# POLITECNICO DI TORINO
## Repository ISTITUZIONALE

Study on electron emission by a spherical object: dynamics of trapped particles

(Article begins on next page)

20 April 2024

# Study on electron emission
# by a spherical object:
# dynamics of trapped particles

## G. Coppa[1]†, R. Mulas[2]

[1]Dipartimento di Elettronica e Telecomunicazioni, Politecnico di Torino, Torino, Italy
[2]Independent Scholar

In the paper, the dynamics of electrons emitted by a spherical object when the total charge of the system is constant is studied in detail. In particular, the condition for which the total electron charge presents damped oscillations is deduced rigorously by considering a perturbation with respect to the steady-state solution. The results obtained can be of utility to simulate the expansion of a spherical plasma by separating ion and electron time scales.

## 1. Introduction

The present work concerns the analysis of the time evolution of the distribution of electrons that are emitted by a sphere of radius $R$ all having same initial velocity $v_0$. A similar study, i.e., the analysis of the electron flow between two electrodes where space-charge effects are taken into account, is a classic problem in plasma physics, with contributions dating back to Child (1911) and to Langmuir & Blodgett (1924), till recent works (in particular, the papers by Akimov *et al.* (2001, 2003) and by Ender *et al.* (2000); Kuznetsov & Ender (2010*a,b*); Ender & Kuznetsov (2014)). Actually, the problem here considered is significantly different, as the total charge of the system (i.e., the charge of the sphere and of the electrons) is fixed, and when the negative charge exceeds a critical value all the new emitted electrons eventually move back and are reabsorbed by the sphere, due to its positive charge. The present analysis was carried out in the framework of a study on the mitigation of effects of electric charge, which accumulates on a satellite when an electron beam is emitted into the ionosphere (Hendrickson *et al.* 1975), by emitting a plasma around the spacecraft (Delzanno *et al.* 2016). Under the hypothesis of spherical symmetry, the numerical simulation of the plasma expansion into space can be carried out in a simple way by means of particle techniques. However, in order to reduce the computational effort, the calculation should somehow take advantage of the different time scales for ions and electrons, e.g., by calculating the equilibrium distribution of the electrons that is reached in a time interval for which the ion motion is negligible; then, the electric field contribution due to the electrons is employed to move the positive ions for a new time step (Colandrea 2018). In order to better understand the details of the numerical method and to obtain reference solutions, the Authors decided to start studying the problem in its simplest configuration, in which only electron charge is present outside the sphere. Notwithstanding its simplicity, the study has other applications beside the one considered above, in particular, in the analysis of the electron emission by a solid nanosphere when it is ionized by a laser beam, which represents the early stage of a

† Email address for correspondence: gianni.coppa@polito.it

Coulomb explosion (Ditmire *et al.* 1996). Also in this case, the electron dynamics is influenced by the positive charge of the sphere (Peano *et al.* 2006).

As mentioned before, the phenomenon is here schematized by considering the emission of monoenergetic electrons moving in radial direction. The electron flux $I_0$ ($I_0 \Delta t$ being the number of electrons emitted in time interval $\Delta t$) can be written as $n_0 v_0 \, 4\pi R^2$, being $n_0$ the density of emitted particles at $r = R$, or, equivalently, as $N_0 v_0$, with $N_0 = 4\pi R^2 n_0$. As long as there are no overtakings (i.e., if electrons emitted at $t_2 > t_1$ have radial coordinate $r$ smaller than those emitted at $t_1$), the electric field acting on the electrons emitted at time $t_0$ is simply given by

$$E\left(r\right) = -\frac{e \, Q_S\left(t_0\right)}{r^2}, \tag{1.1}$$

being $Q_S\left(t_0\right) = Q_S\left(0\right) + eI_0 t_0$ the charge of the body at time of emission (there is not a charge grounding mechanism as the satellite is in space). In fact, for $t > t_0$ the charge of the body will be $Q_S\left(0\right) + eI_0 t$, while the electron charge emitted between $t_0$ and $t$ (which is all inside the sphere of the electrons emitted at $t_0$) is $-eI_0\left(t - t_0\right)$, so the total charge "seen" by the electron emitted at $t_0$ is always $Q_S\left(t_0\right)$. As long as $e \, Q_S\left(t_0\right)/R$ is smaller than $mv_0^2/2$, the electrons are "free" particles, in the sense that their distance from the sphere tends to $+\infty$ as $t \to +\infty$. However, as $Q_S$ grows, beyond a given time the electrons emitted are all trapped. After reaching a turning point, these electrons move back forward the sphere, and eventually they are absorbed by the body. The present work concerns only the dynamics of trapped electrons. This is possible because all the free electrons have larger radial coordinate (since they have been emitted before and the attractive electric field is smaller) and, due to the Gauss's theorem, their charge do not contribute to the electric field in the inner region that is occupied by the trapped particles. In particular, the study shows that the time evolution of the trapped electrons presents periodic oscillations. In some cases the oscillations are damped, leading to a steady state; their determination is investigated in Sect. 2. Instead, for different sets of physical parameters the electron charge presents oscillations of growing amplitude till a saturation. Also in these cases a stationary solution exists, but clearly it is unstable. To discriminate analytically between these two situations a study is presented in Sect. 3, in which the time evolution of a small perturbation of a steady state is considered, so providing a rigorous condition for the rising of the instability.

## 2. Steady-state distribution

In this Section, the problem of finding a stationary distribution for trapped electrons is considered. A possible way of finding a steady-state distribution, if it exists, is considering the transient in which electrons are emitted by the sphere. As the dynamics of free and trapped electrons are completely independent, as discussed in Sect. 1, only the phase in which all the (new) emitted electrons are trapped is considered here. This can be done simply by imposing an initial positive charge on the spherical body, $Q_S\left(0\right) = Q_0 > 0$. In practice, the charge $Q_0$ is due to previous emission of (free) electrons or to other causes (e.g., the emission of a high-energy electron beam, as in space experiments). If $e \, Q_0/R$ is larger than $\frac{1}{2}mv_0^2$, all the electrons are confined by the potential of the sphere. At time $t$, an electric charge, $Q_e\left(t\right) < 0$, due to the electrons will be present outside the body, and, consequently, the charge of the body is $Q_S\left(t\right) = Q_0 - Q_e = Q_0 + |Q_e|$. Typical numerical results are presented in Figs. 1-4. The numerical simulation of the transients has been performed by using the "shell" method (Boella *et al.* 2018). The method uses computational particles in the shape of spherical shells, and the electric
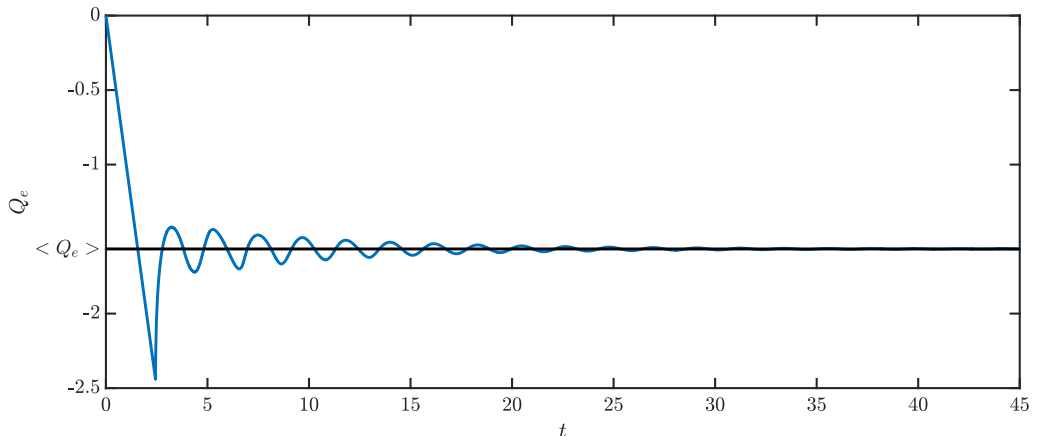
FIGURE 1. Damped oscillations for the total electron charge outside the sphere. Dimensionless units are used, as described in the text. The normalized electron flux, $I_0$, and the normalized initial charge on the sphere, $Q_0$, are both set equal to 1.

field is evaluated with the Gauss's theorem on a shell of radius $r$ by summing the charges of the particles included in the shell without using a computational grid. In this way, no boundary conditions are necessary for the electric potential. The shell method provides accurate, noiseless results that can be regarded as reference.

For different values of the parameters, the evolution of $Q_e$ presents two different situations: damped oscillations [Fig. 1], or oscillating behaviour [Fig.2]. The corresponding phase-space distributions are shown in Figs. 3 and 4 (in these figures, as for all the results presented in the paper, lengths are normalized to $R$, times to $R/v_0$, charges to $mv_0^2 R/e$, particle fluxes to $mv_0^3/e^2$). These results suggest that a stationary solution for the phase-space distribution may exist, at least if the parameters of the problem, $Q_S, v_0, I_0$, are in a given domain. The steady-state solution can be easily found by using the Poisson equation for the electric potential

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\varPhi}{dr}\right) = -4\pi\rho,\tag{2.1}$$

and considering that, for a stationary flow of electrons, one can write the charge density $\rho$ as:

$$\rho = -\frac{1}{4\pi r^2}\frac{2eI_0}{v(r)}\tag{2.2}$$

(the factor 2 is justified by noticing that, for a given $r$, electrons move with velocity $+v(r)$ and $-v(r)$). The velocity $v(r)$ is calculated by considering that $\mathcal{E} = \frac{1}{2}mv_0^2 - e\varPhi(r)$ is a constant of the motion (assuming $\varPhi(R) = 0$, $\mathcal{E}$ is simply equal to $\frac{1}{2}mv_0^2$). Finally, by introducing the potential energy for an electron, $U(r) = -e\varPhi(r)$ and using Eqs. (2.1, 2.2), one eventually obtains the equation:

$$\frac{d}{dr}\left(r^2\frac{dU}{dr}\right) = -\frac{2e^2I_0}{\left[\frac{2}{m}(\mathcal{E} - U)\right]^{1/2}},\tag{2.3}$$

to be solved with the initial conditions:

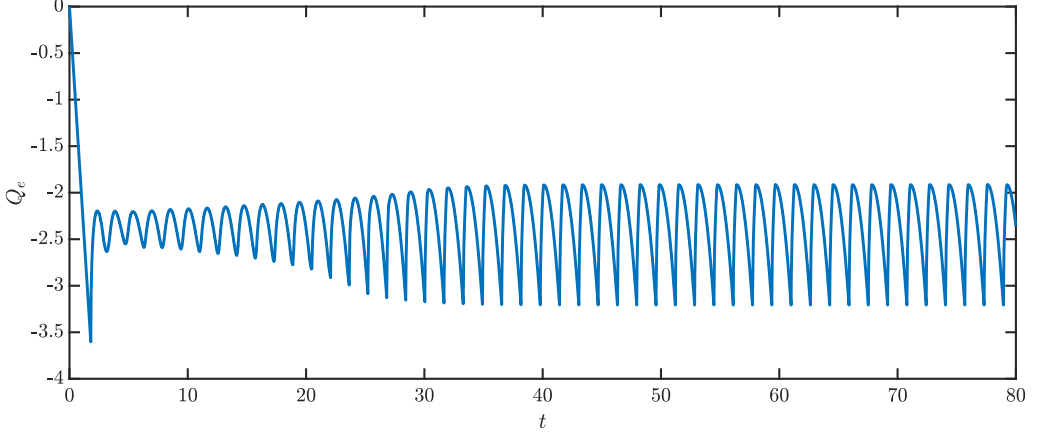$$U(R) = 0, \qquad \frac{dU}{dr}(R) = \frac{e\,Q_S}{R^2},\tag{2.4}$$

FIGURE 2. Oscillating behaviour of the total electron charge. For this case, $I_0 = 2$ and $Q_0 = 1$ have been chosen.

as $\mathrm{d}U/\mathrm{d}r = -e\,\mathrm{d}\Phi/\mathrm{d}r = eE\,(r)$ and $E\,(R) = Q_S/R^2$. Although Eq. (2.3) is well known [it was initially considered by Langmuir & Blodgett (1924) studying an electron diode], in the present case the interest is not the electron flow between two electrodes, but the motion of trapped particles, which return to the same electrode. Therefore the solution of Eq. (2.3) presents some peculiarities, as it must be solved in the $r-$interval $[R, r_{max}]$, being $r_{max}$ the turning point for the electrons, for which $\mathcal{E} - U\,(r_{max}) = 0$. Equivalently, Eq. (2.3) can be cast into two first-order differential equations:

$$\begin{cases} \dfrac{\mathrm{d}U}{\mathrm{d}r} = \dfrac{\Psi}{r^2}, & U\,(R) = 0, \\[3mm] \dfrac{\mathrm{d}\Psi}{\mathrm{d}r} = -\dfrac{2e^2 I_0}{\left[\frac{2}{m}\,(\mathcal{E} - U)\right]^{1/2}}, & \Psi\,(R) = e\,Q_S. \end{cases} \tag{2.5}$$

Equation (2.3), or system (2.5), must be solved numerically, but a direct solution presents two problems: 1) $r_{max}$ is unknown; 2) $(\mathcal{E} - U)^{-1/2}$ diverges when $r \to r_{max}$. On the other hand, the range for $U$ is known, as $U \in [0, \mathcal{E}]$, and therefore $U$ itself can be used as independent variable (i.e., $r = r(U)$ and $\Psi = \Psi\,(U)$), so obtaining

$$\begin{cases} \dfrac{\mathrm{d}r}{\mathrm{d}U} = \dfrac{r^2}{\Psi}, & r\,(U = 0) = R, \\[3mm] \dfrac{\mathrm{d}\Psi}{\mathrm{d}U} = -\dfrac{r^2}{\Psi}\dfrac{2e^2 I_0}{\left[\frac{2}{m}\,(\mathcal{E} - U)\right]^{1/2}}, & \Psi\,(U = 0) = e\,Q_S. \end{cases} \tag{2.6}$$

Finally, the singularity for $U \to \mathcal{E}$ can be eliminated by using the momentum $p = \{2m\,(\mathcal{E} - U)\}^{1/2}$ as independent variable:

$$\begin{cases} \dfrac{\mathrm{d}r}{\mathrm{d}p} = -\dfrac{p}{m}\dfrac{r^2}{\Psi}, & r\,(p_0) = R, \\[3mm] \dfrac{\mathrm{d}\Psi}{\mathrm{d}p} = 2e^2 I_0 \dfrac{r^2}{\Psi}, & \psi\,(p_0) = e\,Q_S, \end{cases} \tag{2.7}$$

with $p \in [0, p_0]$ and $p_0 = mv_0$. An example of solution of this equation is shown in Fig. 3, where the phase-space distribution obtained by Eq. (2.7) is compared with the asymp-
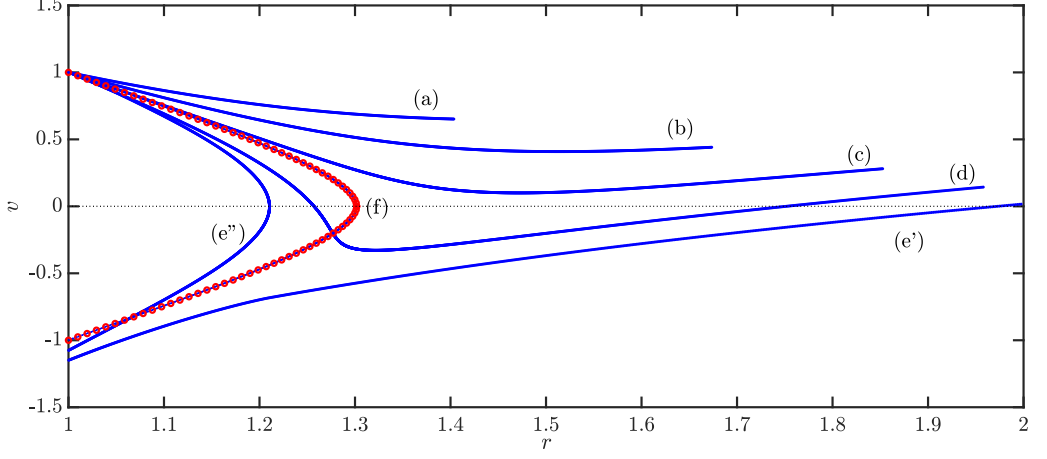
FIGURE 3. Phase-space evolution for the electron distribution in the same case of Fig. 1. The different curves a,b,c,d,e'-e", f, correspond to different times $t = 0.5, 1, 1.5, 2, 2.5, 100$. The steady-state solution (from system (2.7)) is also presented (circles).
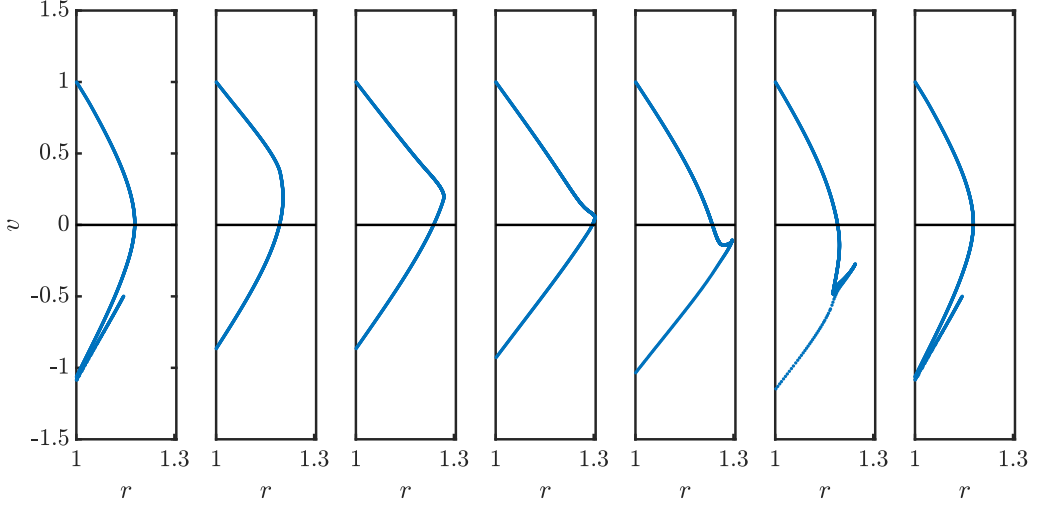


FIGURE 4. Phase-space distribution for the electrons at different times covering a period of an oscillation, for the same case of Fig. 2.

totic distribution as obtained with the shell method. As the system presents damped oscillation, one must expect a complete agreement between these two distributions. This is confirmed by the results presented in the Figure.

System (2.7) can be rewritten by introducing new dimensionless quantities $\bar{r} = r/R$, $\bar{p} = p/p_0$ and $\overline{\Psi} = \Psi/(e\,Q_S)$, as

$$
\begin{cases}
\dfrac{\mathrm{d}\bar{r}}{\mathrm{d}\bar{p}} &= -2\left(\dfrac{e\,Q_S/R}{\mathcal{E}}\right)^{-1} \bar{p}\,\dfrac{\bar{r}^2}{\overline{\Psi}}, & \bar{r}\,(1) = 1, \\[4mm]
\dfrac{\mathrm{d}\overline{\Psi}}{\mathrm{d}\bar{p}} &= 2\left(\dfrac{e\,Q_S/R}{\mathcal{E}}\right)^{-2}\left(\dfrac{e^2 N_0}{\mathcal{E}}\right)\dfrac{\bar{r}^2}{\overline{\Psi}}, & \overline{\Psi}\,(1) = 1,
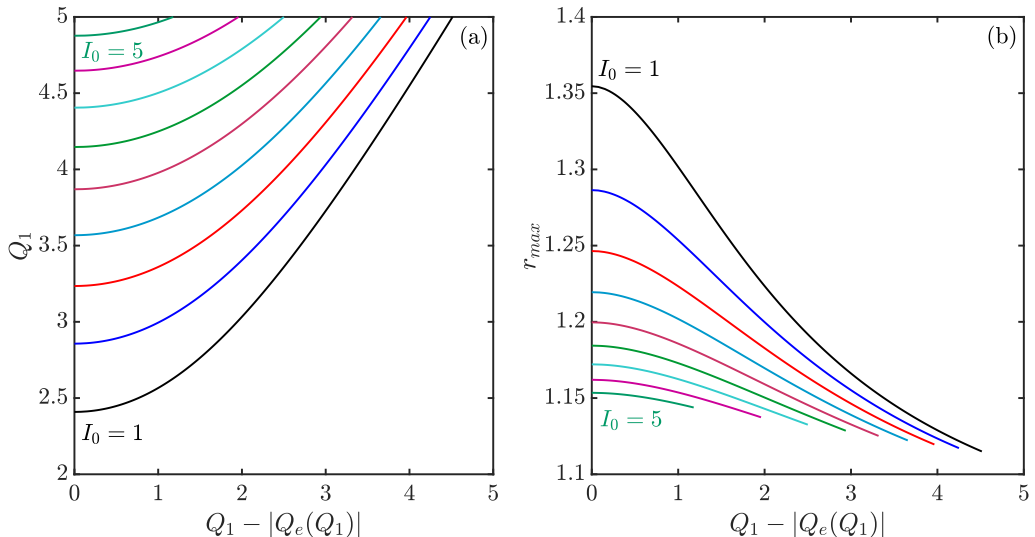\end{cases}
\tag{2.8}
$$

FIGURE 5. (a) - Charge on the sphere, $Q_1$, as a function of the total charge of the system, $Q_1 + Q_e$ for $I_0$ in the range $1 \div 5$ (with step 0.5). (b) - Maximum value of the radial coordinate of the electron distribution, $r_{max}$, as a function of $Q_1 + Q_e$ for the same values of $I_0$.

with $\bar{p} \in [0, 1]$. Equations (2.8) show that the dynamics of the beam depends on two dimensionless parameters: $\dfrac{e\,Q_S/R}{\mathcal{E}}$ and $\dfrac{e^2 N_0}{\mathcal{E}}$.

By solving system (2.7) all the physical quantities, in particular, the electron charge $Q_e = [\Psi(r_{max}) - \Psi(R)]/e$, can be calculated as a function of the charge on the sphere, $Q_S$. In other words, having fixed all the other physical parameters, the function $Q_e(Q_S)$ is obtained, representing the electron charge in equilibrium with the charge $Q_S$ on the sphere. In a real situation, one can imagine an initial condition with a charge $Q_0$ on the sphere and $Q_e = 0$. As the total charge is conserved, after the transient the charge on the sphere becomes $Q_1 = Q_0 + |Q_e(Q_1)|$. If $Q_0$ is known, the equation $Q_1 - |Q_e(Q_1)| = Q_0$ must be solved with respect to $Q_1$. This can be done graphically as shown in Fig. 5.a, where the curves of parametric equation $x = Q_1 - |Q_e(Q_1)|, y = Q_1$ are plotted for different value of $I_0$. In the Figure, each curve represents the value of the charge on the sphere after the transient, $Q_1$, as a function of its initial value $Q_0$. One can observe that the curves are defined if $Q_0 > 0$. When $Q_0 \to 0^+$, the electric field for $r = r_{max}$ tends to 0, and the r.h.s. of the two equations (2.7) becomes singular. For this case, system (2.7) must be solved with "initial" conditions

$$U(r_{max}) = \mathcal{E}, \qquad \Psi(r_{max}) = 0. \tag{2.9}$$

A solution for this particular case is provided in the following. By making use of the kinetic energy $K = \mathcal{E} - U$ instead of $U$, and defining suitable dimensionless variables:

$$s = 1 - r/r_{max}, \quad \widetilde{K} = K/K_1, \quad \widetilde{\Psi} = \Psi/\Psi_1, \tag{2.10}$$

with $K_1 = \left[ \dfrac{2e^2 I_0}{(2/m)^{1/2}} \right]^{2/3}$ and $\Psi_1 = K_1 r_{max}$, system (2.7) becomes:

$$
\begin{cases}
\dfrac{\mathrm{d}\widetilde{K}}{\mathrm{d}s} = \dfrac{\widetilde{\Psi}}{(1-s)^2}, & \widetilde{K}(s=0) = 0, \\[3mm]
\dfrac{\mathrm{d}\widetilde{\Psi}}{\mathrm{d}s} = \dfrac{1}{\widetilde{K}^{1/2}}, & \widetilde{\Psi}(s=0) = 0.
\end{cases}
\tag{2.11}
$$

If $s \ll 1$, the denominator in the first Eq. (2.11) can be approximated by 1, and system (2.11) admits a simple analytic solution:

$$
\widetilde{K}(s) = \left(\dfrac{9}{4}\right)^{\frac{2}{3}} s^{\frac{4}{3}}, \qquad \widetilde{\Psi}(s) = \dfrac{4}{3}\left(\dfrac{9}{4}\right)^{\frac{2}{3}} s^{\frac{1}{3}},
\tag{2.12}
$$

as can be verified by inspection. In general, when $s$ is not negligible, one can write $\widetilde{K}$ and $\widetilde{\Psi}$ as:

$$
\widetilde{K}(s) = \left(\dfrac{9}{4}\right)^{\frac{2}{3}} s^{\frac{4}{3}} \mathcal{T}(s), \quad \widetilde{\Psi}(s) = \dfrac{4}{3}\left(\dfrac{9}{4}\right)^{\frac{2}{3}} s^{\frac{1}{3}} \mathcal{W}(s),
\tag{2.13}
$$

and system (2.11) becomes

$$
\begin{cases}
\dfrac{\mathrm{d}\mathcal{T}}{\mathrm{d}z} = \dfrac{4}{3}\left\{ \dfrac{\mathcal{W}}{(1-\mathrm{e}^z)^2} - \mathcal{T} \right\}, & \mathcal{T}(-\infty) = 1, \\[3mm]
\dfrac{\mathrm{d}\mathcal{W}}{\mathrm{d}z} = \dfrac{1}{3}\left\{ \dfrac{1}{\mathcal{T}^{1/2}} - \mathcal{W} \right\}, & \mathcal{W}(-\infty) = 1.
\end{cases}
\tag{2.14}
$$

having introduced the new variable $z = \log(s)$ in order to eliminate the singularity for $s \to 0$. Now the new system can be readily solved numerically. In addition, for $z \to -\infty$, when $s \ll 1$ and $\mathcal{T}, \mathcal{W} \simeq 1$, one can write $\mathcal{T} = 1 + \tau$, $\mathcal{W} = 1 + w$, with $|\tau|, |w| \ll 1$ and $(1 - \mathrm{e}^z)^{-2} \simeq 1 + 2\mathrm{e}^z$. By linearising the equations, one obtains:

$$
\mathcal{T}(z) \sim 1 + \dfrac{16}{15}\mathrm{e}^z, \quad \mathcal{W}(z) \sim 1 - \dfrac{2}{15}\mathrm{e}^z.
\tag{2.15}
$$

It is worthwhile noticing that $\widetilde{K}(s)$ and $\widetilde{\Psi}(s)$ are "universal" functions, as they do not depend on the parameters of the specific problem. In particular, these functions can be used to calculate the value of $Q_1$ corresponding to $Q_0 = 0$ (as in Fig. 5.a) for different values of $I_0$, by solving the equations:

$$
\dfrac{mv_0^2}{2} = K_1 \widetilde{K}\left(1 - R/r_{max}\right),
\tag{2.16}
$$

$$
eQ_1 = K_1 r_{max} \widetilde{\Psi}\left(1 - R/r_{max}\right).
\tag{2.17}
$$

After obtaining $r_{max}$ from Eq. (2.16), one has $Q_1 (Q_0 = 0)$ from Eq. (2.17). It has been verified that the values of $Q_1 (Q_0 = 0)$ obtained in this way are in perfect agreement with the results showed in Fig. 5.a.

## 3. Transient analysis

The rise of the instability of the steady states, as studied in Sect. 2, can be explained qualitatively in a simple way. Starting from a steady-state configuration with charge

$Q_S$ on the sphere and charge $Q_e(Q_S)$ due to the electrons (thus with total charge $Q_S + Q_e(Q_S)$), one can imagine to add a charge $\delta Q_S$ on the sphere. Now the equilibrium electron charge is $Q_e(Q_S + \delta Q_S) \simeq Q_e(Q_S) + \dfrac{\mathrm{d}Q_e}{\mathrm{d}Q_S}(Q_S) \cdot \delta Q_S$. As the total charge is now $Q_S + Q_e(Q_S) + \delta Q_S$, after a time $\tau$ (approximately of the order of the transit time of the electrons from emission to reabsorption), the charge on the sphere must become $Q_S + \delta Q_S'$, such that

$$Q_S + Q_e(Q_S) + \delta Q_S = Q_S + \delta Q_S' + Q_e(Q_S + \delta Q_S). \tag{3.1}$$

In other terms, after a time interval $\tau$, the charge excess on the sphere is

$$\delta Q_S' = \left[1 - \frac{\mathrm{d}Q_e}{\mathrm{d}Q_S}(Q_S)\right]\delta Q_S \tag{3.2}$$

After a time $n \cdot \tau$, the initial perturbation is multiplied by a factor $\left(1 - \dfrac{\mathrm{d}Q_e}{\mathrm{d}Q_S}\right)^n$. Considering that $\dfrac{\mathrm{d}Q_e}{\mathrm{d}Q_S}$ is always non negative, the perturbation causes damped oscillations if $\dfrac{\mathrm{d}Q_e}{\mathrm{d}Q_S} < 2$, otherwise their amplitude is amplified. Of course, this condition is only approximate, and the real threshold to the instability requires a rigorous analysis of the dynamics of the system, which is the subject of the present Section. The technique here employed represents an extension of the Lagrangian method introduced by Akimov *et al.* (2001, 2003) for the study of the stability of a planar electron diode. However, the spherical geometry makes the problem intrinsically more complicated and, as will be shown in the following, only a mixed analytical-numerical solution is possible.

First of all, the function $\mathcal{R}(t, t_0)$, which represents the radial position at time $t$ of an electron emitted by the sphere at time $t_0$, is introduced, together with its velocity:

$$\mathcal{V}(t, t_0) = \frac{\partial \mathcal{R}}{\partial t}(t, t_0). \tag{3.3}$$

The time evolution of $\mathcal{V}$ is governed by the equation:

$$\frac{\partial \mathcal{V}}{\partial t}(t, t_0) = -\frac{e}{m}E(\mathcal{R}(t, t_0), t), \tag{3.4}$$

being $E(r, t)$ the electric field. As in the time interval $\mathrm{d}t_1$ a number of electrons $I_0 \mathrm{d}t_1$ is emitted and they will have position $\mathcal{R}(t, t_1)$ and velocity $\mathcal{V}(t, t_1)$ at time $t$, a phase-space distribution $f(r, v, t)$ can be defined as

$$f(r, v, t) = \int I_0 \mathrm{d}t_1 \delta(r - \mathcal{R}(t, t_1))\delta(v - \mathcal{V}(t, t_1)). \tag{3.5}$$

The quantity $f(r, v, t)\,\mathrm{d}r\mathrm{d}v$ represents the number of electrons in $[r, r + \mathrm{d}r]$ with radial velocity within $[v, v + \mathrm{d}v]$. From $f(r, v, t)$, the electron charge in the region $[r_1, r_2]$ at time $t$ can be written as

$$Q_e(r_1, r_2, t) = -e\int_{r_1}^{r_2}\mathrm{d}r\int I_0 \mathrm{d}t_1 \delta(r - \mathcal{R}(t, t_1)). \tag{3.6}$$

As the charged contained in a sphere of radius $r > R$ is given by

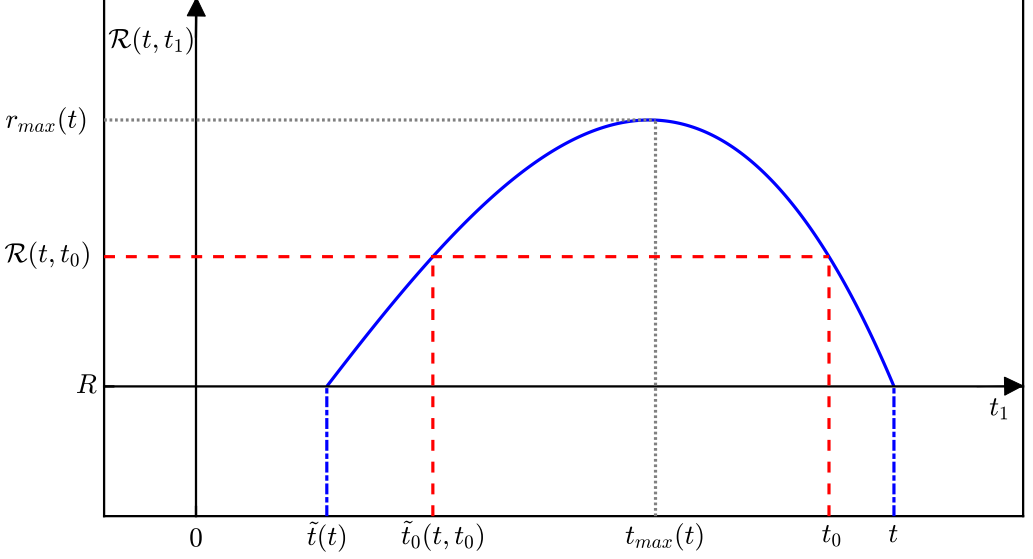$$Q(r, t) = Q_S(t) + Q_e(R, r, t), \tag{3.7}$$

FIGURE 6. Behaviour of $\mathcal{R}(t, t_1)$, for fixed $t$, as a function of $t_1$. Obviously, $\mathcal{R}(t, t_1 = t) = R$ and there are not electrons at time $t$ if the emission time, $t_1$, is larger than $t$. As $t_1$ decreases, $\mathcal{R}(t, t_1)$ increases until the time $t_1 = t_{max}(t)$ for which $\mathcal{R}$ is maximum [in other words, $\mathcal{R}(t, t_{max}(t))$ is the maximum radial distance at time $t$ for any electron of the system]. For $t_1 < t_{max}(t)$, $\mathcal{R}(t, t_1)$ decreases, as electrons are moving back after reaching their maximum distance. Finally, all electrons emitted for $t_1 < \tilde{t}(t)$ have been reabsorbed by the sphere.

where $Q_S(t)$ is the charge of the sphere at time $t$, the electric field $E(\mathcal{R}(t, t_0), t)$ can be written as:

$$E(\mathcal{R}(t, t_0), t) = \frac{Q(\mathcal{R}(t, t_0), t)}{\mathcal{R}^2(t, t_0)}. \tag{3.8}$$

Making use of Eq. (3.6), the contribution of electrons to $Q(\mathcal{R}(t, t_0), t)$ can be expressed as:

$$Q_e(R, \mathcal{R}(t, t_0), t) = -eI_0 \int dt_1 \int_R^{\mathcal{R}(t, t_0)} \delta(r - \mathcal{R}(t, t_1)) \, dr, \tag{3.9}$$

in which the value of the $r-$integral is 1 if $\mathcal{R}(t, t_1) \in [R, \mathcal{R}(t, t_0)]$, otherwise it vanishes. The typical behaviour of $\mathcal{R}(t, t_1)$ as a function of $t_1$ is shown in Fig. 6. Once $t_0 \in (\tilde{t}(t), t)$ is fixed, a new time, $\tilde{t}_0(t, t_0) \neq t_0$ can be determined such that $\mathcal{R}(t, \tilde{t}_0(t, t_0)) = \mathcal{R}(t, t_0)$ (as in Fig. 6). It must be noticed that $\tilde{t}_0 < t_{max}$ if $t_0 > t_{max}$ and vice versa. With reference to Fig. 6, one can conclude that, for $t_0 > t_{max}$:

$$\int_R^{\mathcal{R}(t, t_0)} \delta(r - \mathcal{R}(t, t_1)) \, dr = \begin{cases} 1 & \text{for } t_1 \in [\tilde{t}, \tilde{t}_0] \cup [t_0, t], \\ 0 & \text{otherwise,} \end{cases} \tag{3.10}$$

and, consequently, from Eq. (3.9) one has:

$$Q_e(R, \mathcal{R}(t, t_0), t) = -eI_0 [t - t_0 + \tilde{t}_0(t, t_0) - \tilde{t}(t)]. \tag{3.11}$$

Instead, for $t_0 < t_{max}$, Eq. (3.11) must be modified, as:

$$Q_e(R, \mathcal{R}(t, t_0), t) = -eI_0 [t - \tilde{t}_0(t, t_0) + t_0 - \tilde{t}(t)]. \tag{3.12}$$

Finally, $Q(r,t)$ must be evaluated. To do this, one can start by writing the charge $Q(t,t_0)$ "seen" at time $t$ by the electron emitted at $t_0$ as:

$$Q(t,t_0) = Q_S(t) + Q_e(R, \mathcal{R}(t,t_0), t). \tag{3.13}$$

Moreover, the total charge of the system $Q_{tot} = Q_S(t) + Q_e(R, r_{max}(t), t)$ is constant, and therefore one has:

$$\begin{aligned} Q(t,t_0) &= Q_{tot} + Q_e(R, \mathcal{R}(t,t_0), t) - Q_e(R, r_{max}(t), t) \\ &= Q_{tot} + eI_0 \left| t_0 - \tilde{t}_0(t,t_0) \right|, \end{aligned} \tag{3.14}$$

making use of Eqs. (3.11, 3.12) and noticing that if $\mathcal{R}(t,t_0) = r_{max}(t)$ then $\tilde{t}_0(t,t_0) = t_0 = t_{max}(t)$ (as in Fig. 6). In summary, Eqs. (3.3, 3.4, 3.8, 3.14) determine the electron dynamics.

In a steady-state situation, all the previous quantities depend upon $\lambda \equiv t - t_0$ only. In particular, $\mathcal{R}(t,t_0) = \overline{\mathcal{R}}(\lambda)$ and $\mathcal{V}(t,t_0) = \overline{\mathcal{V}}(\lambda)$. Moreover, in this case the curve of Fig. 6 is symmetric with respect to $t_{max}(t) = t - \mathbb{T}/2$, being $\mathbb{T}$ the total time of the trajectory for a single electron (from emission to absorption) and $t_0 - \tilde{t}_0$ is equal to $\mathbb{T} - 2\lambda$, for $\lambda \in (0, \mathbb{T}/2)$. Therefore, the previous equations become:

$$\begin{cases} \dfrac{d\overline{\mathcal{R}}}{d\lambda} = \overline{\mathcal{V}}, & \overline{\mathcal{R}}(\lambda = 0) = R, \\[3mm] \dfrac{d\overline{\mathcal{V}}}{d\lambda} = -\dfrac{e}{m}\left(\dfrac{Q_0 - 2eI_0\lambda}{\overline{\mathcal{R}}^2}\right), & \overline{\mathcal{V}}(\lambda = 0) = v_0, \end{cases} \tag{3.15}$$

for $\lambda \in [0, \mathbb{T}/2]$, with $Q_0 = Q_{tot} + eI_0\mathbb{T}$. For $\lambda \in [\mathbb{T}/2, \mathbb{T}]$, one has simply

$$\overline{\mathcal{R}}(\lambda) = \overline{\mathcal{R}}(\mathbb{T} - \lambda), \qquad \overline{\mathcal{V}}(\lambda) = -\overline{\mathcal{V}}(\mathbb{T} - \lambda). \tag{3.16}$$

It can be readily proved that system (3.15) is equivalent to the ones in previous Section†. To study a transient, a suitable perturbation is added to the steady-state solution, in the form:

$$\begin{aligned} \mathcal{R}(t,t_0) &= \overline{\mathcal{R}}(\lambda) + \epsilon\widehat{\mathcal{R}}(\lambda)\,e^{i\omega t}, \\ \mathcal{V}(t,t_0) &= \overline{\mathcal{V}}(\lambda) + \epsilon\widehat{\mathcal{V}}(\lambda)\,e^{i\omega t}, \\ E(t,t_0) &= \overline{E}(\lambda) + \epsilon\widehat{E}(\lambda)\,e^{i\omega t}, \end{aligned} \tag{3.17}$$

in which $\epsilon \ll 1$ and $\omega \in \mathbb{C}$. Of course, only the real (or the imaginary) part of $\mathcal{R}$ and $\mathcal{V}$ define the real perturbation. After introducing definitions (3.17) into Eqs. (3.3, 3.4), one obtains:

$$\begin{cases} \dfrac{d\widehat{\mathcal{R}}}{d\lambda} = \widehat{\mathcal{V}} - i\omega\widehat{\mathcal{R}}, \\[3mm] \dfrac{d\widehat{\mathcal{V}}}{d\lambda} = -\dfrac{e}{m}\widehat{E}(\lambda) - i\omega\widehat{\mathcal{V}}. \end{cases} \tag{3.18}$$

† Moreover, by introducing dimensionless quantities: $\widetilde{\mathcal{R}} = \overline{\mathcal{R}}/R$, $\widetilde{\mathcal{V}} = \overline{\mathcal{V}}/v_0$, $\widetilde{\lambda} = \lambda v_0/R$, $\widetilde{Q} = Q\,e/(mv_0^2 R)$, $\widetilde{I}_0 = I_0 e^2/(mv_0^3)$, system (3.15) can be rewritten as:

$$\begin{cases} \dfrac{d\widetilde{\mathcal{R}}}{d\widetilde{\lambda}} = \widetilde{\mathcal{V}}, & \widetilde{\mathcal{R}}(0) = 1, \\[3mm] \dfrac{d\widetilde{\mathcal{V}}}{d\widetilde{\lambda}} = -\dfrac{\widetilde{Q}_0 - 2\widetilde{I}_0\widetilde{\lambda}}{\widetilde{\mathcal{R}}^2}, & \widetilde{\mathcal{V}}(0) = 1, \end{cases}$$

In practice, using the new variables is equivalent to set $v_0$, $R$, $e$, $m$ all equal to 1.

In the following, $E\left(\mathcal{R}\left(t, t_0\right), t\right)$ and $Q\left(\mathcal{R}\left(t, t_0\right), t\right)$ are indicated simply as $E\left(t, t_0\right)$ and $Q\left(t, t_0\right)$, respectively. The value of $\widehat{E}\left(\lambda\right)$ can be calculated by writing the "total" electric field $E\left(t, t_0\right)$ as:

$$
\begin{aligned}
E\left(t, t_0\right) &= \frac{Q\left(t, t_0\right)}{\mathcal{R}^2\left(t, t_0\right)} = \frac{\overline{Q}\left(\lambda\right) + \epsilon\widehat{Q}\left(\lambda\right)\mathrm{e}^{i\omega t}}{\left[\overline{\mathcal{R}}\left(\lambda\right) + \epsilon\widehat{\mathcal{R}}\left(\lambda\right)\mathrm{e}^{i\omega t}\right]^2} \\
&= \frac{\overline{Q}\left(\lambda\right)}{\overline{\mathcal{R}}^2\left(\lambda\right)} + \epsilon\frac{1}{\overline{\mathcal{R}}^2\left(\lambda\right)}\left\{\widehat{Q}\left(\lambda\right) - \frac{2\overline{Q}\left(\lambda\right)}{\overline{\mathcal{R}}\left(\lambda\right)}\widehat{\mathcal{R}}\right\} + O\left(\epsilon^2\right)
\end{aligned}
\tag{3.19}
$$

being

$$
\left\{
\begin{array}{lll}
\overline{Q}\left(\lambda\right) &= Q_0 - 2eI_0\lambda, & \lambda \in \left[0, \mathbb{T}/2\right], \\
\overline{Q}\left(\lambda\right) &= Q\left(\mathbb{T} - \lambda\right), & \lambda \in \left[\mathbb{T}/2, \mathbb{T}\right].
\end{array}
\right.
\tag{3.20}
$$

Thus, one can write:

$$
\widehat{E}\left(\lambda\right) = \frac{1}{\overline{\mathcal{R}}^2\left(\lambda\right)}\left\{\widehat{Q}\left(\lambda\right) - \frac{2\overline{Q}\left(\lambda\right)}{\overline{\mathcal{R}}\left(\lambda\right)}\widehat{\mathcal{R}}\left(\lambda\right)\right\}.
\tag{3.21}
$$

The amplitude of the charge perturbation, $\widehat{Q}\left(\lambda\right)$, can be calculated by means of Eq. (3.14), and therefore $\tilde{t}_0\left(t, t_0\right)$ must be evaluated. From its definition, $\tilde{t}_0$ satisfies the equation:

$$
\overline{\mathcal{R}}\left(t - \tilde{t}_0\right) + \epsilon\widehat{\mathcal{R}}\left(t - \tilde{t}_0\right)\mathrm{e}^{i\omega t} = \overline{\mathcal{R}}\left(\lambda\right) + \epsilon\widehat{\mathcal{R}}\left(\lambda\right)\mathrm{e}^{i\omega t}.
\tag{3.22}
$$

For the steady state, $t - \tilde{t}_0 = \mathbb{T} - \lambda$, as $\frac{1}{2}\left(t - \tilde{t}_0\right) = t - \frac{\mathbb{T}}{2}$. When the transient is considered, a term of order of $\epsilon$ must be added

$$
t - \tilde{t}_0 = \mathbb{T} - \lambda - \epsilon\widehat{\theta}\left(\lambda\right)\mathrm{e}^{i\omega t}.
\tag{3.23}
$$

Introducing definition (3.23) into Eq. (3.22) one obtains:

$$
\begin{aligned}
&\frac{\mathrm{d}\overline{\mathcal{R}}}{\mathrm{d}\lambda}\left(\mathbb{T} - \lambda\right)\left\{-\epsilon\widehat{\theta}\left(\lambda\right)\mathrm{e}^{i\omega t}\right\} + \epsilon\widehat{\mathcal{R}}\left(\mathbb{T} - \lambda\right)\mathrm{e}^{i\omega t} \\
&= \epsilon\widehat{\mathcal{R}}\left(\lambda\right)\mathrm{e}^{i\omega t} + O\left(\epsilon\right),
\end{aligned}
\tag{3.24}
$$

and, eventually,

$$
\widehat{\theta}\left(\lambda\right) = \frac{\widehat{\mathcal{R}}\left(\mathbb{T} - \lambda\right) - \widehat{\mathcal{R}}\left(\lambda\right)}{\dfrac{\mathrm{d}\overline{\mathcal{R}}}{\mathrm{d}\lambda}\left(\mathbb{T} - \lambda\right)}.
\tag{3.25}
$$

Going back to Eq. (3.14), the charge $Q\left(t, t_0\right)$ can be written as

$$
\begin{aligned}
Q\left(t, t_0\right) &= Q_{tot} + eI_0\left|\mathbb{T} - 2\lambda - \epsilon\widehat{\theta}\left(\lambda\right)\mathrm{e}^{i\omega t}\right| \\
&= Q_{tot} + eI_0\left\{\left|\mathbb{T} - 2\lambda\right| \mp \epsilon\widehat{\theta}\left(\lambda\right)\mathrm{e}^{i\omega t}\right\}.
\end{aligned}
\tag{3.26}
$$

The sign - is for $\lambda < \mathbb{T}/2$, the opposite sign for $\lambda > \mathbb{T}/2$. Therefore, one finally obtains:

$$
\widehat{Q}\left(\lambda\right) = \pm eI_0\frac{\widehat{\mathcal{R}}\left(\lambda\right) - \widehat{\mathcal{R}}\left(\mathbb{T} - \lambda\right)}{\widehat{\mathcal{V}}\left(\mathbb{T} - \lambda\right)}.
\tag{3.27}
$$

In summary, Eqs. (3.15, 3.21, 3.27) can be used to determine the behaviour of the perturbation. However, Eq. (3.27) contains both $\widehat{\mathcal{R}}\left(\lambda\right)$ and $\widehat{\mathcal{R}}\left(\mathbb{T} - \lambda\right)$; for this reason,

it turns out useful to define new unknowns only in the range $[0, \mathbb{T}/2]$:

$$
\begin{aligned}
\mathcal{R}_+ (\lambda) &= \widehat{\mathcal{R}} (\lambda), & \mathcal{R}_- (\lambda) &= \widehat{\mathcal{R}} (\mathbb{T} - \lambda), \\
\mathcal{V}_+ (\lambda) &= \widehat{\mathcal{V}} (\lambda), & \mathcal{V}_- (\lambda) &= \widehat{\mathcal{V}} (\mathbb{T} - \lambda),
\end{aligned}
\tag{3.28}
$$

which satisfy the following set of equations:

$$
\begin{cases}
\dfrac{\mathrm{d}\mathcal{R}_+}{\mathrm{d}\lambda} = \mathcal{V}_+ - \mathrm{i}\omega \mathcal{R}_+, \\[2mm]
\dfrac{\mathrm{d}\mathcal{R}_-}{\mathrm{d}\lambda} = -\mathcal{V}_- + \mathrm{i}\omega \mathcal{R}_-, \\[2mm]
\dfrac{\mathrm{d}\mathcal{V}_+}{\mathrm{d}\lambda} = -\dfrac{e}{m} \dfrac{1}{\overline{\mathcal{R}}^2 (\lambda)} \left\{ \widehat{Q} (\lambda) - 2\dfrac{\overline{Q} (\lambda)}{\overline{\mathcal{R}} (\lambda)} \mathcal{R}_+ \right\} - \mathrm{i}\omega \mathcal{V}_+, \\[3mm]
\dfrac{\mathrm{d}\mathcal{V}_-}{\mathrm{d}\lambda} = +\dfrac{e}{m} \dfrac{1}{\overline{\mathcal{R}}^2 (\lambda)} \left\{ -\widehat{Q} (\lambda) - 2\dfrac{\overline{Q} (\lambda)}{\overline{\mathcal{R}} (\lambda)} \mathcal{R}_- \right\} + \mathrm{i}\omega \mathcal{V}_-, \\[3mm]
\widehat{Q} (\lambda) = eI_0 \dfrac{\mathcal{R}_+ (\lambda) - \mathcal{R}_- (\lambda)}{-\overline{\mathcal{V}} (\lambda)}.
\end{cases}
\tag{3.29}
$$

Equations (3.29) represent a linear, homogeneous system of differential equations, to be solved with the boundary conditions:

$$
\begin{array}{rclcrcl}
\mathcal{R}_+ (0) &=& 0, & & \mathcal{V}_+ (0) &=& 0, \\
\mathcal{R}_+ (\mathbb{T}/2) &=& \mathcal{R}_- (\mathbb{T}/2), & & \mathcal{V}_+ (\mathbb{T}/2) &=& \mathcal{V}_- (\mathbb{T}/2)
\end{array}
\tag{3.30}
$$

Two linearly independent solutions can be found numerically, with "initial" conditions:

$$
\begin{array}{llll}
1) & \mathcal{R}_+ (\mathbb{T}/2) = \mathcal{R}_- (\mathbb{T}/2) = A, & \mathcal{V}_+ (\mathbb{T}/2) = \mathcal{V}_- (\mathbb{T}/2) = 0; \\
2) & \mathcal{R}_+ (\mathbb{T}/2) = \mathcal{R}_- (\mathbb{T}/2) = 0, & \mathcal{V}_+ (\mathbb{T}/2) = \mathcal{V}_- (\mathbb{T}/2) = B;
\end{array}
\tag{3.31}
$$

where $A$ and $B$ are arbitrary, non-zero, constant, so obtaining the solutions $\mathcal{R}_\pm^{(1)} (\lambda, \omega)$, $\mathcal{V}_\pm^{(1)} (\lambda, \omega)$, $\mathcal{R}_\pm^{(2)} (\lambda, \omega)$, $\mathcal{V}_\pm^{(2)} (\lambda, \omega)$. The most general solution of system (3.29) satisfying the boundary conditions for $\lambda = \mathbb{T}/2$ can be written as a linear superposition of these two solutions, in particular:

$$
\begin{aligned}
\mathcal{R}_+ (\lambda, \omega) &= c_1 \mathcal{R}_+^{(1)} (\lambda, \omega) + c_2 \mathcal{R}_+^{(2)} (\lambda, \omega), \\
\mathcal{V}_+ (\lambda, \omega) &= c_1 \mathcal{V}_+^{(1)} (\lambda, \omega) + c_2 \mathcal{R}_+^{(2)} (\lambda, \omega).
\end{aligned}
\tag{3.32}
$$

By imposing the boundary conditions for $\lambda = 0$, one obtains a linear homogeneous system that has non-trivial solution only if:

$$
\mathcal{D} (\omega) \equiv \det \begin{pmatrix} \mathcal{R}_+^{(1)} (0, \omega) & \mathcal{R}_+^{(2)} (0, \omega) \\ \mathcal{V}_+^{(1)} (0, \omega) & \mathcal{V}_+^{(2)} (0, \omega) \end{pmatrix} = 0
\tag{3.33}
$$

Equation (3.33) allows one to calculate the set of $\omega$'s for a given set of physical parameters. As the perturbation evolves in time as $\exp (\mathrm{i}\Omega t - \Gamma t)$, being $\omega = \Omega + \mathrm{i}\Gamma$, the steady-state is stable if all $\Gamma$'s are positive. If one of the $\Gamma$'s is negative, a perturbation grows exponentially until a non-linear regime is reached.

To confirm the validity of the theory, a number of tests have been performed. Two of them are presented in the following. The first analysis refers to the case shown in Fig. 1. The behaviour of $Q_e (t)$, as obtained numerically with the shell method, has been fitted by a damped sinusoid $Q_e (t) \simeq A \sin (\Omega_0 t + \alpha) \exp (-\Gamma_0 t)$ so obtaining $\Omega_0 = 2.9206$ and $\Gamma_0 = 0.1218$. Then Eq. (3.33) has been solved numerically, starting from the trial value
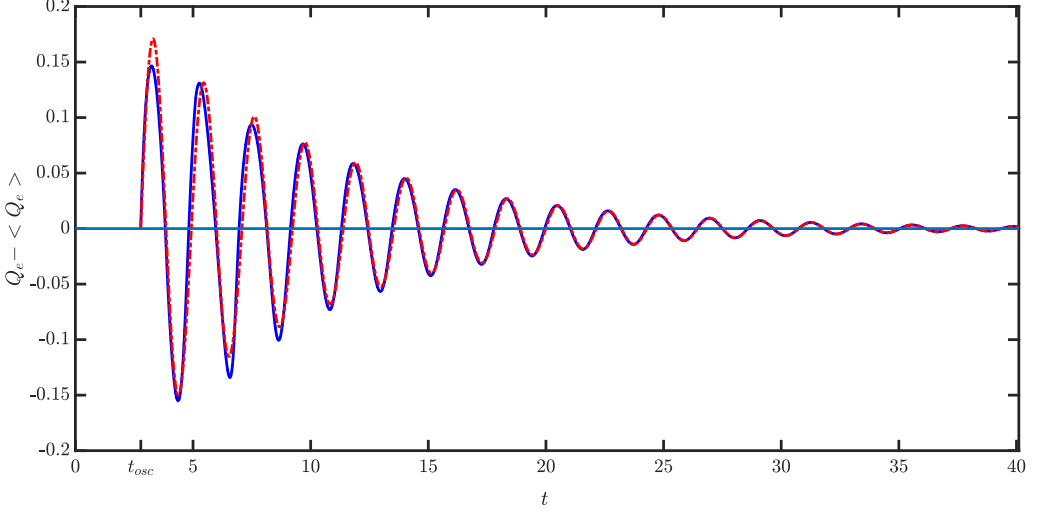
FIGURE 7. Damped oscillations of the electron charge for the same case of Fig. 1 (full line), and best fit as $Q_e - <Q_e> = A \sin(\Omega_0 t + \alpha) \exp(-\Gamma_0 t)$ (dashed line). [$<Q_e>$ is the steady-state for $Q_e$, as obtained by Eq. (2.7)]
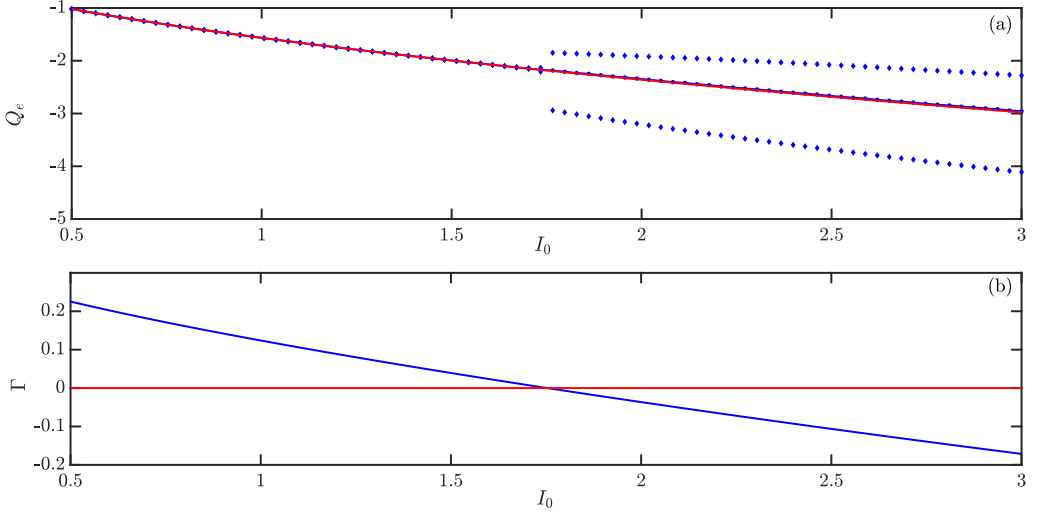


FIGURE 8. (a) Electron charge $Q_e$ as a function of the flux $I_0$ for a fixed value of the total charge $Q_0 = 1$, as obtained by system (2.8) (full line, red). Average electron charge after the transient (averaged over the oscillations), evaluated with the shell method, together with the maximum and the minimum value of $Q_e$ in a period of oscillation, as a function of $I_0$ (dots, blue). (b) Plot of $\Gamma = \text{Im}(\omega)$ as a function of $I_0$, for the same value of $Q_{tot}$, obtained from Eqs. (3.29-3.32).

$\omega_0 = \Omega_0 + i\Gamma_0$, eventually obtaining $\omega = 2.9139 + i\, 0.1237$.

Another study is presented in Fig. 8. By considering a fixed value for the total charge of the system, $Q_{tot}$, the steady-state of $Q_e$, obtained from Eq. (2.7), and the corresponding values from numerical solution with the shell method are reported for different values of $I_0$ (Fig. 8.a). When $I_0$ increases, there is the transition between damped and steady oscillations for the electron charge. For the same values of the parameters, the value of $\Gamma = \text{Im}(\omega)$ is reported (Fig. 8.b). As can be noticed, the value of $I_0$ for the transition corresponds to the condition $\Gamma(I_0) = 0$. In reality, Eq. (3.33) has a complicated set of

eigenvalues $\omega$. Here only one has been calculated for each value of the electron flux $I_0$ with a simple procedure. For the smallest value of $I_0$ considered, for which the oscillations are damped, $\omega$ has been obtained as for the case in Fig. 7. Then, $I_0$ was increased by $\Delta I_0$ and the new $\omega$ was calculated numerically from Eq. (3.33) using the previous $\omega$ as guess. Obtaining the full spectrum of $\omega$'s was beyond the purposes of the present work.

## 4. Discussion and conclusive remarks

One may wonder why the system here considered shows an irreversible behaviour (in particular, the tendency towards a steady-state configuration), which is usually associated with collisional phenomena. In reality, there are well-known situations in which a collisionless plasma presents damped oscillations (e.g., the Landau damping), however in the case studied in this paper the physical cause is completely different, as collisions do exist when the electrons are absorbed by the sphere. In this case, the information about their velocity is lost, and the only way to time-revert the electron dynamics would be reemitting electrons that are absorbed with the right velocity (i.e., the velocity when they hit the sphere), which is unknown after the collision. Also the case of steady oscillations can be regarded as an irreversible behaviour, as it is a limit cycle, which is independent from the initial perturbation of the steady-state equilibrium. With reference to Fig. 4, it can be noticed that the initial velocity of the emitted electrons is a known quantity ($v_0$, a constant in the present model), but the velocity of the particles when they are absorbed is variable (it oscillates in time) as the electric potential is time-dependent and the energy of a single particle is not a constant of the motion.

As mentioned in the Introduction, the initial motivation of the present study was investigating on the possibility of calculating the equilibrium distribution of the electrons when the density of positive charge is known. The answers of the present work are these:

(i) in every situation which is significant from a physical point of view (i.e., a non-negative charge on the sphere before the electron emission) a steady-state distribution for the electron exists;

(ii) the solution either corresponds to a real stationary state (when the oscillations are damped) or it represents the mean value of the charge over an oscillation period. As the period is usually much smaller with respect to the characteristic time of the ion dynamics, also in the latter case the steady-state solution can be usefully employed.

In order to apply these methods to the original problem (i.e., plasma expansion in a vacuum in spherical geometry) two extensions are still necessary:

(i) to include the effect of the ion charge outside the sphere, which can be easily introduced into the equations; this can be done by suitably modifying Eq. (2.5), as

$$
\begin{cases}
\dfrac{\mathrm{d}U}{\mathrm{d}r} & = \dfrac{\Psi_e + eQ_i}{r^2}, \\[2ex]
\dfrac{\mathrm{d}\Psi_e}{\mathrm{d}r} & = -\dfrac{2e^2 I_0}{\left[\frac{2}{m}\left(\mathcal{E} - U\right)\right]^{1/2}}
\end{cases}
\tag{4.1}
$$

where $\Psi_e\left(r\right)/er^2$ is the electric field due to the sphere and to the electrons, while $Q_i\left(r\right)$ is the ion charge in the region $[R, r]$; however, in general $U\left(r\right)$ may not be a monotonic function of $r$, and, consequently, in these cases $U$ (or $p$) cannot be used as independent variables;

(ii) to consider the real velocity distribution (e.g., a Maxwellian) for the electrons when they are emitted. This last point can be dealt with by approximating the energy spectrum of the emitted electrons as a superposition of cold beams of different velocity.

An alternative possibility is suitably adapting the ergodic method (Peano *et al.* 2006) originally introduced to study the dynamics of nanoplasmas; this will be the object of future research.

## REFERENCES

AKIMOV, P. V., SCHAMEL, H., ENDER, A. Y. & KUZNETSOV, V. I. 2003 Switching as a dynamical process in electron diodes. *Journal of Applied Physics* **93** (2), 1246–1256, arXiv: https://doi.org/10.1063/1.1531813.

AKIMOV, P. V., SCHAMEL, H., KOLINSKY, H., ENDER, A. Y. & KUZNETSOV, V. I. 2001 The true nature of space-charge-limited currents in electron vacuum diodes: A lagrangian revision with corrections. *Physics of Plasmas* **8** (8), 3788–3798, arXiv: https://doi.org/10.1063/1.1383287.

BOELLA, E., COPPA, G., D'ANGOLA, A. & PEIRETTI PARADISI, B. 2018 Gridless particle technique for Vlasov-Poisson system in problem with high degree of symmetry. *Comput. Phys. Commun.* **224**, 136–143.

CHILD, C. D. 1911 Discharge From Hot CaO. *Phys. Rev. (Series I)* **32**, 492–511.

COLANDREA, C. 2018 Model to study the expansion of plasma contactor emitted by a magnetospheric spacecraft to mitigate charging effects. Master thesis, Politecnico di Torino, Torino, Italy.

DELZANNO, G. L., BOROVSKY, J. E., THOMSEN, M. F., GILCHRIST, B. E. & SANCHEZ, E. 2016 Can an electron gun solve the outstanding problem of magnetosphere-ionosphere connectivity? *Journal of Geophysical Research: Space Physics* **121** (7), 6769–6773, arXiv: https://agupubs.onlinelibrary.wiley.com/doi/pdf/10.1002/2016JA022728.

DITMIRE, T., DONNELLY, T., RUBENCHIK, A. M., FALCONE, R. W. & PERRY, M. D. 1996 Interaction of intense laser pulses with atomic clusters. *Phys. Rev. A* **53**, 3379–3402.

ENDER, A., KOLINSKY, H., KUZNETSOV, V. & SCHAMEL, H. 2000 Collective diode dynamics: an analytical approach. *Physics Reports* **328** (1), 1 – 72.

ENDER, A. Y. & KUZNETSOV, V. I. 2014 Nonlinear oscillations in the Knudsen plasma diodes. *Journal of Physics: Conference Series* **510** (1), 012046.

HENDRICKSON, R. A., MCENTIRE, R. W. & WINCKLER, J. R. 1975 Echo I: An experimental analysis of local effects and conjugate return echoes from an electron beam injected into the magnetosphere by a sounding rocket. *Planet. Space Sci.* **23**, 1431–1444.

KUZNETSOV, V. I. & ENDER, A. Y. 2010*a* Time-dependent regimes of a Bursian diode I: Stability of steady solutions. *Plasma Physics Reports* **36** (3), 226–235.

KUZNETSOV, V. I. & ENDER, A. Y. 2010*b* Time-dependent regimes of a Bursian diode II: Characteristic features of nonlinear oscillations. *Plasma Physics Reports* **36** (3), 236–249.

LANGMUIR, I. & BLODGETT, K. B. 1924 Currents Limited by Space Charge between Concentric Spheres. *Phys. Rev.* **24**, 49–59.

PEANO, F., PEINETTI, F., MULAS, R., COPPA, G. & SILVA, L. O. 2006 Kinetics of the Collisionless Expansion of Spherical Nanoplasmas. *Phys. Rev. Lett.* **96**, 175002.