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Research

The Generalized Wiener-Hopf Equations for the wave motion in angular regions: electromagnetic application

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In this work, we introduce a general method to deduce spectral functional equations and thus, the Generalized Wiener-Hopf Equations (GWHEs) for the wave motion in angular regions filled by arbitrary linear homogeneous media and illuminated by sources localized at infinity with application to electromagnetics.

The functional equations are obtained by solving vector differential equations of first order that model the problem. The application of the boundary conditions to the functional equations yields GWHEs for practical problems.

This paper shows the general theory and the validity of GWHEs in the context of electromagnetic applications with respect to the current literature. Extension to scattering problems by wedges in arbitrarily linear media in different physics will be presented in future works.

One contribution to a special feature Advances in Wiener-Hopf type techniques: theory and applications

Subject Areas:

applied mathematics, computational physics, mathematical physics, wave motion, electromagnetics

Keywords:

Wave motion, Wedge, Scattering, Wiener-Hopf method, Integral equations, Spectral Domain, Electromagnetics

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1. Introduction

The extension of the Wiener-Hopf (WH) technique in angular regions [1]- [5] demonstrated its efficacy to solve electromagnetic wave scattering problems in presence of geometries containing angular regions and/or stratified planar regions, see for instance [6]- [10] and reference therein.

This technique consists of three steps: 1) the deduction of functional equations in spectral domain of sub-regions that constitute the whole geometry of the problem, 2) the imposition of boundary conditions to get the Generalized Wiener-Hopf Equations (GWHEs) and, 3) the solution of the system of the WH equations using exact or semi-analytical/approximate techniques of factorization as the Fredholm factorization technique [11], [12], [5]- [10].

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This paper is focused on the first and second steps of the procedure showing a new general methodology. In particular, we deduce spectral functional equations and GWHEs for angular regions filled by arbitrary linear homogeneous media in a general framework, following the procedure first proposed in [3] with applications to electromagnetics.

The introduction of the GWHEs in angular region was inspired by Vekua in [13]. This book introduces the Hilbert generalized equations and shows that, with slight modifications, these equations can be solved using the same procedures developed for the solution of the functional equation for classical Hilbert problems. We note that these equations are more general than the ones defined in the WH method.

The GWHEs differ from the Classical Wiener-Hopf equations (CWHEs) for the definitions of the unknowns in spectral domain. While CWHEs introduce plus and minus functions that are always defined in the same complex plane, the GWHEs present plus and minus functions that are defined in different complex planes but related together. However, in several important practical cases, suitable mappings allow to redefine the plus and the minus functions of GWHEs in the same complex plane: for instance, in angular subregions see the mapping reported in [1]- [6]. With this transformation we ensure the remarkable property that GWHEs reduce to CWHEs.

When the problem can be formulated in terms of Helmholtz equations, the GWHEs are related to the difference equation of the Sommerfeld-Malyuzhinets method (SM): see for instance, in wedge problem [10] and references therein. In particular, the mapping $\eta = -k \cos w$ relates the spectral variables η and w, respectively defined in the WH equations using the Laplace transform and in the difference equations using the SM method. Passing from the η plane to the w plane (and vice-versa) is an expedient that allows to exploit solution properties of the same problem with two methods (WH factorization technique and SM difference equations). Hence, the analysis of problems with SM and WH methods determine a helpful synergy. This means that the study of scattering problems in presence of angular regions with different methods is fundamental. In particular, important improvements on the SM method are reported in the books of Babich, Bernard, Budaev, Lyalinov [15]- [18] and reference therein.

The introduction of the GWHEs in scattering problems by angular regions presents some aspects in common with the study of right bounded regions, see [5], [10] and references therein. In particular, several works on right-angled structures have been studied in terms of Riemann-Hilbert (RH) formulations [38]- [40] and relationship between RH and WH methods may be examined in depth. However, Wiener-Hopf and/or Riemann-Hilbert formulations of angular regions have been rarely considered in literature and fully interpreted. For what concerns the WH method, the last equation of the Example 5.15 of [14] at page 219 is a GWHE. In particular, Noble suggested the mapping $\eta = -k \cos w$ as a natural substitution to get the solution.

We also observe that Gautesen in numerous papers (see for example [19]- [22]) proposed the solution of the fundamental scattering problem from an elastic wedge, where the functional angular equations are substantially GWHEs although not defined in this way. This author provides efficient semi-analytical solutions of the spectral equations using the Cauchy decomposition formula in the spectral plane. His method can be considered an efficient technique to approximately solve GWHEs.

GWHEs were also introduced in [23]- [24] for solving the electromagnetic scattering problem of a Perfectly Electrical Conducting (PEC) wedge as well as of an impedance wedge. These authors are aware that their equations might be dealt with the factorization technique, however they proposed a solution based on the SM method and difference equations.

A last set of works concerning the introduction of GWHEs in wedge problems is [25]- [26]. The novelty of these works resides on the application of a mapping that provides a factorization method to solve difference equations in SM method for acoustic impenetrable wedge scalar scattering problems. We recall also that the factorization method to solve difference equations was, for example, proposed in [27]. We note that the mapping used in [25] resembles the one introduced in [1]- [6] but the motivation of its introduction is different. In particular, in [1]- [6] the mapping is introduced to systematically reduce GWHEs of general angular shaped region wave problems defined in Laplace domain to the usual classical WH equations [14].

As per rectangular regions, the WH equations of scattering problems in angular regions can be obtained using two strategies. The first method consists of formulating the problem in terms of integral equations in the natural domain using suitable Green's functions [28]. Since the formulation contains integral representations with convolutional kernels the application of the Fourier or Laplace transforms yields the WH equations in the spectral domain. The second method to obtain the WH equations in spectral domain is proposed by Jones [29], [14]. It is based on the application of the Fourier or the Laplace transforms directly to the partial differential equation formulation of the problem avoiding the necessity to study the Green's function representations in the natural domain. The Jones' procedure is convenient, flexible and applicable to arbitrary media and physics where the evaluation of the Green's function can constitute a cumbersome difficult problem. While the deduction of the functional equations in [19]- [26] is based on the first method also using the second Green's identity, we propose in this paper the Jones' method. We note that, in order to apply the Jones's approach to get the GWHEs in presence of angular region problems, it is important to introduce partial differential equation formulations using oblique Cartesian coordinates as [1]- [5].

We have developed different strategies to apply the Jones's method. In this paper we use a novel general first order differential vector formulation for transverse components of the fields as in [3]- [4] and first proposed as method in [30]- [31] for rectangular problems. The method differs from the one reported in [1], [2], [5] where the second order differential formulation (wave equation) is applied. We claim the superiority of the new procedure (based on first order formulation) to get spectral functional equations in angular regions, since it is capable to model arbitrary linear media in systematic steps as illustrated in the paper. Derivation of explicit equations require the implementation of the procedure reported in the paper, illustrated explicitly for isotropic media and extendable to more complex media (see e.g. Appendix A). While the first order procedure provides a method to get the functional equations for general arbitrary linear media filling the angular region, we note that the second order formulation [2], [5] is unpractical in non-isotropic media since no systematic procedural steps are available. Moreover, the first order differential formulation can be extended also to wave motion problems in different physics.

In this paper, plane wave sources and/or sources localized at infinity are considered in time harmonic electromagnetic field with a time dependence specified by $e^{j\omega t}$ (electrical engineering notation) which is suppressed. The paper is organized into six sections, two appendices and a glossary. The deduction of the GWHEs for scattering problems by wedges in arbitrary linear homogeneous medium is based on applying the boundary conditions to relevant spectral functional equations of angular regions. The main aim of this paper is to get these functional equations by introducing a conceptually simple technique starting from first order differential vector formulation in terms of transverse components of fields (transverse equations). In order to develop this technique, a preliminary study based on an abstract formulation of the Maxwell's equations in an indefinite homogeneous medium is necessary, as reported in Section 2. We recall that this methodology is also useful to study propagation in stratified media.

Using oblique Cartesian coordinates and taking into account the results of Section 2, Section 3 describes the novel application of the method to angular regions with oblique Cartesian coordinates, yielding the oblique transverse equations. The solution of these oblique transverse equations (Section 3), projected on the reciprocal eigenvectors of an algebraic matrix defined in Section 2, provides the functional equations of an arbitrary angular region, reported in Section 4. It is remarkable that we get functional equations independently from the materials and the sources that can be present outside the considered angular region. Properties and validations of functional equations and how to get the GWHEs by imposing the boundary conditions on the two faces of the angular region is finally reported in Section 5 for isotropic media, with conclusions in Section 6. Appendix A reports fundamental explicit matrices to apply the methodology to anisotropic media, while Appendix B justifies the dyadic Green's function formula of Section 4. The Glossary reports the main abbreviations, notations and symbols useful for the readability of the text.

2. First order differential transverse equations for indefinite rectangular regions filled by arbitrary linear homogeneous media

The evaluation of the physical fields in a linear medium can be generally described by a system of partial differential equation of first order. In absence of sources localized at finite or in presence of plane wave sources, the system assumes the homogeneous abstract form:

$$\mathit{\Gamma}_{\nabla} \cdot \psi = \theta \tag{2.1}$$

where $\mathit{\Gamma}_{\nabla}$ is a matrix differential operator that contains partial derivatives of first order, ψ is a vector that defines the field to be evaluated and θ is an additional field that is related to the field ψ through constitutive relations depending on the parameters that define the physical characteristic of the medium where the field is considered. ψ and θ are vectors having the same dimensions and the constitutive relations are defined by the equation

$$\theta = W \cdot \psi \tag{2.2}$$

where the matrix W depends on the medium that is considered.

In electromagnetism, the fields $\mathbf{E}$ and $\mathbf{H}$ in an arbitrary homogeneous linear medium are governed by the Maxwell's equations and present the following constitutive relations

$$\begin{aligned} \mathbf{D} &= \boldsymbol{\varepsilon} \cdot \mathbf{E} + \boldsymbol{\xi} \cdot \mathbf{H} \\ \mathbf{B} &= \boldsymbol{\zeta} \cdot \mathbf{E} + \boldsymbol{\mu} \cdot \mathbf{H} \end{aligned} \tag{2.3}$$

thus, in electromagnetic applications, (2.1) and (2.2) are defined by:

$$\psi = \left| \begin{array}{c} \mathbf{E} \\ \mathbf{H} \end{array} \right|, \; \theta = j\omega \left| \begin{array}{c} \mathbf{D} \\ -\mathbf{B} \end{array} \right|, \; \mathit{\Gamma}_{\nabla} = \left| \begin{array}{cc} 0 & \nabla \times \mathbf{1} \\ \nabla \times \mathbf{1} & 0 \end{array} \right|, \; W = j\omega \left| \begin{array}{cc} \boldsymbol{\varepsilon} & \boldsymbol{\xi} \\ -\boldsymbol{\zeta} & -\boldsymbol{\mu} \end{array} \right| \tag{2.4}$$

where $\mathbf{1}$ is the unit dyadic in the Euclidean space. An extended and detailed treatise about this abstract formulation is reported in [34] that is not easily accessible and not well known in the scientific community; for this reason here we report a short introduction and then our application.

To complete the formulation of the field problem via (2.1)-(2.4) we also need to impose the geometrical domain of the problem, its boundary conditions and the radiation condition.

In our method, first, we derive spectral functional equations avoiding the application of boundary conditions for a particular domain and, then, in practical problems we impose the boundary conditions coupling different regions and yielding the GWHEs of the problem.

For this reason, in the following sections the boundary conditions will appear only at Section 5 where a practical classical problem will be examined as an example of implementation procedure: the Malyuzhinets' problem.

The application of abstract formulation to the electromagnetic study of stratified medium along a direction (say y) is fundamental to introduce several important concepts in wave propagation (see for example [32]- [33]). In particular the introduction of the transverse equations can be used for the analysis of indefinite regions and in Section 3 for the development of the theory for angular regions. The transverse equations of a field are equations that involve only the components of the field ψ, say ψ_t, that remain continuous along the stratification according to the boundary conditions on the interfaces. In [34] the abstract deduction of the transverse equations is obtained starting from the abstract equations (2.1) and (2.2).

In the following, we assume y =const in Cartesian coordinates as the interface among media of rectangular shape (layers). To get the boundary conditions the method resorts to a suitable application of the divergence theorem on the equation (2.1) (see for example [30]). In electromagnetism, the transverse field for a stratification along the y direction is

$$\psi_t = |\mathbf{E}_t\,\mathbf{H}_t|' = |E_z\,E_x\,H_z\,H_x|' \tag{2.5}$$

where $'$ stands for transpose and, $\mathbf{E}_t = \hat{z}E_z + \hat{x}E_x, \mathbf{H}_t = \hat{z}H_z + \hat{x}H_x$ satisfy the boundary condition of continuity on the interfaces of the stratification.

Following [34], we deduce the electromagnetic transverse equations with respect to y starting from (2.1)-(2.4) for a general bianisotropic medium with constitutive parameters W where $\boldsymbol{\varepsilon}, \boldsymbol{\xi}, \boldsymbol{\zeta}, \boldsymbol{\mu}$ are tensors. For practical evaluation we assumes Cartesian coordinates with the ordering (z, x, y). We start from the decomposition of the differential operator

$$\nabla = \nabla_t + \hat{y}\frac{\partial}{\partial y},\ \nabla_t = \hat{z}\frac{\partial}{\partial z} + \hat{x}\frac{\partial}{\partial x} \tag{2.6}$$

that yields

$$\Gamma_\nabla = \Gamma_t + \Gamma_y\frac{\partial}{\partial y} \tag{2.7}$$

with

$$\Gamma_t = \left|\begin{matrix} 0 & \nabla_t\times\mathbf{1} \\ \nabla_t\times\mathbf{1} & 0 \end{matrix}\right|,\ \Gamma_y = \left|\begin{matrix} 0 & \hat{y}\times\mathbf{1} \\ \hat{y}\times\mathbf{1} & 0 \end{matrix}\right|,\ \mathbf{1} = \hat{z}\hat{z} + \hat{x}\hat{x} + \hat{y}\hat{y} \tag{2.8}$$

We observe that the following dyadic relations hold:

$$I_t\cdot\Gamma_t = \Gamma_t\cdot I_y,\quad I_t\cdot\Gamma_y = \Gamma_y\cdot I_t = \Gamma_y,\quad I_y\cdot\Gamma_t = \Gamma_t\cdot I_t,\quad I_y\cdot\Gamma_y = \Gamma_y\cdot I_y = 0 \tag{2.9}$$

where

$$I_t = \left|\begin{matrix} 1_t & 0 \\ 0 & 1_t \end{matrix}\right|,\quad I_y = \left|\begin{matrix} 1_y & 0 \\ 0 & 1_y \end{matrix}\right|,\quad 1_t = \hat{z}\hat{z} + \hat{x}\hat{x},\quad 1_y = \hat{y}\hat{y} \tag{2.10}$$

Taking into account (2.6)- (2.10), the first member of (2.1) becomes:

$$\Gamma_\nabla\cdot\psi = (\Gamma_t + \Gamma_y\frac{\partial}{\partial y})\psi = \Gamma_t\cdot\psi_t + \Gamma_y\frac{\partial}{\partial y}\psi_t + \Gamma_t\cdot\psi_y \tag{2.11}$$

where $\psi_t = |\mathbf{E}_t\,\mathbf{H}_t|' = |E_z\,E_x\,H_z\,H_x|'$ and $\psi_y = |E_y\hat{y}\,H_y\hat{y}|'$ with $\mathbf{E}_t = \hat{z}E_z + \hat{x}E_x, \mathbf{H}_t = \hat{z}H_z + \hat{x}H_x$.

Using the representation

$$W = W_{tt} + W_{ty} + W_{yt} + W_{yy} \tag{2.12}$$

where $W_{tt} = I_t\cdot W\cdot I_t,\quad W_{ty} = I_t\cdot W\cdot I_y, W_{yt} = I_y\cdot W\cdot I_t,\quad W_{yy} = I_y\cdot W\cdot I_y$, we have the following decomposition in transversal and longitudinal components of (2.1)

$$I_y\cdot\Gamma_t\cdot\psi_t = W_{yt}\cdot\psi_t + W_{yy}\cdot\psi_y \tag{2.13}$$

$$I_t\cdot\frac{\partial}{\partial y}\Gamma_y\cdot\psi_t + I_t\cdot\Gamma_t\cdot\psi_y = W_{tt}\cdot\psi_t + W_{ty}\cdot\psi_y \tag{2.14}$$

By substituting the matrix $\hat{W}_y$ defined by

$$\hat{W}_y\cdot W_{yy} = W_{yy}\cdot\hat{W}_y = I_y \tag{2.15}$$

into (2.13), it yields the relation that connects the longitudinal field ψ_y in terms of the transversal field ψ_t:

$$\psi_y = \hat{W}_y\cdot(I_y\cdot\Gamma_t - W_{yt})\cdot\psi_t \tag{2.16}$$

where explicitly

$$\hat{W}_y = \frac{1}{j\omega\,(\varepsilon_y\,\mu_y - \xi_y\,\zeta_y)}\left|\begin{matrix} \mu_y 1_y & \xi_y 1_y \\ -\zeta_y 1_y & -\varepsilon_y 1_y \end{matrix}\right| \tag{2.17}$$

Taking into account that $\mathit{\Gamma}_y^2 = -I_t$, the substitution of (2.16) into (2.14) yields the transversal Maxwell's equations (2.18):

$$\boxed{-\frac{\partial}{\partial y}\psi_t = \mathcal{M}(\frac{\partial}{\partial z}, \frac{\partial}{\partial x}) \cdot \psi_t} \tag{2.18}$$

where the matrix operator of dimension four $\mathcal{M}(\frac{\partial}{\partial z}, \frac{\partial}{\partial x})$ is given by:

$$\mathcal{M}(\frac{\partial}{\partial z}, \frac{\partial}{\partial x}) = -\mathit{\Gamma}_y \cdot \left[I_t \cdot (\mathit{\Gamma}_t - W_{ty}) \cdot \hat{W}_y \cdot (I_y \cdot \mathit{\Gamma}_t - W_{yt}) - W_{tt}\right] \tag{2.19}$$

In the case of isotropic medium i.e. with

$$W = j\omega \begin{vmatrix} \varepsilon\mathbf{1} & 0 \\ 0 & -\mu\mathbf{1} \end{vmatrix} \tag{2.20}$$

we obtain

$$\mathcal{M}(\frac{\partial}{\partial z}, \frac{\partial}{\partial x}) = \begin{pmatrix} 0 & 0 & -\frac{jD_xD_z}{\varepsilon\omega} & \frac{j({D_z}^2+\varepsilon\mu\omega^2)}{\varepsilon\omega} \\ 0 & 0 & -\frac{j({D_x}^2+\varepsilon\mu\omega^2)}{\varepsilon\omega} & \frac{jD_xD_z}{\varepsilon\omega} \\ \frac{jD_xD_z}{\mu\omega} & -\frac{j({D_z}^2+\varepsilon\mu\omega^2)}{\mu\omega} & 0 & 0 \\ \frac{j({D_x}^2+\varepsilon\mu\omega^2)}{\mu\omega} & -\frac{jD_xD_z}{\mu\omega} & 0 & 0 \end{pmatrix} \tag{2.21}$$

where $D_x = \frac{\partial}{\partial x}$, $D_y = \frac{\partial}{\partial y}$, $D_z = \frac{\partial}{\partial z}$. Further specific examples in electromagnetism, elasticity and more general field are reported in [5], [10], [12], [30]- [31], [35].

Here we assume that the geometry of the problem is invariant along the z-direction, thus, without loss of generality, we assume $\psi_t = \psi_t(x, y, z) = f(x, y)e^{-j\alpha_o z}$. It yields $\frac{\partial}{\partial z}\psi_t(x, y, z) = -j\alpha_o\psi_t(x, y, z)$, i.e. $\frac{\partial}{\partial z} \to -j\alpha_o$, thus

$$\mathcal{M}(\frac{\partial}{\partial z}, \frac{\partial}{\partial x}) = \mathcal{M}(-j\alpha_o, \frac{\partial}{\partial x}) = M_o + M_1\frac{\partial}{\partial x} + M_2\frac{\partial^2}{\partial x^2} + M_3\frac{\partial^3}{\partial x^3}\cdots \tag{2.22}$$

Taking into account (2.19), the number of non-null terms at the second member of (2.22) depends on $\mathit{\Gamma}_t$ and thus it is three, i.e. $M_m = 0$ for $m > 2$. The explicit expressions of the matrices M_m are defined by the problem under investigation and, in a general electromagnetic medium, the matrices M_m are of dimension four. In an isotropic medium, from (2.21), we have

$$\begin{gathered} M_o = \begin{pmatrix} 0 & 0 & 0 & \frac{j(-\alpha_0^2+\varepsilon\mu\omega^2)}{\varepsilon\omega} \\ 0 & 0 & -j\mu\omega & 0 \\ 0 & -\frac{j(-\alpha_0^2+\varepsilon\mu\omega^2)}{\mu\omega} & 0 & 0 \\ j\varepsilon\omega & 0 & 0 & 0 \end{pmatrix}, \\ M_1 = \begin{pmatrix} 0 & 0 & -\frac{\alpha_0}{\varepsilon\omega} & 0 \\ 0 & 0 & 0 & \frac{\alpha_0}{\varepsilon\omega} \\ \frac{\alpha_0}{\mu\omega} & 0 & 0 & 0 \\ 0 & -\frac{\alpha_0}{\mu\omega} & 0 & 0 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{j}{\varepsilon\omega} & 0 \\ 0 & 0 & 0 & 0 \\ \frac{j}{\mu\omega} & 0 & 0 & 0 \end{pmatrix} \end{gathered} \tag{2.23}$$

where we have omitted the dependence on $-j\alpha_o$.

The explicit expression of $\mathcal{M}$ (2.19) for a general arbitrary linear medium in electromagnetic applications is reported in [3], while in the Appendix A we report the anisotropic case. For readability, in the following, we will develop explicit expressions in isotropic media even if the theory and the procedure are completely valid for the general case. As shown in [12], [30], [31], [33] the transverse equations are very useful (independently from the application of Section 3) to deduce the WH equation in stratified media with discontinuity at the interfaces.

(a) The eigenvalues and the eigenvectors of $\mathcal{M}$ in spectral domain

By applying Fourier transform along x direction to (2.18) with (2.22)-(2.23) ($M_m = 0,\ m > 2$) in absence of source, we obtain an ordinary vector first order differential equation

$$-\frac{d}{dy}\Psi_t(\eta) = M(\eta) \cdot \Psi_t(\eta) \tag{2.24}$$

where $\psi_t(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Psi_t(\eta) e^{-j\eta x} d\eta$ (notation with omission of y, z dependence) and

$$M(\eta) = \mathcal{M}(-j\alpha_o, -j\eta) = M_o - j\eta M_1 - \eta^2 M_2 \tag{2.25}$$

where $\frac{\partial}{\partial z} \to -j\alpha_o$ for the presence of the field factor $e^{-j\alpha_o z}$ (see also comment before (2.22)) and $\frac{\partial}{\partial x} \to -j\eta$ for the property of Fourier transforms.

Now, let us investigate the properties of the eigenvalue problem (2.26) associated to the differential problem:

$$M(\eta) \cdot u_i(\eta) = \lambda_i(\eta) u_i(\eta) \tag{2.26}$$

We anticipate that the eigenvalues λ_i and the eigenvectors $u_i(\eta)$ ($i = 1..4$) of the matrix $M(\eta)$ (2.26) in rectangular shaped regions will play a fundamental role to get the functional equations of an angular region as solution of the differential problem.

In presence of a passive medium, we observe that two eigenvalues (say $\lambda_1,\ \lambda_2$) present non-negative real part and the other two eigenvalues (say $\lambda_3,\ \lambda_4$) present non-positive real part. While $\lambda_1,\ \lambda_2$ are related to progressive waves, $\lambda_3,\ \lambda_4$ are associated to regressive waves. In this framework we associate the direction of propagation to attenuation phenomena, while we let free of constraint the phase variation to model also left-handed materials.

The eigenvalues of the matrix $M(\eta)$ are

$$\lambda_1 = j\,\xi_1(\eta),\ \lambda_2 = j\,\xi_2(\eta),\ \lambda_3 = -j\,\xi_3(\eta),\ \lambda_4 = -j\,\xi_4(\eta) \tag{2.27}$$

In medium having reflection symmetry we have $\xi_{3,4}(\eta) = \xi_{1,2}(\eta)$. For simplicity and to get explicit simple expressions, let us consider homogeneous isotropic lossy media (see extension to anisotropic media in the Appendix A). For these media we have

$$\xi_i(\eta) = \xi(\eta) = \sqrt{\tau_o^2 - \eta^2}, \quad i = 1, 2, 3, 4 \tag{2.28}$$

where $\tau_o = \sqrt{k^2 - \alpha_o^2}$ with $Im[\tau_o] < 0$ and $k = \omega\sqrt{\varepsilon\mu}$ is the propagation constant with $Im[k] < 0$ (normally $Re[k] > 0$, otherwise $Re[k] < 0$ in left-handed materials). Since $k^2 = k_x^2 + k_y^2 + k_z^2 = \eta^2 + \xi^2 + \alpha_o^2$, $\xi(\eta)$ is a multivalued function of η. In the following we assume as proper sheet of $\xi(\eta)$ the one with $\xi(0) = \tau_o$ and as branch lines the classical line $Im[\xi(\eta)] = 0$ (see in practical engineering estimations Ch. 5.3b of [33]) or the vertical line ($Re[\eta] = Re[\tau_o],\ Im[\eta] < Im[\tau_o]$).

In isotropic media, according to (2.27) and (2.28), the eigenvalue are $\lambda_{1,2} = -\lambda_{3,4} = j\xi(\eta)$. The eigenvectors $u_i(\eta) = u_i$ corresponding to $\lambda_i,\ i = 1, 2, 3, 4$ are

$$u_1 = \left|\begin{array}{c} \frac{\tau_o^2}{\omega\,\varepsilon\,\xi} \\ -\frac{\alpha_o\eta}{\omega\,\varepsilon\,\xi} \\ 0 \\ 1 \end{array}\right|,\ u_2 = \left|\begin{array}{c} \frac{\alpha_o\eta}{\omega\,\varepsilon\,\xi} \\ -\frac{(\xi^2+\alpha_o^2)}{\omega\,\varepsilon\,\xi} \\ 1 \\ 0 \end{array}\right|,\ u_3 = \left|\begin{array}{c} -\frac{\tau_o^2}{\omega\,\varepsilon\,\xi} \\ \frac{\alpha_o\eta}{\omega\,\varepsilon\,\xi} \\ 0 \\ 1 \end{array}\right|,\ u_4 = \left|\begin{array}{c} -\frac{\alpha_o\eta}{\omega\,\varepsilon\,\xi} \\ \frac{(\xi^2+\alpha_o^2)}{\omega\,\varepsilon\,\xi} \\ 1 \\ 0 \end{array}\right| \tag{2.29}$$

We also introduce the reciprocal vectors $\nu_i(\eta)$ of the eigenvectors $u_i(\eta)$ that are the eigenvectors of the transpose of the matrix $M(\eta)$. The vectors $\nu_i(\eta)$ satisfy the bi-orthogonal relations

$$\nu_j \cdot u_i = \delta_{ji} \tag{2.30}$$

or alternatively

$$1_t = u_1\nu_1 + u_2\nu_2 + u_3\nu_3 + u_4\nu_4 \tag{2.31}$$

where δ_{ij} is the Kronecker symbol, 1_t is the identity dyadic such that $1_t \cdot M = M \cdot 1_t$ and in (2.31) we assume dyadic products.

According to the definition reported in (2.30) we obtain from (2.29) the reciprocal vectors $\nu_i(\eta) = \nu_i$

$$\begin{aligned} \nu_1 &= \left| \begin{matrix} \frac{\xi^2+\alpha_o^2}{2\omega\,\mu\,\xi} & \frac{\alpha_0\eta}{2\omega\,\mu\,\xi} & 0 & \frac{1}{2} \end{matrix} \right|, \nu_2 = \left| \begin{matrix} -\frac{\alpha_0\eta}{2\omega\,\mu\,\xi} & -\frac{\tau_o^2}{2\omega\,\mu\,\xi} & \frac{1}{2} & 0 \end{matrix} \right| \\ \nu_3 &= \left| \begin{matrix} -\frac{\xi^2+\alpha_o^2}{2\omega\,\mu\,\xi} & -\frac{\alpha_0\eta}{2\omega\,\mu\,\xi} & 0 & \frac{1}{2} \end{matrix} \right|, \nu_4 = \left| \begin{matrix} \frac{\alpha_0\eta}{2\omega\,\mu\,\xi} & \frac{\tau_o^2}{2\omega\,\mu\,\xi} & \frac{1}{2} & 0 \end{matrix} \right| \end{aligned} \tag{2.32}$$

3. First order differential oblique transverse equations for angular regions filled by arbitrary linear homogeneous media

In this section we introduce the oblique transverse equations using an oblique system of Cartesian axes and applying the properties reported in Section 2 for rectangular regions. In the following sections, first, we deduce spectral functional equations then, by imposing boundary conditions, the GWHEs for any arbitrary medium with angular shape [3]- [4].

With reference to Fig. 1, where angular regions are defined thorough the angle γ $(0 < \gamma < \pi)$, let us introduce the oblique Cartesian coordinates u, v, z in terms of the Cartesian coordinates x, y, z:

$$u = x - y\,\cot\gamma,\; v = \frac{y}{\sin\gamma} \; or \; x = u + v\,\cos\gamma,\; y = v\sin\gamma \tag{3.1}$$

with partial derivatives

$$\begin{aligned} \frac{\partial}{\partial x} &= \frac{\partial u}{\partial x}\frac{\partial}{\partial u} + \frac{\partial v}{\partial x}\frac{\partial}{\partial v} = \frac{\partial}{\partial u}, \quad \frac{\partial}{\partial y} = \frac{\partial u}{\partial y}\frac{\partial}{\partial u} + \frac{\partial v}{\partial y}\frac{\partial}{\partial v} = -\cot\gamma\,\frac{\partial}{\partial u} + \frac{1}{\sin\gamma}\,\frac{\partial}{\partial v} \\ \frac{\partial}{\partial u} &= \frac{\partial x}{\partial u}\frac{\partial}{\partial x} + \frac{\partial y}{\partial u}\frac{\partial}{\partial y} = \frac{\partial}{\partial x}, \quad \frac{\partial}{\partial v} = \frac{\partial x}{\partial v}\frac{\partial}{\partial x} + \frac{\partial y}{\partial v}\frac{\partial}{\partial y} = \cos\gamma\,\frac{\partial}{\partial x} + \sin\gamma\,\frac{\partial}{\partial y} \end{aligned} \tag{3.2}$$

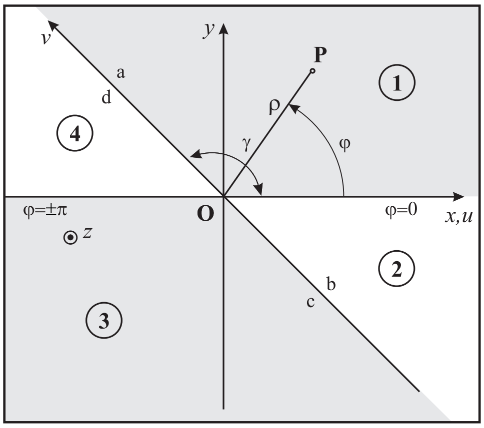

Figure 1. Angular regions and oblique Cartesian coordinates. The figure reports the x, y, z Cartesian coordinates and ρ, φ, z cylindrical coordinates useful to define the oblique Cartesian coordinate system u, v, z with reference to the angular region 1 $0 < \varphi < \gamma$ with $0 < \gamma < \pi$. In the figure, the space is divided into four angular regions delimited by $\varphi = \gamma$ and the face boundaries are labeled a,b,c,d.

In the following, we consider the system of transverse (with respect to y) equations of dimension four for an electromagnetic problem with invariant geometry along z-direction (i.e. $e^{-j\alpha_o z}$ field dependence) in an arbitrary homogeneous linear medium (see Section 2 in particular

(2.18) with (2.22)-(2.23)):

$$-\frac{\partial}{\partial y}\psi_t = \mathcal{M}(-j\alpha_o, \frac{\partial}{\partial x}) \cdot \psi_t = (M_o + M_1\frac{\partial}{\partial x} + M_2\frac{\partial^2}{\partial x^2}) \cdot \psi_t \tag{3.3}$$

Substituting (3.2), in particular $\frac{\partial}{\partial x} = \frac{\partial}{\partial u}$ and $\frac{\partial}{\partial y} = -\cot\gamma\,\frac{\partial}{\partial u} + \frac{1}{\sin\gamma}\,\frac{\partial}{\partial v}$, into (3.3), we obtain

$$-\frac{\partial}{\partial v}\psi_t = \mathcal{M}_e(-j\alpha_o, \frac{\partial}{\partial u}) \cdot \psi_t = (M_{eo} + M_{e1}\frac{\partial}{\partial u} + M_{e2}\frac{\partial^2}{\partial u^2}) \cdot \psi_t \tag{3.4}$$

where

$$M_{eo} = M_o \sin\gamma, \quad M_{e1} = M_1 \sin\gamma - I_t \cos\gamma, \quad M_{e2} = M_2 \sin\gamma \tag{3.5}$$

For the sake of simplicity and in order to get simple explicit expressions, let us consider a homogeneous isotropic medium, even if the procedure is general and applicable to arbitrary linear media (definitions for the anisotropic case are reported in the Appendix A). For isotropic media we have

$$\begin{gathered}
M_{eo} = \begin{pmatrix} 0 & 0 & 0 & \frac{j(-\alpha_0^2+\varepsilon\mu\omega^2)\sin\gamma}{\varepsilon\omega} \\ 0 & 0 & -j\mu\omega\sin\gamma & 0 \\ 0 & -\frac{j(-\alpha_0^2+\varepsilon\mu\omega^2)\sin\gamma}{\mu\omega} & 0 & 0 \\ j\varepsilon\omega\sin\gamma & 0 & 0 & 0 \end{pmatrix}, \\
M_{e1} = \begin{pmatrix} -\cos\gamma & 0 & -\frac{\alpha_0\sin\gamma}{\varepsilon\omega} & 0 \\ 0 & -\cos\gamma & 0 & \frac{\alpha_0\sin\gamma}{\varepsilon\omega} \\ \frac{\alpha_0\sin\gamma}{\mu\omega} & 0 & -\cos\gamma & 0 \\ 0 & -\frac{\alpha_0\sin\gamma}{\mu\omega} & 0 & -\cos\gamma \end{pmatrix}, \\
M_{e2} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{j\sin\gamma}{\varepsilon\omega} & 0 \\ 0 & 0 & 0 & 0 \\ \frac{j\sin\gamma}{\mu\omega} & 0 & 0 & 0 \end{pmatrix}
\end{gathered} \tag{3.6}$$

By applying the Fourier transform along $x = u$ direction to (3.4) (i.e. $\psi_t(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Psi_t(\eta)e^{-j\eta x}d\eta$ with notation omitting v, z dependence), we obtain

$$-\frac{d}{dv}\Psi_t(\eta) = M_e(\gamma, \eta) \cdot \Psi_t(\eta) \tag{3.7}$$

with

$$M_e(\gamma, \eta) = \mathcal{M}_e(-j\alpha_o, -j\eta) = M_{eo} - j\eta M_{e1} - \eta^2 M_{e2} \tag{3.8}$$

since $\frac{\partial}{\partial u} = \frac{\partial}{\partial x} \to -j\eta$.

(a) Link between eigenvalues of $M(\eta)$ and $M_e(\gamma, \eta)$

In oblique coordinates, the solution of (3.7) is related to the eigenvalue problem

$$M_e(\gamma, \eta) \cdot u_{ei}(\gamma, \eta) = \lambda_{ei}(\gamma, \eta)u_{ei}(\gamma, \eta) \tag{3.9}$$

where λ_{ei} and $u_{ei}(\eta)$ $(i = 1..n)$ are respectively the eigenvalues and the eigenvectors of the matrix $M_e(\gamma, \eta)$ of dimension $n = 4$ in our application. Using (3.7) and (3.8), equation (3.9) becomes

$$(M_o \sin\gamma - j\eta M_1 \sin\gamma - \eta^2 M_2 \sin\gamma) \cdot u_{ei}(\gamma, \eta) = (\lambda_{ei}(\gamma, \eta) - j\eta\cos\gamma)u_{ei}(\gamma, \eta) \tag{3.10}$$

and thus

$$M(\eta) \cdot u_{ei}(\gamma, \eta) = \left(\frac{\lambda_{ei}(\gamma, \eta) - j\eta\cos\gamma}{\sin\gamma}\right) u_{ei}(\gamma, \eta) \tag{3.11}$$

Comparing (3.11) with (2.26) we observe the relation among the eigenvalues and the eigenvectors of the two problems. The two problems defined by the matrices $M(\eta)$ and $M_e(\gamma,\eta)$ have same eigenvectors

$$u_{ei}(\gamma,\eta)=u_i(\eta) \tag{3.12}$$

thus same reciprocal vectors and related eigenvalues

$$\frac{\lambda_{ei}(\gamma,\eta)-j\eta\cos\gamma}{\sin\gamma}=\lambda_i(\eta) \tag{3.13}$$

Since $M_e(\gamma,\eta)$ and $M(\eta)$ have same eigenvectors (3.12) and the eigenvectors of $M(\eta)$ are $u_i(\eta)$ reported in (2.29), we note the important property that the eigenvectors of $M_e(\gamma,\eta)$ do not depend on the aperture angle γ (Fig. 1). From (3.13), the eigenvalues λ_{ei} of $M_e(\gamma,\eta)$ can be re-written using the notation (2.27):

$$\begin{array}{l}\lambda_{e1}(\gamma,\eta)=j(\eta\cos\gamma+\sin\gamma\,\xi_1(\eta))\\ \lambda_{e2}(\gamma,\eta)=j(\eta\cos\gamma+\sin\gamma\,\xi_2(\eta))\\ \lambda_{e3}(\gamma,\eta)=j(\eta\cos\gamma-\sin\gamma\,\xi_3(\eta))\\ \lambda_{e4}(\gamma,\eta)=j(\eta\cos\gamma-\sin\gamma\,\xi_4(\eta))\end{array} \tag{3.14}$$

where $\lambda_{e1},\lambda_{e2}$ ($\lambda_{e3},\lambda_{e4}$) are related to progressive (regressive) waves.

For what concerns the specific case of electromagnetic applications with an homogeneous isotropic medium in angular regions, the eigenvalues of the matrix $M_e(\gamma,\eta)$ are

$$\lambda_{e1}=\lambda_{e2}=j\,\eta\,\cos\gamma+j\,\sqrt{{\tau_o}^2-\eta^2}\,\sin\gamma,\quad \lambda_{e3}=\lambda_{e4}=j\,\eta\,\cos\gamma-j\,\sqrt{{\tau_o}^2-\eta^2}\,\sin\gamma \tag{3.15}$$

where k is the propagation constant, $\tau_o=\sqrt{k^2-\alpha_o^2}$ and $\xi=\xi(\eta)=\sqrt{\tau_o^2-\eta^2}$ (2.28) is a multivalued function as discussed in subsection 2(a). Note that, also in the isotropic angular geometries, two independent eigenvectors u_1,u_2 (u_3,u_4) (2.29) correspond to the two equal eigenvalues $\lambda_{e1}=\lambda_{e2}$ ($\lambda_{e3}=\lambda_{e4}$) as reported in (3.15).

4. Solution of the oblique transverse equations

In order to present the general solution procedure, in the following, we consider a system of oblique transverse equations (3.4) of dimension four with matrix operator $\mathcal{M}_e(-j\alpha_o,\frac{\partial}{\partial u})$ with three non-null terms (M_o,M_1,M_2) for a problem with invariant geometry along z-direction. This framework is appropriate for electromagnetic applications in arbitrary linear media and it will be explicitly developed for particular problems in Section 5. In this section, we obtain, as general solution, the spectral functional equations for the four angular regions as identified in Fig. 1. The four angular regions present same equation (3.4) but with different matrices M_{eo},M_{e1},M_{e2} depending on the media as well as on the aperture angle γ.

Let us introduce the Laplace transforms (notation omitting z dependence)

$$\tilde{\psi}_t(\eta,v)=\int_0^\infty e^{j\eta u}\psi_t(u,v)du \tag{4.1}$$

for regions 1,2 and

$$\tilde{\psi}_t(\eta,v)=\int_{-\infty}^0 e^{j\eta u}\psi_t(u,v)du \tag{4.2}$$

for regions 3,4.

The Laplace transforms applied to (3.4) yields:

$$-\frac{d}{dv}\tilde{\psi}_t(\eta,v)=M_e(\gamma,\eta)\cdot\tilde{\psi}_t(\eta,v)+\psi_s(v) \tag{4.3}$$

with

$$M_e(\gamma,\eta)=\mathcal{M}_e(-j\alpha_o,-j\eta)=M_{eo}-j\eta\,M_{e1}-\eta^2 M_{e2} \tag{4.4}$$

Note that (4.4) and (3.8) share the same symbol and explicit mathematical expression, however the first is related to a Fourier transform while the second to a Laplace transform, thus obviously they have the same eigenvalues and eigenvectors. The term $\psi_s(v)$ is obtained from the derivative property of the Laplace transform (initial conditions) and for each angular region we obtain a different expression. In particular, we indicate with $\psi_{as}(v)$ the value of $\psi_s(v)$ on the face a, see Fig. 1, $(0\leq v<+\infty, u=0_+)$, with $\psi_{bs}(v)$ the value of $\psi_s(v)$ on the face b $(-\infty\leq v<0, u=0_+)$, with $\psi_{cs}(v)$ the value of $\psi_s(v)$ on the face c $(-\infty\leq v<0, u=0_-)$ and with $\psi_{ds}(v)$ the value of $\psi_s(v)$ on the face d $(0\leq v<+\infty, u=0_-)$.

Since (4.3) is a system of four ordinary differential equations of first order with constant coefficients in a semi-infinite interval, we have mainly two methods for its solution: 1) to apply the dyadic Green's function procedure in v domain, 2) to apply the Laplace transform in v that yields a linear system of four algebraic equations from which one can write down the general solution in terms of eigenvalues and eigenfunctions. We note that both methods are effective and in particular the second method is more useful for representing the spectral solution in each point of the considered angular region. However, it initially introduces complex functions of two variables. As proposed in the following subsections, we prefer the first method because, by this way, we get the functional equations of the angular regions that involve directly complex functions of one variable.

Using the concept of non-standard Laplace transforms (see section 1.4 of [5]), the validity of (4.3) and (4.4) in absence of sources is extended to the total fields in the presence of plane-wave sources or in general of sources located at infinity.

With reference to Fig. 1, let us now describe the four angular regions in details. The selection of four angular regions as in Fig. 1 related to a unique aperture angle γ does not limit the applicability of the method. In fact all the equations (once derived) can be used with a different appropriate aperture angle just replacing γ with the proper value. The purpose of deriving the functional equations with a unique γ is related to the fact that we formulate and solve the angular region problems by analyzing once and for all the matrix operator $M_e(\gamma,\eta)$ (4.4). We recall also that the imposition of boundary conditions and media for each region will be made only while examining a practical problem and it yields GWHEs.

(a) Region 1: $u>0, v>0$

With reference to Fig. 1, for what concerns region 1 $(u>0, v>0)$, (4.3) holds with

$$\psi_s(v)=\psi_{as}(v)=-M_{e1}\cdot\psi_t(0_+,v)+j\eta\,M_{e2}\cdot\psi_t(0_+,v)-M_{e2}\cdot\left.\frac{\partial}{\partial u}\psi_t(u,v)\right|_{u=0+} \tag{4.5}$$

Equation (4.3) is a system of differential equations of first order (of dimension four in our electromagnetic assumption), whose solution $\tilde{\psi}_t(\eta,v)$ is obtainable as sum of a particular integral $\tilde{\psi}_p(\eta,v)$ with the general solution of the homogeneous equation $\tilde{\psi}_o(\eta,v)$:

$$\tilde{\psi}_t(\eta,v)=\tilde{\psi}_o(\eta,v)+\tilde{\psi}_p(\eta,v) \tag{4.6}$$

The solution of the homogeneous equation must satisfy

$$-\frac{d}{dv}\tilde{\psi}_o(\eta,v)=M_e(\gamma,\eta)\cdot\tilde{\psi}_o(\eta,v) \tag{4.7}$$

Considering the solution form $\tilde{\psi}_o(\eta,v) = C\, e^{-\lambda(\gamma,\eta)v} u(\gamma)$, the most general solution is

$$\tilde{\psi}_o(\eta,v) = C_1 e^{-\lambda_{e1}(\gamma)\,v} u_1(\eta) + C_2 e^{-\lambda_{e2}(\gamma,\eta)\,v} u_2(\eta) + C_3 e^{-\lambda_{e3}(\gamma,\eta)\,v} u_3(\eta) + C_4 e^{-\lambda_{e4}(\gamma,\eta)\,v} u_4(\eta) \tag{4.8}$$

where λ_{ei} and $u_{ei} \equiv u_i$ (i=1,2,3,4) are the eigenvalues and the eigenvectors of the matrix $M_e(\gamma,\eta)$ respectively reported at (3.14) and (3.12).

In presence of a passive medium, we recall that two eigenvalues (say $\lambda_1,\ \lambda_2$) present non-negative real part and the other two eigenvalues (say $\lambda_3,\ \lambda_4$) present non-positive real part. From (3.14), we note that $\lambda_{e1}, \lambda_{e2}$ model progressive waves along positive v direction, while $\lambda_{e1}, \lambda_{e2}$ regressive waves.

The evaluation of the particular integral of (4.3)

$$\tilde{\psi}_p(\eta,v) = -\int_0^\infty G(v,v') \cdot \psi_s(v')dv' \tag{4.9}$$

requires the dyadic Green's function $G(v,v')$ of (4.3), i.e. the solution of

$$\left(\frac{d}{dv} + M_e(\gamma,\eta)\right) \cdot G(v,v') = \delta(v-v')1_t \tag{4.10}$$

with the boundary condition of the problem: in this case the ones of region 1 ($u>0, v>0$). Note that 1_t is the identity dyadic of dimension four in our assumption (2.31).

An original method to get the particular solution is reported in [3], [30], [31]. While in [30]-[31] the method is applied to arbitrary stratified regions with appropriate boundary conditions, in this paper, we apply a slightly different method to the simplified structure constituted of an arbitrary indefinite angular region for the solution of (4.10). According to [36], it is possible to build a Green's function starting from arbitrary solutions of the homogeneous equations without imposing boundary conditions at first. Then, to get the solution of the differential problem with the boundary conditions, the selected form of the particular integral conditions the values of the arbitrary coefficients of the homogeneous solutions for the imposition of the boundary conditions. Finally, the sum of the homogeneous solutions with the particular integrals yields the solution of the problem.

We select progressive and regressive waves in indefinite half-space as homogeneous solutions for building the dyadic Green's function (see Appendix B for justification and properties of the dyadic Green's function). In our framework, we avoid to impose the boundary condition at this point, since we want to find functional equations that are free of this constraint. Only, while investigating a practical problem, we will impose boundary condition to the functional equations (for instance in region 1 at face $\varphi = 0$ i.e. $u>0,\ v=0$ and face $\varphi = \gamma$ i.e. $u=0,\ v>0$) yielding GWHEs of the problem. See Section 5(b) for a practical example of wedge scattering problem.

By applying this method (Appendix B) to the present problem we obtain the dyadic Green's function

$$G(v,v') = \begin{cases} u_1\nu_1 e^{-\lambda_{e1}(\gamma,\eta)(v-v')} + u_2\nu_2 e^{-\lambda_{e2}(\gamma,\eta)(v-v')}, & v>v' \\ -\left[u_3\nu_3 e^{-\lambda_{e3}(\gamma,\eta)(v-v')} + u_4\nu_4 e^{-\lambda_{e4}(\gamma,\eta)(v-v')}\right], & v<v' \end{cases} \tag{4.11}$$

where ν_i are the reciprocal vectors (2.30) of the eigenvectors u_i of $M_e(\gamma,\eta)$ and λ_{ei} are the related eigenvalues. Note that $u_i\nu_i$ in (4.11) are dyadic products.

By substituting (4.8) and (4.9) with (4.11) into (4.6), it yields

$$\tilde{\psi}_t(\eta,v) = \begin{array}{l} C_1 u_1 e^{-\lambda_{e1}(\gamma,\eta)\,v} + C_2 u_2 e^{-\lambda_{e2}(\gamma,\eta)\,v} + C_3 u_3 e^{-\lambda_{e3}(\gamma,\eta)\,v} + C_4 u_4 e^{-\lambda_{e4}(\gamma,\eta)\,v} + \\ -u_1\nu_1 \cdot \int_0^v e^{-\lambda_{e1}(\gamma,\eta)(v-v')}\psi_{as}(v')dv' - u_2\nu_2 \cdot \int_0^v e^{-\lambda_{e2}(\gamma,\eta)(v-v')}\psi_{as}(v')dv' + \\ +u_3\nu_3 \cdot \int_v^\infty e^{-\lambda_{e3}(\gamma,\eta)(v-v')}\psi_{as}(v')dv' + u_4\nu_4 \cdot \int_v^\infty e^{-\lambda_{e4}(\gamma,\eta)(v-v')}\psi_{as}(v')dv' \end{array} \tag{4.12}$$

Looking at the asymptotic behavior of (4.12) for $v \to +\infty$ we have that only the terms $C_3 u_3 e^{-\lambda_{e3}\, v} + C_4 u_4 e^{-\lambda_{e4}\, v}$ are divergent. For this reason we assume $C_3 = C_4 = 0$. Note, in particular, the vanishing of the last two integral terms as $v \to +\infty$.

Setting $v = 0$ in (4.12), we have

$$\tilde{\psi}_t(\eta, 0) = C_1 u_1 + C_2 u_2 + u_3 \nu_3 \cdot \int_0^\infty e^{\lambda_{e3}(\gamma,\eta)\, v'} \psi_{as}(v') dv' + u_4 \nu_4 \cdot \int_0^\infty e^{\lambda_{e4}(\gamma,\eta)\, v'} \psi_{as}(v') dv' \tag{4.13}$$

Multiplying (4.13) by $\nu_i(\eta) = \nu_i$ for $i = 1..4$, we obtain

$$\begin{cases} \nu_1(\eta) \cdot \tilde{\psi}_t(\eta, 0) = C_1 \\ \nu_2(\eta) \cdot \tilde{\psi}_t(\eta, 0) = C_2 \\ \nu_3(\eta) \cdot \tilde{\psi}_t(\eta, 0) = \nu_3(\eta) \cdot \breve{\psi}_{as}(-j\lambda_{e3}(\gamma, \eta)) \\ \nu_4(\eta) \cdot \tilde{\psi}_t(\eta, 0) = \nu_4(\eta) \cdot \breve{\psi}_{as}(-j\lambda_{e4}(\gamma, \eta)) \end{cases} \tag{4.14}$$

due to the property of the reciprocal vectors (2.30) and where $\breve{\psi}_{as}(\chi)$ is the Laplace transform in v along face a ($v = \rho$ in cylindrical coordinates)

$$\breve{\psi}_{as}(\chi) = \int_0^\infty e^{j\chi v} \psi_{as}(v) dv \tag{4.15}$$

The last two equations of (4.14) can be rewritten in the form

$$\nu_3(\eta) \cdot \tilde{\psi}_t(\eta, 0) = \nu_3(\eta) \cdot \breve{\psi}_{as}(-m_{a1}(\gamma, \eta)) \tag{4.16}$$

$$\nu_4(\eta) \cdot \tilde{\psi}_t(\eta, 0) = \nu_4(\eta) \cdot \breve{\psi}_{as}(-m_{a2}(\gamma, \eta)) \tag{4.17}$$

with

$$\begin{aligned} m_{a1}(\gamma, \eta) = j\lambda_{e3}(\gamma, \eta) = -\eta \cos\gamma + \xi_1 \sin\gamma \\ m_{a2}(\gamma, \eta) = j\lambda_{e4}(\gamma, \eta) = -\eta \cos\gamma + \xi_2 \sin\gamma \end{aligned} \tag{4.18}$$

While the first two equations of (4.14) relate the unknowns C_1 and C_2 to the Laplace transform $\tilde{\psi}_t(\eta, 0)$ evaluated in the lower face of the angular region ($u > 0,\ v = 0$), the last two equations of (4.14) provide two important functional equations that relate the Laplace transforms of combinations of the field components on the boundaries of the angular region 1, i.e. $u > 0,\ v = 0$ and $u = 0,\ v > 0$ (face a) in Fig. 1.

These functional equations are the starting point to define the GWHEs of region 1. They are valid for any linear medium filling the region and independently from any boundary conditions surrounding the region.

For example and for simplicity, the explicit form of (4.16)-(4.17) are reported in Section 5 for isotropic media where $\xi_i(\eta) = \xi(\eta) = \sqrt{\tau_o^2 - \eta^2}$ (see definition of the multivalued function $\xi(\eta)$ in subsection 3(a)).

These functional equations are equivalent to (3.3.57) and (3.3.58) of [5] where a completely different method has been applied for the derivation. In fact, in Chapter 3 of [5], the equations are obtained from the second order differential formulation for electromagnetic applications (wave equation). The theory is developed for isotropic medium and cumbersome symmetry properties have been used to develop the equations for the other angular regions with respect to region 1.

In the present work, the theory is more general, applicable to any arbitrary electromagnetic media and extendable to different physics. In particular, the equations of the other regions with respect to region 1 are easily derived as in the following subsections.

(b) Region 2: $u > 0, v < 0$

With reference to Fig. 1, following the procedure reported for region 1 in subsection 4(a), we develop the solution for region 2 ($u > 0, v < 0$). The problem shows same equation (4.3) with

$$\psi_s(v) = \psi_{bs}(v) = -M_{e1}\cdot\psi_t(0_+, v) + j\eta\, M_{e2}\cdot\psi_t(0_+, v) - M_{e2}\cdot\frac{\partial}{\partial u}\psi_t(u, v)\Big|_{u=0+} \tag{4.19}$$

Note the different geometrical support of (4.19) with respect to (4.5), i.e. for region 2 $v < 0$ while for region 1 $v > 0$. As per region 1, the solution of (4.3) is obtained as combination of the homogeneous solution and the particular integral, see (4.6). We note that the particular integral depends on (4.19), while the homogeneous solution depends on the expressions of eigenvalues $\lambda_{ei}(\gamma,\eta)$ and eigenvectors $u_i(\eta)$ of $M_e(\gamma,\eta)$ (4.4) that are the same as for region 1, except for their dependence on the physical constitutive parameters of region 2 that my be inhomogeneous with respect to region 1.

Once obtained the expression of the dyadic Green's function specialized for region 2, we get

$$\tilde{\psi}_t(\eta, v) = \begin{array}{l} C_1 u_1 e^{-\lambda_{e1}(\gamma,\eta)\,v} + C_2 u_2 e^{-\lambda_{e2}(\gamma,\eta)\,v} + C_3 u_3 e^{-\lambda_{e3}(\gamma,\eta)\,v} + C_4 u_4 e^{-\lambda_{e4}(\gamma,\eta)\,v} + \\ -u_1\nu_1\cdot\int_{-\infty}^{v} e^{-\lambda_{e1}(\gamma,\eta)(v-v')}\psi_{bs}(v')dv' - u_2\nu_2\cdot\int_{-\infty}^{v} e^{-\lambda_{e2}(\gamma,\eta)(v-v')}\psi_{bs}(v')dv' + \\ +u_3\nu_3\cdot\int_{v}^{0} e^{-\lambda_{e3}(\gamma,\eta)(v-v')}\psi_{bs}(v')dv' + u_4\nu_4\cdot\int_{v}^{0} e^{-\lambda_{e4}(\gamma,\eta)(v-v')}\psi_{bs}(v')dv' \end{array} \tag{4.20}$$

where λ_{ei} and u_i are reported in (3.14) and (3.12).

Looking at the asymptotic behavior of (4.20) for $v \to -\infty$ we have that only the terms $C_1 u_1 e^{-\lambda_{e1}\,v} + C_2 u_2 e^{-\lambda_{e2}\,v}$ are divergent. For this reason we assume $C_1 = C_2 = 0$. Note, in particular, the vanishing of the first two integral terms as $v \to -\infty$.

Assuming $v = 0$ in (4.20), we have

$$\tilde{\psi}_t(\eta, 0) = C_3 u_3 + C_4 u_4 - u_1\nu_1\int_{-\infty}^{0} e^{-\lambda_{e1}(\gamma,\eta)(-v')}\psi_{bs}(v')dv' - u_2\nu_2\int_{-\infty}^{0} e^{-\lambda_{e2}(\gamma,\eta)(-v')}\psi_{bs}(v')dv' \tag{4.21}$$

Multiplying (4.21) by $\nu_i(\eta) = \nu_i$ for $i = 1..4$, we obtain

$$\begin{cases} \nu_3(\eta)\cdot\tilde{\psi}_t(\eta, 0) = C_3 \\ \nu_4(\eta)\cdot\tilde{\psi}_t(\eta, 0) = C_4 \\ \nu_1(\eta)\cdot\tilde{\psi}_t(\eta, 0) = -\nu_1(\eta)\cdot\breve{\psi}_{bs}(j\lambda_{e1}(\gamma,\eta)) \\ \nu_2(\eta)\cdot\tilde{\psi}_t(\eta, 0) = -\nu_2(\eta)\cdot\breve{\psi}_{bs}(j\lambda_{e2}(\gamma,\eta)) \end{cases} \tag{4.22}$$

where

$$\breve{\psi}_{bs}(\chi) = \int_{-\infty}^{0} e^{-j\chi v}\psi_{bs}(v)dv = \int_{0}^{\infty} e^{j\chi\rho}\psi_{bs}(-\rho)d\rho \tag{4.23}$$

is the left Laplace transform of $\psi_{bs}(v)$ in v along face b (Fig. 1) or the Laplace transform in ρ of $\psi_{bs}(-\rho)$ in cylindrical coordinates (ρ, φ, z).

As stated for region 1, in media with reflection symmetry ($\xi_{3,4}(\eta) = \xi_{1,2}(\eta)$), the last two equations of (4.22) can be rewritten in the form

$$\nu_1(\eta)\cdot\tilde{\psi}_t(\eta, 0) = -\nu_1(\eta)\cdot\breve{\psi}_{bs}(-m_{b1}(\gamma,\eta)) \tag{4.24}$$

$$\nu_2(\eta)\cdot\tilde{\psi}_t(\eta, 0) = -\nu_2(\eta)\cdot\breve{\psi}_{bs}(-m_{b2}(\gamma,\eta)) \tag{4.25}$$

with

$$\begin{array}{l} m_{b1}(\gamma,\eta) = -j\lambda_{e1}(\gamma,\eta) = \eta\cos\gamma + \xi_1\sin\gamma \\ m_{b2}(\gamma,\eta) = -j\lambda_{e2}(\gamma,\eta) = \eta\cos\gamma + \xi_2\sin\gamma \end{array} \tag{4.26}$$

While the first two equations of (4.22) relate the unknowns C_3 and C_4 to the Laplace transform $\tilde{\psi}_t(\eta, 0)$ evaluated at the face of the angular region ($u > 0$, $v = 0$), the last two equations of (4.22) provide two important functional equations that relate the Laplace transforms of combinations of field components on the boundaries of the angular region 2, i.e. $u>0,\ v=0$ and $u=0,\ v<0$ (face b) in Fig. 1. These functional equations are the starting point to define the GWHEs of region 2. As stated for region 1, they are valid for any linear medium filling the region and and independently from any boundary conditions surrounding the region. They agree with the ones proposed in chapter 3 of [5] in case of isotropic medium for electromagnetic applications.

Note that, in view of dealing with scattering problems by wedges, see Section 5(b), the aperture angle of region 2 is usually different from γ. This difference modifies the equations only in (4.26) for the dependence on a different aperture angle. We recall that the motivation of deriving the functional equations with a unique γ is related to the fact that we formulate and solve the angular region problems by analyzing once and for a single matrix operator $M_e(\gamma,\eta)$ (4.4).

(c) Region 4: $u<0, v>0$

With reference to Fig. 1, following the procedure reported for region 1 in subsection 4(a), we develop the solution for region 4 ($u<0, v>0$). Applying the the Laplace transform

$$\begin{array}{l}\tilde{\psi}_t(\eta,0)=\int_{-\infty}^{0} e^{j\eta\, u}\psi_t(u,0)du=\tilde{\psi}_{\pi t}(-\eta,0)\\ \tilde{\psi}_{\pi t}(\eta,0)=\int_{0}^{\infty} e^{j\eta\, u}\psi_t(-u,0)du\end{array} \tag{4.27}$$

to (3.4), the problem in region 4 shows the same equation (4.3)

$$-\frac{d}{dv}\tilde{\psi}_t=M_e(\gamma,\eta)\cdot\tilde{\psi}_t+\psi_s(v) \tag{4.28}$$

with $M_e(\gamma,\eta)$ reported in (4.4) and with the different definition of

$$\psi_s(v)=\psi_{ds}(v)=M_{e1}\cdot\psi_t(0_-,v)-j\eta\, M_{e2}\cdot\psi_t(0_-,v)+M_{e2}\cdot\left.\frac{\partial}{\partial u}\psi_t(u,v)\right|_{u=0_-} \tag{4.29}$$

that is related to the derivative property of the Laplace transform (4.27) along face d (see Fig. 1).

The application of the method used for region 1 yields the two functional equations

$$\nu_3(\eta)\cdot\tilde{\psi}_t(\eta,0)=\nu_3(\eta)\cdot\breve{\psi}_{ds}(-j\lambda_{e3}(\gamma,\eta)) \tag{4.30}$$

$$\nu_4(\eta)\cdot\tilde{\psi}_t(\eta,0)=\nu_4(\eta)\cdot\breve{\psi}_{ds}(-j\lambda_{e4}(\gamma,\eta)) \tag{4.31}$$

where we have defined the Laplace transform

$$\breve{\psi}_{ds}(\chi)=\int_0^\infty e^{j\chi v}\psi_{ds}(v)dv \tag{4.32}$$

The other difference with respect to the last two equations of (4.14) is the definition of

$$\tilde{\psi}_t(\eta,u)=\int_{-\infty}^{0} e^{j\eta\, u}\psi_t(u,v)du \tag{4.33}$$

that is a minus function (left Laplace transform). Changing η with $-\eta$ we rewrite (4.30) and (4.31) as

$$\nu_3(-\eta)\cdot\tilde{\psi}_{\pi t}(\eta,0)=\nu_3(-\eta)\cdot\breve{\psi}_{ds}(-j\lambda_{e3}(\gamma,-\eta) \tag{4.34}$$

$$\nu_4(-\eta)\cdot\tilde{\psi}_{\pi t}(\eta,0)=\nu_4(-\eta)\cdot\breve{\psi}_{ds}(-j\lambda_{e4}(\gamma,-\eta)) \tag{4.35}$$

with the plus function (right Laplace transform)

$$\tilde{\psi}_{\pi t}(\eta,0)=\int_0^\infty e^{j\eta\, u}\psi_t(-u,0)du \tag{4.36}$$

(d) Region 3: $u < 0, v < 0$

As already done for regions 1,2,4 we repeat the procedure. We get the same equation (4.3) with the definition $\tilde{\psi}_t(\eta, 0)$ (4.27) except for

$$\psi_s(v) = \psi_{cs}(v) = M_{e1} \cdot \psi_t(0_-, v) - j\eta\, M_{e2} \cdot \psi_t(0_-, v) + M_{e2} \cdot \frac{\partial}{\partial u}\psi_t(u, v)\bigg|_{u=0_-} \tag{4.37}$$

that is related to the derivative property of the Laplace transform (4.27) along face c (see Fig. 1).

It yields the two functional equations

$$\nu_1(\eta) \cdot \tilde{\psi}_t(\eta, 0) = -\nu_1(\eta) \cdot \breve{\psi}_{cs}(j\lambda_{e1}(\gamma, \eta)) \tag{4.38}$$

$$\nu_2(\eta) \cdot \tilde{\psi}_t(\eta, 0) = -\nu_2(\eta) \cdot \breve{\psi}_{cs}(j\lambda_{e2}(\gamma, \eta)) \tag{4.39}$$

where we have defined the Laplace transform

$$\breve{\psi}_{cs}(\chi) = \int_{-\infty}^{0} e^{-j\chi v}\psi_{cs}(v)dv = \int_{0}^{\infty} e^{j\chi\rho}\psi_{cs}(-\rho)d\rho \tag{4.40}$$

The other difference with respect to the last two equations of (4.14) is the definition of

$$\tilde{\psi}_t(\eta, u) = \int_{-\infty}^{0} e^{j\eta\, u}\psi_t(u, v)du \tag{4.41}$$

that is a minus function (left Laplace transform). Changing η with $-\eta$ we rewrite (4.38) and (4.39) as

$$\nu_1(-\eta) \cdot \tilde{\psi}_{\pi t}(\eta, 0) = -\nu_1(-\eta) \cdot \breve{\psi}_{cs}(j\lambda_{e1}(\gamma, -\eta)) \tag{4.42}$$

$$\nu_2(-\eta) \cdot \tilde{\psi}_{\pi t}(\eta, 0) = -\nu_2(-\eta) \cdot \breve{\psi}_{cs}(j\lambda_{e2}(\gamma, -\eta)) \tag{4.43}$$

with the plus function (right Laplace transform)

$$\tilde{\psi}_{\pi t}(\eta, 0) = \int_{0}^{\infty} e^{j\eta\, u}\psi_t(-u, 0)du \tag{4.44}$$

5. Properties and validation of the functional equations

(a) Explicit form for regions 1-2 and validation

Using the concept of non-standard Laplace transforms (see section 1.4 of [5]), the validity of the functional equations (4.16)-(4.17), (4.24)-(4.25),(4.34)-(4.35),(4.42)-(4.43) obtained in absence of sources is extended to the total fields in presence of plane-wave sources or in general of sources located at infinity.

In order to validate the functional equations obtained in this paper (4.16)-(4.17), (4.24)-(4.25),(4.34)-(4.35),(4.42)-(4.43) we demonstrate that they are equivalent to the ones proposed in Chapter 3 of [5] for electromagnetic applications with angular regions filled by an isotropic medium with permittivity ε and permeability μ. Let us consider, for simplicity, region 1 with (4.16)-(4.17), i.e.

$$\nu_3(\eta) \cdot \tilde{\psi}_t(\eta, 0) = \nu_3(\eta) \cdot \breve{\psi}_{as}(-m_{a1}(\gamma, \eta)) \tag{5.1}$$

$$\nu_4(\eta) \cdot \tilde{\psi}_t(\eta, 0) = \nu_4(\eta) \cdot \breve{\psi}_{as}(-m_{a2}(\gamma, \eta)) \tag{5.2}$$

These equations need to be compared to (3.3.57) and (3.3.58) of [5] that for readability are reported here using the notation of this paper:

$$\xi\tilde{E}_z(\eta,0)-\frac{\tau_o^2}{\omega\varepsilon}\tilde{H}_x(\eta,0)-\frac{\alpha_o\eta}{\omega\varepsilon}\tilde{H}_z(\eta,0)=-n\breve{E}_z\left(-m,\gamma\right)-\frac{\tau_o^2}{\omega\varepsilon}\breve{H}_\rho\left(-m,\gamma\right)+\frac{\alpha_o m}{\omega\varepsilon}\breve{H}_z\left(-m,\gamma\right) \tag{5.3}$$

$$\xi\tilde{H}_z(\eta,0)+\tfrac{\tau_o^2}{\omega\varepsilon}\tilde{E}_x(\eta,0)+\tfrac{\alpha_o\eta}{\omega\varepsilon}\tilde{E}_z(\eta,0)=-n\breve{H}_z\left(-m,\gamma\right)+\tfrac{\tau_o^2}{\omega\varepsilon}\breve{E}_\rho\left(-m,\gamma\right)-\tfrac{\alpha_o m}{\omega\varepsilon}\breve{E}_z\left(-m,\gamma\right) \tag{5.4}$$

where for the isotropy of media

$$m=m(\gamma,\eta)=m_{a1}(\gamma,\eta)=m_{a2}(\gamma,\eta)=-\eta\cos\gamma+\xi\sin\gamma=j\lambda_{e3}(\gamma,\eta)=j\lambda_{e4}(\gamma,\eta) \tag{5.5}$$

$$n=n(\gamma,\eta)=-\eta\sin\gamma-\xi\cos\gamma \tag{5.6}$$

In (5.3)-(5.4) we have used Laplace transforms in η along $u>0, v=0$ in the LHS and in $-m$ along $u=0, v>0$ in the RHS respectively denoted by $\sim$ and $\smile$ symbols and reported in (4.1) and (4.15).

To explicitly represent (5.1) and (5.2) we apply on the LHS the definitions of $\psi_t = |E_z\ E_x\ H_z\ H_x|'$ and the reciprocal vectors reported in (2.32).

On the RHS we use the source term $\psi_{as}(v)$ (4.5) of the differential equation (4.3), where, substituting the explicit expressions of M_{e1} and M_{e2} reported in (3.6), it yields

$$\psi_{as}(v)=\begin{vmatrix} E_z\cos(\gamma)+\frac{\alpha_o H_z\sin(\gamma)}{\omega\epsilon} \\ E_x\cos(\gamma)+\frac{jD_u H_z\sin(\gamma)-H_x\alpha_o\sin(\gamma)+H_z\eta\sin(\gamma)}{\omega\epsilon} \\ H_z\cos(\gamma)-\frac{\alpha_o E_z\sin(\gamma)}{\mu\omega} \\ H_x\cos(\gamma)+\frac{-jD_u E_z\sin(\gamma)+\alpha_o E_x\sin(\gamma)-E_z\eta\sin(\gamma)}{\mu\omega} \end{vmatrix} \tag{5.7}$$

where $D_u=\frac{\partial}{\partial u}$ and the field quantities are defined for $u=0_+$ and depends on $v>0$.

We observe that, while $\tilde{\psi}_t(\eta,0)$ is continuous at $\varphi=0$ by definition (2.5), we need to apply mathematical manipulations to demonstrate the continuity of $\psi_{as}(v)$ (5.7) at face a for an arbitrary aperture angle γ. In fact, $\psi_{as}(v)$ shows possible discontinuous terms at face a ($u=0_+, v>0$) due to the presence of $D_u H_z$ and $D_u E_z$.

For this purpose, we resort to the Maxwell's equations

$$D_u H_z = j\frac{-kE_y-H_x Z_o\alpha_o}{Z_o},\quad D_u E_z = j(kZ_o H_y-\alpha_o E_x) \tag{5.8}$$

Substituting (5.8) into (5.7)

$$\psi_{as}(v)=\begin{vmatrix} E_z\cos(\gamma)+\frac{\alpha_o H_z\sin(\gamma)}{\omega\epsilon} \\ E_x\cos(\gamma)+\frac{(kE_y+H_z\eta Z_o)\sin(\gamma)}{k} \\ H_z\cos(\gamma)-\frac{\alpha_o E_z\sin(\gamma)}{\mu\omega} \\ H_x\cos(\gamma)+\frac{(H_y kZ_o-E_z\eta)\sin(\gamma)}{kZ_o} \end{vmatrix} \tag{5.9}$$

where the field quantities are defined for $u=0_+$ and depends on $v>0$. The next step is to rewrite E_x, E_y, H_x and H_y in terms of the components (E_v, H_v) and (E_n, H_n) respectively tangential and normal to the face a (outward normal with respect to region 1). We have:

$$\begin{aligned} E_x &= -E_n\sin(\gamma)+E_v\cos(\gamma) \\ H_x &= -H_n\sin(\gamma)+H_v\cos(\gamma) \\ E_y &= E_v\sin(\gamma)+E_n\cos(\gamma) \\ H_y &= H_v\sin(\gamma)+H_n\cos(\gamma) \end{aligned} \tag{5.10}$$

Substituting (5.10) into (5.9), we have

$$\psi_{as}(v) = \left| \begin{array}{c} E_z \cos(\gamma) + \frac{\alpha_o H_z \sin(\gamma)}{\omega\epsilon} \\ E_v + \frac{H_z \eta Z_o \sin(\gamma)}{k} \\ H_z \cos(\gamma) - \frac{\alpha_o E_z \sin(\gamma)}{\mu\omega} \\ H_v + \frac{-E_z \eta \sin(\gamma)}{k Z_o} \end{array} \right| \tag{5.11}$$

Note that the discontinuous components of fields (i.e. the normal components of electromagnetic field **E,H**) are canceled by substitution in (5.11), thus $\psi_{as}(v)$ is continuous at face a. The absence of the discontinuous components E_n, H_n in (5.11) is justified by the equivalence theorem of electromagnetism, i.e. the field in the region 1 can be computed and depends only on the field components continuous at the boundaries: for face a the tangential components of electromagnetic field are E_v, H_v, E_z, H_z in u, v, z.

Now, substituting the Laplace transforms $\tilde{\psi}_t(\eta, 0)$ (4.1) of $\psi_t(u, 0)$ and $\breve{\psi}_{as}(-m)$ (4.15) of $\psi_{as}(v)$ (5.11) with (5.5) into (5.1)-(5.2), and using (2.32), it yields the two functional equations

$$\begin{aligned} &-\alpha_o \eta \tilde{E}_x + (\eta^2 - k^2)\tilde{E}_z + k\xi Z_o \tilde{H}_x = \\ &\qquad = -\alpha_o \eta \breve{E}_v - [\eta\xi \sin(\gamma) + \cos(\gamma)(k^2 - \eta^2)]\breve{E}_z + k\xi Z_o \breve{H}_v - \sin(\gamma)\alpha_o k Z_o \breve{H}_z \end{aligned} \tag{5.12}$$

$${\tau_o}^2 \tilde{E}_x + \alpha_o \eta \tilde{E}_z + k\xi Z_o \tilde{H}_z = {\tau_o}^2 \breve{E}_v + \alpha_o[\cos(\gamma)\eta - \sin(\gamma)\xi]\breve{E}_z + kZ_o[\sin(\gamma)\eta + \cos(\gamma)\xi]\breve{H}_z \tag{5.13}$$

that we have normalized by the multiplying factor $2kZ_o\xi$. In (5.12)-(5.13) the field quantities on the LHS are Laplace transforms in η along $u > 0, v = 0$ (symbol $\sim$), while the field quantities on the RHS are Laplace transforms in $-m$ along $v > 0, u = 0$ (symbol $\smile$). As a consequence, the field components on the LHS are plus functions in η, while the ones on the RHS are minus functions in m. We also observe that v components of field in oblique Cartesian coordinates are equivalent to ρ components in cylindrical coordinates.

Eqs. (5.12)-(5.13) are explicit expression of functional equations of region 1 filled by an isotropic medium.

We note that (5.3)-(5.4) and (5.12)-(5.13) are obtained from completely different methods and therefore equivalence is not immediate in the general case $\alpha_o \neq 0$. However, each of (5.12)-(5.13) is a linear combination of (5.3)-(5.4) and vice-versa.

For simplicity, we explicitly report the equivalence between (5.3) and a linear combination of (5.12)-(5.13). First, we demonstrate the equivalence of the left member of (5.3) to the left member of a linear combination between (5.12) and (5.13), imposing

$$2kZ_o\xi(\mathcal{C}_1\, \nu_3(\eta) \cdot \tilde{\psi}_t(\eta, 0) + \mathcal{C}_2\, \nu_4(\eta) \cdot \tilde{\psi}_t(\eta, 0)) = \xi \tilde{E}_z(\eta, 0) - \frac{\tau_o^2}{\omega\varepsilon}\tilde{H}_x(\eta, 0) - \frac{\alpha_o \eta}{\omega\varepsilon}\tilde{H}_z(\eta, 0) \tag{5.14}$$

To evaluate the linear combination constants $\mathcal{C}_1$ and $\mathcal{C}_2$ in (5.14), first we impose that the coefficients of $\tilde{H}_x$ in both the members of (5.14) are the same. It yields:

$$\mathcal{C}_1 = -\frac{\tau_o^2}{k^2\xi} \tag{5.15}$$

Second, we need to eliminate the component $\tilde{E}_x$ from the first member of (5.14) since no $\tilde{E}_x$ component is present at the second member, therefore

$$\mathcal{C}_2 = \mathcal{C}_1 \frac{\alpha_o \eta}{\tau_o^2} \tag{5.16}$$

With the above values of $\mathcal{C}_1$ and $\mathcal{C}_2$ the identity (5.14) holds.

Finally, we simply prove by substitution that the constants (5.15) and (5.16) enforce the same equality on the right members of the two formulations, i.e.

$$\begin{aligned} 2kZ_o\xi(\mathcal{C}_1\,\nu_3(\eta)\cdot\breve{\psi}_{as}(-m)+\mathcal{C}_2\,\nu_4(\eta)\cdot\breve{\psi}_{as}(-m)) &= \\ = -n\breve{E}_z\,(-m,\gamma)-\frac{\tau_o^2}{\omega\varepsilon}\breve{H}_\rho\,(-m,\gamma)+\frac{\alpha_{o1}m_1}{\omega\varepsilon}\breve{H}_z\,(-m,\gamma) \end{aligned} \tag{5.17}$$

Due to the structure of (5.4) that is similar to the one of (5.3), it is possible to demonstrate the equivalence of (5.4) to a linear combination of (5.15) and (5.16) with the same procedure, that we omit here.

Analogously to region 1, we can derive the explicit form of functional equations (4.24)-(4.25) for region 2 filled by an isotropic medium with permittivity ε and permeability μ:

$$\nu_1(\eta)\cdot\tilde{\psi}_t(\eta,0)=-\nu_1(\eta)\cdot\breve{\psi}_{bs}(-m_{b1}(\gamma,\eta)) \tag{5.18}$$

$$\nu_2(\eta)\cdot\tilde{\psi}_t(\eta,0)=-\nu_2(\eta)\cdot\breve{\psi}_{bs}(-m_{b2}(\gamma,\eta)) \tag{5.19}$$

Regions 1 and 2 share the same procedure to get the explicit form of the functional equations. In particular, we note the following analogies and differences: 1) the source term assumes the same form $\psi_{bs}(v)=\psi_{as}(v)$ (5.11) with the exception for the dependence on the constitutive parameters ε,μ, 2) while applying the Maxwell's equations (5.8) to represent the field components in terms of face a(b) tangential (E_v,H_v) and normal (E_n,H_n) components we need to consider the outward normal of region 1(2).

Focusing the attention on region 2 and substituting the Laplace transforms $\tilde{\psi}_t(\eta,0)$ (4.1) of $\psi_t(u,0)$ and $\breve{\psi}_{bs}(-m_b)$ (4.23) of $\psi_{bs}(v)$ with

$$m_b=m_b(\gamma,\eta)=m_{b1}(\gamma,\eta)=m_{b2}(\gamma,\eta)=\eta\cos\gamma+\xi\sin\gamma=j\lambda_{e1}(\gamma,\eta)=j\lambda_{e2}(\gamma,\eta) \tag{5.20}$$

into (5.18)-(5.19), and using (2.32), it yields the two functional equations

$$\begin{aligned} +\alpha_o\eta\tilde{E}_x-(\eta^2-k^2)\tilde{E}_z+k\xi Z_o\tilde{H}_x &= \\ =-\alpha_o\eta\breve{E}_v-[-\eta\xi\sin(\gamma)+\cos(\gamma)(k^2-\eta^2)]\breve{E}_z-k\xi Z_o\breve{H}_v-\sin(\gamma)\alpha_o kZ_o\breve{H}_z \end{aligned} \tag{5.21}$$

$$-{\tau_o}^2\tilde{E}_x-\alpha_o\eta\tilde{E}_z+k\xi Z_o\tilde{H}_z={\tau_o}^2\breve{E}_v+\alpha_o[\cos(\gamma)\eta+\sin(\gamma)\xi]\breve{E}_z+kZ_o[\sin(\gamma)\eta-\cos(\gamma)\xi]\breve{H}_z \tag{5.22}$$

that we have normalized by the multiplying factor $2kZ_o\xi$. Eqs. (5.21)-(5.22) show change in sign with respect to (5.12)-(5.13) of region 1. In (5.21)-(5.22) the field quantities on the LHS are Laplace transforms in η along $u>0, v=0$ (symbol $\sim$), while the field quantities on the RHS are Laplace transforms in $-m_b$ along $v<0, u=0$ (symbol $\smile$). As a consequence, the field components on the LHS are plus functions in η, while the ones on the RHS are minus functions in m_b. We also observe that v components of field in oblique Cartesian coordinates are equivalent to ρ components with opposite sign in cylindrical coordinates (the sign is due to the face b orientation, see Fig. 1). Equivalence of (5.21)-(5.22) to (3.3.59) and (3.3.60) of [5] can be accomplished as already done for (5.3) that is a linear combination of (5.12)-(5.13). In this case we need to pay attention that γ in (5.21)-(5.22) must be substituted with $\pi-\gamma_b$ for the equivalence with (3.3.59) and (3.3.60) of [5], since Fig. 1 of this paper describes a region 2 that is different from the one of Fig. 3.3.2 used in [5]. Moreover, explicit expressions of functional equations for more complex media can be derived starting from the definitions of M_m matrices in (2.22): in the Appendix A we report the matrices for the anisotropic case.

(b) A classical example of GWHEs for the validation of functional equations: the Malyuzhinets' problem

In this subsection, to further convince the readers about the validity and the correctness of the proposed procedure based on matrix first order differential formulation (Section 4), we derive the GWHEs for a classical scalar problem: the Malyuzhinets' problem.

The general derivation of functional equations of the angular regions do not depend on the materials, the sources located outside the considered angular region and the boundary conditions.

By imposing on them the constitutive parameters of the media and the boundary conditions on the faces we get GWHEs that in general are coupled to the electromagnetic equations present in the regions outside the considered angular region.

We affirm that, in particular, the functional equations are useful to derive GWHEs for wedge problems with impenetrable boundaries as with penetrable ones, see for instance applications in [6]- [7]. Moreover, the functional equations of angular regions can be used to describe more complex scattering problems where angular regions are coupled with stratified planar regions, see for instance [8]- [9].

If we are interested in decoupling the evaluation of the electromagnetic field in a region from the equations that hold outside, we can resort to impenetrable approximate boundary conditions.

For instance, we can assume Leontovich boundary conditions that impose impedance surfaces on the faces of the angular region [37]. In this context, several studies have been developed based on higher order approximate boundary conditions that involves derivatives of the components of the field on the faces. In particular these enhanced version of boundary conditions have been examined in right angled structures [38]- [40] yielding Riemann-Hilbert problems with exact solutions.

In this section, we report as simple demonstration of the method, the classical impenetrable wedge scattering problem known as the Malyuzhinets' problem [41] and extensively studied in literature with different methods. We start from the functional equations and we derive the GWHEs of the problem.

With reference to Fig. 2, the Malyuzhinets' problem is constituted of an impenetrable wedge structure immersed in an isotropic medium and illuminated by a plane wave at normal incidence ($\alpha_o = 0$) where the following scalar boundary conditions are imposed in cylindrical coordinates:

$$\begin{bmatrix} E_z(\rho,\gamma) \\ E_\rho(\rho,\gamma) \end{bmatrix} = Z_a \begin{bmatrix} H_\rho(\rho,\gamma) \\ -H_z(\rho,\gamma) \end{bmatrix}, \quad \begin{bmatrix} E_z(\rho,-\gamma) \\ E_\rho(\rho,-\gamma) \end{bmatrix} = -Z_b \begin{bmatrix} H_\rho(\rho,-\gamma) \\ -H_z(\rho,-\gamma) \end{bmatrix} \tag{5.23}$$

In Fig. 2, with respect to Fig. 1, we identify two symmetrical homogeneous isotropic regions respectively with aperture angle γ and $\pi-\gamma$, while regions 3 and 4 are not physically considered.

The functional equations of region 1 are reported in (5.12)-(5.13), before the application of the boundary conditions of the problem. For region 2, we note the difference on the aperture angle of Fig. 1 with respect to the aperture angle of Fig. 2. For this reason to derive the functional equations of region 2 of Fig. 2 we need to replace γ with $\pi-\gamma$ in (5.21)-(5.22).

At normal incidence ($\alpha_o = 0$), the functional equations of region 1 take the following form

$$\xi \tilde{E}_z + k Z_o \tilde{H}_\rho = -[\eta \sin(\gamma) + \cos(\gamma)\xi]\breve{E}_z + k Z_o \breve{H}_\rho \tag{5.24}$$

$$k \tilde{E}_\rho + \xi Z_o \tilde{H}_z = k \breve{E}_\rho + Z_o[\sin(\gamma)\eta + \cos(\gamma)\xi]\breve{H}_z \tag{5.25}$$

with direction vectors $\hat{v} = \hat{\rho}$ for $\varphi = \gamma$ (face a) and $\hat{x} = \hat{\rho}$ for $\varphi = 0$. The field quantities on the LHS of (5.24)-(5.25) depends on η and are evaluated for $\varphi = 0$, i.e. $\tilde{F} = \tilde{F}(\eta, \varphi = 0)$, while the field quantities on RHS depends on $-m$ and are evaluated for $\varphi = \gamma$, i.e. $\breve{F} = \breve{F}(-m, \varphi = +\gamma)$.

The functional equations of region 2 take the following form

$$\xi \tilde{E}_z + k Z_o \tilde{H}_\rho = [\eta \sin(\gamma) + \cos(\gamma)\xi]\breve{E}_z + k Z_o \breve{H}_\rho \tag{5.26}$$

$$-k \tilde{E}_\rho + \xi Z_o \tilde{H}_z = -k \breve{E}_\rho + Z_o[\sin(\gamma)\eta + \cos(\gamma)\xi]\breve{H}_z \tag{5.27}$$

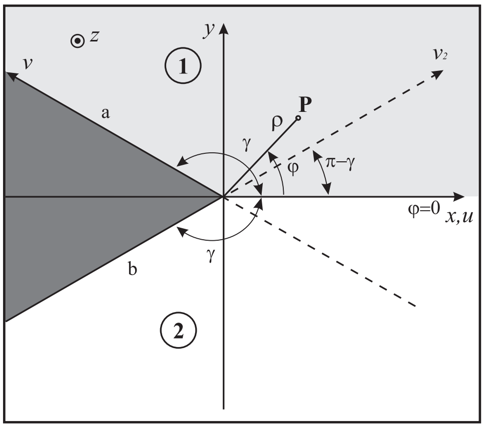

Figure 2. Impenetrable wedge problem with surrounding space made by a homogeneous isotropic medium divided into angular region 1 and 2. Cartesian coordinates (x, y, z) and cylindrical coordinates (ρ, φ, z) are reported. For each angular region a local oblique Cartesian coordinate system is defined: for region 1 u, v, z with aperture angle γ, for region 2 u, v_2, z with aperture angle $\pi - \gamma$. With respect to Fig. 1, regions 3 and 4 are not physically considered. Boundary conditions are imposed at face a and b.

with direction vectors $\hat{v}_2 = -\hat{\rho}$ for $\varphi = -\gamma$ (face b) and $\hat{x} = \hat{\rho}$ for $\varphi = 0$ (see Fig. 2). Note also that for region 2 of Fig. 2 we have from (5.20) and (5.5)

$$m_b(\pi - \gamma, \eta) = -\eta \cos\gamma + \xi \sin\gamma = m \tag{5.28}$$

In this case, while the field quantities on the LHS of (5.26)-(5.27) are the same of the LHS of (5.24)-(5.25), i.e. $\tilde{F} = \tilde{F}(\eta, \varphi = 0)$, the field quantities on RHS of (5.26)-(5.27) depends on $-m$ and are evaluated for $\varphi = -\gamma$, i.e. $\breve{F} = \breve{F}(-m, \varphi = -\gamma)$.

For simplicity, focusing the attention on E_z polarization we use only (5.24) and (5.26). By imposing the boundary conditions (5.23) and eliminating $\breve{E}_z$, we obtain the following system of equations after some mathematical manipulation:

$$-\xi \tilde{E}_z(\eta, 0) + kZ_o \tilde{H}_\rho(\eta, 0) = (kZ_o + nZ_a)\breve{H}_\rho(-m, +\gamma) \tag{5.29}$$

$$\xi \tilde{E}_z(\eta, 0) + kZ_o \tilde{H}_\rho(\eta, 0) = (kZ_o + nZ_b)\breve{H}_\rho(-m, -\gamma) \tag{5.30}$$

with $n = -\eta \sin(\gamma) - \cos(\gamma)\xi$. Finally, (5.29)-(5.30) can be reduced in the normal form to

$$\mathcal{G}(\eta) F_+(\eta) = F_-(m) \tag{5.31}$$

with

$$\mathcal{G}(\eta) = \begin{vmatrix} -\frac{\xi}{Z_a(n_a+n)} & \frac{kZ_o}{Z_a(n_a+n)} \\ \frac{\xi}{Z_b(n_b+n)} & \frac{kZ_o}{Z_b(n_b+n)} \end{vmatrix}, \quad F_+(\eta) = \begin{vmatrix} \tilde{E}_z(\eta, 0) \\ \tilde{H}_\rho(\eta, 0) \end{vmatrix}, \quad F_-(m) = \begin{vmatrix} \breve{H}_\rho(-m, +\gamma) \\ \breve{H}_\rho(-m, -\gamma) \end{vmatrix} \tag{5.32}$$

and where $n_{a,b} = kZ_o/Z_{a,b}$. Note that (5.31) is a matrix GWHE with kernel $\mathcal{G}(\eta)$, plus functions $F_+(\eta)$ in η and minus functions $F_-(m)$ in m. Solutions of the GWHEs of the Malyuzhinets' problem can be found in [2]- [6], [10] using analytical and/or semi-analytical procedure after their reduction to classical WH equations (CWHEs) in a new complex plane $\bar{\eta}$ using the special mapping [5]:

$$\eta(\bar{\eta}) = -k \cos\left(-\frac{\gamma}{\pi} \arccos\left(-\frac{\bar{\eta}}{k}\right)\right), \tag{5.33}$$

(c) Remarks on the functional equations to get GWHEs

In general, the functional equations (4.16)-(4.17), (4.24)-(4.25),(4.34)-(4.35),(4.42)-(4.43) respectively for regions 1,2,3,4 (Fig. 1) are the starting point to derive the GWHEs of arbitrary angular regions (aperture angle, material) in complex scattering problems. In order to obtain the GWHEs for a practical problem we need to define the media and to enforce the boundary conditions at the interfaces of the regions. For instance, see electromagnetic scattering problems by anisotropic impedance wedges in [4], [6], section 5.2 of [10] and more complex problems in [7]- [10].

With reference to Fig. 1, we observe that the *axial spectra* $\tilde{\psi}_t(\eta,0)$ and $\tilde{\psi}_{\pi t}(\eta,0)$ at the interfaces respectively between regions 1 and 2 and between regions 3 and 4 are defined in terms of only continuous components of the fields satisfying the boundary conditions in electromagnetic problems. Meanwhile, the *face spectra* $\breve{\psi}_s(\chi)$ on the interface between the regions 1 and 4 (2 and 3) could present discontinuous components and/or derivative of the fields: see face a and d (face b and c) in Fig. 1. To check the continuity of the face spectra we have re-written the component of $\breve{\psi}_s(\chi)$ in terms of continuous components of the field in the case of isotropic media. In practical case, according to our experience, we note that appropriate relations are always available in arbitrary linear media.

Once obtained the GWHEs from the functional equations of an angular region problem, an important aspect is their reduction to CWHEs by using a suitable mapping as the one reported in (5.33).

The introduction of the complex angular plane w

$$\eta = -k\cos w \tag{5.34}$$

helps the analysis of asymptotic solution of practical problems by allowing analytical extension of approximate solutions [5]- [10]. In fact the application of (5.34) to GWHEs allows to get difference equations useful for recursive applications. We further note that the difference equations relate GWHEs to the SM method for an valuable synergy between the two methods.

The text reports explicit expressions of functional equations for isotropic media. However the procedure is general and applicable to more complex media starting from the definitions of M_m matrices in (2.22): in the Appendix A we report the matrices for the anisotropic media.

6. Conclusion

In this work, we have introduced a general method for the deduction of spectral functional equations in angular regions filled by arbitrary linear homogeneous media. These equations are obtained by solving vector differential equations of first order using dyadic Green's function and then by projecting the solution on reciprocal eigenvectors of an algebraic matrix related to the medium filling the angular region. The fundamental starting point to derive equations in arbitrary linear media is the derivation of matrices M_o, M_1, M_2. From a practical point of view, we have reported these matrices for anisotropic media in the Appendix A, while the main text contains the ones for isotropic media. Derivation of explicit equations requires the implementation of the procedure reported in the paper, illustrated explicitly for isotropic media. The application of the boundary conditions to the functional equations yields GWHEs for practical problems. In this paper, the method is applied to electromagnetic applications and the functional equations are explicitly derived and verified in the case of isotropic media with respect to the current literature.

The efficacy of the GWHE formulation has been demonstrated in several practical electromagnetic engineering works by the authors, see references. We assert that the proposed method to get spectral functional equations in arbitrary angular regions for wave motion problem is general and it is applicable to different physics.

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Appendix A

In this Appendix we report the explicit definitions of the fundamental matrices M_o, M_1, M_2 (2.22) useful to develop applications of the method in electromagnetic anisotropic media, i.e. $\boldsymbol{\xi} = 0$, $\boldsymbol{\zeta} = 0$ in (2.3)-(2.4) (we avoid to report the matrices for the bi-anisotropic case due to their length). In particular to develop the procedure it is sufficient to replace (2.23) of the isotropic case with

$$M_o = \begin{vmatrix} -\frac{j\alpha_o\varepsilon_{yz}}{\varepsilon_{yy}} & -j\alpha_o\left(\frac{\varepsilon_{yx}}{\varepsilon_{yy}} - \frac{\mu_{xy}}{\mu_{yy}}\right) & \frac{j\omega(\mu_{xz}\mu_{yy}-\mu_{xy}\mu_{yz})}{\mu_{yy}} & j\omega\left(\mu_{xx} - \frac{\mu_{xy}\mu_{yx}}{\mu_{yy}}\right) - \frac{j{\alpha_o}^2}{\varepsilon_{yy}\omega} \\ 0 & -\frac{j\alpha_o\mu_{zy}}{\mu_{yy}} & \frac{j\omega(\mu_{yz}\mu_{zy}-\mu_{yy}\mu_{zz})}{\mu_{yy}} & -j\omega\left(\mu_{zx} - \frac{\mu_{yx}\mu_{zy}}{\mu_{yy}}\right) \\ -j\omega\left(\varepsilon_{xz} - \frac{\varepsilon_{xy}\varepsilon_{yz}}{\varepsilon_{yy}}\right) & \frac{j{\alpha_o}^2}{\mu_{yy}\omega} + \frac{j\omega(\varepsilon_{xy}\varepsilon_{yx}-\varepsilon_{xx}\varepsilon_{yy})}{\varepsilon_{yy}} & -\frac{j\alpha_o\mu_{yz}}{\mu_{yy}} & j\alpha_o\left(\frac{\varepsilon_{xy}}{\varepsilon_{yy}} - \frac{\mu_{yx}}{\mu_{yy}}\right) \\ \frac{j\omega(\varepsilon_{yy}\varepsilon_{zz}-\varepsilon_{yz}\varepsilon_{zy})}{\varepsilon_{yy}} & \frac{j\omega(\varepsilon_{yy}\varepsilon_{zx}-\varepsilon_{yx}\varepsilon_{zy})}{\varepsilon_{yy}} & 0 & -\frac{j\alpha_o\varepsilon_{zy}}{\varepsilon_{yy}} \end{vmatrix} \tag{A.1}$$

$$M_1 = \begin{vmatrix} -\frac{j\mu_{xy}}{\mu_{yy}} & 0 & \frac{j\alpha_o}{\varepsilon_{yy}\omega} & 0 \\ \frac{j(\varepsilon_{yy}\mu_{zy}-\varepsilon_{yz}\mu_{yy})}{\varepsilon_{yy}\mu_{yy}} & -\frac{j\varepsilon_{yx}}{\varepsilon_{yy}} & 0 & -\frac{j\alpha_o}{\varepsilon_{yy}\omega} \\ -\frac{j\alpha_o}{\mu_{yy}\omega} & 0 & -\frac{j\varepsilon_{xy}}{\varepsilon_{yy}} & 0 \\ 0 & \frac{j\alpha_o}{\mu_{yy}\omega} & \frac{j(\varepsilon_{zy}\mu_{yy}-\varepsilon_{yy}\mu_{yz})}{\varepsilon_{yy}\mu_{yy}} & -\frac{j\mu_{yx}}{\mu_{yy}} \end{vmatrix} \tag{A.2}$$

$$M_2 = \begin{vmatrix} -\frac{j\mu_{xy}}{\mu_{yy}} & 0 & \frac{j\alpha_o}{\varepsilon_{yy}\omega} & 0 \\ \frac{j(\varepsilon_{yy}\mu_{zy}-\varepsilon_{yz}\mu_{yy})}{\varepsilon_{yy}\mu_{yy}} & -\frac{j\varepsilon_{yx}}{\varepsilon_{yy}} & 0 & -\frac{j\alpha_o}{\varepsilon_{yy}\omega} \\ -\frac{j\alpha_o}{\mu_{yy}\omega} & 0 & -\frac{j\varepsilon_{xy}}{\varepsilon_{yy}} & 0 \\ 0 & \frac{j\alpha_o}{\mu_{yy}\omega} & \frac{j(\varepsilon_{zy}\mu_{yy}-\varepsilon_{yy}\mu_{yz})}{\varepsilon_{yy}\mu_{yy}} & -\frac{j\mu_{yx}}{\mu_{yy}} \end{vmatrix} \tag{A.3}$$

As a practical propagation example, by restricting the case to $\alpha_o = 0$ and diagonal $\boldsymbol{\varepsilon}$, $\boldsymbol{\mu}$ we compute easily the eigenvalues (2.27) of $M(\eta)$ (2.25), yielding

$$\xi_1 = \xi_3 = \frac{\sqrt{\mu_{xx}}}{\sqrt{\mu_{yy}}}\sqrt{\omega^2\varepsilon_{zz}\mu_{yy} - \eta^2}, \quad \xi_2 = \xi_4 = \frac{\sqrt{\varepsilon_{xx}}}{\sqrt{\varepsilon_{yy}}}\sqrt{\omega^2\varepsilon_{yy}\mu_{zz} - \eta^2} \tag{A.4}$$

442 that constitutes two propagation modalities, the ordinary and extraordinary waves.

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Appendix B

In this Appendix we report the justification and the properties of the dyadic Green?s function (4.11) to get the particular solution (4.9) of (4.3). The dyadic Green's function is solution of the dyadic equation

$$\frac{d}{dv}G(v,v') + M_e(\gamma,\eta)G(v,v') = \delta(v - v')1_t \tag{B.1}$$

where 1_t is the unitary dyadic (2.31). According to [36], we select as solutions of the homogeneous equations to build the dyadic Green's function progressive and regressive waves in an indefinite region. Moreover the dyadic Green's functions need to model the behavior at $v = v'$ of (B.1) to allow the particular solution (4.9) be solution of (4.3). Using dyadic notation, for $v > v'$ we have the set of progressive waves ($i = 1, 2$), while for $v < v'$ regressive waves ($i = 3, 4$), i.e.

$$G_i(v,v') = u_i A_i(v')e^{-\lambda_{ei}(\gamma,\eta)v}, \ i = 1..4 \tag{B.2}$$

444 where $\lambda_{ei}(\gamma,\eta)$ and u_i are the eigenvalues and the eigenvectors of the matrix of dimension four
445 $M_e(\gamma,\eta)$ and, $A_i(v')$ are arbitrary vector coefficients.

The most general solution of (B.1) is expressed by the dyadics

$$G(v,v') = \begin{cases} \vec{G}(v,v') = u_1 A_1(v') e^{-\lambda_{e1}(\gamma,\eta)v} + u_2 A_2(v') e^{-\lambda_{e2}(\gamma,\eta)v} & v > v' \\ \overleftarrow{G}(v,v') = u_3 A_3(v') e^{-\lambda_{e3}(\gamma,\eta)v} + u_4 A_4(v') e^{-\lambda_{e4}(\gamma,\eta)v} & v < v' \end{cases} \tag{B.3}$$

In order to find the vectors $A_i(v')$, $G(v,v')$ must satisfy (B.1) also at $v = v'$ by imposing the fundamental jump condition

$$\vec{G}(v'_+, v') - \overleftarrow{G}(v'_-, v') = 1_t \tag{B.4}$$

It yields

$$u_1 A_1(v') e^{-\lambda_{e1}(\gamma,\eta)v'} + u_2 A_2(v') e^{-\lambda_{e2}(\gamma,\eta)v'} - (u_3 A_3(v') e^{-\lambda_{e3}(\gamma,\eta)v'} + u_4 A_4(v') e^{-\lambda_{e4}(\gamma,\eta)v'}) = 1_t \tag{B.5}$$

Pre-multiplying (B.5) by the reciprocal eigenvectors ν_i (2.30)-(2.31), we get

$$\begin{array}{ll} A_i(v') = \nu_i e^{\lambda_{ei}(\gamma,\eta)v'} & (i = 1,2) \\ A_i(v') = -\nu_i e^{\lambda_{ei}(\gamma,\eta)v'} & (i = 3,4) \end{array} \tag{B.6}$$

Substituting (B.6) into (B.3), we get the dyadic Green's function (4.11).

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Glossary

In the following table 1 we report a glossary of main abbreviations, notations and symbols introduced in the paper and useful for its readability.

Table 1. Main Abbreviations, Notations and Symbols introduced in the paper

Notation	**Description**
WH	Wiener-Hopf
GWHEs	Generalized Wiener-Hopf Equations
CWHEs	Classical Wiener-Hopf Equations
SM	Sommerfeld-Malyuzhinets (method)
PEC	Perfect Electrical Conductor
(x, y, z)	Cartesian coordinates
(ρ, φ, z)	cylindrical coordinates
(u, v, z)	oblique Cartesian coordinates
E, H, D, B	electric field, magnetic field, dielectric induction, magnetic induction
k	propagation constant
Z_o	free space impedance
$\boldsymbol{\varepsilon}, \boldsymbol{\mu}$ and $\boldsymbol{\xi}, \boldsymbol{\zeta}$	tensor constitutive parameters (electric permittivity, magnetic permeability and magnetoelectric tensors)
$e^{j\omega t}$	time dependence of harmonic field
Γ_∇	matrix differential operator in abstract notation
ψ, θ	vector fields in abstract notation
W	matrix constitutive parameters of media
ψ_t	transverse field for a stratification along the y direction
ψ_y	longitudinal field for a stratification along the y direction
$\mathcal{M}(\frac{\partial}{\partial z}, \frac{\partial}{\partial x})$	transversal operator for Maxwell's equations
$D_x = \frac{\partial}{\partial x}$	alternative partial derivative notation
α_o	due to invariance along the z-direction, without loss of generality, we suppose that a field dependence specified by the factor $e^{-j\alpha_o z}$
η	Fourier or Laplace spectral variable according to the position on the text
$\Psi_t(\eta)$	Fourier transform along $x = u$ direction (y, z or v, z dependence is omitted)
$M(\eta)$	matrix operator in Fourier/Laplace domain in indefinite rectangular region
λ_i, u_i	eigenvalues and eigenvector of $M(\eta)$
ν_i	reciprocal vectors of u_i
ξ_i	different notation of λ_i for propagation's properties, multivalued function
γ	aperture angle of angular regions (Fig. 1)
$M_e(\gamma, \eta)$	matrix operator in Fourier/Laplace domain in indefinite angular region
λ_{ei}	eigenvalues of $M_e(\gamma, \eta)$
$\tilde{\psi}_t(\eta, v)$	Laplace transform along $x \equiv u$ of $\psi_t(u, v)$ (omitting z dependence)
$\psi_s(v)$	field components on the face of an angular region in Laplace domain
$\psi_{as}(v)$	specialized expression of $\psi_s(v)$ on face a
$G(v, v')$	dyadic Green's function in Laplace domain for an angular region
$\breve{\psi}_{as}(\chi)$	Laplace transform in v along face a ($v = \rho$)
m_{a1}, m_{a2}	spectral variables for the evaluation of $\breve{\psi}_{as}(\chi)$ in functional equations