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1 Model order reduction: basic concepts and notation

Abstract: This is the first chapter of a three-volume series dedicated to theory and application of Model Order Reduction (MOR). We motivate and introduce the basic concepts and notation, with reference to the two main cultural approaches to MOR: the system-theoretic approach employing state-space models and transfer function concepts (Volume 1), and the numerical analysis approach as applied to partial differential operators (Volume 2), for which projection and approximation in suitable function spaces provide a rich set of tools for MOR. These two approaches are complementary but share the main objective of simplifying numerical computations while retaining accuracy. Despite the sometimes different adopted language and notation, they also share the main ideas and key concepts, which are briefly summarized in this chapter. The material is presented so that all chapters in this three-volume series are put into context, by highlighting the specific problems that they address. An overview of all MOR applications in Volume 3 is also provided.

Keywords: model order reduction, (Petrov–)Galerkin projection, snapshots, parametric operator equation, transfer function

MSC 2010: 35B30, 37M99, 41A05, 65K99, 93A15, 93C05

1.1 Overview

The ever-increasing demand for realistic simulations of complex products and processes places a heavy burden on the shoulders of mathematicians and, more generally, researchers and engineers working in the area of computational science and engineering (CSE). Realistic simulations imply that the errors of the virtual models should be small, and that different aspects of the product or process must be taken into account, resulting in complex coupled simulations.

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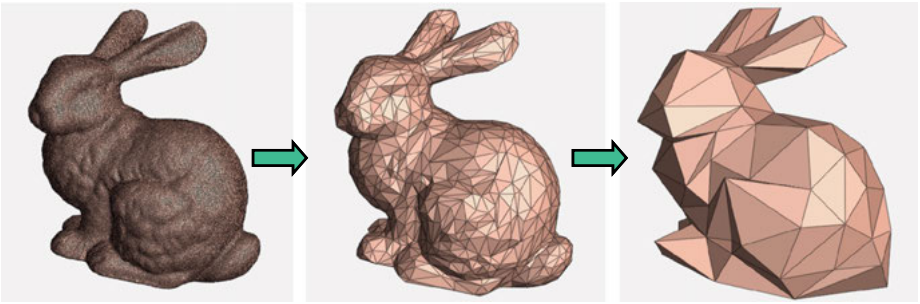
Often, there is a lot of superfluous detail in these simulations that is not needed to provide accurate results. With the current advances in new computer architectures, viz. the availability of many processors, one might be tempted to just use the abundant computational resources. However, this could lead to enormous energy consumption for the simulations, which should be avoided if possible. Besides, it could still lead to very lengthy and time-consuming simulations. Hence, it seems wise to use methods that can reduce the size of such huge problems, and which are able to get rid of the superfluous and unnecessary detail, still guaranteeing the accuracy of solutions.

An example is provided by the co-simulation of an electronic circuit with the interconnect structure that is mounted on top of the circuit to provide all desired connections. The metallic interconnect structure causes electromagnetic effects that may influence the behavior of the underlying circuit. There is no need, however, to solve the Maxwell equations for this complex three-dimensional structure in full detail, only the most dominant effects causing delays need to be included. This is where model order reduction (MOR) comes into play; MOR methods are able to extract the dominant behavior by reducing the size of the system to be solved.

To explain what MOR is, we often use the following picture:



If we showed the picture on the right, everyone would recognize that this is a rabbit. Hence, we do not need the detailed representation in the left picture to describe the animal. Maybe we would need a slightly more detailed description as shown in the middle picture, depending on the demands.[1] MOR works in the same way: the original problem is reduced, the representation of the solution is given with far less variables, and the hope is that this is sufficient to guarantee an accurate solution. If more accuracy is needed, clearly the problem size should be reduced to a lesser extent.

1 Disclaimer: though our picture might indicate this, note that simply using a coarser mesh in a discretization of a continuous model is not a competitive MOR technique in general!

By now, MOR is a very active and relatively mature field of research. Much progress has been made in the past 40 years, and in many different directions. The seminal papers by Feldmann and Freund (1994/95) [9, 10], simultaneously with [12], sparked many development in the area of Krylov-type methods, strongly related to the field of numerical linear algebra. Mimetic elements were considered, leading to passivity-preserving and structure-preserving methods, necessary to retain vital properties of the original system of equations in electronics and electromagnetics. First textbooks appeared with a special focus on applications in this area, [11] even before the above papers, and after the field had matured, new textbooks [13, 22] and several collections [3, 5] were published.

Within the systems and control area, one concentrates mainly on balancing techniques, involving the solution of Lyapunov equations. The main ideas center on preserving those states in a dynamical system that can be reached with the least amount of energy, and on the other hand, those states that provide the most information when observed. In balanced coordinates, both sets of states coincide. Here, the seminal paper of [16] has to be mentioned which rendered these ideas computationally feasible. A first textbook in this area was published in 2001 by Obinata and Anderson [17].

Also, the need for MOR was already discussed in the 1960s in mechanical engineering, from which the technique of modal truncation emerged, in combination with substructuring and with further developments like component mode synthesis. These are nowadays standard techniques, found in many variants in structural analysis software, and covered by many textbooks in numerical mechanics. A comprehensive textbook focusing on this area is [18].

First textbooks and collections of tutorials that made connections between the MOR techniques developed mainly in the above-mentioned application areas started appearing in the mid-2000s, including the fundamental textbook [1] by Antoulas in 2005, and the edited volumes [6, 21].

Later, researchers started to consider parametric model order reduction, especially within the area of reduced basis methods, focusing on the fast solution of parametric partial differential equations (PDEs) [14, 19, 20]. A related, but somewhat different approach to parametric PDEs was developed in the framework of the proper generalized decomposition [8]. One can also find a collection of articles on MOR for parametric problems in [7].

Methods for nonlinear problems were also considered, important developments being the empirical interpolation methods and other so-called hyperreduction techniques. But also methods like proper orthogonal decomposition (POD), where snapshots of the solution of a nonlinear problem are used to create a basis for solutions, became popular for nonlinear problems. Basic concepts of these approaches, including also the MOR techniques already mentioned above, also with some historical perspectives, can be found in the collection of tutorials [4].

The demand for more realistic simulations led to the development of MOR methods for interconnected and coupled systems. Extensions to descriptor systems, or alternatively differential-algebraic systems, led to the creation of index-preserving methods and to the development of an interpolatory projection framework for model reduction of descriptor systems. An entirely different approach is provided by data-driven methods, in which the Loewner framework plays an important role; see the recent textbook on the interpolatory framework including the Loewner framework for more details [2].

Most recently, focus also turned to data-driven and non-intrusive MOR methods, requiring no or only incomplete knowledge of the physical model, but merely relying on tools or software to produce relevant data from which a model description can be inferred. One prominent technique in this area is dynamic mode decomposition [15], with many new methods emerging even more recently, often making connections to techniques from machine and deep learning.

The three volumes constituting this handbook of *Model Order Reduction* discuss many of the aforementioned developments and methods. This first volume contains theoretical expositions of system-theoretic, interpolatory, and data-driven methods and algorithms. The second volume treats snapshot-based methods and algorithms for parametric PDEs. The mathematical strategy behind these methods relies on Galerkin projection on finite-dimensional subspaces generated by snapshot solutions corresponding to a specific choice of parameters. The third volume contains a large variety of applications of MOR. Originally, the fields of mechanical engineering, automation and control, as well as the electronics industry were the main driving forces for the developments of MOR methods, but in recent years, MOR has been introduced in many other fields (not all covered in Volume 3, though), like chemical and biomedical engineering, life sciences, geosciences, finance, fluid mechanics and aerodynamics, to name a few. Moreover, Volume 3 also provides a chapter surveying the landscape of existing MOR software.

1.2 A quick tour

MOR is a multidisciplinary topic, which has been developed over the last decades by mathematicians, scientists and engineers from widely different communities. Although the main ideas in MOR can be classified in a relatively small set of fundamental problem statements and related reduction approaches, these ideas have been developed by different communities with different languages, notation, and scope. One of the purposes of the *Model Order Reduction* handbook project is in fact to provide a comprehensive overview of MOR approaches, hopefully forming bridges that cross different disciplines.

Several classifications can be attempted in the world of MOR. Probably the most natural high-level classification distinguishes between the two main cultural approaches of system theory on one side, and numerical analysis as applied to solving PDEs on the other. Other classifications may be considered, for instance based on the various classes of reduction approaches, which may be model-driven or data-driven, optimal or heuristic, deterministic or stochastic, or alternatively on the type of the system being addressed, which can be linear or nonlinear, uniquely defined or parameterized by some geometrical or material variable, deterministic or stochastic, finite- or infinite-dimensional. The specific methods that apply in each of these cases will be discussed in detail in the various chapters of this three-volume series. In this initial chapter we mainly distinguish the two major cultural approaches to MOR, for which reduction methods, notation and language are sometimes quite different in the existing literature.

System-theoretic approaches usually deal with a system under investigation described as a large-scale set of Ordinary Differential Equations (ODEs) or Differential-Algebraic Equations (DAEs), whose dynamics is expressed in terms of a set of state variables. The main objective is to derive some compact Reduced-Order Model (ROM) with the same structure, characterized by a significantly smaller number of states, and whose response approximates the true system response according to well-defined criteria. Very often the ROM represents a component of a larger system that is impossible or impractical to solve in its full-size formulation. In this setting, reduction is required in order to replace the original large-scale description of individual components with accurate and robust ROMs, so that a global system-level numerical simulation becomes feasible.

A second major approach to MOR addresses fast numerical solution of PDEs. In this setting, the field problem of a PDE is taken as the starting point. In the snapshot-based methods, the full-order variational form is often retained by the MOR process. This allows projection-based methods to utilize the parametric operator equations and define a reduced-order operator of the same parametric dependency. Since the starting point is the PDE form, a discretization in space and time is required, leading to a large-scale discretized model. Various methods exist to control the approximation error, often balancing rigorousness with computability.

We see that the above two approaches share their main objective of speeding up numerical computations with control over approximation errors, although the starting points are different. We should, however, consider that the two approaches practically coincide once a field problem described in terms of PDEs is discretized in its space variables in terms of suitable coefficients, which basically play the same role of the state variables in system-theoretical approaches.

We address the two approaches in Sections 1.2.1 and 1.2.2, by introducing basic notation and concepts that will be used extensively throughout Volumes 1 and 2 of this book series. Section 1.2.3 provides a glimpse at the MOR applications that are extensively discussed in Volume 3.

1.2.1 The system-theoretic approach

System-theoretic approaches consider models whose dynamics is expressed in terms of internal *state variables*, which in the finite-dimensional setting are denoted as $x(t) \in \mathcal{X} \subset \mathbb{R}^n$. These states evolve with time $t \in [t_0, T]$ according to dynamic equations which are driven by some *inputs* or *control signals* $u(t) \in \mathcal{U} \subset \mathbb{R}^m$, whereas the *quantities of interest* or *outputs* are $y(t) \in \mathcal{Y} \subset \mathbb{R}^q$, usually with $q \ll n$. Denoting the "true" system as $\mathcal{S}$, the MOR objective is to obtain an approximate system $\widehat{\mathcal{S}}$ with a small number $r \ll n$ of internal states $\hat{x}(t) \in \widehat{\mathcal{X}} \subset \mathbb{R}^r$. Reduction is conducted by enforcing appropriate approximation conditions such that, given input signals $u(t)$, the ROM $\widehat{\mathcal{S}}$ provides an output $\hat{y}(t)$ that is "close" in some sense to the corresponding output $y(t)$ of the original system $\mathcal{S}$.

1.2.1.1 Standard system descriptions: the LTI case

The simplest system description assumes Linearity and Time-Invariance (LTI) and is provided by a set of ODEs in *state-space* form

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \quad x(t_0) = x_0, \\ y(t) &= Cx(t) + Du(t), \end{aligned} \tag{1.1}$$

where $\dot{x}(t)$ denotes the time derivative of $x(t)$, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{q\times n}$, $D \in \mathbb{R}^{q\times m}$ are constant matrices, and x_0 is a prescribed initial condition. A more general formulation of LTI system dynamics can be expressed in *descriptor* form,

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + Bu(t), \quad x(t_0) = x_0, \\ y(t) &= Cx(t) + Du(t), \end{aligned} \tag{1.2}$$

where an additional and possibly singular matrix $E \in \mathbb{R}^{n\times n}$ enters the state equations. Casting (1.2) in the Laplace domain and assuming vanishing initial conditions, $x_0 = 0$, leads to

$$Y(s) = H(s)U(s), \quad H(s) = C(sE - A)^{-1}B + D, \tag{1.3}$$

where $H(s)$ is the transfer function of the system and $s \in \mathbb{C}$ is the Laplace variable. Well-posedness of (1.3) requires that $\det(sE - A) \neq 0$ for some s, i. e., that the pencil (A, E) is *regular*. In most cases also an (asymptotic) *stability* requirement is established, so that all finite eigenvalues of the pencil (A, E) have a (strictly) negative real part.

This system description forms the basis of most of the following chapters in this volume.

1.2.1.2 Approximation criteria

Some common approximation criteria that are appropriate for LTI systems are listed now:

- The quantities of interest of both full-scale system $\mathcal{S}$ and reduced system $\widehat{\mathcal{S}}$ are the outputs $y(t)$ and $\widehat{y}(t)$, respectively. Therefore, it is natural to bound the output error defined as $\|\widehat{y}-y\|_{\mathcal{L}}$ within a suitable function space $\mathcal{L}$, with the natural choice being the Hilbert space of square integrable signals $L_2(t_0, T)$, with

$$\|y\|^2_{L_2(t_0,T)} = \int_{t_0}^{T} \|y(t)\|_2^2 \, dt. \tag{1.4}$$

- An alternative is to control the error of the ROM transfer function $\widehat{H}(s)$ by minimizing $\|\widehat{H} - H\|_{\mathcal{H}}$, where $\mathcal{H}$ is an appropriate function space. Common choices are the Hardy spaces $\mathcal{H}_2$ and $\mathcal{H}_\infty$, which are adequate under asymptotic stability assumptions, for which

$$\|H\|^2_{\mathcal{H}_2} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \|H(\jmath\omega)\|_F^2 \, d\omega, \quad \|H\|_{\mathcal{H}_\infty} = \sup_{s\in\mathbb{C}_+} \|H(s)\|_2, \tag{1.5}$$

  where F denotes the Frobenius norm and $\jmath = \sqrt{-1}$. We refer to Chapter 2 in this volume for more precise definitions and for an introduction of the main system-theoretic properties that are relevant for error control in MOR.
- Data-driven approaches aim at enforcing suitable interpolation or approximation conditions starting from available samples of the original transfer function $H_k = H(s_k)$ at a set of complex frequencies s_k for $k = 1, \ldots, \bar{k}$. Interpolation methods (see Chapter 6, where the Loewner framework is introduced and discussed) enforce

$$\widehat{H}(s_k) = H_k, \quad \forall k = 1, \ldots, \bar{k}, \tag{1.6}$$

  possibly extending this exact matching also to higher derivatives

$$\left.\frac{d^\nu \widehat{H}}{ds^\nu}\right|_{s_k} = H_k^{(\nu)}, \quad \forall \nu = 0, \ldots, \bar{\nu}_k, \quad \forall k = 1, \ldots, \bar{k}, \tag{1.7}$$

  giving rise to so-called *moment-matching* methods (see Chapter 3). In some cases, the point and moment matching is performed at adaptively selected frequencies $s_k \in \mathbb{C}$; see, e. g., the IRKA algorithm in Chapter 3. Moments can also be matched implicitly through projection of the original system onto suitably-defined Krylov subspaces, also discussed in Chapter 3.
- A relaxed version of the above matching conditions involves minimization of the least squares error,

$$\sum_{k=1}^{\bar{k}} \|\widehat{H}(s_k) - H_k\|_F^2. \tag{1.8}$$

Curve fitting approaches, including the Vector Fitting (VF) methods (see Chapter 8) fall into this class. When data H_k come from measurements, only purely imaginary frequencies $s_k = \jmath\omega_k$ are available and used.

- A fundamental class of system-theoretic approaches for MOR are based on *truncation* of state-space or descriptor systems, where those state variables that are poorly coupled to the inputs or which provide negligible contribution to the outputs are discarded. Balanced truncation methods (see Chapter 2) and modal methods (Chapter 4) belong to this class.
- Some applications require additional constraints to be enforced during reduction. A notable case is enforcements of passivity and of dissipativity, which are appropriate for systems that are unable to generate energy on their own. Dissipativity conditions for state-space systems are reviewed in Chapters 5 and 2, together with appropriate methods for their enforcement, either as a feature of the MOR scheme or as a postprocessing.

1.2.1.3 Parameterized LTI systems

An additional layer of complexity is introduced by allowing the system $\mathcal{S}$ to be *parameterized* by some deterministic and/or stochastic variables $\mu \in \mathcal{P} \subset \mathbb{R}^p$. Assuming that the input signals u are not parameter-dependent, we can write (1.2) in the parameterized form

$$\begin{aligned} E(\mu)\dot{x}(t,\mu) &= A(\mu)x(t,\mu) + B(\mu)u(t), \quad x(t_0,\mu) = x_0(\mu), \\ y(t,\mu) &= C(\mu)x(t,\mu) + D(\mu)u(t), \end{aligned} \tag{1.9}$$

with the corresponding transfer function

$$H(s,\mu) = C(\mu)\big(sE(\mu) - A(\mu)\big)^{-1}B(\mu) + D(\mu). \tag{1.10}$$

In this parameterized setting, one is usually interested in preserving a closed-form parameterization also in the ROM, so that the corresponding transfer function must match (1.10) not only throughout the frequency band of interest, but also throughout the parameter space. Chapter 3 provides an overview of moment-matching parameterized MOR (PMOR) in the case of affine dependence of $E(\mu)$ and $A(\mu)$ on the parameters. The so-called reduced basis methods discussed in Chapter 4 of Volume 2 would provide the counterpart of PMOR in the PDE reduction setting, which is extensively treated in all chapters of Volume 2.

1.2.1.4 Nonlinear systems

Generalization to nonlinear systems is also possible, although effectiveness of MOR strongly depends on the class of systems being considered. Several results are avail-

able for systems that can be cast in the form

$$\begin{aligned} \dot{x}(t) &= f(x(t)) + g(x(t))\, u(t), \\ y(t) &= h(x(t)), \end{aligned} \tag{1.11}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$, $g : \mathbb{R}^n \to \mathbb{R}^{n\times m}$ and $h : \mathbb{R}^n \to \mathbb{R}^q$ are smooth functions. A notable particular case is the *quadratic-bilinear* form, for which the nonlinear functions can be written and/or approximated as quadratic polynomials in their variables and compactly expressed, e. g., as

$$f(x) = f(0) + A_1 x + A_2 (x \otimes x), \tag{1.12}$$

where $\otimes$ is the Kronecker product and $f(0) \in \mathbb{R}^n$, $A_1 \in \mathbb{R}^{n\times n}$, $A_2 \in \mathbb{R}^{n\times n^2}$ are constant matrices. A discussion of methods applicable to MOR of such systems is available in Chapters 2 and 3.

In more general settings, supporting algorithms providing interpolation/approximation of high-dimensional nonlinear multivariate functions are indeed available. We mention the Empirical Interpolation Methods in their various formulations introduced in Chapter 1 of Volume 2 and manifold interpolation (Chapter 7 in this volume), which provides a general framework for interpolation of orthogonal bases, subspaces or positive definite system matrices. Both these approaches are recurrent tools in several modern MOR frameworks.

1.2.1.5 Surrogate modeling

Extending the framework of classical MOR, which in the system-theoretic approach is usually applied to a state-spate description, surrogate modeling approaches provide tools for processing sequences of input-output data points and constructing an approximate metamodel that explains and reproduces their relationship. The last two chapters in this volume describe two alternative approaches for surrogate modeling. Chapter 9 presents an overview of the celebrated kernel methods, an approach that is very popular in the machine learning community, both for acceleration of complex simulation models, but also for classification and signal processing. Chapter 10 discusses Kriging methods or Gaussian Processes (GPs), with emphasis on design and analysis of computer experiments. These extensions of MOR bridge the gap between control and system theory with statistics, computer science, and (big) data science, further demonstrating how pervasive the key objectives are that characterize MOR.

1.2.2 The PDE approach

The second major approach to MOR starts from a field problem defined over a continuous domain Ω. Thus, a parametric PDE is given as starting point of the MOR procedure.

Two main steps are performed: the numerical discretization in space and time and the projection of the discretized form onto a reduced-order space. The projection space is chosen such that the field variable is well approximated in a natural PDE norm or it is chosen with respect to a given output quantity of interest. These basic tools are discussed in more detail in Chapter 1 of Volume 2.

The variational or weak form of a parametric linear PDE in the continuous setting is posed over a suitable Hilbert space $V(\Omega)$ and given as

$$a(u(\boldsymbol{\mu}), v; \boldsymbol{\mu}) = f(v; \boldsymbol{\mu}) \quad \forall v \in V, \tag{1.13}$$

with bilinear form $a : V \times V \times \mathcal{P} \to \mathbb{R}$ and linear form $f : V \times \mathcal{P} \to \mathbb{R}$. The parameter vector is denoted $\boldsymbol{\mu}$ and is an element of the parameter space $\mathcal{P}$. In many application scenarios, a particular output of interest $s : \mathcal{P} \to \mathbb{R}$ is sought, given by the linear form $l : V \times \mathcal{P} \to \mathbb{R}$ as

$$s(\boldsymbol{\mu}) = l(u(\boldsymbol{\mu}); \boldsymbol{\mu}). \tag{1.14}$$

The case of a coercive and continuous bilinear form is the setting for many introductory examples but does not cover all PDE settings. E. g., in electromagnetics, i. e., when solving Maxwell's equations, an inf-sup stable sesquilinear form is often considered. In unsteady problems, the time-dependence is often made explicit and time is treated differently from other parameters in the ROM setting; see the POD-greedy algorithm for example. Nonlinear problems require particular care and methods, which are often adapted to the particular type of nonlinearity.

A suitable discretization method is chosen to approximate the field variable u, defining a corresponding discrete space V_h. The method of weighted residuals is invoked to turn the continuous form (1.13) into a discrete variational formulation.

The weak form in the discrete setting is given as

$$a(u_h(\boldsymbol{\mu}), v_h; \boldsymbol{\mu}) = f(v_h; \boldsymbol{\mu}) \quad \forall v_h \in V_h, \tag{1.15}$$

with bilinear form $a : V_h \times V_h \times \mathcal{P} \to \mathbb{R}$ and linear form $f : V_h \times \mathcal{P} \to \mathbb{R}$. The space of all v_h is the test space, while the space of u_h is the trial space.

A discrete solution is found by invoking Galerkin orthogonality, by enforcing that the test space is orthogonal to the residual. In Ritz–Galerkin methods, the residual is tested against the same set of functions as the ansatz functions, i. e., the test space is the same as the ansatz or trial space. In a more general Petrov–Galerkin method, test space and trial space are chosen as different spaces.

Starting from the discrete high-fidelity formulation (1.15), another Galerkin projection is invoked to arrive at the reduced-order formulation. A set of solutions is computed at parameter values $S_{N_{\max}} = \{\boldsymbol{\mu}^1, \ldots, \boldsymbol{\mu}^{N_{\max}}\}$, either by pre-specifying $S_{N_{\max}}$ or using an iterative algorithm such as the greedy sampling. These solutions are often called 'snapshots'. A projection space V_N is determined by a suitable method. The dif-

ferent methods are briefly introduced below and discussed in much detail in dedicated chapters of Volume 2.

The reduced-order variational formulation is to determine $u_N(\boldsymbol{\mu}) \in V_N$, such that

$$a(u_N(\boldsymbol{\mu}), v_N; \boldsymbol{\mu}) = f(v_N; \boldsymbol{\mu}) \quad \forall v_N \in V_N. \tag{1.16}$$

With matrix $\mathbb{A}_h$ assembling the bilinear form and the load vector $\mathbf{f}_h$, let $\mathbb{V} \in \mathbb{R}^{N_h \times N}$ denote the matrix of basis vectors, derived from the snapshot solutions and project (1.15) onto the reduced-order space as

$$\mathbb{V}^T \mathbb{A}_h \mathbb{V} \mathbf{u}_N = \mathbb{V}^T \mathbf{f}_h. \tag{1.17}$$

The high-order solution is then approximated as

$$\mathbf{u}_h \approx \mathbb{V} \mathbf{u}_N. \tag{1.18}$$

Typical ROM ingredients are an affine parameter dependency, an offline–online decomposition and error bounds, which are explained in Chapter 1 of Volume 2.

Pointers to subsequent chapters for accurate ROMs in the PDE setting are given in this section for Volume 2. Each chapter explains in a detailed way a different method of how to obtain the projection spaces or follows an alternate route altogether. Numerical examples can be found in the respective chapters.

- *Proper Orthogonal Decomposition*
  In the Proper Orthogonal Decomposition (POD), the projection space is determined from the principal modes of the *singular value decomposition* of sampled field solutions. The sampling is uniform over the parameter domain in many cases. POD is covered in depth in Chapter 2 of Volume 2.
- *Proper Generalized Decomposition*
  The Proper Generalized Decomposition (PGD) assumes a separated representation, in which all variables, i. e., space, time and parameters, can be treated in the same way; see Chapter 3 of Volume 2. Error indicators and error bounds serve to iteratively build the approximation spaces.
- *Reduced Basis Method*
  Reduced Basis (RB) MOR uses residual-based error indicators and error estimators to determine the projection space by a greedy sampling; see Chapter 4 of Volume 2. It is not uncommon in the literature to consider POD as a RB method.
- *Hyperreduction*
  Hyperreduction techniques are related to the *Empirical Interpolation Method* (EIM) which generally aims to approximate an affine parameter dependency for an originally non-affine problem. The EIM is introduced in Chapter 1 of Volume 2 while the chapter on hyperreduction (Chapter 5 of Volume 2) details how these techniques can be used for ROM generation.

- *Localized Model Order Reduction*
  The localized model reduction aims to determine local ROMs valid over parts of the computational domain and construct a global approximation through suitable couplings of local ROMs. The localized ROMs are usually generated with POD and RB techniques; see Chapter 6 of Volume 2.
- *Dynamic Mode Decomposition*
  The Dynamic Mode Decomposition (DMD) is also based on the *singular value decomposition*; see Chapter 7 of Volume 2. The starting points are measurements of the time-trajectory which aim to approximate the time-advancement operator. The DMD is thus understood as a data-driven approach, since it does not project an affinely expanded system matrix.

1.2.3 Applications

In this section, we briefly introduce the several MOR applications that are collected in the third volume of this book series. Several early developments in MOR originated in the exponentially growing field of microelectronics during the last few decades of the 20th century. The enormous growth in complexity in designing microprocessors and computing systems was requiring scalable, efficient, and especially automated design and verification methods. This necessity provided a fertile ground for research on MOR, so that many contributors from mathematics, system and control theory, and electronics engineering proposed several key ideas and algorithms that are still widely adopted in modern tools. Chapter 4 of Volume 3 reviews some of these steps and provides an overview of MOR applications in microelectronics. It is not a surprise that MOR proves very successful also in electromagnetics, since electric/electronic circuits are just a lumped approximation of the more general Maxwell's field formulations. Applications of MOR in electromagnetics are discussed in detail in Chapter 5 of Volume 3.

Not long after the initial developments, the MOR field became more and more mature, with consolidated approaches both in the system-theoretic and in the PDE communities. This enabled reaching cross- and multidisciplinary applications. Volume 3 of this book series reports on several such applications of MOR, in particular: chemical process optimization (Chapter 1 of Volume 3), mechanical engineering (Chapter 2 of Volume 3), acoustics and vibration (Chapter 3 of Volume 3), computational aerodynamics (Chapter 6 of Volume 3) and fluid dynamics (Chapter 9 of Volume 3). These chapters build on the methods discussed in the first two volumes, in some cases proposing application-driven customized versions, and testify that pervasivity of MOR exists in practically all fields of applied engineering.

Consolidation of MOR theory made algorithms more and more reliable. Therefore, unexpected applications started to be pursued even on biological systems. One of the most striking yet successful extensions is cardiovascular modeling (Chapter 8 of Volume 3), which attempts a quantitative prediction of the behavior of the most existing

complex "system", the human body. The same objective is shared by Chapter 7 of Volume 3 on MOR applications to the neurosciences.

MOR continues its mainstream advancement in those areas, such as mathematics and control, where methodological aspects have been introduced and are still continuously refined. Chapter 11 of Volume 3 combines classical reduction approaches with graph theory for the reduction of network systems. This contribution is quite timely nowadays, when relations between distributed systems, agents, individuals at physical or social level are often described and explained based on their networked interconnection structure. Another timely application of MOR is described in Chapter 10 of Volume 3, discussing the very important aspects of uncertainty quantification, which play a fundamental role in all applications when the description of the systems in terms of their constitutive parameters is not deterministic but subject to stochastic variations.

Chapter 12 of Volume 3 confirms the relevance of MOR in industrial production settings. The recent paradigm shift towards "Industry 4.0" augmented the requirements for sophisticated prediction methods and tools. It is now conceivable that suitably constructed abstraction layers can be devised to build so-called "digital twins", with the objective of mimicking the behavior of actual physical systems in real time and during their lifetime. This chapter provides an overview of the state of the art in this respect, where MOR plays once again a key role.

We conclude this introduction advising the reader to check Chapter 13 of Volume 3, which provides an overview of existing MOR software. Several commercial and academic software packages are reviewed, suitably classified with respect to the type of problems being addressed. Many of the latter packages can be freely downloaded, used, and possibly extended by active MOR researchers with new features and functionalities.

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