_0}) \to \mathcal{E}xt^1_X(\mathcal{O}_{X_0}, \mathcal{O}_{X_0}).$$

Since the ideal sheaf $\mathcal{I}_{X_0|X}$ has rank at least one, the sheaf $\mathcal{E} = \mathcal{H}om_X(\mathcal{I}_{X_0|X}, \mathcal{O}_{X_0})$ is (torsion-free) of rank at least one over X_0 as well. Therefore, since $m \geq 1$, for $q \gg 0$, the dimension of $\mathrm{H}^0(X, \mathcal{E}(q))$ is unbounded, and thus also the dimension of $\mathrm{H}^0(X, \mathcal{E}xt^1_X(\mathcal{O}_{X_0}, \mathcal{O}_{X_0}(q)))$.

We use now the exact sequence of lower degree terms of the local-to-global spectral sequence, together with the fact that $\mathcal{O}_{X_0} \simeq \mathcal{H}om_X(\mathcal{O}_{X_0}, \mathcal{O}_{X_0})$. This gives an exact sequence:

$$\mathrm{Ext}^1_X(\mathcal{O}_{X_0}, \mathcal{O}_{X_0}(q)) \to \mathrm{H}^0(X, \mathcal{E}xt^1_X(\mathcal{O}_{X_0}, \mathcal{O}_{X_0}(q)) \to \mathrm{H}^2(X, \mathcal{O}_{X_0}(q))$$

Now by Serre's vanishing $\mathrm{H}^2(X, \mathcal{O}_{X_0}(q)) = 0$ for $q \gg 0$, so the dimension of $\mathrm{Ext}^1_X(\mathcal{O}_{X_0}, \mathcal{O}_{X_0}(q))$ is unbounded for $q \gg 0$. The conclusion again follows from item i) of Theorem A, applied to $\mathcal{B} = \mathcal{O}_{X_0}$ and $\mathcal{A} = \mathcal{O}_{X_0}(q)$ with $q \gg 0$. □

7. Varieties of minimal degree

Assume $\boldsymbol{k}$ is algebraically closed and $\mathrm{char}(\boldsymbol{k}) \neq 2$. Let $X \subset \mathbb{P}^n$ be a reduced closed ACM subscheme of dimension $m \geq 1$ and degree d. The subscheme X is thus linearly normal and, without loss of generality, we may assume throughout that X spans $\mathbb{P}^n$, so that X is embedded by the complete linear series of a very ample line bundle $\mathcal{O}_X(1)$. We argue on the *sectional genus p* of X, that is, the arithmetic genus of a reduced 1-dimensional linear section of X. We also introduce the *Δ-genus* of X, defined as $\Delta(X) = d - n + m - 1$. Since X is connected in codimension one (see [Har62] for the definition and the result), a theorem of Xambó (see [XD81]) generalizing the classical lower bound on d (see for example [Mum76, Corollary 5.13]), asserts that $\Delta(X) \geq 0$, or in other words $d \geq n - m + 1$. If equality is attained, X is said to be of *minimal degree.* This happens if and only if $\Delta(X) = 0$, and also if and only if $p = 0$.

Before proceeding, a few words for the non-reduced case are in order: if X is ACM but not reduced, of dimension $m \geq 1$ and of degree $n - m + 1$, it still makes sense to ask about the representation type of X. In this case, X is a 2-regular scheme (see [EGHP06] and references therein for this notion and related results) and therefore X has a non-reduced irreducible component whose reduction X_0 is integral of minimal degree (see [EGHP06, Corollary 0.8 and Theorem 0.4]); therefore X_0 is ACM, in which case X is CM-wild by Theorem 6.3.

Let us return to the reduced case: we can suppose $m \geq 2$, which is harmless since the representation type of curves is well-known, see [DG01, DG93, BBDG06]. Here is the main result of this section.

Theorem 7.1. *Let X be a non-degenerate ACM closed subscheme of minimal degree and dimension $m \geq 2$. Then X is CM-wild, except if X a linear space, a quadric hypersurface of corank at most 1, the Veronese surface in $\mathbb{P}^5$, or a smooth rational normal surface scroll of degree 3 or 4.*

By [EGHP06, Theorem 0.4 and 1.4] and [XD81, Theorem 1], if the scheme X is reducible, then it has at least two reduced irreducible components, both of minimal degree, meeting along a divisor, so that X is CM-wild by Theorem 6.2.

Therefore it only remains to see what happens when X is integral. In this case X fits in the classification of del Pezzo and Bertini, see [EH87]. If X is smooth, then X is either a quadric, or a Veronese surface, or a rational normal scroll, and the representation type of all these varieties is known. Indeed, X is of finite CM-type (see [AR87, EH88]) if it is a linear space (see [Hor64]), or a smooth quadric (see [Knö87]), or the Veronese surface in $\mathbb{P}^5$, or a smooth cubic scroll in $\mathbb{P}^4$ (see [AR89], see also [Fae15]), or a rational normal curve. Also, X is of tame CM-type if it is a rational normal surface scroll of degree 4, see [FM17]. Besides these cases, X is strictly Ulrich wild, as we see by applying Theorem A to the Ulrich line bundles considered in [MR13, FM17]. For the reader's convenience (and because these sheaves will play a role further on), we recall that, if the scroll $X \subset \mathbb{P}^n$ has degree d, the Ulrich line bundles are the ideal of a fibre of the scroll twisted by $\mathcal{O}_X(1)$, and the dual of the ideal of $d-1$ fibres.

It remains to understand what happens when X is an integral but singular scheme of minimal degree. The goal of the rest of this section is to settle this point.

Theorem 7.2. *An integral, non-degenerate, singular variety X of minimal degree and dimension $m \geq 2$ is of wild CM type unless X is a quadric of corank 1.*

According to del Pezzo and Bertini, singular varieties of minimal degree are cones over smooth varieties of minimal degree, so we start by studying in some detail the behavior of sheaves defined on cones.

7.1. Extension of sheaves over cones

Let $X \subset \mathbb{P}^n$ be a closed non-degenerate subscheme. Fix a linear subspace $\Lambda \subset \mathbb{P}^n$. Let $\Lambda^0 \subset \mathbb{P}^n$ be a linear subspace disjoint from Λ such that Λ and Λ^0 span $\mathbb{P}^n$, and consider a subscheme $X^0 \subset \Lambda^0$.

Definition 7.3. We say that X is a *cone* with vertex (or apex) Λ and base X^0 if X is the union of all lines joining a point of Λ and a point of X^0.

When X is a cone with vertex $\Lambda \subset \mathbb{P}^n$ of codimension n_0+1, any subspace Λ^0 disjoint from Λ and of dimension n_0 provides $X^0 = X \cap \Lambda^0$ as base of X. To write the equations of a cone, choose coordinates so that Λ^0 is defined by the vanishing of the linear forms $x_{n_0+1}, \ldots, x_n$. We denote $\lambda_i = x_{n_0+1+i}$ the coordinates of Λ so that $\boldsymbol{k}[\Lambda] \simeq \boldsymbol{k}[\lambda_0, \ldots, \lambda_{n-n_0-1}]$. Setting $S^0 = \boldsymbol{k}[x_0, \ldots, x_{n_0}]$, the ideals $I_{X|S}$ of X in S and $I_{X^0|S^0}$ of X^0 in S^0 are generated by the same minimal set of polynomials. Put $R = \boldsymbol{k}[X]$ and $R^0 = \boldsymbol{k}[X^0]$. In terms of graded rings:

$$R \simeq R^0 \otimes_{\boldsymbol{k}} \boldsymbol{k}[\Lambda].$$

Lemma 7.4. *Given finitely generated R^0-modules E^0 and F^0, set $E = E^0 \otimes_{R^0} R$ and $F = F^0 \otimes_{R^0} R$. Then, for all $i \geq 0$, we have an isomorphism of graded $\boldsymbol{k}[X]$-modules:*

$$\mathrm{Ext}^i_R(E,F) \simeq \mathrm{Ext}^i_{R^0}(E^0,F^0) \otimes \boldsymbol{k}[\Lambda].$$

Proof. We have

$$\mathrm{Ext}^i_R(E,F) \simeq \mathrm{Ext}^i_{R^0}(E^0,F^0 \otimes_{R^0} R) \simeq \mathrm{Ext}^i_{R^0}(E^0,F^0) \otimes_{R^0} \boldsymbol{k}[X]$$

where the first isomorphism is [Rot09, Theorem 11.65], using that R is a flat R^0-module. In order to finish, we need only to observe the standard isomorphism:

$$\begin{aligned}\mathrm{Ext}^i_{R^0}(E^0,F^0) \otimes_{R^0} (R^0 \otimes_{\boldsymbol{k}} \boldsymbol{k}[\Lambda]) &\simeq (\mathrm{Ext}^i_{R^0}(E^0,F^0) \otimes_{R^0} R^0) \otimes_{\boldsymbol{k}} \boldsymbol{k}[\Lambda] \\ &\simeq \mathrm{Ext}^i_{R^0}(E^0,F^0) \otimes_{\boldsymbol{k}} \boldsymbol{k}[\Lambda].\end{aligned}$$

□

7.2. The cubic cone

We focus now on the only case where all construction methods of representation embeddings seen so far fail, for two reasons. First, no extension group grows enough to use Theorem A. Second, all the ACM sheaves that do have deformations have many endomorphisms, which is another obstruction to use Theorem A. We develop a specific technique to study this very intriguing case. For this subsection, the field $\boldsymbol{k}$ is arbitrary.

Theorem 7.5. *The cone $X \subset \mathbb{P}^4$ over a rational normal cubic curve in $\mathbb{P}^3$ is CM-wild.*

Proof. We divide the proof into eight steps.

Step 1. *Define the sheaf $\mathcal{F}^0$ on the twisted cubic and compute its self-extensions.*

Let us write X^0 for the base of the cone X. Using the convention of §7.1 let us put $R = \boldsymbol{k}[X]$, $R^0 = \boldsymbol{k}[X^0]$ so that $R \simeq \mathbf{R}^0 \otimes \boldsymbol{k}[\lambda_0]$, and let us abbreviate $\lambda = \lambda_0$. Define $\mathcal{F}^0$ to be the line bundle of degree 2 on X^0, that is, $\mathcal{F}^0 \simeq \mathcal{O}_{\mathbb{P}^1}(2)$. This is a stable Ulrich sheaf on X^0. We write F^0 for its associated $\boldsymbol{k}[X^0]$-module. Set W for the 2-dimensional vector space $W = \mathrm{Ext}^1_{X^0}(\mathcal{F}^0(1), \mathcal{F}^0) \simeq \mathrm{H}^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-3))$. By Lemma 2.4 with $\mathcal{E}^0 = \mathcal{F}^0(s)$ for all $s \in \mathbb{Z}$ we get a graded isomorphism:

$$\mathrm{Ext}^1_{R^0}(F^0(s)_{\geq 0}, F^0)_{\geq 0} \simeq \bigoplus_{t \geq 0} \mathrm{Ext}^1_{X^0}(\mathcal{F}^0(s), \mathcal{F}^0(t)) \simeq \bigoplus_{0 \leq t \leq s-1} S^{3(s-t)-2} W,$$

where by convention a symmetric power with negative exponent is zero.

Step 2. *Define the sheaf $\mathcal{F}$ on X and compute its self-extensions.*

We now pay attention to X. Let $\mathcal{F}$ be the ideal sheaf of a ray of the cone X, twisted by $\mathcal{O}_X(1)$; this is a stable Ulrich sheaf of rank 1 on X. The R-module of global sections F associated with $\mathcal{F}$ satisfies $F \simeq F^0 \otimes \boldsymbol{k}[\lambda]$. By Lemma 7.4 we have, for all $q \in \mathbb{Z}$:

$$\mathrm{Ext}^1_R(F(1), F)_q \simeq \mathrm{Ext}^1_{R^0}(F^0(1), F^0) \otimes \boldsymbol{k}[\lambda]_q.$$

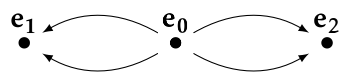
Using Lemma 2.4 and setting $\lambda^q = 0$ by convention for $q < 0$, we conclude, for all $q \in \mathbb{Z}$:

$$\mathrm{Ext}^1_X(\mathcal{F}(1), \mathcal{F}(q)) \simeq \mathrm{Ext}^1_R(F(1), F)_q \simeq W \cdot \lambda^q.$$

Step 3. *Define the quiver Θ and a functor* $\mathbf{Rep}_\Theta \to \mathbf{ACM}_X$.

We introduce now the quiver Θ, which we depict as follows.

Θ:



Let $\mathcal{R}$ be a representation of Θ with dimension vector (a_0, a_1, a_2), so that $\mathcal{R}$ consists of two pencils of linear maps $A_1 \leftleftarrows A_0 \rightrightarrows A_2$, where A_i is a vector space of dimension a_i. As we did in §3.1 we associate a linear map with each of these two pencils by indexing the arrows of Θ with basis elements of W. This way, the datum or $\mathcal{R}$ is tantamount to:

$$(\xi_1, \xi_2) \in \mathrm{Hom}_{\boldsymbol{k}}(A_0, A_1) \otimes W \cdot \lambda^2 \times \mathrm{Hom}_{\boldsymbol{k}}(A_0, A_2) \otimes W \cdot \lambda^3.$$

In other words, we identify $\mathcal{R}$ with:

$$(\xi_1, \xi_2) \in \mathrm{Ext}^1_X(A_0 \otimes \mathcal{F}, (A_1 \otimes \mathcal{F}(1)) \oplus (A_2 \otimes \mathcal{F}(2))).$$

By the procedure described in §3.1, this gives a sheaf $\mathcal{E}$ fitting into the exact sequence:

$$0 \to \begin{matrix} A_1 \otimes \mathcal{F}(1) \\ \oplus \\ A_2 \otimes \mathcal{F}(2) \end{matrix} \to \mathcal{E} \to A_0 \otimes \mathcal{F} \to 0.$$

The sheaf $\mathcal{E}$ is thus clearly ACM and has a Jordan-Hölder filtration whose associated graded object is $\mathcal{F}^{a_0} \oplus \mathcal{F}(1)^{a_1} \oplus \mathcal{F}(2)^{a_2}$. The procedure of constructing $\mathcal{E}$ from ξ, or equivalently from $\mathcal{R}$, is functorial, which can be seen readily following the proof of Theorem A.

Step 4. *Define a functor* $\Phi : \mathbf{FMod}_{\boldsymbol{k}[x_1,x_2]} \to \mathbf{ACM}_X$ *factoring through* $\mathbf{Rep}_\Theta \to \mathbf{ACM}_X$.

Put $\Sigma = \boldsymbol{k}[x_1,x_2]$. We define a functor from $\mathbf{FMod}_\Sigma$ towards the category of ACM sheaves over X. Choose a basis (w_0,w_1) of W. A finite-dimensional Σ-module is a finite dimensional vector space M together with two commuting endomorphisms x_1 and x_2. We define two representations ξ_1 and ξ_2 by setting $A_0 = M \oplus M$, $A_1 = A_2 = M$ and:

$$\xi_1 = (\mathrm{id}_M w_0 + \mathsf{x}_1 w_1, 0), \qquad \xi_2 = (0, \mathrm{id}_M w_0 + \mathsf{x}_2 w_1).$$

The sheaf $\mathcal{E}$ associated with M is defined by the pair $\xi = (\xi_1,\xi_2)$ as in Step 3.

Step 5. *Prove that, if* $\mathcal{E} = \Phi(M)$ *and* $\mathcal{E}' = \Phi(M')$ *are isomorphic, then* $M \simeq M'$ *as* $\boldsymbol{k}[x_1]$*-modules.*

Given two finite-dimensional Σ-modules M and M', we have two sheaves $\mathcal{E}$ and $\mathcal{E}'$. Assume that these sheaves are isomorphic. By the Harder-Narasimhan filtrations of $\mathcal{E} \simeq \mathcal{E}'$ we get that $\mathcal{F}(2)^{a_2}$ is a maximal destabilizing subsheaf of both $\mathcal{E}$ and $\mathcal{E}'$, which implies that the quotient sheaves $\mathcal{E}_1 = \mathcal{E}/\mathcal{F}(2)^{a_2}$ and $\mathcal{E}_1' = \mathcal{E}'/\mathcal{F}(2)^{a_2}$ are isomorphic. These sheaves are given by the extension classes ξ_1 and ξ_1', which are thus isomorphic by the argument we used in the proof of Theorem A.

We identify M and M' as vector spaces and consider x_1 and x_1' as endomorphisms of M. Recall the expressions $\xi_1 = (\mathrm{id}_M w_0 + \mathsf{x}_1 w_1, 0)$ and $\xi_1' = (\mathrm{id}_M w_0 + \mathsf{x}_1' w_1, 0)$. Because ξ_1 and ξ_1' are isomorphic, there are linear isomorphisms $\alpha_0 \in \mathrm{End}_{\boldsymbol{k}}(M \oplus M)$, $\alpha_1 \in \mathrm{End}_{\boldsymbol{k}}(M)$ such that $\alpha_1 \otimes \mathrm{id}_W \circ \xi_1' = \xi_1 \circ \alpha_0$. Decomposing α_0 as a block matrix of endomorphisms $\alpha_0^{i,j}$ of M, we rewrite this as

$$\alpha_1(\mathrm{id}_M w_0 + \mathsf{x}_1' w_1, 0) = (\mathrm{id}_M w_0 + \mathsf{x}_1 w_1, 0)\begin{pmatrix} \alpha_0^{1,1} & \alpha_0^{1,2} \\ \alpha_0^{2,1} & \alpha_0^{2,2} \end{pmatrix}$$

In particular we get:

$$\alpha_1 w_0 + \alpha_1 \mathsf{x}_1' w_1 = \alpha_0^{1,1} w_0 + \mathsf{x}_1 \alpha_0^{1,1} w_1,$$

so $\alpha_1 = \alpha_0^{1,1}$ and α_1 conjugates x_1' to x_1. Then M and M' are isomorphic as $\boldsymbol{k}[x_1]$-modules.

Step 6. *Prove that, if the sheaves* $\mathcal{E} = \Phi(M)$ *and* $\mathcal{E}' = \Phi(M')$ *are isomorphic, then* $M \simeq M'$.

Suppose again $\mathcal{E} \simeq \mathcal{E}'$ and assume now $\xi_1 = \xi_1'$, which can be achieved after linear automorphisms by the previous point.

By definition, $\mathcal{E}$ and $\mathcal{E}'$ are obtained from $\mathcal{E}_1 = \mathcal{E}_{\xi_1}$ by using ξ_2 and ξ_2', which are both linear maps $A_0 \to A_2 \otimes W$. Let us look more closely at how this is achieved. Start with the extension ξ_1 and the sheaf $\mathcal{E}_1$ fitting into:

$$(\xi_1) \qquad 0 \to A_1 \otimes \mathcal{F}(1) \to \mathcal{E}_1 \to A_0 \otimes \mathcal{F} \to 0.$$

Apply to this the functor $\mathrm{Hom}_X(-, A_2 \otimes \mathcal{F}(2))$. We get:

$$\mathrm{Hom}_X(A_1 \otimes \mathcal{F}(1), A_2 \otimes \mathcal{F}(2)) \to \mathrm{Ext}^1_X(A_0 \otimes \mathcal{F}, A_2 \otimes \mathcal{F}(2)) \to \mathrm{Ext}^1_X(\mathcal{E}_1, A_2 \otimes \mathcal{F}(2)). \tag{7.1}$$

This is rewritten in the form:

$$\mathrm{Hom}_{\boldsymbol{k}}(A_1, A_2) \cdot \lambda \to \mathrm{Hom}_{\boldsymbol{k}}(A_0, A_1) \otimes W \cdot \lambda^3 \to \mathrm{Ext}^1_X(\mathcal{E}_1, A_2 \otimes \mathcal{F}(2)). \tag{7.2}$$

Given a linear map $\gamma : A_1 \to A_2$, the leftmost map in the previous diagram sends $\gamma \cdot \lambda$ to $\gamma \otimes \mathrm{id}_W \circ \xi_1 \cdot \lambda^3$. The isomorphism class of the sheaf $\mathcal{E}$ determines an element of $\mathrm{Ext}^1_X(\mathcal{E}_1, A_2 \otimes \mathcal{F}(2))$ coming from a pair (ξ_1, ξ_2), so the previous diagram shows that $\mathcal{E}$ determines the isomorphism class of ξ_2 up to adding any map of the form $\gamma \otimes \mathrm{id}_W \circ \xi_1$, for $\gamma \in \mathrm{Hom}_{\boldsymbol{k}}(A_1, A_2)$.

This says that an isomorphism $\mathcal{E}' \to \mathcal{E}$ exists if and only if there exist a linear isomorphism $\alpha_2 \in \mathrm{End}(A_2)$ and a linear map $\gamma : A_1 \to A_2$ such that ξ_2 is carried to ξ_2' by $\alpha_2 + \gamma \otimes \mathrm{id}_W$, after composition with ξ_1, that is:

$$\alpha_2 \otimes \mathrm{id}_W \circ \xi_2' = \xi_2 + \gamma \otimes \mathrm{id}_W \circ \xi_1. \tag{7.3}$$

The sheaves $\mathcal{E}$ and $\mathcal{E}'$ determine isomorphic extensions and therefore equal elements in the group $\mathrm{Ext}^1_X(\mathcal{E}_1, A_2 \otimes \mathcal{F}(2))$. This implies that there exist a linear isomorphism $\alpha_2 \in \mathrm{End}_{\boldsymbol{k}}(M)$ and a linear map $\gamma : A_1 \to A_2$ such that (7.3) holds. According to the decomposition $A_0 = M \oplus M$, we rewrite this as:

$$(0, \alpha_2 w_0 + \alpha_2 \mathsf{x}_2' w_1) = (0, \mathrm{id}_M w_0 + \mathsf{x}_2 w_1) + (\gamma w_0 + \gamma \mathsf{x}_1 w_1, 0).$$

In particular we obtain $\alpha_2 = \mathrm{id}_M$ and $\mathsf{x}_2' = \mathsf{x}_2$. The modules M and M' are thus isomorphic.

Step 7. *Prove that, if $\mathcal{E} = \Phi(M)$ is decomposable, then M is decomposable as $\boldsymbol{k}[x_1]$-module.*

We write $\mathcal{E} = \mathcal{E}' \oplus \mathcal{E}''$. The Harder-Narasimhan filtration of $\mathcal{E}'$ must be compatible with that of $\mathcal{E}$ and therefore its associated graded object must be $\mathcal{F}^{a_0'} \oplus \mathcal{F}(1)^{a_1'} \oplus \mathcal{F}(2)^{a_2'}$, and similarly for $\mathcal{E}''$. Then, the quotient $\mathcal{E}_1' = \mathcal{E}'/\mathcal{F}(2)^{a_2'}$ has a graded object of the form $\mathcal{F}(1)^{a_1'} \oplus \mathcal{F}(2)^{a_2'}$ and therefore, as in the proof of Theorem A, we must have $\mathcal{E}_1' \simeq \mathcal{E}_{\xi_1'}$ for some $\xi_1' \in \mathrm{Hom}_{\boldsymbol{k}}(A_0', A_1') \otimes W$, where A_0' and A_1' are vector spaces of dimension a_0 and a_1 appearing in the vector space decompositions $A_0 = A_0' \oplus A_0''$ and $A_1 = A_1' \oplus A_1''$. Likewise we have $\mathcal{E}_1'' \simeq \mathcal{E}_{\xi_1''}$ for some $\xi_1'' \in \mathrm{Hom}_{\boldsymbol{k}}(A_0'', A_1'') \otimes W$ and $\mathcal{E}_1 = \mathcal{E}_1' \oplus \mathcal{E}_1''$, which implies that ξ_1 has a block-diagonal form in terms of ξ_1' and ξ_1''.

Now we have $M = A_1$ decomposed as vector space as $A_1' \oplus A_1''$. We put $M' = A_1'$, $M'' = A_1''$ and we write $M = M' \oplus M''$, so we decompose $A_0 = M \oplus M$ as $A_0 = M' \oplus M'' \oplus M' \oplus M''$. By definition we have $\xi_1 = (\mathrm{id}_M w_0 + \mathsf{x}_1 w_1, 0)$ so the expression $\mathrm{id}_M w_0 + \mathsf{x}_1 w_1$ gives a map $M' \oplus M'' \to (M' \oplus M'') \otimes W$ which, in view of the decomposition of ξ_1 in diagonal form in terms of ξ_1' and ξ_1'', takes the form:

$$\mathrm{id}_M w_0 + \mathsf{x}_1 w_1 = \begin{pmatrix} \mathrm{id}_{M'} w_0 + \mathsf{x}_1' w_1 & 0 \\ 0 & \mathrm{id}_{M''} w_0 + \mathsf{x}_1'' w_1 \end{pmatrix},$$

for some linear maps $\mathsf{x}_1' : M' \to M'$ and $\mathsf{x}_1'' : M'' \to M''$.

Step 8. *Prove that, if $\mathcal{E} = \Phi(M)$ is decomposable, then M is decomposable.*

We proved that $M = M' \oplus M''$ as $\boldsymbol{k}[x_1]$-module. Now we have to use ξ_2 to prove that the splitting $\mathcal{E} = \mathcal{E}' \oplus \mathcal{E}'$ provides a second pair of endomorphisms $\mathsf{x}_2' : M' \to M'$ and $\mathsf{x}_2'' : M'' \to M''$ compatible with the decomposition $M = M' \oplus M''$ induced by $\mathcal{E}_1 = \mathcal{E}_1' \oplus \mathcal{E}_1''$. Again the Harder-Narasimhan filtration induces a decomposition $A_2 = A_2' \oplus A_2''$. The exact sequence defining $\mathcal{E}$ as an extension of $\mathcal{F}_1$ by $A_2 \otimes \mathcal{F}(2)$ together with the direct sum decompositions of $\mathcal{E}_1$ and A_2 provides elements ζ' and ζ'' of the Ext groups:

$$\zeta' \in \mathrm{Ext}^1_X(\mathcal{E}_1', A_2' \otimes \mathcal{F}(2)), \qquad \zeta'' \in \mathrm{Ext}^1_X(\mathcal{E}_1'', A_2'' \otimes \mathcal{F}(2)),$$

and the extension providing $\mathcal{E}$ is the block-diagonal sum of ζ' and ζ''. This means that, denoting by $\iota : \mathcal{E}_1' \to \mathcal{E}_1$ the obvious injection, the map

$$\iota^* : \mathrm{Ext}^1_X(\mathcal{E}_1, (A_2' \oplus A_2'') \otimes \mathcal{F}(2)) \to \mathrm{Ext}^1_X(\mathcal{E}_1', (A_2' \oplus A_2'') \otimes \mathcal{F}(2))$$

must map the class ζ of $\mathcal{E}$ to $(\zeta', 0)$, and similarly the map induced by the injection $\mathcal{E}_1'' \to \mathcal{E}_1$ must map ζ to $(0, \zeta'')$.

In view of the description of the Ext groups we have given in (7.1) and (7.2), and because ι corresponds to the inclusion of M' into M, this implies that, up to adding $\gamma \otimes \mathrm{id}_W \circ \xi_1$ for some $\gamma : M' \to A_2' \oplus A_2''$, the map

$$\xi_2 \circ \iota : M' \oplus M' \to (A_2' \oplus A_2'') \otimes W$$

must have a vanishing component in $A_2'' \otimes W$. Write γ as the transpose of (γ', γ''), where $\gamma' \in \mathrm{Hom}_{\boldsymbol{k}}(M', A_2')$ and $\gamma'' \in \mathrm{Hom}_{\boldsymbol{k}}(M', A_2'')$.

Denote by $\pi' : M \to A_2'$ and $\pi'' : M \to A_2''$ the obvious projections and recall that, by definition, we have $\xi_2 = (0, \mathrm{id}_M w_0 + \mathrm{x}_2 w_1)$ and $\xi_1 = (\mathrm{id}_M w_0 + \mathrm{x}_1 w_1, 0)$. Evaluating at $(w_0, w_1) = (1, 0)$, we get maps $M' \oplus M' \to A_2''$ satisfying the following equality:

$$(\gamma'' \circ \mathrm{id}_{M'}, 0) = (0, \pi'' \circ \iota).$$

In particular $M' = \mathrm{Im}(\iota) \subset \ker(\pi'') = A_2'$. Similarly we get $M'' \subset \ker(\pi') = A_2''$ and in view of the equalities $M' \oplus M'' = M = A_2' \oplus A_2''$, we obtain $M' = A_2'$ and $M'' = A_2''$.

Evaluating at $(w_0, w_1) = (0, 1)$ we get $\pi_2'' \circ \mathrm{x}_2 \circ \iota = 0$, meaning that x_2 maps $M' = \mathrm{Im}(\iota)$ to $M' = \ker(\pi'')$, namely M' is stable for x_2. One proves similarly the same statement for M''. Therefore $\mathrm{x}_2 : M \to M$ is the block-diagonal sum of $\mathrm{x}_2' : M' \to M'$ and $\mathrm{x}_2'' : M'' \to M''$ so that M is the direct sum of M' and M'' as Σ-modules.

Note that, if the decomposition of $\mathcal{E}$ is non-trivial (that is to say, if $\mathcal{E}' \neq 0 \neq \mathcal{E}''$) then at least one of the decompositions of A_0, A_1 and A_2 is non-trivial, and therefore all of them are by what we have just seen, so the decomposition of M as Σ-module is non-trivial too. □

7.3. Proof of Theorem 7.2

The variety X is a cone over a base X^0 which is a smooth irreducible variety of minimal degree. We adopt the convention of §7.1 and write $R = \boldsymbol{k}[X]$ and $R^0 = \boldsymbol{k}[X^0]$ so $R \simeq R^0 \otimes \boldsymbol{k}[\Lambda]$. We always put a 0 superscript to sheaves on X^0 and modules on R^0, remove the superscript to indicate the corresponding object on X, and put a calligraphic letter for the coherent sheaf associated with a module denoted by that letter.

We have three cases to check according to whether X^0 is a quadric, or a Veronese surface in $\mathbb{P}^5$, or a scroll. These are treated in a conceptually unified way by Lemma 7.4, Lemma 2.4 and Theorem A: only the choice of the basic sheaves on X^0 obliges us to separate them. Note that, once the ACM (or Ulrich) sheaves $\mathcal{E}^0$ and $\mathcal{F}^0$ are stable on X^0, their lifts $\mathcal{E}$ and $\mathcal{F}$ are ACM (or Ulrich) and stable on X. Indeed, the ACM and Ulrich conditions are obvious as they can be read on the minimal graded free resolutions of E^0 and E or of F^0 and F, which are unchanged on S or S^0. Stability is also clear, as any destabilizing subsheaf, restricted to a generic linear space of dimension n_0, would destabilize $\mathcal{E}^0$ or $\mathcal{F}^0$.

7.3.1. Quadric cones.— Here, X^0 is a quadric of corank greater than one, in other words $\dim(\Lambda) \geq 1$. We take $\mathcal{S}^0$ to be a spinor bundle on X^0, see [Ott88, BGS87, Knö87]. Then, $\mathcal{E}^0 = \mathcal{S}^0(1)$ is an Ulrich bundle and it sits in a short exact sequence

$$0 \to \mathcal{F}^0 \to \mathcal{O}_{X^0}^{2\,\mathrm{rk}\,\mathcal{S}^0} \to \mathcal{E}^0 \to 0,$$

where $\mathcal{F}^0$ is again a spinor bundle (isomorphic to $\mathcal{S}^0$ if and only if the dimension of X^0 is odd). Both are stable sheaves on X^0. We have, by Lemma 2.4:

$$\bigoplus_{t \in \mathbb{N}} \mathrm{Ext}^1_{X^0}(\mathcal{E}^0, \mathcal{F}^0(t)) \simeq \mathrm{Ext}^1_{R^0}(E^0, F^0) \simeq \boldsymbol{k},$$

where $E^0 = \Gamma_*(\mathcal{E}^0)$ and $F^0 = \Gamma_*(\mathcal{F}^0)$. Therefore, defining $\mathcal{E}$ and $\mathcal{F}$ as sheafifications of $E = E^0 \otimes_{\boldsymbol{k}} \boldsymbol{k}[\Lambda]$ and $F = F^0 \otimes_{\boldsymbol{k}} \boldsymbol{k}[\Lambda]$, by Lemma 7.4, we obtain an isomorphism of R-modules:

$$\bigoplus_{t \in \mathbb{N}} \mathrm{Ext}^1_X(\mathcal{E}, \mathcal{F}(t)) \simeq \boldsymbol{k}[\Lambda].$$

In particular the component of degree t, for $t \geq 2$, of this extension space has dimension at least 3. Applying item i) of Theorem A to $\mathcal{B} = \mathcal{E}$ and $\mathcal{A} = \mathcal{F}(2)$ gives the result.

7.3.2. Cones over the Veronese surface.— If X^0 is the Veronese image of $\mathbb{P}^2$ in $\mathbb{P}^5$, we take $\mathcal{E}^0$ to be the tangent bundle $T_{\mathbb{P}^2}$. This is a stable Ulrich bundle with respect to $\mathcal{O}_{X^0}(1) \simeq \mathcal{O}_{\mathbb{P}^2}(2)$. Write W for the 3-dimensional space $\mathrm{H}^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))$. Letting $\Omega_{\mathbb{P}^2}$ be the cotangent bundle on $\mathbb{P}^2$ and tensoring the Euler sequence with $\Omega_{\mathbb{P}^2}(t)$ we get, for $t = 0$:

$$\mathrm{H}^1(\mathbb{P}^2, \Omega_{\mathbb{P}^2} \otimes T_{\mathbb{P}^2}(-2)) \simeq \mathrm{H}^2(\mathbb{P}^2, \Omega_{\mathbb{P}^2}(-2)) \simeq W.$$

In general for $t \in \mathbb{Z}$, we obtain:

$$\mathrm{Ext}^1_{X^0}(\mathcal{E}^0(1), \mathcal{E}^0(t)) \simeq \begin{cases} W & \text{if } t = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore we have isomorphisms of graded R-modules:

$$\bigoplus_{t \in \mathbb{N}} \mathrm{Ext}^1_X(\mathcal{E}(1), \mathcal{E}(t)) \simeq \mathrm{Ext}^1_{X^0}(\mathcal{E}^0(1), \mathcal{E}^0) \otimes_{\boldsymbol{k}} \boldsymbol{k}[\Lambda] \simeq W \otimes_{\boldsymbol{k}} \boldsymbol{k}[\Lambda].$$

Applying item i) of Theorem A to the pair of sheaves $\mathcal{B} = \mathcal{E}(1)$ and $\mathcal{A} = \mathcal{E}(2)$ gives the result.

7.3.3. Cones over scrolls.— Finally, assume that X^0 is a scroll of degree d. By Theorem 7.5 we may suppose $d \geq 4$ or $m \geq 3$. If $m = 2$ (and hence $X^0 \simeq \mathbb{P}^1$) we work like in Theorem 7.5 and take $\mathcal{F}$ to be the ideal sheaf of a ray of the cone X. We have $\mathcal{F}^0 \simeq \mathcal{O}_{\mathbb{P}^1}(-1)$. The vector space

$$W = \mathrm{Ext}^1_{X^0}(\mathcal{F}^0(1), \mathcal{F}^0) \simeq \mathrm{H}^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-d))$$

has dimension $d - 1 \geq 3$. Moreover, by Lemma 2.4, Case iii), we get that $\mathrm{Ext}^1_{R^0}(F^0(1), F)_0 \simeq W$. Then, we are in position to apply Theorem A because, for all $q \in \mathbb{N}$, the following space has dimension at least 3:

$$\mathrm{Ext}^1_X(\mathcal{F}(1), \mathcal{F}(q)) \simeq \mathrm{Ext}^1_R(F(1), F)_q \simeq W \cdot \lambda_0^q.$$

If $m \geq 3$, either $\dim(\Lambda) \geq 1$ or $\dim(X^0) \geq 2$. In the former case, we may assume $\dim(X^0) = 1$ so again $X^0 \simeq \mathbb{P}^1$ and we choose $\mathcal{F}^0$ as before. The space $W = \mathrm{Ext}^1_{X^0}(\mathcal{F}^0(1), \mathcal{F}^0)$ has positive dimension and we get, for $q \geq 0$:

$$\mathrm{Ext}^1_X(\mathcal{F}(1), \mathcal{F}(q)) \simeq \mathrm{Ext}^1_R(F(1), F)_q \simeq W \cdot \boldsymbol{k}[\Lambda]_q,$$

and this space has unbounded dimension for $q \gg 0$ as $\dim(\Lambda) \geq 1$. In the latter case, we may assume $\dim(\Lambda) = 0$, and choose $\mathcal{F}^0$ to be the ideal of a fibre of the scroll, twisted by $\mathcal{O}_{X^0}(1)$, and $\mathcal{E}^0$ to be the line bundle associated with the divisor of $d - 1$ fibres. These are both (obviously stable) Ulrich line bundles. Also, $W_{-1} := \mathrm{Ext}^1_{X^0}(\mathcal{E}^0(1), \mathcal{F}^0) \simeq \mathrm{H}^1(\mathcal{O}_{\mathbb{P}^1}(-d)) \simeq \boldsymbol{k}^{d-1}$, while $W_0 := \mathrm{Ext}^1_{X^0}(\mathcal{E}^0, \mathcal{F}^0)$ has dimension at least 1, see [MR13, Lemma 3.1]. Therefore, we get:

$$\mathrm{Ext}^1_X(\mathcal{E}, \mathcal{F}) \simeq \mathrm{Ext}^1_X(E, F)_0 \simeq \mathrm{Ext}^1_{X^0}(E^0, F^0)_{-1} \cdot \lambda_0 \oplus \mathrm{Ext}^1_{X^0}(E^0, F^0)_0 \simeq W_{-1} \cdot \lambda_0 \oplus W_0.$$

So $\dim_{\boldsymbol{k}} \mathrm{Ext}^1_X(\mathcal{E}, \mathcal{F}) \geq 3$ and Theorem A allows us to conclude.

Remark 7.6. If the base field $\boldsymbol{k}$ is algebraically closed and $\mathrm{char}(\boldsymbol{k}) = 2$, all the statements of Theorem 7.2 and hence of Theorem 7.1 remain true, except perhaps the fact that quadric cones of corank one are CM-countable.

All the proofs remain the same except when X is a quadric hypersurface. If the quadric X is smooth, then X is CM-finite by [BEH87]. If X is not smooth, then X is a cone over a smooth quadric X^0, the vertex of the cone being a linear subspace $\Lambda \subset \mathbb{P}^n$ (recall that $\boldsymbol{k}$ is algebraically closed). If $\dim(\Lambda) \geq 1$, again [BEH87] provides the sheaves $\mathcal{E}^0$ and $\mathcal{F}^0$ as is §7.3.1, and these sheaves are Ulrich, hence semistable, as we shall see in Lemma 8.3. Since any destabilizing subsheaf should be Ulrich, CM-finiteness of X^0 implies that $\mathcal{E}^0$ and $\mathcal{F}^0$ can be chosen to be stable and hence simple. So X is CM-wild in this case as in §7.3.1.

We do not know if a quadric cone X of corank 1 is CM-countable when $\mathrm{char}(\boldsymbol{k}) = 2$. Indeed, our construction provides countably many ACM sheaves on X also in this case, but Knörrer periodicity does not apply directly to show that these are the only indecomposable ACM sheaves on X up to isomorphism.

8. Varieties of almost minimal degree

From now on the field $\boldsymbol{k}$ is algebraically closed of arbitrary characteristic. Let us take a further step in the proof of our main result. In view of Theorem 6.3, we will assume from now on that the subscheme $X \subset \mathbb{P}^n$ is reduced, and keep the usual assumption that X is closed, non-degenerate and ACM of dimension $m \geq 1$. In this section we pay attention to varieties of *almost minimal degree*, namely the degree of X is $d = n-m+2$. If X is irreducible and normal then X is usually called a del Pezzo variety (terminology may differ slightly in the literature). Our goal in this section is to prove the following result.

Theorem 8.1. *Any reduced, non-degenerate ACM scheme $X \subset \mathbb{P}^n$ of dimension $m \geq 2$ and almost minimal degree is of wild CM-type. For $m = 2$, X is strictly Ulrich wild.*

Let us first look briefly at reducible (and reduced) ACM subschemes of almost minimal degree and then focus on the irreducible ones, normal or not.

8.1. Reducible subschemes of almost minimal degree

Let us start by assuming that X is reducible, namely $X = X_1 \cup X_2$ where X_1 and X_2 are closed subschemes of $\mathbb{P}^n$ of degree d_1 and d_2. We claim that not both of them have degree $d_i \geq n_i - m + 2$ at the same time, where n_i denotes the dimension of the linear span $\langle X_i \rangle$ of X_i inside $\mathbb{P}^n$. Otherwise,

$$n - m + 2 = d = d_1 + d_2 \geq n_1 + n_2 - 2m + 4.$$

On the other hand $n = n_1 + n_2 - l$, with $l = \dim(\langle X_1 \rangle \cap \langle X_2 \rangle)$. From here we would obtain that $l \leq m-2$, in contradiction with the fact that, since the subscheme X is ACM, it must be connected in codimension one (again, see [Har62] for the definition and the result) and therefore X_1 and X_2, both of dimension m should meet along a Weil divisor.

By induction on the number of components of X, we deduce that we can find two irreducible components X_1 and X_2 of X, of dimension m, such that either both of them are of minimal degree in their linear span or one of them, say X_1, is of minimal degree on its linear span and X_2 is of quasi-minimal degree.

In both cases, the subscheme X_1, being of minimal degree, is ACM in its linear span, while X_2 is either ACM or the image of a (finite) projection of a variety $\bar{X}_2$ of minimal degree (see [BS07, Theorem 1.2]). Since $\bar{X}_2$ supports an ACM sheaf (see §7), so does X_2, because taking the direct image via a finite map preserves the ACM property. In all these circumstances, Theorem 6.2 applies to show that $X_1 \cup X_1$ is CM-wild, and by Lemma 6.1, X is CM-wild as well.

8.2. Reduction to surfaces

In view of the previous discussion, we may assume from now on that X is irreducible and of almost minimal degree. In other words, X has Δ-genus one. This condition amounts to asking that the sectional p_X of X is one, see [Fuj90, page 45]. Essentially, these varieties are completely classified: see [Fuj82] for the case of normal varieties, and [Rei94, BS07] for the non-normal case. For surfaces of degree 3 and 4 we refer to [LPS11, LPS12]. For the roots of this classification, originated from work of Schläfli and Cayley, see for instance [Abh60, BW79].

Since we are assuming that X is ACM, it turns out that X is arithmetically Gorenstein (AG), which is to say, $R = \boldsymbol{k}[X]$ is a graded Gorenstein ring (see [BS07, Remark 4.5]). Therefore the canonical sheaf satisfies $\omega_X \simeq \mathcal{O}_X(m-1)$: indeed, by [Mig98, Corollary 4.1.5], ω_X must be of the form $\mathcal{O}_X(t)$ for some $t \in \mathbb{Z}$; on the other hand restricting to a generic one-dimensional linear section Y gives $\omega_Y \simeq \mathcal{O}_Y(t+m-1)$ by adjunction because $p_Y = 1$. Therefore $\chi(\omega_Y) = 0$ so $t = 1 - m$.

Let us quickly rule out the case $m = 1$. We have that X is of tame CM-type if it smooth (namely X is an elliptic curve), by classical work of Atiyah, see [Ati57]. If X is singular, we know by [DG01] that X is CM-tame if X a cycle of rational curves with ordinary double points, and that X is CM-wild otherwise.

The goal of this section is to deal with higher dimensions. Namely we want to prove that, if X is ACM of almost minimal degree, then X is CM-wild as soon as $m \geq 2$.

The idea is to use our reduction to linear sections. Let $Y \subset \mathbb{P}^d$ be a linear section of X with $\dim(Y) = 2$. By Bertini's theorem, we may assume that Y is also an irreducible ACM subscheme of almost minimal degree, with $\omega_Y \simeq \mathcal{O}_Y(-1)$. Note that the codimension c of Y in X is $m-2$ so $\omega_Y(m-c-1) \simeq \mathcal{O}_Y$. Therefore, Theorem C applies to prove that X is CM-wild, as soon as we show that Y supports two simple Ulrich sheaves $\mathcal{A}$ and $\mathcal{B}$ such that:

$$\mathrm{Hom}_Y(\mathcal{A},\mathcal{B}) = 0 = \mathrm{Hom}_Y(\mathcal{B},\mathcal{A}), \quad \text{and} \quad \dim_{\boldsymbol{k}} \mathrm{Ext}^1_Y(\mathcal{B},\mathcal{A}) \geq 3.$$

Finding $\mathcal{A}$ and $\mathcal{B}$ as above will be our task. It is natural to look for $\mathcal{A}$ and $\mathcal{B}$ among sheaves of low rank. However, in general it will not be possible to obtain sheaves of rank one. Indeed, Ulrich sheaves of rank one may not exist for certain del Pezzo surfaces, for example cubic surfaces with an E_6 singularity (see [Dol12, Theorem 9.3.6]).

Therefore we move forward to construct rank two sheaves using the Hartshorne-Serre correspondence. Of course this idea is not new, as for instance it is widely used in [CH11] precisely to construct families of Ulrich bundles on smooth cubic surfaces, a special case of surfaces of almost minimal degree. The construction can be performed in quite a general setup; for instance, for cubic surfaces one knows, even if $\boldsymbol{k}$ is not algebraically closed, the degree of the field extension needed to construct $\mathcal{E}_Z$, see [Tan14]. However, in our setting we have to be a bit more careful since the surfaces under consideration may be badly singular.

8.3. Surface cones

Let us quickly rule out the case of cones, namely assume that $Y \subset \mathbb{P}^d$ is a cone over an integral curve $Y^0 \subset \mathbb{P}^{d-1}$ with trivial canonical sheaf, the vertex of the cone being a single point. In this case, using [AK80, Proposition 3.5], we may choose non-isomorphic Ulrich line bundles $\mathcal{F}^0$ and $\mathcal{E}^0$ on the curve Y^0, put $\mathcal{M} = \mathcal{E}^0 \otimes (\mathcal{F}^0)^\vee$ and observe:

$$\mathrm{Ext}^1_{Y^0}(\mathcal{E}^0, \mathcal{F}^0(-1))^* \simeq \mathrm{H}^0(Y^0, \mathcal{M}(1)) \simeq \boldsymbol{k}^d.$$

because $\mathcal{M}(1)$ is a line bundle of degree d, and clearly $d \geq 3$. We also have $\mathrm{Ext}^1_{Y^0}(\mathcal{E}^0, \mathcal{F}^0)^* \simeq \mathrm{H}^0(Y^0, \mathcal{M}) = 0$ because $\mathcal{E}^0$ and $\mathcal{F}^0$ are not isomorphic. By the same reason we have $\mathrm{Hom}_{Y^0}(\mathcal{E}^0, \mathcal{F}^0) = \mathrm{Hom}_{Y^0}(\mathcal{F}^0, \mathcal{E}^0) = 0$.

We use the approach of §7.1. The coordinate ring $R = \boldsymbol{k}[X]$ takes the form $R^0 \otimes_{\boldsymbol{k}} \boldsymbol{k}[\lambda]$, with $R^0 = \boldsymbol{k}[X^0]$ and the modules $E^0 = \Gamma_*(\mathcal{E}^0)$ and $F^0 = \Gamma_*(\mathcal{F}^0)$ give rise, by tensoring with $\boldsymbol{k}[\lambda]$, to Ulrich R-modules F and E (as their S-resolutions are still linear) and thus to Ulrich sheaves $\mathcal{E}$ and $\mathcal{F}$ over X. By Lemma 7.4 and Lemma 2.4, we have:

$$\mathrm{Ext}^1_Y(\mathcal{E}, \mathcal{F}) \simeq \mathrm{Ext}^1_{Y^0}(\mathcal{E}^0, \mathcal{F}^0(-1)) \cdot \lambda,$$

so this space has dimension at least 3. The same argument shows $\mathrm{Hom}_Y(\mathcal{E}, \mathcal{F}) = \mathrm{Hom}_Y(\mathcal{F}, \mathcal{E}) = 0$. Then $\mathcal{A} = \mathcal{E}$ and $\mathcal{B} = \mathcal{F}$ are the desired sheaves to apply Theorem C.

8.4. Hartshorne-Serre correspondence

At this point we may assume that the surface $Y \subset \mathbb{P}^d$ is not a cone. Let us consider a set $Z \subset Y \subset \mathbb{P}^d$ of $d+2$ distinct points points in general linear position and disjoint from $\mathrm{Sing}(Y)$. We have:

$$\mathrm{Ext}^1_Y(\mathcal{I}_{Z|Y}(2), \mathcal{O}_Y) \simeq \mathrm{Ext}^1_Y(\mathcal{O}_Y, \mathcal{I}_{Z|Y}(1))^* \simeq \mathrm{H}^1(Y, \mathcal{I}_{Z|Y}(1))^* \simeq \boldsymbol{k},$$

where we used that $\omega_Y \simeq \mathcal{O}_Y(-1)$ and that Z spans the whole $\mathbb{P}^d$. A non-zero element $\lambda \in \mathrm{H}^1(Y, \mathcal{I}_{Z|Y}(1))^*$ provides a coherent sheaf $\mathcal{F}_Z$ of rank 2 that fits into the short exact sequence:

$$0 \to \mathcal{O}_Y \to \mathcal{F}_Z \to \mathcal{I}_{Z|Y}(2) \to 0. \tag{8.1}$$

Given a set of points Z of $d+2$ points of Y, write $[Z]$ for the corresponding element of the Hilbert scheme $\mathrm{Hilb}^{d+2}(Y)$ of subschemes of length $d+2$ of Y. In the next lines, stability is always with respect to $\mathcal{O}_Y(1)$.

Lemma 8.2. *The sheaf $\mathcal{F}_Z$ is Ulrich and locally free of rank 2.*

Proof. Let us prove that $\mathcal{F}_Z$ is locally free. We need only prove this around any point z of Z, as $\mathcal{I}_{Z|Y}$ is already free of rank 1 away from Z. First of all, taking the dual of the short exact sequence

$$0 \to \mathcal{I}_{Z|Y}(2) \to \mathcal{O}_Y(2) \to \mathcal{O}_Z(2) \to 0,$$

we deduce, since Z is smooth and zero-dimensional, that

$$\mathcal{E}xt^1_{\mathcal{O}_Y}(\mathcal{I}_{Z|Y}(2), \mathcal{O}_Y) \simeq \mathcal{E}xt^2_{\mathcal{O}_Y}(\mathcal{O}_Z, \mathcal{O}_Y) \simeq \omega_Z.$$

Next, note that $\mathcal{H}om_{\mathcal{O}_Y}(\mathcal{I}_{Z|Y}(2), \mathcal{O}_Y)) \simeq \mathcal{O}_Y(-2)$. In view of the vanishing $\mathrm{H}^1(Y, \mathcal{O}_Y(-2)) = 0$, by the local-to-global spectral sequence we get an exact sequence:

$$0 \to \mathrm{Ext}^1_Y(\mathcal{I}_{Z|Y}(2), \mathcal{O}_Y) \to \mathrm{H}^0(Y, \mathcal{E}xt^1_{\mathcal{O}_Y}(\mathcal{I}_{Z|Y}(2), \mathcal{O}_Y)) \to \mathrm{H}^2(Y, \mathcal{O}_Y(-2)) \to 0.$$

Using Serre duality and the above isomorphisms we rewrite this as

$$0 \to \mathrm{Ext}^1_Y(\mathcal{I}_{Z|Y}(2), \mathcal{O}_Y) \to \mathrm{H}^0(Y, \omega_Z)^* \to \mathrm{H}^0(Y, \mathcal{O}_Y(1))^* \to 0.$$

We may choose coordinates so that Z is the union of $d+2$ points of a projective coordinate system, so that λ is the vector $(1, \dots, 1, -1)$ in $\mathrm{H}^0(Y, \omega_Z)^*$, which shows that λ is non-zero at any point of Z.

Therefore, λ corresponds to a global section:

$$\mathcal{O}_Y \to \mathcal{E}xt^1_{\mathcal{O}_Y}(\mathcal{I}_{Z|Y}(2), \mathcal{O}_Y) \simeq \omega_Z,$$

which is non-zero at any point z of Z. Since z is locally defined by two equations, the sheaf $\mathcal{E}xt^1_{\mathcal{O}_Y}(\mathcal{I}_{z|Y}(2), \mathcal{O}_Y)$ is one-dimensional at z, generated by the extension given by the Koszul complex of these equations; so the middle term of such extension is $\mathcal{F}_Z$. Hence $\mathcal{F}_Z$ is locally free around any point z of Z.

The fact that the sheaf $\mathcal{F}_Z$ is Ulrich follows from [ESW03, Proposition 2.1]. Indeed, we need to prove that $\mathrm{H}^i(Y, \mathcal{F}_Z(-1)) = \mathrm{H}^i(Y, \mathcal{F}_Z(-2)) = 0$ for all i. Since $\mathcal{F}_Z$ is locally free of rank 2 and clearly $\wedge^2 \mathcal{F}_Z \simeq \mathcal{O}_Y(2)$, we have:

$$\mathrm{H}^{2-i}(Y, \mathcal{F}_Z(-2))^* \simeq \mathrm{H}^i(Y, \mathcal{F}_Z^\vee(2) \otimes \omega_Y) \simeq \mathrm{H}^i(Y, \mathcal{F}_Z(-1)),$$

so it will be enough to prove one set of vanishing conditions. This amounts to checking that the map $\mathrm{H}^1(Y, \mathcal{I}_{Z|Y}(1)) \to \mathrm{H}^2(Y, \mathcal{O}_Y(-1))$ is an isomorphism. We observe that this map is Serre-dual of the map $\mathrm{H}^0(Y, \mathcal{O}_Y) \to \mathrm{Ext}^1_Y(\mathcal{I}_{Z|Y}(2), \mathcal{O}_Y)$ that sends the identity to λ, and therefore it is an isomorphism.

Otherwise, one may deduce that $\mathcal{F}_Z$ is Ulrich by the form of the minimal graded free resolution of the ideal of Z, which can be extracted from [MRPL12]. ☐

Lemma 8.3. *Let $\mathcal{F}$ be an Ulrich sheaf on an m-dimensional closed subscheme $X \subset \mathbb{P}^n$. Then $\mathcal{F}$ is semistable and any destabilizing subsheaf of $\mathcal{F}$ is Ulrich.*

Proof. This follows again from [ESW03, Proposition 2.1]. Indeed, first note that, since $\mathcal{F}$ is Ulrich, it is also locally Cohen-Macaulay and therefore pure. Next, choosing a finite linear projection $\pi : X \to \mathbb{P}^m$, we have $\pi_*(\mathcal{F}) \simeq \mathcal{O}^u_{\mathbb{P}^m}$ for some integer u. Put $\chi_m = \mathrm{p}(\mathcal{O}_{\mathbb{P}^m})$ and $d = \deg(X)$.

Suppose that $\mathcal{F}'$ is a proper subsheaf of $\mathcal{F}$ with $\mathrm{p}(\mathcal{F}') > \mathrm{p}(\mathcal{F})$. Since π is finite, we have:

$$P(\mathcal{F}', t) \frac{\mathrm{rk}(\mathcal{F})}{u\, \mathrm{rk}(\mathcal{F}')} > \chi_m(t),$$

and, because $\mathcal{O}^u_{\mathbb{P}^m}$ is semistable, $P(\mathcal{F}', t)/\mathrm{rk}(\pi_*(\mathcal{F}')) \leq \chi_m(t)$. But this contradicts the equality

$$d = \frac{\mathrm{rk}(\pi_*(\mathcal{F}'))}{\mathrm{rk}(\mathcal{F}')} = \frac{u}{\mathrm{rk}(\mathcal{F})}.$$

This shows that $\mathcal{F}$ is semistable. Moreover, if $\mathrm{p}(\mathcal{F}') = \mathrm{p}(\mathcal{F})$, then $\mathrm{p}(\pi_*(\mathcal{F}')) = \chi_m(t)$ so $\pi_*(\mathcal{F}') \simeq \mathcal{O}^{u'}_{\mathbb{P}^m}$ for some integer u' because $\mathcal{O}^u_{\mathbb{P}^m}$ is polystable, which implies that $\mathcal{F}'$ is Ulrich. ☐

Proposition 8.4. *The set Z can be chosen so that $\mathcal{F}_Z$ is stable.*

Proof. We know that $\mathcal{F}_Z$ is Ulrich so either $\mathcal{F}_Z$ is stable or there exist $\mathcal{A}$ and $\mathcal{B}$ Ulrich sheaves of rank 1 such that $\mathcal{F}_Z$ fits into:

$$0 \to \mathcal{A} \to \mathcal{F}_Z \to \mathcal{B} \to 0.$$

Note that $\mathcal{A}$ is reflexive as $\mathcal{B}$ is torsion-free and $\mathcal{F}_Z$ is locally free. Therefore, a global section of $\mathcal{A}$ vanishes along a subscheme of Y which is Cohen-Macaulay of dimension one (see [Har94, Proposition 2.8 and 2.9]).

By construction, the global section $s \in \mathrm{H}^0(Y, \mathcal{F}_Z)$ associated with Z vanishes precisely on Z. Therefore s cannot lie in $\mathrm{H}^0(Y, \mathcal{A}) \subset \mathrm{H}^0(Y, \mathcal{F}_Z)$, as s would then vanish in codimension 1. Hence we can construct the following commutative diagram:

$$\begin{array}{ccccccccc} & & & & \mathcal{O}_Y & = & \mathcal{O}_Y & & \\ & & & & \downarrow s & & \downarrow & & \\ 0 & \to & \mathcal{A} & \to & \mathcal{F}_Z & \to & \mathcal{B} & \to & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \to & \mathcal{A} & \to & \mathcal{I}_{Z|Y}(2) & \to & \mathcal{T} & \to & 0 \end{array}$$

where $\mathcal{T}$ is a torsion sheaf defined by the diagram. This tells us that:

$$\mathrm{H}^0(Y, \mathcal{I}_{Z|Y} \otimes \mathcal{A}^\vee(2)) \neq 0,$$

namely Z lies on a divisor D from the linear system $|\mathcal{A}^\vee(2)|$. Our goal is to prove that Z can be chosen so that it lies on no such divisor. Notice that, by [CHGS12, Lemma 2.4], $\mathcal{A}^\vee(2)$ is also an Ulrich sheaf of rank one.

Assume first that Y is normal. For each fixed Ulrich sheaf $\mathcal{L}$ of rank one, we know that $\dim_{\boldsymbol{k}} \mathrm{H}^0(Y, \mathcal{L}) = d$, and each non-zero global section of $\mathcal{L}$ vanishes along a Weil divisor $D \subset Y$. We view isomorphism classes of such sheaves as elements of the divisor class group of Y, which is identified with the group $\mathrm{APic}(Y)$ of generalized divisors on Y, see [Har86, §2].

Each linear system $|\mathcal{L}|$ has dimension $d-1$. For each D in $|\mathcal{L}|$, taking all non-degenerate smooth subschemes $Z \subset Y$ lying in D we obtain a non-empty open subset of the Hilbert scheme $\mathrm{Hilb}_{d+2}(D)$ which is $(d+2)$-dimensional. The union of these such Z for all choices of D in $|\mathcal{L}|$ forms a subscheme of $\mathrm{Hilb}_{d+2}(Y)$ which is of dimension at most $(d+2)+\dim|\mathcal{L}| = 2d+1$. But the main component of $\mathrm{Hilb}_{d+2}(Y)$ (that is, the component containing smooth subschemes) has dimension $2d+4$, so we may choose Z not lying in any $D \in |\mathcal{L}|$. Finally, since the divisor class group of a normal del Pezzo surface is discrete, we may choose Z away from the union, over all divisor classes arising from Ulrich sheaves $\mathcal{L}$, of the subschemes of $\mathrm{Hilb}_{d+2}(Y)$ associated with D lying in $|\mathcal{L}|$. So $\mathcal{F}_Z$ is stable if Z is general enough.

Now let us assume that Y is not normal. We know by [Rei94] that Y is normalized by a surface $\bar{Y} \subset \mathbb{P}^{d+1}$ of minimal degree d, the normalization map $\bar{Y} \to Y$ being induced by a projection $\mathbb{P}^{d+1} \to \mathbb{P}^d$. The normalization is an isomorphism away from a conic in $\bar{Y}$ which is mapped onto the singular locus of Y, which in turn is a line L. Moreover, the surface $\bar{Y}$ is smooth as otherwise, being of minimal degree, it would have to be a cone, but then Y would be a cone too, which we excluded.

Given an Ulrich sheaf $\mathcal{L}$ of rank one, again we have $\dim_{\boldsymbol{k}} \mathrm{H}^0(Y, \mathcal{L}) = d$, and we choose a non-zero global section of $\mathcal{L}$. This vanishes along a Weil divisor $D \subset Y$ of degree d, which contains a structure of multiplicity $e \leq d$ over L. Removing this structure from D we obtain an effective generalized divisor D_0, whose class lies in $\mathrm{APic}(Y)$, see [Har86, Proposition 2.12]. Since Z is disjoint from L, it will be enough to prove that we may choose Z away from any divisor D_0 of degree $d-e$, and obviously it suffices to show that this holds for D_0 of degree d.

To do this we use the explicit description of $\mathrm{APic}(\bar{Y})$ given by [HP15, Theorem 4.1 and Proposition 4.2]. Indeed, $\bar{Y}$ is either a Veronese surface in $\mathbb{P}^5$ (and thus $d = 4$) or a rational normal scroll of degree $d \geq 3$, and D_0 is the image of an effective divisor D_0' of degree d in $\bar{Y}$. Also, taking a hyperplane section $C \simeq \mathbb{P}^1$ of $\bar{Y}$, we get:

$$0 \to \mathcal{O}_{\bar{Y}}(D' - H) \to \mathcal{O}_{\bar{Y}}(D') \to \mathcal{O}_C(D') \to 0,$$

so $\dim_{\boldsymbol{k}} \mathrm{H}^0(\bar{Y}, \mathcal{O}_{\bar{Y}}(D')) \leq d + 2$, as $\deg(D' - H) \leq 0$ and $\mathcal{O}_C(D') \simeq \mathcal{O}_{\mathbb{P}^1}(\deg(D'))$, with $\deg(D') \leq d$.

Therefore, since there are finitely many effective divisor classes of degree at most d in $\bar{Y}$, if we choose $Z' \subset \bar{Y}$ to be a non-degenerate set of $d + 2$ distinct points lying away from all divisors D' in those classes, the image Z of Z' in Y will be contained in no generalized divisor D_0 of degree at most d. We conclude that $\mathcal{F}_Z$ is stable. □

Lemma 8.5. *We may choose Z and Z' sets of $d + 2$ points of Y such that $\mathcal{E} = \mathcal{F}_Z$, $\mathcal{F} = \mathcal{F}_{Z'}$ are non-isomorphic stable Ulrich bundles of rank 2 on Y. In this case:*

$$\mathrm{Hom}_Y(\mathcal{E}, \mathcal{F}) = \mathrm{Hom}_Y(\mathcal{F}, \mathcal{E}) = 0,$$
$$\dim_{\boldsymbol{k}} \mathrm{Ext}^1_Y(\mathcal{E}, \mathcal{F}) = 4.$$

Proof. We have proved so far that, for Z general enough in the main component of $\mathrm{Hilb}_{d+2}(Y)$, the sheaf $\mathcal{F}_Z$ is a stable locally free Ulrich sheaf of rank 2. Given such Z, choosing a non-zero global section of $\mathcal{F}_Z$ gives a map from an open dense subset of $\mathbb{P}(\mathrm{H}^0(Y, \mathcal{F}_Z)) \simeq \mathbb{P}^{2d-1}$ to the main component of $\mathrm{Hilb}_{d+2}(Y)$ associating with the section its vanishing locus. This map cannot be surjective by dimension reasons, so we can take Z' general enough, lying away from the image of this map and such that $\mathcal{F}_{Z'}$ is also a stable locally free Ulrich sheaf of rank 2. Because Z' is not the vanishing locus of a global section of $\mathcal{F}_Z$, we have that $\mathcal{F}_Z$ and $\mathcal{F}_{Z'}$ are not isomorphic.

The first two statements concerning morphisms are clear since $\mathcal{E}$ and $\mathcal{F}$ are stable with the same slope and not isomorphic. For the last statement, since $\mathcal{E}$ is locally free, we have $\mathrm{Ext}^1_Y(\mathcal{E}, \mathcal{F}) \simeq \mathrm{H}^1(Y, \mathcal{E}^\vee \otimes \mathcal{F})$. Tensoring the short exact sequence 8.1 by $\mathcal{E}^\vee$ and considering the associated long exact sequence of global sections, since $\mathrm{H}^1(Y, \mathcal{E}^\vee) = \mathrm{H}^2(Y, \mathcal{E}^\vee) = 0$, we get an isomorphism

$$\mathrm{Ext}^1_Y(\mathcal{E}, \mathcal{F}) \simeq \mathrm{H}^1(Y, \mathcal{E}^\vee \otimes \mathcal{I}_{Z|Y}(2)). \tag{8.2}$$

On the other hand, the exact sequence defining $Z \subset Y$ twisted by $\mathcal{O}_Y(2)$ reads:

$$0 \to \mathcal{I}_{Z|Y}(2) \to \mathcal{O}_Y(2) \to \mathcal{O}_Z(2) \to 0. \tag{8.3}$$

Taking into account that $\mathcal{E}^\vee \simeq \mathcal{E}(-2)$ we obtain $\mathrm{H}^0(Y, \mathcal{E}^\vee \otimes \mathcal{I}_{Z|Y}(2)) = 0$. Then, tensoring (8.3) by $\mathcal{E}^\vee$, taking global sections and combining with (8.2) we get:

$$0 \to \mathrm{H}^0(Y, \mathcal{E}^\vee(2)) \to \mathrm{H}^0(Y, \mathcal{E}^\vee \otimes \mathcal{O}_Z(2)) \to \mathrm{Ext}^1_Y(\mathcal{E}, \mathcal{F}) \to 0.$$

Now we know that $\mathcal{E}^\vee(2) \simeq \mathcal{E}$ has $2d$ independent global sections, as $\mathcal{E}$ is Ulrich. On the other hand, since Z has length $d + 2$, so that $\mathcal{E}^\vee \otimes \mathcal{O}_Z(2)$ is just a vector space of rank 2 concentrated at $d + 2$ points, hence $\dim_{\boldsymbol{k}} \mathrm{H}^0(Y, \mathcal{E}^\vee \otimes \mathcal{O}_Z(2)) = 2d + 4$. We conclude that $\dim_{\boldsymbol{k}} \mathrm{Ext}^1_Y(\mathcal{E}, \mathcal{F}) = 4$. □

Theorem C combined with the results of this section yields the proof of Theorem 8.1.

9. Varieties of higher degree

In this final section we prove our main theorem for subschemes of degree higher than almost minimal, that is, in the range $\Delta(X) > 1$. Let again $X \subset \mathbb{P}^n$ be a reduced ACM subscheme of dimension $m \geq 1$ and degree $d > n - m + 2$, or in other words $\Delta(X) > 1$. Thus $p \geq 2$ (see [Fuj90, (6.4.5)]).

We take a linear section Y of dimension 1, which we may assume to be reduced, so Y is an ACM curve of arithmetic genus $p \geq 2$. We note that the proof of [AK80, Proposition 3.5] applies to Y as it only uses the fact that the projective curve Y is reduced and connected. Then, we may find a line bundle $\mathcal{L}_1$ on Y satisfying:

$$\mathrm{H}^0(Y,\mathcal{L}_1) = \mathrm{H}^1(Y,\mathcal{L}_1) = 0.$$

Therefore, $\mathcal{L}_1(1)$ is an Ulrich line bundle by [ESW03, Theorem 4.3] Clearly, $\mathrm{Hom}_Y(\mathcal{L}_1,\mathcal{L}_1) \simeq \mathrm{H}^0(Y,\mathcal{O}_Y) \simeq \boldsymbol{k}$ because $\mathcal{L}_1$ is invertible, so $\mathcal{L}_1$ is simple. Moreover the space $\mathrm{Ext}^1_Y(\mathcal{L}_1,\mathcal{L}_1) \simeq \mathrm{H}^1(Y,\mathcal{O}_Y)$ has dimension $p \geq 2$ and $\mathrm{Ext}^2_Y(\mathcal{L}_1,\mathcal{L}_1) = 0$. So we may take general flat deformations of $\mathcal{L}_1$ to get sheaves $\mathcal{L}_2$, $\mathcal{L}_3$, $\mathcal{L}_4$, not isomorphic to one another nor to $\mathcal{L}_1$, which will also be invertible (hence simple) and satisfy $\mathrm{H}^0(Y,\mathcal{L}_i) = \mathrm{H}^1(Y,\mathcal{L}_i) = 0$ for all i by semicontinuity.

We claim that we may assume $\mathrm{Hom}_Y(\mathcal{L}_i,\mathcal{L}_j) = 0$ for $i \neq j$. Indeed, first note that the degree of the line bundle $\mathcal{L}_1$ on each irreducible component of Y is constant along small deformations so we may assume that all the sheaves $\mathcal{L}_i$ have the same degree along each component. Put $\mathcal{M} = \mathcal{L}_i^\vee \otimes \mathcal{L}_j$, take a non-zero morphism $\mathcal{L}_i \to \mathcal{L}_j$ and rewrite it as a nonzero global section $\varphi : \mathcal{O}_Y \to \mathcal{M}$. The restriction of $\mathcal{M}$ to any irreducible component of Y is a line bundle of degree 0, so φ is an isomorphism as soon as its restriction to all such components is non-zero. Set Y' for the union in Y of the irreducible components of Y where φ is non-zero (and hence an isomorphism) and put Y'' for the closure in Y of $Y \setminus Y'$, $\mathcal{M}'' = \mathcal{M}|_{Y''}$. By assumption $Y' \neq \emptyset \neq Y''$. Then we have the commutative exact diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{I}_{Y'|Y} & \longrightarrow & \mathcal{O}_Y & \longrightarrow & \mathcal{O}_{Y''} & \longrightarrow & 0 \\ & & \downarrow \varphi' & & \downarrow \varphi & & \downarrow \varphi'' & & \\ 0 & \longrightarrow & \mathcal{I}_{Y'|Y} \otimes \mathcal{M} & \longrightarrow & \mathcal{M} & \longrightarrow & \mathcal{M}'' & \longrightarrow & 0 \end{array}$$

Here, $\varphi'' = \varphi|_{\mathcal{O}''_Y}$ is zero, while φ' is the restriction of φ to Y', tensored with the identity over $\mathcal{I}_{Y'|Y}$, and therefore is an isomorphism. Hence the snake lemma gives a splitting $\mathcal{O}_{Y''} \to \mathcal{O}_Y$ of the surjection $\mathcal{O}_Y \to \mathcal{O}_{Y''}$, so Y cannot be connected unless Y' or Y'' are empty, a contradiction.

As a consequence, we get that the space $\mathrm{Ext}^1(\mathcal{L}_i,\mathcal{L}_j) \simeq \mathrm{H}^1(Y,\mathcal{L}_i^\vee \otimes \mathcal{L}_j)$ has dimension $p-1$ for $i \neq j$. Now we may choose $\mathcal{A}$ and $\mathcal{B}$ as two sheaves given by non-trivial extensions:

$$0 \to \mathcal{L}_1(1) \to \mathcal{A} \to \mathcal{L}_2(1) \to 0,$$
$$0 \to \mathcal{L}_3(1) \to \mathcal{B} \to \mathcal{L}_4(1) \to 0.$$

It is clear that $\mathcal{A}$ and $\mathcal{B}$ are locally free Ulrich sheaves of rank 2. Also, the sheaves $\mathcal{A}$ and $\mathcal{B}$ are simple and satisfy (see for instance [PLT09, Proposition 5.1.3]:

$$\mathrm{Hom}_Y(\mathcal{A},\mathcal{B}) = \mathrm{Hom}_Y(\mathcal{B},\mathcal{A}) = 0.$$
$$\chi(\mathcal{A},\mathcal{B}) = \chi(\mathcal{B},\mathcal{A}) = 4(1-p).$$

We obtain the following:

$$\dim_{\boldsymbol{k}} \mathrm{Ext}^1_Y(\mathcal{A},\mathcal{B}) = \dim_{\boldsymbol{k}} \mathrm{Ext}^1_Y(\mathcal{B},\mathcal{A}) = 4(p-1) \geq 4.$$

Note that the non-vanishing condition of Theorem B reduces to $\mathrm{H}^0(Y,\omega_Y) \neq 0$, which is true because $p \geq 2$. Now Theorem C implies that X is of wild CM representation type.

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