A note on the natural density of product sets

Bull. London Math. Soc. 53 (2021) 1407–1413 doi:10.1112/blms.12506

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Abstract

Given two sets of natural numbers $\mathcal{A}$ and $\mathcal{B}$ of natural density 1, we prove that their product set $\mathcal{A}\cdot\mathcal{B} := \{ab : a \in \mathcal{A},\, b \in \mathcal{B}\}$ also has natural density 1. On the other hand, for any $\varepsilon > 0$, we show there are sets $\mathcal{A}$ of density $> 1-\varepsilon$ for which the product set $\mathcal{A}\cdot\mathcal{A}$ has density $< \varepsilon$. This answers two questions of Hegyvári, Hennecart and Pach.

1. *Introduction*

Given two sets of natural numbers $\mathcal{A}$ and $\mathcal{B}$, let $\mathcal{A}\cdot\mathcal{B} := \{ab : a \in \mathcal{A},\, b \in \mathcal{B}\}$ be their *product set*. Also, for any positive integer k, let $\mathcal{A}^k$ denote the k-fold product $\mathcal{A}\cdots\mathcal{A}$.

The problem of studying the cardinality of product sets has long been of interest in mathematics. The classic *multiplication table problem* due to Erdős [2, 3] asks for bounds on the cardinality M_n of the $n \times n$ multiplication table, that is, of the set $\{1,\dots,n\}^2$. Erdős showed that $M_n = o(n^2)$ and Ford [5], following earlier results of Tenenbaum [11], determined the exact order of magnitude of M_n. More recently [7], the second author of the present paper provided uniform bounds for $\#(\{1,\dots,n_1\}\cdots\{1,\dots,n_s\})$ holding for a wide range of $n_1,\dots,n_s \in \mathbb{N}$.

For more general sets $\mathcal{A}$, the problem of estimating $\#(\mathcal{A}\cap[1,x])^2$ was studied by Cilleruelo, Ramana, and Ramaré [1]. For example, they studied this problem when $\mathcal{A}$ is the set of shifted primes, the set of sums of two squares, and the set of shifted sums of two squares. Moreover, they computed the (almost sure) asymptotic behavior for $\#\mathcal{A}^2$ when $\mathcal{A}$ is a random subset of $\{1,\dots,n\}$ that contains each element of $\{1,\dots,n\}$ independently with probability $\delta \in (0,1)$. The third author of the present paper [10] extended this last result to the product of arbitrarily many sets, and Mastrostefano [9] gave a necessary and sufficient condition for having $\#\mathcal{A}^2 \sim (\#\mathcal{A})^2/2$ almost surely.

Hegyvári, Hennecart and Pach [6] considered the analogous problem for infinite sets of natural numbers. In this context, the role of the cardinality is played by the *natural density* $\mathbf{d}(\mathcal{A})$ of a set $\mathcal{A}$, defined as usual by

$$\mathbf{d}(\mathcal{A}) = \lim_{x\to\infty} \frac{\#\mathcal{A}\cap[1,x]}{x}.$$

They asked the following questions ([6, Questions 3 and 2], respectively):

Received 30 June 2020; published online 7 May 2021.

2020 *Mathematics Subject Classification* 11B05 (primary).

S.B. is partially supported by PRIN 2017 'Geometric, algebraic and analytic methods in arithmetic' and by INdAM. D.K. is partially supported by the Natural Sciences and Engineering Research Council of Canada (Discovery Grant 2018-05699) and by the Fonds de recherche du Québec – Nature et technologies (projet de recherche en équipe - 256442). C.S. was supported by an INdAM postdoctoral fellowship.

Question 1. If $\mathcal{A}$ is a set of natural numbers of density 1, is it true that $\mathcal{A}^2$ also has density 1?

Question 2. Is it true that $\inf_{\mathcal{A}\subset\mathbb{N}:\ \mathbf{d}(\mathcal{A})=\alpha}\mathbf{d}(\mathcal{A}^2)=0$ for any $\alpha\in[0,1)$, or at least for $\alpha\in[0,\alpha_0)$ for some $\alpha_0\in(0,1)$?

Clearly, Question 1 has an affirmative answer if $1\in\mathcal{A}$, and Hegyvári, Hennecart and Pach showed that it also suffices that $\mathcal{A}$ contains an infinite subset of mutually coprime integers $a_1 < a_2 < \cdots$ such that $\sum_{i=1}^{\infty} a_i^{-1} = +\infty$. Here, we show that the answer is 'yes' in full generality.

Theorem 1. *Let* $\mathcal{A},\mathcal{B}\subseteq\mathbb{N}$. *If* $\mathbf{d}(\mathcal{A})=\mathbf{d}(\mathcal{B})=1$, *then* $\mathbf{d}(\mathcal{A}\cdot\mathcal{B})=1$.

Corollary. *If* $\mathcal{A}\subset\mathbb{N}$ *is such that* $\mathbf{d}(\mathcal{A})=1$, *then* $\mathbf{d}(\mathcal{A}^k)=1$ *for each* $k=2,3,\ldots$

Remark. In fact, the case $\mathcal{A}=\mathcal{B}$ of Theorem 1 implies easily the general case. Indeed, if $\mathbf{d}(\mathcal{A})=\mathbf{d}(\mathcal{B})=1$, then $\mathbf{d}(\mathcal{A}\cap\mathcal{B})=1$. In addition, if $(\mathcal{A}\cap\mathcal{B})^2$ has density 1, then so does $\mathcal{A}\cdot\mathcal{B}$.

As it will be clear from the proof, the difference in the density of $\mathbf{d}(\mathcal{A}^2)$ with respect to Erdős's multiplication table problem lies in the fact that many elements of $\mathcal{A}^2$ come from very 'unbalanced' products, meaning products ab such that the sizes of a and b are completely different.

Let us now turn to Question 2. We will answer it in a strong form that shows, among other things, that the condition that $\mathbf{d}(\mathcal{A})=1$ in Theorem 1 cannot be relaxed.

Theorem 2. *For* $\alpha\in[0,1]$, *we have*

$$\inf_{\mathcal{A}\subseteq\mathbb{N}:\ \mathbf{d}(\mathcal{A})=\alpha}\mathbf{d}(\mathcal{A}^2)=\begin{cases}0 & \text{if } \alpha<1,\\ 1 & \text{if } \alpha=1.\end{cases}$$

2. *Preliminaries*

Notation. We employ Landau's notation $f=O(g)$ and Vinogradov's notation $f\ll g$ both to mean that $|f|\leqslant C|g|$ for a some constant $C>0$. Moreover, we write $f\asymp g$ to mean that $f\ll g$ and $g\ll f$. The notation $f=o(g)$ as $x\to a$ (respectively, $f\sim g$ as $x\to a$) means that $\lim_{x\to a} f(x)/g(x)=0$ (respectively, $=1$). Given an integer n, we write $P^-(n)$ and $P^+(n)$ for its smallest and largest prime factors, respectively, with the convention that $P^-(1)=\infty$ and $P^+(1)=1$. If $P^+(n)\leqslant y$, we say that n is *y-smooth*, and if $P^-(n)>y$, we say that it is *y-rough*. As usual, we let $\Phi(x,y)$ denote the number of y-rough numbers in $[1,x]$. Given any integer n, we may write it uniquely as $n=ab$ with $P^+(a)\leqslant y<P^-(b)$. We then call a and b the *y-smooth and y-rough part* of n, respectively. Finally, we let $\Omega(n)$ denote the number of prime factors of n counted with multiplicity.

We need some standard lemmas. We give their proofs for the sake of completeness.

Lemma 2.1. *For* $x\geqslant y>1$, *we have* $\Phi(x,y)\ll x/\log y$.

Proof. This follows for example from [8, Theorem 14.2] with $f(n)=1_{P^-(n)>y}$. □

Lemma 2.2. *Uniformly for* $x\geqslant y^2\geqslant 1$ *and* $u\geqslant 1$, *we have*

$$\#\{n\leqslant x : \exists d|n \text{ such that } P^+(d)\leqslant y^{1/u} \text{ and } d>y\}\ll x\cdot(e^{-u}+y^{-1/3}).$$

Proof. Without loss of generality, $u \geqslant 4$. Let $\mathcal{B}$ denote the set of $n \in \mathbb{Z} \cap [1, x]$ that have a $y^{1/u}$-smooth divisor $d > y$. Given $n \in \mathcal{B}$, let $p_1 \leqslant p_2 \leqslant \cdots \leqslant p_k$ be the sequence of prime factors of n of size $\leqslant y^{1/u}$ listed in increasing order and according to their multiplicity. By our assumption on n, we must have $p_1 \cdots p_k > y$. Let j be the smallest integer such that $p_1 \cdots p_j > y$. We must have $j \geqslant 5$ because all factors p_i are $\leqslant y^{1/u} \leqslant y^{1/4}$. We then set $a = p_1 \cdots p_{j-2}$, $p = p_{j-1}$, and $b = n/(ap)$, so that $a > y/(p_{j-1}p_j) \geqslant \sqrt{y}$, $ap \leqslant y$, and $P^+(a) \leqslant p \leqslant P^-(b)$. Consequently,

$$\#\mathcal{B} \leqslant \sum_{p \leqslant y^{1/u}} \sum_{\substack{P^+(a) \leqslant p \\ \sqrt{y} < a \leqslant y/p}} \sum_{\substack{b \leqslant x/(ap) \\ P^-(b) \geqslant p}} 1 \ll \sum_{p \leqslant y^{1/u}} \sum_{\substack{P^+(a) \leqslant p \\ a > \sqrt{y}}} \frac{x}{ap \log p} \tag{1}$$

by Lemma 2.1. If we let $\varepsilon_p = \min\{2/3, 2/\log p\}$, then Rankin's trick implies

$$\frac{\#\mathcal{B}}{x} \ll \sum_{p \leqslant y^{1/u}} \sum_{P^+(a) \leqslant p} \frac{(a/\sqrt{y})^{\varepsilon_p}}{ap \log p} = \sum_{p \leqslant y^{1/u}} \frac{y^{-\varepsilon_p/2}}{p \log p} \sum_{P^+(a) \leqslant p} \frac{1}{a^{1-\varepsilon_p}}.$$

The sum over a equals $\prod_{q \leqslant p} (1 - q^{-1+\varepsilon_p})^{-1}$ with q denoting a prime number. Since $q^{\varepsilon_p} = 1 + O(\log q/\log p)$ for $q \leqslant p$, Mertens' estimates [8, Theorem 3.4] imply that the sum over a is $\ll \log p$. We conclude that

$$\frac{\#\mathcal{B}}{x} \ll y^{-1/3} + \sum_{100 < p \leqslant y^{1/u}} \frac{e^{-\log y/\log p}}{p} \leqslant y^{-1/3} + \sum_{j \geqslant 1} \sum_{y^{1/(u(j+1))} < p \leqslant y^{1/(uj)}} \frac{e^{-ju}}{p}$$
$$\ll y^{-1/3} + \sum_{j \geqslant 1} e^{-ju} \ll y^{-1/3} + e^{-u}$$

using Mertens' estimates once again. This completes the proof. □

LEMMA 2.3. *Let $y \geqslant 2$ and $\lambda \in [0, 1.99]$, and set $Q(\lambda) = \lambda \log \lambda - \lambda + 1$ for $\lambda > 0$ and $Q(0) = 0$. If $0 \leqslant \lambda \leqslant 1$, then*

$$\prod_{p \leqslant y} \left(1 - \frac{1}{p}\right) \sum_{\substack{P^+(m) \leqslant y \\ \Omega(m) \leqslant \lambda \log\log y}} \frac{1}{m} \ll (\log y)^{-Q(\lambda)},$$

whereas if $1 \leqslant \lambda \leqslant 1.99$, then

$$\prod_{p \leqslant y} \left(1 - \frac{1}{p}\right) \sum_{\substack{P^+(m) \leqslant y \\ \Omega(m) \geqslant \lambda \log\log y}} \frac{1}{m} \ll (\log y)^{-Q(\lambda)}.$$

Proof. The result is trivial if $\lambda = 0$ by Mertens' estimates [8, Theorem 3.4], so assume $\lambda > 0$. If $0 < \lambda \leqslant 1$, then

$$\sum_{\substack{P^+(m) \leqslant y \\ \Omega(m) \leqslant \lambda \log\log y}} \frac{1}{m} \leqslant \sum_{P^+(m) \leqslant y} \frac{\lambda^{\Omega(m) - \lambda \log\log y}}{m} = (\log y)^{-\lambda \log \lambda} \prod_{p \leqslant y} \left(1 - \frac{\lambda}{p}\right)^{-1}$$
$$\asymp (\log y)^{-Q(\lambda)} \prod_{p \leqslant y} \left(1 - \frac{1}{p}\right)^{-1}$$

where we used Mertens' estimates once again. Similarly, if $1 \leqslant \lambda \leqslant 1.99$, then

$$\sum_{\substack{P^+(m)\leqslant y \\ \Omega(m)\geqslant \lambda \log\log y}} \frac{1}{m} \leqslant \sum_{P^+(m)\leqslant y} \frac{\lambda^{\Omega(m)-\lambda\log\log y}}{m} \asymp (\log y)^{-Q(\lambda)} \prod_{p\leqslant y}\left(1-\frac{1}{p}\right)^{-1}.$$

This completes the proof. □

LEMMA 2.4. *Let $\mathcal{P}$ be a set of primes such that $\sum_{p\in\mathcal{P}} 1/p < \infty$. Then*

$$\mathbf{d}(\{n \in \mathbb{N} : p|n \ \Rightarrow p \notin \mathcal{P}\}) = \prod_{p\in\mathcal{P}}\left(1-\frac{1}{p}\right).$$

Proof. The number of integers $n \leqslant x$ with a prime divisor $p > \log x$ from $\mathcal{P}$ is

$$\leqslant \sum_{p>\log x,\, p\in\mathcal{P}} \frac{x}{p} = o(x) \qquad \text{as} \quad x \to \infty,$$

because $\sum_{p\in\mathcal{P}} 1/p$ converges. Hence, if we write $\mathcal{P}' = \mathcal{P} \cap [1, \log x]$, then

$$\#\{n \leqslant x : p|n \ \Rightarrow p \notin \mathcal{P}\} = \#\{n \leqslant x : p|n \ \Rightarrow p \notin \mathcal{P}'\} + o(x) = x \prod_{p\in\mathcal{P}'}\left(1-\frac{1}{p}\right) + o(x)$$

from the inclusion–exclusion principle that has $\leqslant 2^{\#\mathcal{P}'} \leqslant 2^{\log x} = o(x)$ steps (for example, see [8, Theorem 2.1]). Since $\prod_{p\in\mathcal{P}\setminus\mathcal{P}'}(1-1/p) \sim 1$ by our assumption that $\sum_{p\in\mathcal{P}} 1/p < \infty$, the proof is complete. □

3. *Proof of Theorem 1*

Assume x is sufficiently large and let $y = y(x)$ and $u = u(x)$ to be chosen later, with $y, u \to +\infty$ slowly as $x \to +\infty$. In particular, $y \leqslant \sqrt{x}$. In the following, for the sake of notation, we will often omit the dependence on x, y, u.

With a small abuse of notation, given an integer n, let n_{smooth} denote its $y^{1/u}$-smooth part and let n_{rough} denote its $y^{1/u}$-rough part. We then set

$$\mathcal{N} = \{n \leqslant x : n_{\text{smooth}} \leqslant y\}.$$

By Lemma 2.2, we have $\#\mathcal{N} \sim x$ as $x \to \infty$. Therefore, in order to prove Theorem 1, it is enough to show that

$$\#\mathcal{C} = o(x), \quad \text{where} \quad \mathcal{C} := \mathcal{N} \setminus (\mathcal{A}\cdot\mathcal{B}).$$

Let $n \in \mathcal{C}$. Since $n = n_{\text{smooth}} \cdot n_{\text{rough}}$, we must have that either $n_{\text{smooth}} \notin \mathcal{A}$ or $n_{\text{rough}} \notin \mathcal{B}$. Consequently,

$$\#\mathcal{C} \leqslant S_1 + S_2$$

with

$$S_1 := \#\{n \in \mathcal{N} : n_{\text{smooth}} \notin \mathcal{A}\} \quad \text{and} \quad S_2 := \#\{n \in \mathcal{N} : n_{\text{rough}} \notin \mathcal{B}\}.$$

Let us first bound S_1. Letting $m = n_{\mathrm{smooth}}$, we have

$$S_1 \leqslant \sum_{m\leqslant y,\, m\notin\mathcal{A}} \Phi(x/m, y^{1/u}) \ll \frac{ux}{\log y} \sum_{m\leqslant y,\, m\notin\mathcal{A}} \frac{1}{m}$$

by Lemma 2.1. Since we have assumed that $\mathbf{d}(\mathcal{A}) = 1$, we must have that $\mathbf{d}(\mathbb{N}\setminus\mathcal{A}) = 0$ and thus

$$\alpha(t) := \frac{1}{\log t} \sum_{m\leqslant t,\, m\notin\mathcal{A}} \frac{1}{m} \ \to\ 0 \qquad \text{as} \quad t\to\infty.$$

Hence, setting $u = u(y) := \alpha(y)^{-1/2}$, we have $u \to +\infty$ and $S_1 = o(x)$ as $x \to +\infty$.

Let us now bound S_2. Writing $m' = n_{\mathrm{rough}}$, we have

$$S_2 \leqslant \sum_{m\leqslant y} \#\{m' \leqslant x/m : m' \notin \mathcal{B}\}.$$

By hypothesis, we have $\mathbf{d}(\mathcal{B}) = 1$, so that $\mathbf{d}(\mathbb{N}\setminus\mathcal{B}) = 0$. Thus

$$\beta(t) := \sup_{s\,\geqslant\, t} \frac{\#((\mathbb{N}\setminus\mathcal{B})\cap[1,s])}{s} \ \to\ 0 \qquad \text{as} \quad t\to\infty.$$

Hence, setting $y := \min(x^{1/2}, \exp(\beta(x^{1/2})^{-1/2}))$, we have $y \to +\infty$ as $x \to +\infty$ and

$$S_2 \leqslant \sum_{d\,\leqslant\, y} \beta(x/d)\cdot\frac{x}{d} \leqslant x\beta(x/y)\sum_{d\,\leqslant\, y}\frac{1}{d} \ll x\beta(x^{1/2})\log y \leqslant x\beta(x^{1/2})^{1/2} = o(x).$$

In conclusion, $\#\mathcal{C} = o(x)$, as desired.

REMARK. The proof of Theorem 1 can be made quantitative. For example, if one has $\#\{n\leqslant x : n\notin\mathcal{A}\}, \#\{n\leqslant x : n\notin\mathcal{B}\} \ll x(\log x)^{-a}$ for some fixed $0<a<1$, then taking $y = \exp((\log x)^{\frac{a}{1+a}})$ and $u = \log\log x$ in the above argument yields

$$\#\{n\leqslant x : n\notin\mathcal{A}\cdot\mathcal{B}\} \ll xe^{-u} + \frac{xu}{(\log y)^a} + \frac{x\log y}{(\log x)^a} \ll x(\log x)^{-\frac{a^2}{1+a}+o(1)}.$$

An interesting question is to determine the optimal exponent of $\log x$ in this upper bound.

4. *Proof of Theorem* 2

The case $\alpha = 1$ follows from Theorem 1, whereas for the case $\alpha = 0$ one can just observe that $\mathbf{d}(\emptyset) = \mathbf{d}(\emptyset^2) = 0$. We may thus assume $\alpha \in (0,1)$. Given any $\varepsilon > 0$, we need to construct a set $\mathcal{A}$ of density α such that the density of $\mathcal{A}^2$ exists and is smaller than ε.

Let $k\in\mathbb{N}$, $y\geqslant 1$ and a set of primes $\mathcal{P}\subset(y,+\infty)$ with $\sum_{p\in\mathcal{P}} 1/p < \infty$ to be chosen later. Using the notation $\Omega_y(n) = \sum_{p^a|n,\, p\leqslant y} 1$, let us consider the sets

$$\mathcal{B}_{y,k,\mathcal{P}} := \big\{n\in\mathbb{N} : \Omega_y(n)\geqslant k,\ (n,p)=1\ \forall p\in\mathcal{P}\big\}.$$

The key property these sets have is that $\mathcal{B}_{y,k,\mathcal{P}}^2 = \mathcal{B}_{y,2k,\mathcal{P}}$.

Now, using Lemma 2.4 twice (once, with $\mathcal{P}_{\text{Lemma 2.4}} = \mathcal{P} \cup \{p \leqslant y\}$ and once with $\mathcal{P}_{\text{Lemma 2.4}} = \{p \leqslant y\}$), we find that

$$\mathbf{d}(\mathcal{B}_{y,k,\mathcal{P}}) = \prod_{p\in\mathcal{P}}\left(1-\frac{1}{p}\right)\prod_{p\leqslant y}\left(1-\frac{1}{p}\right)\sum_{\substack{P^+(m)\leqslant y\\ \Omega(m)\geqslant k}}\frac{1}{m} = \mathbf{d}(\mathcal{B}_{y,k,\emptyset})\prod_{p\in\mathcal{P}}\left(1-\frac{1}{p}\right).$$

Similarly,

$$\mathbf{d}(\mathcal{B}^2_{y,k,\mathcal{P}}) = \mathbf{d}(\mathcal{B}_{y,2k,\mathcal{P}}) = \prod_{p\in\mathcal{P}}\left(1-\frac{1}{p}\right)\mathbf{d}(\mathcal{B}_{y,2k,\emptyset}).$$

Now, take $y := \exp(\exp(4k/3))$, so that $k = \frac{3}{4}\log\log y$. For any fixed $\varepsilon > 0$, Lemma 2.3 implies that if k is sufficiently large in terms of α and ε, then $\mathbf{d}(\mathcal{B}_{y,k,\emptyset}) > \alpha$ and $\mathbf{d}(\mathcal{B}_{y,2k,\emptyset}) < \varepsilon$. Let us fix for the remainder of the proof such a choice of k. We then construct $\mathcal{P}$ in the following way: we take $p_1 > y$ to be the smallest prime such that $(1-1/p_1)\mathbf{d}(\mathcal{B}_{y,k,\emptyset}) > \alpha$, $p_2 > p_1$ the smallest prime such that $(1-1/p_1)(1-1/p_2)\mathbf{d}(\mathcal{B}_{y,k,\emptyset}) > \alpha$ and so on. Taking $\mathcal{P} := \{p_1, p_2, \dots\}$ we clearly have $\mathbf{d}(\mathcal{B}_{y,k,\emptyset})\prod_{p\in\mathcal{P}}(1-1/p) = \alpha$. Thus, $\mathbf{d}(\mathcal{B}_{y,k,\mathcal{P}}) = \alpha$ and $\mathbf{d}(\mathcal{B}^2_{y,k,\mathcal{P}}) < \varepsilon$, as desired.

REMARK. If $\mathbf{d}(\mathcal{A}^2)$ in Theorem 2 is replaced by the upper density $\overline{\mathbf{d}}(\mathcal{A}^2)$, then one could just take $\mathcal{A}$ to be any density α subset of $\{n \in \mathbb{N} : \Omega_y(n) \geqslant \frac{3}{4}\log\log y\}$ for y large enough. However, in general there is no guarantee that $\mathcal{A}^2$ has asymptotic density. For this reason, in order to prove Theorem 2, it is more convenient to construct explicit suitable sets $\mathcal{A}$.

Acknowledgements. The third author wishes to thank Paolo Leonetti for bringing the article of Hegyvári-Hennecart-Pach [6] to his attention. In addition, the first two authors of the paper would like to thank the Mathematisches Forschungsinstitut Oberwolfach for the hospitality; the paper was partially written there while attending a workshop in November 2019. S.B. is member of the INdAM group GNAMPA. C.S. is a member of the INdAM group GNSAGA, and of CrypTO, the Cryptography and Number Theory group of Politecnico di Torino.

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