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Self-synchronization of unbalanced rotors and the swing equation

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Abstract: We consider a problem of self-synchronization in a system of vibro-exciters (rotors) installed on a common oscillating platform. This problem was studied by I.I. Blekhman and later by L. Sperling. Extending their approach, we derive the equations for a system of n rotors and show that, separating the slow and fast motions, the "slow" dynamics of this systems reduces to a special case of a so-called swing equation that is well studied in theory of power networks. On the other hand, the system may be considered as "pendulum-like" system with multidimensional periodic nonlinearities. Using the theory of such systems developed in our previous works, we derive an analytic criteria for synchronization of two rotors. Unlike synchronization criteria available in mechanical literature, our criterion ensures *global* convergence of every trajectory to the synchronous manifold.

Keywords: Synchronization, stability of nonlinear systems, vibrational mechanics

1. INTRODUCTION

Synchronization is a fundamental principle that explains many natural phenomena and lies in the heart of numerous technical designs (Pikovsky et al., 2001; Néda et al., 2000; Strogatz, 2003). Although general mathematical definitions of synchronous processes in dynamical systems have appeared quite recently Blekhman et al. (1997, 2002), special problems of synchronization in ensembles of coupled oscillators date back to the seminal experiment with clocks reported by Huygens in 1665 (Bennett et al., 2002; Czolczynski et al., 2011; Pikovsky et al., 2001).

In Huygens' framework, two clocks suspended on a common beam *self-synchronize* in the absence of external control. Similar phenomena of self-synchronization play an important role in vibrational mechanics (Blekhman, 2000), in particular, design of automatic dynamic balances to prevent harmful vibrations (Thearle and Schenectady, 1932; Hedaya and Sharp, 1977; Blekhman, 1988). In this paper, we address the problem that was pioneered by Blekhman (Blekhman, 1953) and is concerned with self-synchronization of two vibro-exciters (eccentric rotors) installed on a common rigid platform. The analysis of this phenomenon is important not only to understand mechanisms of synchronization in natural and technical systems, but also for engineering applications. On one hand, self-synchronization between multiple eccentric rotors driving a vibration mill allows to simplify the overall mill's construction and get rid of additional synchronization controllers (Chen et al., 2016), on the other hand, self-synchronizing dynamic balancers can be used to cancel harmful vibrations (Sperling et al., 2000).

Following the framework introduced in (Sperling, 1994a,b; Sperling et al., 1997), we consider a system of n rotors installed on an rigid platform with one degree of freedom. Using the standard mechanical approach, the phase (angle) and frequency variables of each rotor are decomposed into a slowly changing and a fast changing components, and the problem in question is when the slow components of the rotors' frequencies are asymptotically synchronous. It should be noted that this problem is highly non-trivial even for the case of two rotors. The results published in mechanical journals are typically confined to analysis of the conditions under which the synchronous mode is possible and *local* stability analysis of the synchronous motion (Sperling et al., 1997; Chen et al., 2016; Zhang et al., 2013; Fang et al., 2014, 2019).

The contribution of this paper is threefold. First, we derive the equations for a system of n rotors on a rigid platforms and show that the model coincides with the special case of a so-called *swing equation* that arises in analysis of synchronization and transient stability of power networks (Dörfler and Bullo, 2012; Varaiya et al., 1985). This opens up a perspective of obtaining *global* stability criteria elaborated in control theory for power systems. Second, we show that, on the other hand, the slow dynamics of the coupled rotors can be reduced to the class of systems with periodic nonlinearities examined in our previous works (Leonov et al., 1992, 1996; Smirnova and Proskurnikov, 2019b). Although the frequency-domain criterion is difficult to validate in the case of multiple nonlinearities, the theory developed in these previous works can

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be applied to derive global stability of the synchronous motion for two rotors, which is our third contribution.

2. THE SYSTEM OF COUPLED ROTORS

Consider the system of n rotors (Fig. 1) installed on a rigid platform with one degree of freedom (Blekhman, 2000; Sperling et al., 1997). The rigid platform can move in the fixed direction Ox and connected to a stationary support by the elastic element. The axes of the rotors are orthogonal to the direction Ox. The rotors are brought into force by the asynchronous electric motors.

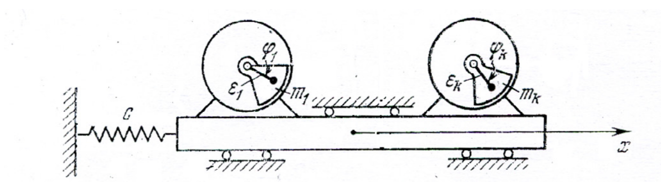

Fig. 1. Rotors situated on a rigid platform

2.1 Equations of motion. Synchronization problem.

We exploit here the mathematical model of the system borrowed from Sperling (1994a,b):

$$I_i\ddot{\varphi}_i = L_i(\dot{\varphi}_i) + m_i\varepsilon_i\ddot{x}\sin\varphi_i \quad (i = 1, 2, \dots, n), \tag{1}$$

$$M\ddot{x} = -cx + \sum_{i=1}^{n} m_i\varepsilon_i(\dot{\varphi}_i^2\cos\varphi_i + \ddot{\varphi}_i\sin\varphi_i)$$
$$(M = M_0 + \sum_{i=1}^{n} m_i;\ I_i = J_i + m_i\varepsilon_i^2). \tag{2}$$

Here x is the displacement of the platform, φ_i ($i = 1, 2, \dots, n$) is the angle of the i-th rotor counted from Ox-axis. The constants J_i, m_i, ε_i stand for, respectively, the i-th rotor's moment of inertia, mass and eccentricity, M_0 is the mass of the platform and c is the elasticity coefficient of the spring supporting the platform.

The value of $L_i(\dot{\varphi}_i)$ is the rotation moment of the motor, represented as follows

$$L_i = L_i^0 - k_i\dot{\varphi}_i \quad (L_i^0, k_i = const). \tag{3}$$

The system (1)-(3) can be now transformed by the method of "direct partition" of motion, separating the slow and fast components. We reproduce the arguments from Sperling et al. (1997). It is supposed that

$$\varphi_i(t) = \Omega t + \alpha_i(t) + \Psi_i(t, \Omega t) \quad (\Omega = const); \tag{4}$$

$$\frac{1}{2\pi}\int_0^{2\pi} \Psi_i(t, \Omega t)\, d\Omega t = 0, \tag{5}$$

where α_i and Ψ_i are, respectively, the slow and the fast components on the phase φ_i.

It is also assumed that Ψ_i is small and in the equation (2)

$$\dot{\varphi}_i \approx \Omega, \quad \ddot{\varphi}_i \approx 0. \tag{6}$$

Then (1) transforms into

$$I_i\ddot{\varphi}_i + k_i\dot{\varphi}_i = L_i^0 + m_i\varepsilon_i\ddot{x}\sin\varphi_i \quad (i = 1, 2, \dots, n), \tag{7}$$

and (2) takes the form

$$M\ddot{x} + cx = \sum_{i=1}^{n} f_i\cos(\Omega t + \alpha_i) \quad (f_i = m_i\varepsilon_i\Omega^2). \tag{8}$$

The linear equation (8) has a solution

$$x = A_{xx}\sum_{i=1}^{n} f_i\cos(\Omega t + \alpha_i) \tag{9}$$

where

$$A_{xx} = \frac{1}{M(\omega^2 - \Omega^2)}, \quad \omega^2 = \frac{c}{M}, \tag{10}$$

whence

$$\ddot{x} = -A_{xx}\Omega^2\sum_{i=1}^{n} f_i\cos(\Omega t + \alpha_i). \tag{11}$$

Using equation (7), we can obtain that

$$I_i\ddot{\alpha}_i + k_i\dot{\alpha}_i = k_i(\Omega_i - \Omega) + V_i, \tag{12}$$

$$\Omega_i := \frac{L_i^0}{k_i}. \tag{13}$$

$$V_i := -\frac{A_{xx}f_i}{2\pi}\int_0^{2\pi}\sum_{s=1}^{n} f_s\cos(\Omega t + \alpha_s)\sin(\Omega t + \alpha_i)\, d(\Omega t). \tag{14}$$

Taking into account that

$$\frac{1}{2\pi}\int_0^{2\pi}\cos(\Omega t + \alpha_s)\sin(\Omega t + \alpha_i)\, d(\Omega t) = $$
$$= \frac{1}{2}\sin(\alpha_i - \alpha_s), \tag{15}$$

it can be easily seen that

$$V_i = -\frac{A_{xx}}{2}\sum_{s=1}^{n} f_s f_i\sin(\alpha_i - \alpha_s) \tag{16}$$

Equation (12) thus shapes into

$$I_i\ddot{\alpha}_i + k_i\dot{\alpha}_i = k_i\bar{\Omega}_i - \frac{A_{xx}}{2}\sum_{s=1}^{n} f_s f_i\sin(\alpha_i - \alpha_s)$$
$$\bar{\Omega}_i := \Omega_i - \Omega, \quad i = 1, 2, \dots, n. \tag{17}$$

Summing up the equation over all $i = 1, \dots, n$, it can be seen that the equilibrium $\alpha_i \equiv const$ may exist only when

$$\Omega = \frac{k_1\Omega_1 + \dots + k_n\Omega_n}{k_1 + \dots + k_n} = \frac{\sum_{i=1}^n L_i^0}{\sum_{i=1}^n k_i}. \tag{18}$$

Note that (18) is only a necessary condition yet not sufficient. A simple sufficient condition for the equilibrium's existence is obtained by noticing that

$$\Omega_i - \Omega = \frac{\sum_{\substack{s=1\\ s\neq i}}^n k_s(\Omega_i - \Omega_s)}{\sum_{i=1}^n k_i}, \tag{19}$$

and hence equations (17) can be rewritten as

$$I_i\ddot{\alpha}_i + k_i\dot{\alpha}_i = \sum_{\substack{s=1\\ s\neq i}}^{n} F_{is},$$

$$F_{is}(t) = \frac{k_i k_s(\Omega_i - \Omega_s)}{\sum_{i=1}^n k_i} - \frac{A_{xx}}{2} f_s f_i\sin(\alpha_i(t) - \alpha_s(t)). \tag{20}$$

Hence, the equilibrium automatically exists if for each pair of indices $i = 1, \dots, n$ and $s = i+1, \dots, n$ the following equation is solvable

$$\sin(\alpha_i - \alpha_s) = \gamma_{is} := \frac{2k_i k_s(\Omega_i - \Omega_s)}{A_{xx}f_s f_i\sum_{i=1}^n k_i}, \tag{21}$$

or, equivalently, $|\gamma_{is}| \leq 1$ for all i, s.

The problem of the rotors' slow dynamics synchronization is as follows (Blekhman, 2000):

Problem. To find conditions on the parameters I_i, k_i, L_i^0 such that every solution of equations (17) (with parameters defined in (13) and (18)) converges to one of the equilibria and, additionally,

$$\dot{\alpha}_i - \dot{\alpha}_s \xrightarrow[t\to+\infty]{} 0 \quad \forall s, i. \tag{22}$$

The convergence of all solutions to equilibria is known as the *gradient-like behavior* of the system (Leonov et al., 1992). It should be noted that the convergence is required for all possible initial conditions and not "locally" (in a sufficiently small vicinity of the synchronous motion). Such a property cannot be guaranteed by standard methods of local stability analysis such as e.g. the Routh-Hurwitz criterion typically employed in mechanical literature.

2.2 Slow dynamics of rotors and swing equations

One may notice that equations (17) have the same structure as the *swing equations* describing the dynamics of multi-machine power networks in the so-called *lossless* (zero transfer conductance) case (Baillieul and Byrnes, 1982; Varaiya et al., 1985; Dörfler and Bullo, 2012). Considering (17) as equations describing a power network, I_i, k_i stand for the inertia and damping constant of the ith generator, f_i is the internal voltage of the ith oscillator and $A_{xx}/2$ stands for the constant transfer susceptance between each pair of generators. The value $\bar{\Omega}_i$ is interpreted as a "natural frequency" or, physically, the effective power input to generator i (Dörfler and Bullo, 2012).

The parallel with power networks, which has not been realized in vibrational mechanics literature, opens up the perspective of employing control-theoretic tools proposed in the literature on power systems control, in particular, special energy-based Lyapunov functions (Varaiya et al., 1985; Bretas and Alberto, 2003) and other methods such as e.g. singular perturbation theory (Dörfler and Bullo, 2012). It appears, in particular, that in the situation where $I_i \ll k_i$ equations (17) are efficiently approximated by the non-uniform *Kuramoto* network that admits a more complete analysis (Dörfler and Bullo, 2012, 2014).

2.3 Slow dynamics of rotors as a pendulum-like systems

On the other hand, equations (17) can be written as a feedback superposition of stable linear time-invariant (LTI) systems and a (multidimensional) periodic nonlinearity. Systems of this type have been thoroughly studied in the works by Leonov (Gelig et al., 2004; Leonov, 2006; Leonov et al., 1992, 1996); the most recent progress in their analysis is reported in our previous work (Smirnova and Proskurnikov, 2019b). Systems with periodic nonlinearities are often referred to as "pendulum-like" (as they include the classical pendulum as a special case) or "synchronization" systems (because they described dynamics of phase-locked loops and other circuits providing synchronization of signals).

Equations (20) can be rewritten as

$$\dot{\alpha}_i(t) = \dot{\alpha}_i(0)e^{-\frac{k_i}{I_i}t} + \frac{1}{I_i}\int\limits_0^t e^{-\frac{k_i}{I_i}(t-\tau)} \sum_{\substack{s=1\\ s\neq i}}^{n} F_{is}(\tau)\, d\tau \tag{23}$$

Introduce three $l = \frac{(n-1)n}{2}$ – vector-functions

$$\sigma = (\alpha_1 - \alpha_2, \ldots, \alpha_1 - \alpha_n, \alpha_2 - \alpha_3, \ldots, \alpha_{n-1} - \alpha_n)^\top,$$
$$F(\sigma) = (\Phi_{12}, \ldots, \Phi_{1n}, \Phi_{23}, \ldots, \Phi_{(n-1)n})^\top$$

with $\Phi_{is} := \gamma_{is} - \sin(\alpha_i - \alpha_s)$ (and γ_{is} from (21)), and

$$b^T = \Big(\dot{\alpha}_1(0)e^{-\frac{k_1}{I_1}t} - \dot{\alpha}_2(0)e^{-\frac{k_2}{I_2}t}, \ldots$$
$$\ldots, \dot{\alpha}_{n-1}(0)e^{-\frac{k_{n-1}}{I_{n-1}}t} - \dot{\alpha}_n(0)e^{-\frac{k_n}{I_n}t}\Big),$$

the equations are written as follows

$$\dot{\sigma}(t) = b(t) - \int\limits_0^t \Gamma(t-\tau)F(\sigma(\tau))\, d\tau. \tag{24}$$

Here $\Gamma \in \mathbb{R}^{l\times l}$ is a matrix function. Its elements are either zeros or decreasing exponents.

Assuming that $|\gamma_{is}| < 1$, system (24) has a countable set of equilibria. In (Leonov et al., 1992, 1996; Smirnova and Proskurnikov, 2019b), frequency-domain conditions for the gradient-like behavior have been established.

3. FREQUENCY-ALGEBRAIC CRITERION FOR GRADIENT-LIKE BEHAVIOR OF SYNCHRONIZATION SYSTEMS

Consider the synchronization system described by the system of integro-differential Volterra equations

$$\dot{\sigma}(t) = b(t) - \int\limits_0^t \Gamma(t-\tau)F(\sigma(\tau))\, d\tau \quad (t > 0) \tag{25}$$

where $b : R_+ \to \mathbb{R}^l$, $\Gamma : R_+ \to \mathbb{R}^{l\times l}$ and $F : \mathbb{R}^l \to \mathbb{R}^l$. System (25) may be considered as a feedback superposition of the LTI system

$$\dot{\sigma}(t) = b(t) - \int\limits_0^t \Gamma(t-\tau)\xi(\tau)\, d\tau \tag{26}$$

and the nonlinear block $\xi(\tau) = F(\sigma(\tau))$.

We suppose that the following assumptions hold.

A1. The function $b(t)$ is continuous for $t \geq 0$, the matrix-function $\Gamma(t)$ is piece-wise continuous for $t \geq 0$ and

$$|b(t)| + |\Gamma(t)| < Ce^{-rt} \quad (C > 0,\, r > 0) \tag{27}$$

A2. The map $F^T = (F_1, \ldots, F_l)$ is $\mathbb{C}^1$–smooth. Furthermore, $F_j = F_j(\sigma_j)$ depends only on σ_j and is Δ_j-periodic

$$F_j(\zeta + \Delta_j) = F_j(\zeta) \quad (\Delta_j > 0).$$

Also, F_j has *simple* zeros:

$$F_j^2 + (F_j')^2 \neq 0 \quad (j = 1, \ldots, l).$$

Asymptotic behavior of control systems described by integral or integro-differential Volterra equations often is successfully investigated by Popov's method of a priory integral indices (Rasvan, 2006; Popov, 1973). Sufficient conditions for any type of stability are formulated then by means of transfer function of the linear part of the system. They have the form frequency-domain inequalities with varying parameters. Because of specific character of synchronization systems, traditional Popov's functionals are of no use here. That is why for synchronization systems the method of a priori integral indices was combined with special procedure (Bakaev and Guzh, 1965; Leonov et al., 1996; Smirnova and Proskurnikov, 2019b). As a result frequency-domain inequalities were supplemented by some algebraic restrictions on varying parameters, which gave rise to a number of frequency-algebraic stability criteria. In particular, rather tight estimates for stability domains of phase-locked loops (PLLs) with time delay have been obtained in (Smirnova and Proskurnikov, 2019a,b; Proskurnikov and Smirnova, 2020).

The transfer matrix of the linear part (26) from $(-\xi)$ to $(\dot\sigma)$ is defined as

$$K(p) = \int_0^\infty \Gamma(t)e^{-pt}\,dt \quad (p \in \mathbb{C}). \tag{28}$$

Further for complex-valued matrices H we shall use the denotation

$$Re\,H = \frac{1}{2}(H + H^*),$$

where $(*)$ stands for Hermitian conjugation.

Define numbers

$$\mu_{1j} \triangleq \inf_{\zeta\in[0,\Delta_j)} F_j'(\zeta); \quad \mu_{2j} \triangleq \sup_{\zeta\in[0,\Delta_j)} F_j'(\zeta) \quad (\mu_{1j}\mu_{2j} < 0) \tag{29}$$

and introduce the $l \times l$ – diagonal matrices

$$\begin{aligned} M_1 &\triangleq diag\{\alpha_{11}, \ldots, \alpha_{1l}\}, \quad \alpha_{1j} \le \mu_{1j}, \forall j,\\ M_2 &\triangleq diag\{\alpha_{21}, \ldots, \alpha_{2l}\}, \quad \alpha_{2j} \ge \mu_{2j}, \forall j. \end{aligned} \tag{30}$$

Introduce the functions

$$\Phi_j(\zeta) \triangleq \sqrt{(1-\alpha_{1j}^{-1}F_j'(\zeta))(1-\alpha_{2j}^{-1}F_j'(\zeta))}, \tag{31}$$

(recall that $\alpha_1 \le \mu_1 < 0, \alpha_2 \ge \mu_2 > 0$), and the constants

$$\nu_j = \frac{\int_0^{\Delta_j} F_j(\zeta)\,d\zeta}{\int_0^{\Delta_j} |F_j(\zeta)|\,d\zeta}, \quad \nu_{0j} = \frac{\int_0^{\Delta_j} F_j(\zeta)\,d\zeta}{\int_0^{\Delta_j} |F_j(\zeta)|\,\Phi_j(\zeta)\,d\zeta}. \tag{32}$$

Theorem 1. (Smirnova and Proskurnikov, 2019b) Suppose that diagonal matrices $\varkappa, \varepsilon, \delta > 0$ and numbers $\alpha_{1j} \le \mu_{1j}$, $\alpha_{2j} \ge \mu_{2j}$ $(j = 1, \ldots, l)$ and numbers $a_j \in [0, 1]$ $(j = 1, \ldots, l)$ exist such that

1) the frequency-domain inequality holds as follows

$$\begin{aligned} &Re\{\varkappa K(i\omega) - K^*(i\omega)\varepsilon(K(i\omega)-\\ &-[K(i\omega) + i\omega M_1^{-1}]^*\tau[K(i\omega) + i\omega M_2^{-1}] - \delta \ge 0; \end{aligned} \tag{33}$$

2) the quadratic forms

$$Q_j(x, y, z) = \varepsilon_j x^2 + \delta_j y^2 + \tau_j z^2 + \varkappa_j\nu_j a_j xy + \varkappa_j\nu_{0j}(1-a_j)yz$$

are positive definite. Here $\varkappa_j, \varepsilon_j, \delta_j > 0$ denote the jth diagonal entry of the corresponding matrix.

Then, every solution converges to an equilibrium, i.e.,

$$\sigma_j(t) \to q_j \text{ as } t \to +\infty, \quad F_j(q_j) = 0, \tag{34}$$

$$\dot\sigma_j(t) \to 0 \text{ as } t \to +\infty. \tag{35}$$

4. THE APPLICATION OF FREQUENCY-ALGEBRAIC STABILITY CRITERION TO THE SYSTEM OF TWO ROTORS

In this section we use the frequency-algebraic criterion in order to establish the conditions for synchronization of two rotors $(n = 2)$ in the space of parameters of the system.

In this case system (17) takes the form

$$\begin{cases} I_1\ddot\alpha_1 + k_1\dot\alpha_1 + AF(\alpha_1 - \alpha_2) = 0,\\ I_2\ddot\alpha_2 + k_2\dot\alpha_2 - AF(\alpha_1 - \alpha_2) = 0 \end{cases} \tag{36}$$

where A, F are defined as follows

$$A = \frac{1}{2}f_1 f_2 A_{xx}, \tag{37}$$

$$F(\sigma) = \sin\sigma - \frac{\beta}{A}, \quad \beta = \frac{k_1k_2}{k_1+k_2}(\Omega_1 - \Omega_2). \tag{38}$$

Without loss of generality, assume that $\Omega_1 \ge \Omega_2$. Also, it suffices to consider the case where $A > 0$, otherwise one can replace in (36) A by $|A|$ and $F(\sigma)$ by

$$\bar F(\sigma) = -\sin\sigma - \frac{\beta}{|A|}.$$

The system (36) can be reduced to integro-differential equation

$$\begin{aligned} \dot\sigma(t) &= \left(\dot\alpha_1(0)e^{-\frac{k_1}{I_1}t} - \dot\alpha_2(0)e^{-\frac{k_2}{I_2}t}\right) -\\ &-A\int_0^t \left(\frac{1}{I_1}e^{-\frac{k_1}{I_1}(t-\tau)} + \frac{1}{I_2}e^{-\frac{k_2}{I_2}(t-\tau)}\right)F(\sigma(\tau))\,d\tau. \end{aligned} \tag{39}$$

It is obvious that Assumptions A1 and A2 are fulfilled for the equation (39). In this case $l = 1$. So we need no changing indices for functions F, Φ, σ and constants $a, \mu_1, \mu_2, \alpha_1, \alpha_2, \nu, \nu_0$. Similarly the matrices $K(p), \varepsilon, \delta, \tau, \varkappa$ become scalar values. Notice that in case $\beta = 0$ condition 2) of Theorem 1 is fulfilled automatically. In case $\beta \ne 0$ this condition can be modified to guarantee the optimal value of the varying parameter a.

So in this section we use the simplified frequency-algebraic criterion.

Theorem 2. (Proskurnikov and Smirnova, 2020) Suppose there exist numbers $\varepsilon, \delta, \tau > 0$, $\varkappa = \{-1, 0, 1\}$, $\alpha_1 \le \mu_1$, $\alpha_2 \ge \mu_2$, such that

1)the frequency-domain inequality is valid

$$\begin{aligned} &Re\{\varkappa K(i\omega) - \tau(K(i\omega) + \alpha_1^{-1}i\omega)^*\cdot\\ &\cdot(K(i\omega) + \alpha_2^{-1}i\omega) - \varepsilon|K(i\omega)|^2 \ge \delta, \ \forall\omega \ge 0; \end{aligned} \tag{40}$$

2) and, additionally, the algebraic condition holds

$$\delta > \varkappa^2\frac{\nu_0^2\nu^2}{4(\varepsilon\nu_0^2 + \tau\nu^2)}.$$

Then, every solution of (39) converges to an equilibrium

$$\sigma(t) \to q \text{ as } t \to +\infty, \quad F(q) = 0 \tag{41}$$

$$\dot\sigma(t) \to 0 \text{ as } t \to +\infty, \tag{42}$$

The transfer function of the linear part of (39) is as follows

$$K(p) = A\left(\frac{1}{I_1p + k_1} + \frac{1}{I_2p + k_2}\right) \quad (p \in \mathbb{C}). \tag{43}$$

Let $\varkappa = 1$, $\alpha_2 = -\alpha_1 = 1$.

Then the frequency-domain inequality (40) has the form:

$$\tau\omega^2 + Re\,K(i\omega) - (\varepsilon+\tau)|K(i\omega)|^2 \geq \delta \quad \forall\omega \geq 0. \tag{44}$$

For the transfer function (43) the inequality (44) is equivalent to the following

$$\begin{aligned} S(y) &:= (\tau y - \delta)(k_1^2 + I_1^2 y)(k_2^2 + I_2^2 y)+\\ &+A(y(I_1^2 k_2 + I_2^2 k_1) + (k_1 k_2^2 + k_2 k_1^2))-\\ &-(\varepsilon+\tau)A^2((k_1+k_2)^2 + (I_1+I_2)^2 y) \geq 0 \;\; \forall y \geq 0. \end{aligned} \tag{45}$$

Notice that

$$S(y) = S_1(y)\cdot y + S_2 y + S_3,$$

where

$$\begin{aligned} S_1(y) &= \tau I_1^2 I_2^2 y^2 + (-\delta I_1^2 I_2^2+\\ &+\tau(k_2^2 I_1^2 + k_1^2 I_2^2))y + \tau k_1^2 k_2^2,\\ S_2 &= -\delta(k_2^2 I_1^2 + k_1^2 I_2^2) + A(I_2^2 k_1 + I_1^2 k_2)-\\ &-A^2(\varepsilon+\tau)(I_1+I_2)^2,\\ S_3 &= -\delta k_1^2 k_2^2 - A^2(\varepsilon+\tau)(k_1+k_2)^2 + A k_1 k_2(k_1+k_2). \end{aligned}$$

Suppose the following inequalities are true:

$$\varepsilon + \tau \leq \frac{k_1 k_2}{2A(k_1+k_2)}, \tag{46}$$

$$\delta \leq \frac{A(k_1 I_2^2 + k_2 I_1^2)}{2(k_1^2 I_2^2 + k_2^2 I_1^2)}, \tag{47}$$

$$\delta \leq \frac{\tau(I_1 k_2 + I_2 k_1)^2}{I_1^2 I_2^2}. \tag{48}$$

Then we have

$$\begin{aligned} S_3 &\geq -\frac{A}{2}\frac{(k_1 I_2^2 + k_2 I_1^2)k_1^2 k_2^2}{k_1^2 I_2^2 + k_2^2 I_1^2} + \frac{A}{2}k_1 k_2(k_1+k_2) > 0\\ S_2 &\geq \frac{A}{2}[(I_2^2 k_1 + I_1^2 k_2) - \frac{(I_1+I_2)^2 k_1 k_2}{k_1+k_2}] >\\ &> \frac{A(I_2 k_1 - I_1 k_2)^2}{2(k_1+k_2)} \geq 0\\ S_1 &\geq \tau(I_1 I_2 y - k_1 k_2)^2. \end{aligned}$$

So inequalities (46) (47) (48) guarantee that the frequency-domain inequality (44) holds.

To guarantee the algebraic condition 2) of Theorem 2 we calculate the constants ν and ν_0:

$$\nu = \frac{\int_0^{2\pi}\left(\sin\sigma - \frac{\beta}{A}\right)d\sigma}{\int_0^{2\pi}\left|\sin\sigma - \frac{\beta}{A}\right|d\sigma} = \frac{-\pi\beta}{2\left(\beta\arcsin\frac{\beta}{A} + \sqrt{A^2-\beta^2}\right)}; \tag{49}$$

$$\begin{aligned} \nu_0 &= \frac{\int_0^{2\pi}\left(\sin\sigma - \frac{\beta}{A}\right)d\sigma}{\int_0^{2\pi}\left|\sin^2\sigma - \frac{\beta}{A}\sin\sigma\right|d\sigma} =\\ &= \frac{-A\pi\beta}{A^2\frac{\pi}{2} + 2\beta A - \beta\sqrt{A^2-\beta^2} - A^2\arcsin\frac{\beta}{A}}. \end{aligned} \tag{50}$$

Theorem 3. Every solution of (39) converges to an equilibrium provided that either

$$\begin{aligned} &A \leq \sqrt{\frac{k_1 k_2(k_1^2 I_2^2 + k_2^2 I_1^2)}{2(k_1+k_2)(k_1 I_2^2 + k_2 I_1^2)}}\cdot\frac{I_1 k_2 + I_2 k_1}{I_1 I_2},\\ &\frac{\nu_0^2\nu^2}{(\nu_0^2+\nu^2)} < \frac{k_1 k_2(k_1 I_2^2 + k_2 I_1^2)}{2(k_1+k_2)(k_1^2 I_2^2 + k_2^2 I_1^2)} \end{aligned} \tag{51}$$

or, alternatively,

$$\begin{aligned} &A > \sqrt{\frac{k_1 k_2(k_1^2 I_2^2 + k_2^2 I_1^2)}{2(k_1+k_2)(k_1 I_2^2 + k_2 I_1^2)}}\cdot\frac{I_1 k_2 + I_2 k_1}{I_1 I_2}, \;//\\ &\frac{\nu_0^2\nu^2}{(\nu_0^2+\nu^2)} < \frac{k_1^2 k_2^2(k_1 I_2 + k_2 I_1)^2}{4A^2 I_1^2 I_2^2(k_1+k_2)^2}. \end{aligned} \tag{52}$$

Proof. The proof is immediate from Theorem 2. We choose

$$\varepsilon = \tau = \frac{k_1 k_2}{4A(k_1+k_2)}.$$

In the first case, let

$$\delta = \frac{A(k_1 I_2^2 + k_2 I_1^2)}{2(k_1^2 I_2^2 + k_2^2 I_1^2)}.$$

Due to (51) the inequality (48) holds, so condition 1) of Theorem 2 is satisfied. The condition 2) of Theorem 2 takes the form

$$4\varepsilon\delta = 4\tau\delta > \frac{\nu_0^2\nu^2}{(\nu_0^2+\nu^2)} \tag{53}$$

which follows from (51).

In the second case, we choose

$$\delta = \frac{\tau(k_2 I_1 + k_1 I_2)^2}{I_1^2 I_2^2}.$$

In view of (52), the inequality (47) is valid. Then the second inequality in (52) is equivalent to the condition 2) of Theorem 2. ■

Example. Following (Tomchina, 2020) consider the model of two rotors with $J_1 = J_2 = J$, $\varepsilon_1 = \varepsilon_2 = \varepsilon$, $m_1 = m_2 = m$, $k_1 = k_2 = k$. In this case all the formulas can be simplified. It can be seen that

$$\frac{\nu_0^2\nu^2}{(\nu_0^2+\nu^2)} < \frac{\pi^2\beta^2}{A^2(4+\frac{\pi^2}{4})}.$$

So conditions (51), (52) are implied by inequalities

$$A \leq \frac{k^2}{I};\; \beta < \frac{A\sqrt{16+\pi^2}}{4\pi} \tag{54}$$

and

$$A > \frac{k^2}{I};\; \beta < \frac{k^2\sqrt{16+\pi^2}}{4I\pi} \tag{55}$$

respectively.

Let $m = 1.5\,kg$, $M_0 = 9.0\,kg$, $\varepsilon = 0.04\,m$, $k = 0.05\,kg\cdot m^2/sec$, $c = 1800\,kg\cdot m/sec^2$, $J = 0.012\,kg\cdot m^2$, $L_1^o = 1.56\,kg\cdot m^2/sec^2$, $L_2^o = 1.44\,kg\cdot m^2/sec^2$. In this case the condition (54) is fulfilled and every solution of (39) converges.

5. CONCLUSION

In this paper we consider the problem of synchronization between several vibro-exciters (eccentric rotors) installed on a common oscillating platform. We show that this problem can be solved in the framework of stability theory for pendulum-like systems (Leonov et al., 1996; Smirnova and Proskurnikov, 2019b). We use this theory to derive an analytic criterion for self-synchronization of two rotors.

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