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Abstract. We describe the fermionic and bosonic Fock representation of endomorphisms of the exterior algebra of a $\mathbb{Q}$- vector space of infinite countable dimension. We achieve our goal by exploiting the extension of certain Schubert derivations, originally defined for exterior algebras only, to the fermionic Fock space.

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Contents

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Introduction

0.1 The Goal. Let $B := \mathbb{Q}[\mathbf{x}]$ be the polynomial ring in the infinitely many indeterminates $\mathbf{x} := (x_1, x_2, \ldots)$ and $B(\xi) := B \otimes_{\mathbb{Q}} \mathbb{Q}[\xi^{-1}, \xi]$, where ξ is one further indeterminate over $\mathbb{Q}$. The purpose of this paper is to describe $B(\xi)$, which we refer to as the *bosonic Fock space*, as a (product of) vertex operator representation of the Lie superalgebra

$$gl(\bigwedge \mathcal{V}) \cong \bigwedge \mathcal{V} \otimes \bigwedge \mathcal{V}^*,$$

where $\mathcal{V} := \bigoplus_{i\in\mathbb{Z}} \mathbb{Q} \cdot b_i$ is a $\mathbb{Q}$-vector space, with basis $\mathbf{b} := (b_i)_{i\in\mathbb{Z}}$, parametrised by the integers, and $V^* := \bigoplus_{j\in\mathbb{Z}} \mathbb{Q}\beta_j$ is its restricted dual, with basis $\boldsymbol{\beta} := (\beta_j)_{j\in\mathbb{Z}}$, where $\beta_j(b_i) = \delta_{ij}$. Our goal is achieved in our Theorem 3.6, already anticipated at the end of this introduction. It generalises, and further enhances, a classical result which, for convenience, we shall refer to as DJKM representation of $gl(\mathcal{V})$ (after Date, Jimbo, Kashiwara and Miwa). The latter describes the bosonic Fock space $B(\xi) \otimes_{\mathbb{Q}} \mathbb{C}$ as a representation of the Lie algebra $gl_\infty(\mathbb{C}) \cong gl(\mathcal{V}) \otimes_{\mathbb{Q}} \mathbb{C}$, of all the complex valued matrices $(a_{ij})_{i,j\in\mathbb{Z}}$, whose entries are all zero but finitely many.

0.2 Some motivations and background. The representation theory of Lie algebras of endomorphisms of infinite dimensional vector spaces, often phrased through the physicist's jargon of *charged free fermions* [13, p. 28] which, in down to the earth terms, are basis elements of a canonical Clifford algebra supported on $\mathcal{V} \oplus \mathcal{V}^*$), got a tremendous impulse from the theory of *solitons*, as faced by the Sato's japanese school of algebraic analysis. In important pioneering work on the subject [3] (see also [11, Section 1] and [12, Theorem 6.1]), Date, Jimbo, Kashiwara and Miwa deduce a bosonic representation of the central extension $\mathfrak{a}_\infty(\mathbb{C})$ of the Lie algebra of all complex valued matrices $(a_{ij})_{i,j\in\mathbb{Z}}$ with finitely many nonzero diagonals. Its elegant shape remarkably involves a discrete version of the vertex operators occurring in the Skyrme model of self-interacting meson-like fields [16]. The bosonic vertex representation of $gl_\infty(\mathbb{C}) := gl_\infty(\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C}$, as in the cited reference [14, Proposition 5.2], can be regarded as a particular case of the DJKM's one. Its expression is defined over the rational numbers. That is why throughout the paper we considers our scalars in the field $\mathbb{Q}$ only. On one hand it is more than enough for our purposes, because the theory could be more generally developed over the integers, like in [10]. On the other hand, the rational field is big enough to enable the use of exponential functions to spell our main formulas, turning easier the comparison with earlier related literature. Since $gl(\mathcal{V})$ is a Lie subalgebra of $gl(\bigwedge \mathcal{V})$, one must expect our Theorem 3.6 generalising, and therefore recovering, as a particular case, the DJKM representation of $gl(\mathcal{V})$, as it is the case.

0.3 Generating functions of bases. The idea to compactly describe $B(\xi)$ as a module over the Lie algebra $gl(\mathcal{V})$, amounts to a convenient phrasing of the generating function

$$\mathcal{E}(z, w) = \sum_{i,j\in\mathbb{Z}} b_i \otimes \beta_j z^i w^{-j}, \tag{1}$$

of the natural basis $\mathbf{b} \otimes \boldsymbol{\beta} := (b_i \otimes \beta_j)_{i,j\in\mathbb{Z}}$ of $\mathcal{V} \otimes \mathcal{V}^*$. The program then consists in identifying a suitable extension of the generating function (1) to a natural basis of $gl(\bigwedge \mathcal{V})$. But first, we must let ta new character coming into play.

To this end, let $\mathcal{P}$ denote the set of all partitions (non increasing sequences of non negative integers all zero but finitely many). By *fermionic Fock space* we shall mean a $\mathbb{Z}$-graded $\mathbb{Q}$–vector space $\mathcal{F} :=$

$\mathcal{F}(\mathcal{V})=\bigoplus_{m\in\mathbb{Z}}\mathcal{F}_m$ which, like $B(\xi)$, the bosonic one, possesses a basis $[\mathbf{b}]_{m+\boldsymbol{\lambda}}$ parametrised by $\mathbb{Z}\times\mathcal{P}$. More than that, it is essential, from our point of view, to think of $\mathcal{F}$ as a $B(\xi)$-module of rank 1 generated by $[\mathbf{b}]_0 := b_0\wedge b_{-1}\wedge b_{-2}\wedge\cdots$, such that

$$\xi^m S_{\boldsymbol{\lambda}}(\mathbf{x})[\mathbf{b}]_0=[\mathbf{b}]_{m+\boldsymbol{\lambda}}=b_{m+\lambda_1}\wedge\cdots\wedge b_{m-r+1+\lambda_r}\wedge b_{m-r}\wedge b_{m-r-1}\wedge\cdots\,, \tag{2}$$

where $S_{\boldsymbol{\lambda}}(\mathbf{x})$ denotes the Schur polynomial associated to the partition $\boldsymbol{\lambda}$ and to the sequence $\mathbf{x}$. Equality (2) can be understood either as a Giambelli's formula for Schubert Calculus on infinite Grassmannian (see [9]), or as a Jacobi-Trudy-like formula (see [5, p. 32 and 34] and also [15]). To follow more closely the reference [14, Theorem 6.1], and being more adherent to the subject of the paper, we call (2) the *boson-fermion correspondence*. Our starting point is the obvious remark that $\bigwedge\mathcal{V}$ is a (irreducible) representation of the Lie superalgebra $gl(\bigwedge\mathcal{V})$ of all endomorphisms vanishing at all but finitely many basis elements of $\bigwedge\mathcal{V}$. An explicit generating function encoding the $gl(\bigwedge\mathcal{V})$-module structure of $\bigwedge\mathcal{V}$ has already been proposed in [1] (see also [2] for a finite dimensional example), where the vertex operators shaping the boson-fermion correspondence spontaneously arise in all their splendour, although in a more classical framework. In addition, as noticed in [10], little effort is needed to extend the $\bigwedge\mathcal{V}$-representation to $\mathcal{F}$, mainly because the latter is a module over the former. This reflects in the fact that each degree $\mathcal{F}_m$ of $\mathcal{F}$, as formula (2) suggests, can be thought of as a semi-infinite exterior power. Finally, one just pulls back on $B(\xi)$ the $\mathcal{F}$–representation of $gl(\bigwedge\mathcal{V})$, invoking the boson-fermion correspondence. The program still demands, however, to identify a convenient generalisation of the DJKM generating function (1). Last, but not the least, one is left to determine explicitly its action on $\bigwedge\mathcal{V}$. This is the point that, as in our previous contribution, the flexible formalism of Schubert derivations (a distinguished kind of Hasse-Schmidt derivation on an exterior algebra), extended to $\mathcal{F}$, enters the game.

0.4 To pursue our program. we use the basis of $\bigwedge\mathcal{V}\otimes\bigwedge\mathcal{V}^*=\bigoplus_{k,l\geq 0}\bigwedge^k\mathcal{V}\otimes\bigwedge^l\mathcal{V}^*$ given by the union of those induced on $\bigwedge^k\mathcal{V}\otimes\bigwedge^l\mathcal{V}^*$ by $\mathbf{b}$ and $\boldsymbol{\beta}$, for all $k,l\geq 0$. This is quite straightforward, up to getting aware of one main combinatorial point, i.e. that they are best parametrised by the set $\overline{\mathcal{P}}$ of what, in Definition 2.1, lacking of a better terminology, we called *bilateral partitions*. More precisely, given $r\geq 0$, we stipulate to denote by $\overline{\mathcal{P}}_r$ the set of all r-tuples $\boldsymbol{\lambda}=(\lambda_1,\ldots,\lambda_r)\subseteq\mathbb{Z}^r$, such that $\lambda_1\geq\cdots\geq\lambda_r$. We so have

$$\bigwedge^k\mathcal{V}=\bigoplus_{\boldsymbol{\mu}\in\overline{\mathcal{P}}_k}\mathbb{Q}[\mathbf{b}]^k_{\boldsymbol{\mu}}\qquad\text{and}\qquad\bigwedge^l\mathcal{V}^*=\bigoplus_{\boldsymbol{\nu}\in\overline{\mathcal{P}}_l}\mathbb{Q}[\boldsymbol{\beta}]^l_{\boldsymbol{\mu}},$$

where

$$[\mathbf{b}]^k_{\boldsymbol{\mu}}=b_{k-1+\mu_1}\wedge\cdots\wedge b_{\mu_k}\qquad\text{and}\qquad[\boldsymbol{\beta}]^l_{\boldsymbol{\nu}}=\beta_{l-1+\nu_1}\wedge\cdots\wedge\beta_{\nu_l}.$$

Then

$$\mathcal{E}(\mathbf{z}_k,\mathbf{w}_l^{-1})=\sum_{\boldsymbol{\mu},\boldsymbol{\nu}\in\overline{\mathcal{P}}_k\otimes\overline{\mathcal{P}}_l}[\mathbf{b}]^k_{\boldsymbol{\mu}}\otimes[\boldsymbol{\beta}]^l_{\boldsymbol{\nu}}\mathbf{s}_{\boldsymbol{\mu}}(\mathbf{z}_k)\mathbf{s}_{\boldsymbol{\nu}}(\mathbf{w}_l^{-1}), \tag{3}$$

is the generating function of the distinguished basis $[\mathbf{b}]^k_{\boldsymbol{\mu}}\otimes[\boldsymbol{\beta}]^l_{\boldsymbol{\nu}}$ of $\bigwedge^k\mathcal{V}\otimes\bigwedge^l\mathcal{V}^*$, where $\mathbf{z}_k$ and $\mathbf{w}_l^{-1}$ are, respectively, k-tuples $(z_1,\ldots,z_k)$ and l-tuples $(w_1^{-1},\ldots,w_l^{-1})$ of formal variables. Abusing notation, we have chosen to denote by the same symbols $\mathbf{s}_{\boldsymbol{\mu}}(\mathbf{z}_k)$ and $\mathbf{s}_{\boldsymbol{\nu}}(\mathbf{w}_l^{-1})$ natural extensions of the classical Schur polynomials occurring in the theory of symmetric functions as in, e.g., [6, Section 3] and/or [4, Section 2.2.]. The difference with the classical ones is that they are symmetric *rational functions*. They do coincide with the usual Schur symmetric polynomials whenever $\boldsymbol{\lambda} \in \mathcal{P}_r = \overline{\mathcal{P}} \cap \mathbb{N}^r$. We are now in position to anticipate the statement of our main result.

Theorem 3.6. *The* (DJKM bosonic) *action of* $\mathcal{E}(\mathbf{z}_k, \mathbf{w}_l^{-1})$ *on* $B(\xi)$ *is given by*

$$\mathcal{E}(\mathbf{z}_k, \mathbf{w}_l^{-1}) = \exp\left(\sum_{n\geq 1} \frac{1}{n} p_n(\mathbf{z}_k^{-1}) p_n(\mathbf{w}_l)\right) \Gamma(\mathbf{z}_k, \mathbf{w}_l), \tag{4}$$

where

i) the expression $p_n(\mathbf{z}_k^{\pm})$ *and* $p_n(\mathbf{w}_l^{\pm 1})$ *denote the Newton powers sums symmetric polynomials, in the variables* $\mathbf{z}_k^{\pm 1}$ *and* $\mathbf{w}_l^{\pm 1}$*, i.e. more explicitly*

$$p_n(\mathbf{z}_k^{\pm 1}) := z_1^{\pm n} + \cdots + z_k^{\pm n} \qquad \text{and} \qquad p_n(\mathbf{w}_l^{\pm 1}) := w_1^{\pm n} + \cdots + w_l^{\pm n};$$

ii) the map $\Gamma(\mathbf{z}_k, \mathbf{w}_l) : B(\xi) \to B(\xi)[\![\mathbf{z}_k^{\pm 1}, \mathbf{w}_l^{\pm 1}]\!]$ *is the* vertex operator

$$R(\mathbf{z}_k, \mathbf{w}_l^{-1}) \exp\left(\sum_{n\geq 1} x_n (p_n(\mathbf{z}_k) - p_n(\mathbf{w}_l))\right) \exp\left(\sum_{n\geq 1} \frac{p_n(\mathbf{z}_k^{-1}) - p_n(\mathbf{w}_l^{-1})}{n} \frac{\partial}{dx_n}\right); \tag{5}$$

iii) the map $R(\mathbf{z}_k, \mathbf{w}_l^{-1}) : B(\xi)[\![\mathbf{z}_k, \mathbf{w}_l^{-1}]\!] \to B(\xi)[\![\mathbf{z}_k, \mathbf{w}_l^{-1}]\!]$ *is the unique* $B[\![\mathbf{z}_k, \mathbf{w}_l^{-1}]\!]$*-linear extension of*

$$\xi^m \mapsto \xi^{m+k-l} \prod_{\substack{1 \leq i \leq k \\ 1 \leq j \leq l}} \frac{z_i^{m-l+1}}{w_j^{m-l+1}}.$$

The meaning of formula (5) is that if $P(\mathbf{x}, \xi) \in B(\xi)$ is any polynomial, then its "multiplication" by $[\mathbf{b}]^k_{\boldsymbol{\mu}} \otimes [\boldsymbol{\beta}]^l_{\boldsymbol{\nu}}$, is the coefficient of $\mathbf{s}_{\boldsymbol{\mu}}(\mathbf{z}_k)\mathbf{s}_{\boldsymbol{\nu}}(\mathbf{w}_l^{-1})$ in the expansion $\mathcal{E}(\mathbf{z}_k, \mathbf{w}_l^{-1})P(\mathbf{x}, \xi)$. This may seem tricky. However multiplying the resulting expression by the product of the two Vandermonde determinants, $\Delta_0(\mathbf{x}_k)\Delta_0(\mathbf{w}_l^{-1})$, it is sufficient to look at the coefficient of the less intimidating monomial $z_k^{k-1+\mu_1} \cdots z_1^{\mu_k} \cdot w_1^{-l+1-\nu_1} \cdots w_l^{-\nu_k}$.

To end up, reading formula (5) for $k = l = 1$, putting $z_1 = z$ and $w_1 = w$, one has $\mathbf{s}_{(i)}(z) = z^i$ and $\mathbf{s}_{(j)}(w^{-1}) = w^{-j}$, for all $i, j \in \mathbb{Z}$. By the definition of the logarithm of an invertible formal power series:

$$\exp\left(\sum_{n\geq 1} \frac{1}{n} \frac{w^n}{z^n}\right) = \frac{1}{1 - \dfrac{w}{z}}.$$

The fact that, in this case, $R(z, w^{-1})\xi^m = \xi^m \dfrac{z^m}{w^m}$, equality (4) simplifies into

$$\mathcal{E}(z, w^{-1})_{|B\xi^m} = \frac{z^m}{w^m} \frac{1}{1 - \dfrac{w}{z}} \exp\left(\sum_{n\geq 1} x_n (z^n - w^n)\right) \exp\left(-\sum_{n\geq 1} \frac{z^{-n} - w^{-n}}{n} \frac{\partial}{\partial x_n}\right), \tag{6}$$

which is precisely the original DJKM formula for the bosonic representation of $gl(\mathcal{V})$ (like in [14, Proposition 5.2]. This may look surprising indeed, because comparing (5) with (6), it is apparent that the former can be obtained from the latter by simply replacing the variables z, w in (6) by the power sums of the indeterminates $(z_1, \ldots, z_k)$ and $(w_1, \ldots, w_l)$, respectively, used to define the generating function (3). As in our previous references [1, 9, 10], we have borrowed methods from the theory of *Hasse-Schmidt derivations on an exterior algebra*, like in the book [7]. The similarity of DJKM formula with our (4), however, makes us wonder whether there is any other argument to deduce our Theorem 3.6, bypassing our methods.

0.5 Organisation of the paper. In the first section we recall some more or less known pre-requisites. We revise, in particular, the construction of the fermionic Fock space following [10, Section 5], as well as the way to extend the Schubert derivation on it. A little background on Schur polynomials, mainly following [6] but also [14, Lecture 6], is included as well. Section 2 is devoted to carefully define the generating function of the basis elements of $\bigwedge^k \mathcal{V} \otimes \bigwedge^l \mathcal{V}^*$, that is best suited to describe the fermionic and bosonic representation of $gl(\bigwedge \mathcal{V})$. In this same section the natural notion of bilateral partition is also introduced. It is reasonable to suspect that it is somewhere hidden in pieces of less known literature. Section 3 eventually concerns the statement and proof of our main theorem which, as announced, supplies the expression of both the fermionic and the bosonic expression of $gl(\bigwedge \mathcal{V})$. The two cases are treated in a unified way, reflecting the fact inspiring the references [7, 9, 10] that there is a very little, if not any at all, substantial difference between the two spaces. Indeed, as explained in [1], the vertex operators occurring in the representation theory of the Heisenberg algebra, come naturally to life, exactly the same, already at the level of multivariate Schubert derivations on exterior algebras. With no serious need, at least for the focused purposes of our research, to cross the walls to enter into the realm of the infinite wedge powers, as however we did in the present contribution.

1 Background and notation

1.1 We shall deal with a $\mathbb{Q}$-vector space $\mathcal{V} := \bigoplus_{i\in\mathbb{Z}} \mathbb{Q} \cdot b_i$ and its restricted dual $\mathcal{V}^* := \bigoplus_{j\in\mathbb{Z}} \mathbb{Q} \cdot \beta_j$, where $\beta_j \in \mathrm{Hom}_{\mathbb{Q}}(\mathcal{V}, \mathbb{Q})$ is the unique linear form such that $\beta_j(b_i) = \delta_{ji}$. The generating series of the basis elements of $\mathcal{V}$ and $\mathcal{V}^*$ are, respectively:

$$\mathbf{b}(z) = \sum_{i\in\mathbb{Z}} b_i z^i \in V[\![z^{-1}, z]\!] \qquad \text{and} \qquad \boldsymbol{\beta}(w^{-1}) = \sum_{j\in\mathbb{Z}} \beta_j w^{-j} \in V^*[\![w, w^{-1}]\!]. \tag{7}$$

1.2 Hasse-Schmidt Derivations on $\bigwedge \mathcal{V}$. A map $\mathcal{D}(z) : \bigwedge \mathcal{V} \to \bigwedge \mathcal{V}[\![z]\!]$ is said to be *Hasse-Schmidt* (HS) derivation on $\bigwedge \mathcal{V}$ if $\mathcal{D}(z)(\mathbf{u} \wedge \mathbf{v}) = \mathcal{D}(z)\mathbf{u} \wedge \mathcal{D}(z)\mathbf{v}$, for all $\mathbf{u}, \mathbf{v} \in \bigwedge \mathcal{V}$. Write $\mathcal{D}(z)$ in the form $\sum_{j\ge 0} D_j z^j$, with $D_j \in \mathrm{End}_{\mathbb{Q}}(\bigwedge \mathcal{V})$. Then $\mathcal{D}(z)$ is invertible in $\mathrm{End}_{\mathbb{Q}}(\bigwedge \mathcal{V})[\![z]\!]$ if and only if D_0 is invertible. In this case $\mathcal{D}(z)$ is invertible and its inverse $\overline{\mathcal{D}}(z)$ in $\mathrm{End}_{\mathbb{Q}}(\bigwedge \mathcal{V})[\![z]\!]$ is a HS–derivation as well.

1.3 Schubert derivations. Consider the shifts endomorphisms $\sigma_{\pm 1} \in gl(\bigwedge \mathcal{V})$ given by $\sigma_{\pm 1} b_j = b_{j\pm 1}$. By [7, Proposition 4.1.13], there exist unique HS derivations on $\sigma_\pm(z) : \bigwedge \mathcal{V} \to \bigwedge \mathcal{V}[\![z^{\pm 1}]\!]$ such that

$$\sigma_\pm(z) b_j = \sum_{i\ge 0} b_{j\pm i} z^{\pm i}.$$

Let us denote by $\overline{\sigma}_\pm(z)$ their inverses in $\bigwedge \mathcal{V}[\![z^{\pm 1}]\!]$. Restricted to $\mathcal{V}$ they work as follows

$$\overline{\sigma}_+(z) b_j = b_j - b_{j+1} z \qquad \text{and} \qquad \overline{\sigma}_-(z) b_j = b_j - b_{j-1} z^{-1}. \tag{8}$$

They are called *Schubert derivations* in the references [7, 9, 10].

1.4 Fermionic Fock space. We quickly summarise the definition of the fermionic Fock space borrowed from [10]. Let $[\mathcal{V}]$ be a copy of $\mathcal{V}$ (framed by square bracket to distinguish by the original $\mathcal{V}$ itself). It is the $\mathbb{Q}$-vector space with basis $([\mathbf{b}]_m)_{m\in\mathbb{Z}}$. Identify $[\mathcal{V}]$ with a sub-module of the tensor product $\bigwedge\mathcal{V}\otimes_{\mathbb{Q}}[\mathcal{V}]$ via the map $[\mathbf{b}]_m\mapsto 1\otimes[\mathbf{b}]_m$, seen as a left $\bigwedge\mathcal{V}$-module. Let W be the left $\bigwedge\mathcal{V}$–submodule of $\bigwedge\mathcal{V}\otimes_{\mathbb{Q}}[\mathcal{V}]$ generated by all the expressions $\{b_m\otimes[\mathbf{b}]_{m-1}-[\mathbf{b}]_m, b_m\otimes[\mathbf{b}]_m\}_{m\in\mathbb{Z}}$. In formulas:

$$W:=\bigwedge\mathcal{V}\big(b_m\otimes[\mathbf{b}]_{m-1}-[\mathbf{b}]_m\big)+\bigwedge\mathcal{V}\big(b_m\otimes[\mathbf{b}]_m\big).$$

where the module structure is given by $\mathbf{u}(\mathbf{v}\otimes[\mathbf{b}]_m)=(\mathbf{u}\wedge\mathbf{v})\otimes[\mathbf{b}]_m$.

1.5 Definition. *The fermionic Fock space is the* $\bigwedge\mathcal{V}$*- module*

$$\mathcal{F}:=\mathcal{F}(\mathcal{V}):=\frac{\bigwedge\mathcal{V}\otimes_{\mathbb{Q}}[\mathcal{V}]}{W}. \tag{9}$$

Let $\bigwedge\mathcal{V}\otimes_{\mathbb{Q}}[\mathcal{V}]\to\mathcal{F}$ be the canonical projection. The class of $\mathbf{u}\otimes[\mathbf{b}]_m$ in $\mathcal{F}$ will be denoted $\mathbf{u}\wedge[\mathbf{b}]_m$. Thus the equalities $b_m\wedge[\mathbf{b}]_m=0$ and $b_m\wedge[\mathbf{b}]_{m-1}=[\mathbf{b}]_m$ hold in $\mathcal{F}$. For all $m\in\mathbb{Z}$ and $\boldsymbol{\lambda}\in\mathcal{P}$ let, by definition

$$[\mathbf{b}]_{m+\boldsymbol{\lambda}}:=\mathbf{b}^r_{m+\boldsymbol{\lambda}}\wedge[\mathbf{b}]_{m-r}=b_{m+\lambda_1}\wedge b_{m-1+\lambda_2}\wedge\cdots\wedge b_{m-r+1+\lambda_r}\wedge[\mathbf{b}]_{m-r}$$

where r is any positive integer such that $\ell(\boldsymbol{\lambda})\leq r$, which implicitly defines $\mathbf{b}^r_{m+\boldsymbol{\lambda}}$ as an element of $\bigwedge^r\mathcal{V}_{\geq m-r+1}$, where by $\mathcal{V}_{\geq j}$ we understand $\bigoplus_{i\geq j}\mathbb{Q}\cdot b_i$. It turns out that $\mathcal{F}$ is a graded $\bigwedge\mathcal{V}$-module:

$$\mathcal{F}:=\bigoplus_{m\in\mathbb{Z}}\mathcal{F}_m,$$

where

$$\mathcal{F}_m:=\bigoplus_{\boldsymbol{\lambda}\in\mathcal{P}}\mathbb{Q}[\mathbf{b}]_{m+\boldsymbol{\lambda}}=\bigoplus_{r\geq 0}\bigoplus_{\boldsymbol{\lambda}\in\mathcal{P}_r}\mathbb{Q}\mathbf{b}^r_{m+\boldsymbol{\lambda}}\wedge[\mathbf{b}]_{m-r}, \tag{10}$$

is the *fermionic Fock space* of charge m [14, p. 36].

1.6 Proposition.

i) The equality $b_j\wedge[\mathbf{b}]_m=0$ holds for all $j\leq m$;

ii) The image of the map $\bigwedge^r\mathcal{V}\otimes\mathcal{F}_m\to\mathcal{F}$ given by $(\mathbf{u},\mathbf{v})\mapsto u\wedge v$ is contained in $\mathcal{F}_{m+r}$.

Proof. They are [10, Proposition 4.4 and 4.5]. ∎

1.7 Extending Schubert derivations to $\mathcal{F}$. We now extend the Schubert derivations, in principle only defined on $\bigwedge\mathcal{V}$, on $\mathcal{F}$ according to [10] to which we refer to for more details. First we define their action on elements of the form $[\mathbf{b}]_m$ by setting:

$$\overline{\sigma}_-(z)[\mathbf{b}]_m=\sigma_-(z)[\mathbf{b}]_m:=[\mathbf{b}]_m,\qquad \sigma_+(z)[\mathbf{b}]_m:=\sigma_+(z)b_m\wedge[\mathbf{b}]_{m-1}$$

and

$$\overline{\sigma}_+(z)[\mathbf{b}]_m:=\sum_{j\geq 0}[\mathbf{b}]_{m+(1^j)}z^j$$

where (1^j) denotes the partition with j parts equal to 1. Finally, we set

$$\sigma_\pm(z)[\mathbf{b}]_{m+\boldsymbol{\lambda}} = \sigma_\pm(z)\mathbf{b}^r_{m+\boldsymbol{\lambda}} \wedge \sigma_\pm(z)[\mathbf{b}]_{m-r} \quad \text{and} \quad \overline{\sigma}_\pm(z)[\mathbf{b}]_{m+\boldsymbol{\lambda}} = \overline{\sigma}_\pm(z)\mathbf{b}^r_{m+\boldsymbol{\lambda}} \wedge \overline{\sigma}_\pm(z)[\mathbf{b}]_{m-r}. \tag{11}$$

1.8 Proposition. *For all $m \in \mathbb{Z}$, Giambelli's formula for the Schubert derivation $\overline{\sigma}_+(z)$ holds:*

$$[\mathbf{b}]_{m+\boldsymbol{\lambda}} = \det(\sigma_{\lambda_j - j + i})[\mathbf{b}]_m \tag{12}$$

Proof. See [10, Proposition 5.13]. ∎

We introduce now an operator on $\mathcal{F}$ which, in a sense, plays the role of the determinant of the shift endomorphism σ_1. We denote it by ξ. We shall understand it as the unique algebra endomorphism of $\bigwedge \mathcal{V}$ such that $\xi \cdot b_j = b_{j+1}$. Being an algebra homomorphism implies that

$$\xi \mathbf{b}_{m+\boldsymbol{\lambda}} = \mathbf{b}_{m+1+\boldsymbol{\lambda}}$$

It is clearly invertible. Its inverse ξ^{-1} is such that $\xi^{-1} b_j = b_{j-1}$. Secondly, we extend it to $\mathcal{F}$ as follows:

$$\xi[\mathbf{b}]_{m+\boldsymbol{\lambda}} = \xi(\mathbf{b}^r_{m+\boldsymbol{\lambda}}) \wedge [\mathbf{b}]_{m+1+\boldsymbol{\lambda}}, \tag{13}$$

where r is any integer greater than the length of the partition $\boldsymbol{\lambda}$. It is trivial to check that such a definition does not depend on the choice of $r > \ell(\boldsymbol{\lambda})$. So for instance

$$\xi^{m'}[\mathbf{b}]_{m+\boldsymbol{\lambda}} = [\mathbf{b}]_{m+m'+\boldsymbol{\lambda}}.$$

1.9 Bosonic Fock space. Let $B := \mathbb{Q}[\mathbf{x}]$, the polynomial ring in infinitely many indeterminates $\mathbf{x} := (x_1, x_2, \ldots)$. As a $\mathbb{Q}$–vector space it possesses a basis of Schur polynomials parametrised by the set $\mathcal{P}$ of all partitions. Moreover, $(S_1(\mathbf{x}), S_2(\mathbf{x}), \ldots)$ generate B as a $\mathbb{Q}$-algebra, because $S_i(\mathbf{x})$ is a polynomial of degree i, for all $i \geq 0$. If $\boldsymbol{\lambda} \in \mathcal{P}$ one sets

$$S_{\boldsymbol{\lambda}}(\mathbf{x}) = \det(S_{\lambda_j - j + i}(\mathbf{x})) \tag{14}$$

where the sequence $(S_1(\mathbf{x}), S_2(\mathbf{x}), \ldots)$ is defined by

$$\sum_{j \in \mathbb{Z}} S_j(\mathbf{x}) z^j = \exp(\sum_{i \geq 1} x_i z^i). \tag{15}$$

Let $B(\xi) := B \otimes_{\mathbb{Q}} \mathbb{Q}[\xi^{-1}, \xi]$ be the $\mathbb{Q}[\xi]$-algebra of B-valued Laurent polynomials in ξ. We shall refer to $B(\xi)$ as the *bosonic Fock space.* It follows that

$$B(\xi) = \bigoplus_{\substack{m \in \mathbb{Z},\\ \boldsymbol{\lambda} \in \mathcal{P}}} \mathbb{Q} \cdot \xi^m S_{\boldsymbol{\lambda}}(\mathbf{x})$$

1.10 The space $\mathcal{F}$ can be endowed with a structure of free $B(\xi)$-module generated by $[\mathbf{b}]_0$ of rank one generated by $[\mathbf{b}]_0$ such that $\xi^m S_{\boldsymbol{\lambda}}(\mathbf{x})[\mathbf{b}]_0 = [\mathbf{b}]_{\boldsymbol{\lambda}}$, by simply declaring

$$\xi^m S_i(\mathbf{x})[\mathbf{b}]_{\boldsymbol{\lambda}} \;:=\; \sigma_i[\mathbf{b}]_{m+\boldsymbol{\lambda}}. \tag{16}$$

In fact

$$\begin{aligned}
[\mathbf{b}]_{m+\boldsymbol{\lambda}} &= \xi^m[\mathbf{b}]_{\boldsymbol{\lambda}} && \text{(Equation (13))}\\
&= \xi^m \det(\sigma_{\lambda_j-j+i})[\mathbf{b}]_0 && \text{(Giambelli's formula for Schubert derivations)}\\
&= \xi^m \det(S_{\lambda_j-j+i})[\mathbf{b}]_0 && \text{(by equality (16))}\\
&= \xi^m S_{\boldsymbol{\lambda}}(\mathbf{x})[\mathbf{b}]_0 && \text{(Definition of } S_{\boldsymbol{\lambda}}(\mathbf{x})\text{)).}
\end{aligned}$$

Equality (16) can be also phrased by saying that $S_i(\mathbf{x})$ is an eigenvalue of the $\mathbb{Q}(\xi)$-linear map $\sigma_i : \mathcal{F} \to \mathcal{F}$ with $\mathcal{F}_m$ as eigenspaces. It implies that

$$\sigma_+(z)[\mathbf{b}]_{m+\boldsymbol{\lambda}} = \exp\left(\sum_{i\geq 1} x_i z^i\right)[\mathbf{b}]_{m+\boldsymbol{\lambda}}, \tag{17}$$

i.e., abusing terminology, $\exp(\sum_{i\geq 1} x_i z^i)$ is an eigenvalue of $\sigma_+(z)$.

1.11 Lemma.

i) The Schubert derivations $\sigma_\pm(z), \overline{\sigma}_\pm(z)$ commute with multiplication by ξ, i.e.

$$\xi\sigma_\pm(z) = \sigma_\pm(z)\xi \qquad \text{and} \qquad \xi\overline{\sigma}_\pm(z) = \overline{\sigma}_\pm(z)\xi; \tag{18}$$

ii) by regarding the Schubert derivation $\sigma_-(z)$ (resp. $\overline{\sigma}_-(z)$) as a map $B \to B[z^{-1}]$ by setting $(\sigma_-(z)S_{\boldsymbol{\lambda}}(\mathbf{x}))[\mathbf{b}]_m = \sigma_-(z)[\mathbf{b}]_{m+\boldsymbol{\lambda}}$ (resp. $(\overline{\sigma}_-(z)S_{\boldsymbol{\lambda}}(\mathbf{x}))[\mathbf{b}]_m = \overline{\sigma}_-(z)[\mathbf{b}]_{m+\boldsymbol{\lambda}}$, one has:

$$\overline{\sigma}_-(z)S_i(\mathbf{x}) = S_i(\mathbf{x}) - \frac{S_{i-1}(\mathbf{x})}{z} \tag{19}$$

$$\sigma_-(z)S_i(\mathbf{x}) = \sum_{j=0}^{i} \frac{S_{i-j}(\mathbf{x})}{z^j}; \tag{20}$$

iii) the maps $\sigma_-(z)$ and $\overline{\sigma}_-(z)$ are $\mathbb{Q}(\xi)$-algebra endomorphism of $B(\xi)$. In particular

$$\sigma_-(z)S_{\boldsymbol{\lambda}}(\mathbf{x}) = \det(\sigma_-(z)S_{\lambda_j-j+i}(\mathbf{x})) \tag{21}$$

and

$$\overline{\sigma}_-(z)S_{\boldsymbol{\lambda}}(\mathbf{x}) = \det(\overline{\sigma}_-(z)S_{\lambda_j-j+i}(\mathbf{x})); \tag{22}$$

iv) the maps $\sigma_-(z)$ and $\overline{\sigma}_-(z)$ act on B as exponential of a first order differential operators, namely:

$$\sigma_-(z)S_{\boldsymbol{\lambda}}(\mathbf{x}) = \exp\left(\sum_{n\geq 1} \frac{1}{nz^n}\frac{\partial}{\partial x_n}\right) S_{\boldsymbol{\lambda}}(\mathbf{x}) \tag{23}$$

and

$$\overline{\sigma}_-(z)S_{\boldsymbol{\lambda}}(\mathbf{x}) = \exp\left(-\sum_{n\geq 1} \frac{1}{nz^n}\frac{\partial}{\partial x_n}\right) S_{\boldsymbol{\lambda}}(\mathbf{x}). \tag{24}$$

Proof. i) First we show that the commutation holds on the exterior algebra $\bigwedge \mathcal{V}$. This is nearly obvious, because

$$\sigma_\pm(z)\xi b_j = \sigma_\pm(z) b_{j+1} = \sum_{i\geq 0} b_{j+1\pm i} z^{\pm i} = \xi \sum_{i\geq 0} b_{j\pm i} z^{\pm i} = \xi \sigma_\pm(z) b_j$$

The same holds for $\overline{\sigma}_\pm(z)$. We have

$$\overline{\sigma}_\pm(z)\xi b_j = \overline{\sigma}_\pm(z) b_{j+1} = b_{j+1} - b_{j+1\pm 1} z^{\pm 1} = \xi(b_j - b_{j\pm 1} z^{\pm 1}) = \xi\, \overline{\sigma}_\pm(z) b_j.$$

Secondly, the commutation rules hold for elements of the form $[\mathbf{b}]_m$. In fact:

$$\begin{aligned}
\sigma_-(z)\xi[\mathbf{b}]_m &= \sigma_-(z)[\mathbf{b}]_{m+1} && \text{(Definition of } \xi\text{)}\\
&= [\mathbf{b}]_{m+1} && (\sigma_-(z) \text{ acts as the identity})\\
&= \xi[\mathbf{b}]_m = \xi\sigma_-(z)[\mathbf{b}]_m && \text{(Definition of } \xi \text{ and } \sigma_-(z) \text{ acts as the identity on } [\mathbf{b}]_m)
\end{aligned}$$

Similarly one sees that $\overline{\sigma}_-(z)\xi = \xi\overline{\sigma}_-(z)$. The check for $\sigma_+(z)$ and $\overline{\sigma}_+(z)$ works analogously as follows.

$$\begin{aligned}
\sigma_+(z)\xi[\mathbf{b}]_m &= \sigma_+(z)[\mathbf{b}]_{m+1} && \text{(Definition of } \xi\text{)}\\
&= \sigma_+(z) b_{m+1} \wedge [\mathbf{b}]_m && \text{(Definition of } \sigma_+(z)[\mathbf{b}]_m\text{)}\\
&= \textstyle\sum_{i\geq 0} b_{m+1+i} z^i \wedge [\mathbf{b}]_m && \text{(Definition of } \sigma_+(z) b_m\text{)}\\
&= \textstyle\sum_{i\geq 0} \xi b_{m+i} \wedge \xi[\mathbf{b}]_{m-1} = \xi\sigma_+(z)[\mathbf{b}]_m
\end{aligned}$$

and

$$\begin{aligned}
\overline{\sigma}_+(z)\xi[\mathbf{b}]_m &= \overline{\sigma}_+(z)[\mathbf{b}]_{m+1} && \text{(Definition of } \xi\text{)}\\
&= \textstyle\sum_{j\geq 0}(-1)^j b_{m+1+(1^j)} \wedge [\mathbf{b}]_{m-j} z^j && \text{(Definition of } \overline{\sigma}_+(z)[\mathbf{b}]_{m+1}\text{)}\\
&= \textstyle\sum_{j\geq 0}(-1)^j \xi b_{m+(1^j)} \wedge \xi[\mathbf{b}]_{m-1-j} z^j && \text{(Definition of multiplying by } \xi\text{)}\\
&= \xi \textstyle\sum_{j\geq 0}(-1)^j b_{m+(1^j)} \wedge [\mathbf{b}]_{m-1-j} z^j = \xi\overline{\sigma}_+(z)[\mathbf{b}]_m
\end{aligned}$$

Let us show now that (18) holds when evaluated against a general element of $\mathcal{F}$. We check for $\sigma_+(z)$, the others being analogous and even easier. Let $\boldsymbol{\lambda}$ be any partition and r any integer such that $\ell(\boldsymbol{\lambda}) < r$. Then:

$$
\begin{aligned}
\sigma_\pm(z)(\xi[\mathbf{b}]_{m+\boldsymbol{\lambda}}) &= \sigma_\pm(z)[\mathbf{b}]_{m+1+\boldsymbol{\lambda}} && \text{(definition of multiplication by } \xi) \\
&= \sigma_\pm(z)(\mathbf{b}^r_{m+1+\boldsymbol{\lambda}} \wedge [\mathbf{b}]_{m+1-r}) && \text{(decomposition of } [\mathbf{b}]_{m+1+\boldsymbol{\lambda}}) \\
&= \sigma_\pm(z)\mathbf{b}^r_{m+1+\boldsymbol{\lambda}} \wedge \sigma_\pm(z)[\mathbf{b}]_{m+1-r} && (\sigma_\pm(z) \text{ is a derivation}) \\
&= \sigma_\pm(z)\xi\mathbf{b}^r_{m+\boldsymbol{\lambda}} \wedge \sigma_\pm(z)\xi[\mathbf{b}]_{m-r} && \text{(definition of multiplication by } \xi) \\
&= \xi\sigma_\pm(z)\mathbf{b}^r_{m+\boldsymbol{\lambda}} \wedge \xi\sigma_\pm(z)[\mathbf{b}]_{m-r} && \text{(Lemma 1.11, item i))} \\
&= \xi\sigma_\pm(z)[\mathbf{b}]_{m+\boldsymbol{\lambda}}.
\end{aligned}
$$

The proof for the Schubert derivations $\sigma_-(z)$ and $\overline{\sigma}_\pm(z)$ works the same.

ii) The proof of this second statement works verbatim as in [8, Proposition 5.3], where the $S_i(\mathbf{x})$ are denoted by h_i;

iii) In this case the check follows by combining [8, Proposition 7.1] and [8, Corollary 7.3];

iv) Recall that $B(\xi) = \mathbb{Q}(\xi)[S_1(\mathbf{x}), S_2(\mathbf{x}), \ldots]$. Equation (15) implies that

$$\frac{\partial S_i(\mathbf{x})}{\partial x_j} = S_{i-j}(\mathbf{x}),$$

Then (19), e.g., says that

$$\overline{\sigma}_-(z)S_i(\mathbf{x}) = \left(1 - \frac{1}{z}\frac{\partial}{\partial x_1}\right) S_i(\mathbf{x}) = \exp\left(-\sum_{n\geq 1} \frac{1}{nz^n}\frac{\partial^n}{\partial x_1^n}\right) S_i(\mathbf{x}) \tag{25}$$

Now $\dfrac{\partial^n}{\partial x_1^n}S_i(\mathbf{x}) = \dfrac{\partial}{\partial x_n}S_i(\mathbf{x})$. Since $S_i(\mathbf{x})$ generate B as a $\mathbb{Q}$-algebra and $\overline{\sigma}_-(z)$ are algebra homomorphisms coinciding on generators, (24) follows. The proof of (23) is analogous, but it also follows from inverting both members of the equality (24), obtaining

$$\sigma_-(z) = \exp\left(\sum_{n\geq 1} \frac{1}{nz^n}\frac{\partial}{\partial x_n}\right). \qquad \blacksquare$$

1.12 In the sequel we will need the following observation. Suppose that ϕ is anyone among the endomorphism $\sigma_{\pm i}$ of $\overline{\sigma}_{\pm j}$, for i and j arbitrary non negative integers. Suppose further that

$$\phi[\mathbf{b}]_{m+\boldsymbol{\lambda}} = \sum_{\boldsymbol{\mu}} a_{\boldsymbol{\mu}}[\mathbf{b}]_{m+\boldsymbol{\mu}}.$$

Then, for any $m' \in \mathbb{Z}$,

$$\sum_{\boldsymbol{\mu}} a_{\boldsymbol{\mu}}[\mathbf{b}]_{m+m'+\boldsymbol{\mu}} = \phi[\mathbf{b}]_{m+m'+\boldsymbol{\lambda}}.$$

The proof is based on the definition of multiplication by ξ.

$$
\begin{aligned}
\sum_{\boldsymbol{\mu}} a_{\boldsymbol{\mu}}[\mathbf{b}]_{m+m'+\boldsymbol{\mu}} &= \sum_{\boldsymbol{\mu}} a_{\boldsymbol{\mu}}\xi^{m'}[\mathbf{b}]_{m+\boldsymbol{\mu}} = \xi^{m'}\sum_{\boldsymbol{\mu}} a_{\boldsymbol{\mu}}[\mathbf{b}]_{m+\boldsymbol{\mu}} \\
&= \xi^{m'}\phi[\mathbf{b}]_{m+\boldsymbol{\lambda}} = \phi\xi^{m'}[\mathbf{b}]_{m+\boldsymbol{\lambda}} = \phi[\mathbf{b}]_{m+m'+\boldsymbol{\lambda}}.
\end{aligned}
$$

2 The generating functions of the bases of $\bigwedge^k \mathcal{V}$ and $\bigwedge^l \mathcal{V}^*$

Let $\bigwedge \mathcal{V} = \bigoplus_{k\geq 0}\bigwedge^k \mathcal{V}$ and $\bigwedge \mathcal{V}^* = \bigoplus_{l\geq 0}\bigwedge^l \mathcal{V}^*$ be the exterior algebra of $\mathcal{V}$ and $\mathcal{V}^*$ respectively. To describe the bases of $\bigwedge^k \mathcal{V}$ and $\bigwedge^l \mathcal{V}^*$ induced by the basis $\mathbf{b}$ of $\mathcal{V}$ and of $\boldsymbol{\beta}$ of $\mathcal{V}^*$ (Cf. Section 1.1), we need to explain what we shall mean by *bilateral partition*.

2.1 Definition. *A* bilateral partition *of length at most $r \geq 1$ is an element of the set:*

$$\overline{\mathcal{P}}_r := \{\boldsymbol{\lambda} := (\lambda_1, \lambda_2, \dots, \lambda_r) \in \mathbb{Z}^r \mid \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r\}\,.$$

Clearly, $\mathcal{P}_r := \overline{\mathcal{P}}_r \cap \mathbb{N}^r$ is the set of the usual partitions of length at most r, namely the non–increasing sequences of non–negative integers with at most r non zero parts. If $i_1 > \cdots > i_k$ is a decreasing sequence of integers, there exists one and only one bilateral partition $\boldsymbol{\mu} \in \overline{\mathcal{P}}_k$ such that $i_j = k - j + \mu_j$. Therefore $([\mathbf{b}]^k_{\boldsymbol{\mu}})_{\boldsymbol{\mu}\in\overline{\mathcal{P}}_k}$ and $([\boldsymbol{\beta}]^l_{\boldsymbol{\nu}})_{\boldsymbol{\nu}\in\overline{\mathcal{P}}_l}$ where:

$$[\mathbf{b}]^k_{\boldsymbol{\mu}} = b_{k-1+\mu_1} \wedge \ldots \wedge b_{\mu_k} \qquad \text{and} \qquad [\boldsymbol{\beta}]^l_{\boldsymbol{\nu}} = \beta_{l-1+\nu_1} \wedge \ldots \wedge \beta_{\nu_l},$$

are $\mathbb{Q}$-bases of $\bigwedge^k \mathcal{V}$ and $\bigwedge^l \mathcal{V}^*$ respectively. Let $\mathbf{z}_k := (z_1, \dots, z_k)$ and $\mathbf{w}_k^{-1} := (w_1^{-1}, \dots, w_k^{-1})$ be two ordered finite sequences of formal variables. The $\bigwedge^k \mathcal{V}$-valued formal power series

$$\mathbf{b}(z_k) \wedge \cdots \wedge \mathbf{b}(z_1)$$

vanishes whenever $z_i = z_j$, for all $1 \leq i < j \leq k$. Therefore it is divisible by the Vandermonde determinant $\Delta_0(\mathbf{z}_k) = \prod_{1<\leq i<j\leq k}(z_j - z_i)$. We then define, for all $\boldsymbol{\lambda} \in \overline{\mathcal{P}}_k$, the extended Schur polynomial

$$\mathbf{s}_{\boldsymbol{\lambda}}(\mathbf{z}_k)$$

through the equality

$$\sum_{\boldsymbol{\mu}\in\overline{\mathcal{P}}} [\mathbf{b}]^k_{\boldsymbol{\mu}} \mathbf{s}_{\boldsymbol{\mu}}(\mathbf{z}_k)\Delta_0(\mathbf{z}_k) := \mathbf{b}(z_k) \wedge \cdots \wedge \mathbf{b}(z_1), \tag{26}$$

and therefore the expression

$$[\mathbf{b}]^k(\mathbf{z}_k) := \sum_{\boldsymbol{\mu}\in\overline{\mathcal{P}}_k} [\mathbf{b}]^k_{\boldsymbol{\mu}} \mathbf{s}_{\boldsymbol{\mu}}(\mathbf{z}_k) \tag{27}$$

is a generating function of the basis elements of $\bigwedge^k \mathcal{V}$ induced by the given basis $\mathbf{b}$ of $\mathcal{V}$. Similarly, a generating function for the basis elements $([\boldsymbol{\beta}]^l_{\boldsymbol{\nu}})_{\boldsymbol{\nu}\in\overline{\mathcal{P}}_l}$ is given by

$$[\boldsymbol{\beta}]^l(\mathbf{w}_l^{-1}) := \sum_{\boldsymbol{\nu}\in\overline{\mathcal{P}}_{\boldsymbol{\nu}}} [\boldsymbol{\beta}]^l_{\boldsymbol{\nu}} \cdot \mathbf{s}_{\boldsymbol{\nu}}(\mathbf{w}_l^{-1}), \tag{28}$$

where $\mathbf{s}_{\boldsymbol{\nu}}(\mathbf{w}_l^{-1})$ is now defined, for all $\boldsymbol{\nu} \in \overline{\mathcal{P}}_l$, via the equality

$$\sum_{\boldsymbol{\nu}\in\overline{\mathcal{P}}} [\boldsymbol{\beta}]^l_{\boldsymbol{\nu}} \mathbf{s}_{\boldsymbol{\nu}}(\mathbf{w}_l^{-1})\Delta_0(\mathbf{w}_l^{-1}) := \boldsymbol{\beta}(w_1^{-1}) \wedge \cdots \wedge \boldsymbol{\beta}(w_l^{-1}), \tag{29}$$

where

$$\Delta_0(\mathbf{w}_l^{-1}) = \prod_{1<\leq i<j\leq l} (w_j^{-1} - w_i^{-1}) = \frac{\prod_{1\leq i<j\leq l}(w_i - w_j)}{\prod_{i=1}^{l} w_i^{l-1}}. \tag{30}$$

Notice the different numbering adopted for the variables $\mathbf{z}$ (formula (26)) and the variables $\mathbf{w}^{-1}$ (formula (29))

2.2 Remark. If $\boldsymbol{\lambda} \subseteq \mathbb{N}^k$, then $\mathbf{s}_{\boldsymbol{\lambda}}(\mathbf{z}_k)$ is the usual Schur symmetric polynomial in $(z_1, z_2, \ldots, z_k)$. If $\boldsymbol{\lambda} := (\lambda_1, \lambda_2, \ldots, \lambda_k) \in \overline{\mathcal{P}}_k$, with all $\lambda_i < 0$, then

$$\mathbf{s}_{\boldsymbol{\lambda}}(\mathbf{z}_k) = \frac{\mathbf{s}_{-\boldsymbol{\lambda}}(\mathbf{z}_k^{-1})}{z_1^{k-1} \cdots z_k^{k-1}}. \tag{31}$$

where $-\boldsymbol{\lambda} = (-\lambda_k, -\lambda_{k-1}, \ldots, -\lambda_1)$. If $\lambda_1 > 0$ and $\lambda_k < 0$, instead

$$\mathbf{s}_{\boldsymbol{\lambda}_k}(\mathbf{z}_k) = \frac{\mathbf{s}_{(\lambda_1+\lambda_k, \cdots, \lambda_{k-1}+\lambda_k, 0)}(\mathbf{z}_k)}{\prod_{j=0}^{k} z_j^{\lambda_k}}. \tag{32}$$

It is then clear that all $\mathbf{s}_{\boldsymbol{\lambda}}(\mathbf{z})$, where $\boldsymbol{\lambda}$ runs on $\mathcal{P}_k$, are $\mathbb{Q}$-linearly independent. The same holds true for $\Delta_0(\mathbf{w}_l^{-1})$.

2.3 Let $\beta \in \mathcal{V}^*$. The contraction $\beta \lrcorner : \bigwedge \mathcal{V} \to \bigwedge \mathcal{V}$ can be depicted via the following diagram:

$$\begin{vmatrix} \beta(b_{r-1+\lambda_1}) & \beta(b_{r-2+\lambda_2}) & \ldots & \beta(b_{\lambda_r}) \\ \\ b_{r-1+\lambda_1} & b_{r-2+\lambda_2} & \ldots & b_{\lambda_r} \end{vmatrix} \tag{33}$$

to be read as follows. The scalar $\beta(b_{r-j+\lambda_j})$ is the coefficient of the element of $\bigwedge^{r-1} \mathcal{V}$ obtained by removing the j-th exterior factor from $[\mathbf{b}]^r_{\boldsymbol{\lambda}}$.

The contraction of $\bigwedge^r \mathcal{V}$ against $[\boldsymbol{\beta}]^l_{\boldsymbol{\nu}} \in \bigwedge^l \mathcal{V}^*$ is well defined as well. It is an element of $\bigwedge^{r-l} \mathcal{V}$ which can be represented as (See [1]):

$$[\boldsymbol{\beta}]^l_{\boldsymbol{\nu}} \lrcorner [\mathbf{b}]^r_{\boldsymbol{\lambda}} = \begin{vmatrix} \beta_{l-1+\nu_1}(b_{r-1+\lambda_1}) & \ldots & \beta_{l-1+\nu_1}(b_{\lambda_r}) \\ \vdots & \ddots & \vdots \\ \beta_{\nu_l}(b_{r-1+\lambda_1}) & \ldots & \beta_{\nu_l}(b_{\lambda_r}) \\ \\ b_{r-1+\lambda_1} & \ldots & b_{\lambda_r} \end{vmatrix} \tag{34}$$

to be read as follows. The Laplace-like expansion of the array (34) along the first row is an alternating linear combination of contractions of elements of $\bigwedge^{k-1} \mathcal{V}$ against elements of $\bigwedge^{l-1} \mathcal{V}^*$. Having already set the case $k = 1$ in (33), we have described it completely.

2.4 Although it may be easily guessed, let us now make precise the definition of the contraction of an element of $\mathcal{F}$ against an element of $\bigwedge^l \mathcal{V}^*$. Giving the definition on bases elements $[\mathbf{b}]_{m+\boldsymbol{\lambda}}$ of $\mathcal{F}$ and $[\boldsymbol{\beta}]^l_{\boldsymbol{\nu}}$ ($\boldsymbol{\nu} := (\nu_1 \geq \ldots \geq \nu_l)$ of $\bigwedge^l \mathcal{V}^*$ will suffice. Let $r \geq 0$ such that $\ell(\boldsymbol{\lambda}) \leq r$ and $\nu_l \geq m - r$ and define:

$$[\boldsymbol{\beta}]^l_{\boldsymbol{\nu}} \lrcorner [\mathbf{b}]_{m+\boldsymbol{\lambda}} := ([\boldsymbol{\beta}]^l_{\boldsymbol{\nu}} \lrcorner [\mathbf{b}]^r_{m+\boldsymbol{\lambda}}) \wedge [\mathbf{b}]_{m-r}.$$

It is straightforward to see that the definition does not depend on the choice of the non-negative integer $r > \ell(\boldsymbol{\lambda})$.

2.5 Let

$$\mathcal{E}(\mathbf{z}_k, \mathbf{w}_l^{-1}) = [\mathbf{b}]^k(\mathbf{z}_k) \otimes [\boldsymbol{\beta}]^l(\mathbf{w}_l^{-1}) = \sum_{\boldsymbol{\mu}, \boldsymbol{\nu}} [\mathbf{b}]^k_{\boldsymbol{\mu}} \otimes [\boldsymbol{\beta}]^l_{\boldsymbol{\nu}} \mathbf{s}_{\boldsymbol{\mu}}(\mathbf{z}_k) \mathbf{s}_{\boldsymbol{\nu}}(\mathbf{w}_l^{-1}), \tag{35}$$

be the generating function of the basis of $\bigwedge^k \mathcal{V} \otimes \bigwedge^l \mathcal{V}^*$. It defines two maps

$$\mathcal{E}_f(\mathbf{z}_k, \mathbf{w}_l^{-1}) : \mathcal{F} \to \mathcal{F}[\![\mathbf{z}_k, \mathbf{w}_l, \mathbf{z}_k^{-1}, \mathbf{w}_l^{-1}]\tag{36}$$

and

$$\mathcal{E}_b(\mathbf{z}_k, \mathbf{w}_l^{-1}) := B(\xi) \to B(\xi)[\![\mathbf{z}_k, \mathbf{w}_l, \mathbf{z}_k^{-1}, \mathbf{w}_l^{-1}]\tag{37}$$

which we distinguish by putting a subscript in the notation and satisfying the compatibility relation imposed by the boson-fermion correspondence. More precisely we define:

$$\mathcal{E}_f(\mathbf{z}_k, \mathbf{w}_l^{-1})[\mathbf{b}]_{m+\boldsymbol{\lambda}} := [\mathbf{b}]^k(\mathbf{z}_k) \wedge [\boldsymbol{\beta}]^l(\mathbf{w}_l^{-1}) \lrcorner [\mathbf{b}]_{m+\boldsymbol{\lambda}}\tag{38}$$

and

$$\big(\mathcal{E}_b(\mathbf{z}_k, \mathbf{w}_l^{-1})\xi^m S_{\boldsymbol{\lambda}}(\mathbf{x})\big)[\mathbf{b}]_0 = \mathcal{E}_f(\mathbf{z}_k, \mathbf{w}_l^{-1})[\mathbf{b}]_{m+\boldsymbol{\lambda}}\tag{39}$$

where we have used the notation of (27) and (29).

2.6 Products of Schubert derivations. To further elaborate the shape of (38) and (39), we need to introduce the following new piece of notation. Let

$$\sigma_+(\mathbf{z}_k) = \sigma_+(z_1)\cdots\sigma_+(z_k), \qquad \overline{\sigma}_+(\mathbf{z}_k) = \overline{\sigma}_+(z_1)\cdots\overline{\sigma}_+(z_k),\tag{40}$$

and

$$\sigma_-(\mathbf{w}_l) = \sigma_-(w_1)\cdots\sigma_-(w_l), \qquad \overline{\sigma}_-(\mathbf{z}_l) = \overline{\sigma}_-(w_1)\cdots\overline{\sigma}_-(w_l).\tag{41}$$

Equalities (40) and (41) must be read in $\mathrm{End}_{\mathbb{Q}}(\bigwedge \mathcal{V})[\![\mathbf{z}_k]\!]$ and $\mathrm{End}_{\mathbb{Q}}(\bigwedge \mathcal{V})[\![\mathbf{w}_l^{-1}]\!]$ respectively. They are *multivariate HS-derivations* of $\bigwedge \mathcal{V}$ in the following sense: i) they are *multi-variate* because are $\mathrm{End}_{\mathbb{Q}}(\bigwedge \mathcal{V})$ formal power series in more than one indeterminate, namely $\mathbf{z}_k := (z_1, \ldots, z_k)$ and $\mathbf{w}_l^{-1} := (w_1^{-1}, \ldots, w_l^{-1})$, and ii) are *HS derivations*, being compatible with the wedge product:

$$\sigma_\pm(\mathbf{z}_k)(\mathbf{u} \wedge \mathbf{v}) = \sigma_\pm(\mathbf{z}_k)\mathbf{u} \wedge \sigma_\pm(\mathbf{z}_k)\mathbf{v} \qquad \text{and} \qquad \overline{\sigma}_\pm(\mathbf{z}_k)(\mathbf{u} \wedge \mathbf{v}) = \overline{\sigma}_\pm(\mathbf{z}_k)\mathbf{u} \wedge \overline{\sigma}_\pm(\mathbf{z}_k)\mathbf{v}.$$

2.7 Lemma. *The following commutation rule holds:*

$$\sigma_-(w_1)\overline{\sigma}_+(w_2) = \left(1 - \frac{w_2}{w_1}\right)\overline{\sigma}_+(w_2)\sigma_-(w_1),\tag{42}$$

in $\mathrm{End}_{\mathbb{Q}}(\mathcal{F})[w_1^{-1}, w]]$

Proof. First of all we notice that

$$\sigma_-(w_1)\overline{\sigma}_+(w_2)\mathbf{u} = \sigma_+(w_2)\overline{\sigma}_-(w_1)\mathbf{u},\tag{43}$$

for all $\mathbf{u} \in \mathcal{V}$. It is sufficient to check for one basis element. One one hand

$$\sigma_-(w_1)\overline{\sigma}_+(w_2)b_j \quad = \quad \sigma_-(w_1)(b_j - b_{j+1}w_2) = \sum_{i\geq 0} b_{j-i}w_1^{-i} - \sum_{i\geq 0} b_{j+1-i}w_1^{-i}w_2$$

$$= \sum_{i\geq 0} b_{j-i}w_1^{-i} - \sigma_1(\sum_{i\geq 0} b_{j-i}w_1^{-i})w = \overline{\sigma}_+(w_2)\sigma_-(w_1)b_j$$

and (43) is proven. Now we prove that (42) holds for elements of the form $[\mathbf{b}]_m$. In fact

$$\sigma_-(w_1)\overline{\sigma}_+(w_2)[\mathbf{b}]_m = \sigma_-(w_1)\big([\mathbf{b}]_m - (b_{m+1}\wedge[\mathbf{b}]_{m-1})w_2 + (b_{m+1}\wedge b_m\wedge[\mathbf{b}]_{m-2})w_2^2 + \cdots\big)$$

$$= [\mathbf{b}]_m - \left(b_{m+1} - \frac{b_m}{w_1}\right)\wedge[\mathbf{b}]_{m-1}w_2 + b_{m+1}\wedge\left(b_m - \frac{b_{m-1}}{w_1}\right)\wedge[\mathbf{b}]_{m-2})w_2^2 + \cdots$$

$$= \overline{\sigma}_+(w_2)[\mathbf{b}]_m - \frac{w_2}{w_1}\overline{\sigma}_+(w_2)[\mathbf{b}]_m = \left(1-\frac{w_2}{w_1}\right)\overline{\sigma}_+(w_2)[\mathbf{b}]_m = \left(1-\frac{w_2}{w_1}\right)\overline{\sigma}_+(w_2)\sigma_-(w_1)[\mathbf{b}]_m$$

To conclude the proof we must check it on the general basis element $[\mathbf{b}]_{m+\boldsymbol{\lambda}} \in \mathcal{F}$. One has:

$$\begin{aligned}
\sigma_-(w_1)\overline{\sigma}_+(w_2)[\mathbf{b}]_{m+\boldsymbol{\lambda}} &= \sigma_-(w_1)\overline{\sigma}_+(w_2)([\mathbf{b}]^r_{m+\boldsymbol{\lambda}}\wedge[\mathbf{b}]_{m-r})\\
&= \sigma_-(w_1)\overline{\sigma}_+(w_2)[\mathbf{b}]^r_{m+\boldsymbol{\lambda}}\wedge\sigma_-(w_1)\overline{\sigma}_+(w_2)[\mathbf{b}]_{m-r}\\
&= \overline{\sigma}_+(w_1)\sigma_-(w_2)[\mathbf{b}]^r_{m+\boldsymbol{\lambda}}\wedge\left(1-\frac{w_2}{w_1}\right)\overline{\sigma}_+(w_2)\sigma_-(w_1)[\mathbf{b}]_{m-r}\\
&= \left(1-\frac{w}{w_1}\right)\overline{\sigma}_+(w_1)\sigma_-(w_2)([\mathbf{b}]^r_{m+\boldsymbol{\lambda}}\wedge[\mathbf{b}]_{m-r})\\
&= \left(1-\frac{w}{w_1}\right)\overline{\sigma}_+(w_1)\sigma_-(w_2)[\mathbf{b}]_{m+\boldsymbol{\lambda}}
\end{aligned}$$

and the Lemma is proven. ■

2.8 Corollary. *Let* $\mathbf{w}_l\setminus w_1 := (w_2,\ldots,w_l)$ *and* $\mathbf{w}_l^{-1}\setminus w_1^{-1} := (w_2^{-1},\ldots,w_l^{-1})$. *Then*

$$\sigma_-(w_1)\overline{\sigma}_+(\mathbf{w}_l\setminus w_1) = \prod_{j=2}^{l}\left(1-\frac{w_j}{w_1}\right)\cdot\overline{\sigma}_+(\mathbf{w}_l\setminus w_1)\sigma_-(w_1). \tag{44}$$

Proof. Induction on $l\geq 2$. Lemma 2.7 sets the case $l=2$. Suppose that the property holds for all $l-1\geq 2$. Then

$$\begin{aligned}
\sigma_-(w_1)\overline{\sigma}_+(\mathbf{w}_l\backslash w_1) &= \sigma_-(w_1)\overline{\sigma}_+(\mathbf{w}_{l-1}\backslash w_1)\sigma_-(w_l)\\
&= \prod_{j=2}^{l-1}\left(1-\frac{w_j}{w_1}\right)\cdot\overline{\sigma}_+(\mathbf{w}_{l-1}\backslash w_1)\sigma_-(w_1)\overline{\sigma}_+(w_l)\\
&= \prod_{j=2}^{l-1}\left(1-\frac{w_j}{w_1}\right)\overline{\sigma}_+(\mathbf{w}_{l-1}\backslash w_1)\left(1-\frac{w_l}{w_1}\right)\overline{\sigma}_+(w_l)\sigma_-(w_1)
\end{aligned}$$

$$= \prod_{j=2}^{l}\left(1-\frac{w_j}{w_1}\right)\cdot\overline{\sigma}_+(\mathbf{w}_l\backslash w_1)\sigma_-(w_1)$$

because $\overline{\sigma}_+(\mathbf{w}_{l-1}\backslash w_1)\overline{\sigma}_+(w_l)=\overline{\sigma}_+(\mathbf{w}_l\backslash w_1)$, by definition. ∎

2.9 Proposition. *The following equality holds:*

$$\boldsymbol{\beta}(w_1^{-1})\wedge\cdots\wedge\boldsymbol{\beta}(w_l^{-1})\lrcorner[\mathbf{b}]_{m+\boldsymbol{\lambda}}=\frac{\Delta_0(\mathbf{w}_l^{-1})}{\prod_{j=1}^{l}w_j^{m-l+1}}\overline{\sigma}_+(\mathbf{w}_l)\sigma_-(\mathbf{w}_l)[\mathbf{b}]_{m-l+\boldsymbol{\lambda}}. \tag{45}$$

Proof. If $l=1$ formula (46) reads as

$$\beta(w_1^{-1})\lrcorner[\mathbf{b}]_{m+\boldsymbol{\lambda}}=w_1^{-m}\overline{\sigma}_+(w_1)\sigma_-(w_1)[\mathbf{b}]_{m-1+\boldsymbol{\lambda}}$$

and this is precisely [10, Proposition 6.13]. Assume the formula holds for $l-1\geq 0$. For notational simplicity let $\mathbf{w}_l\backslash w_1:=(w_2,\ldots,w_l)$ and $\mathbf{w}_l^{-1}\backslash w_1^{-1}:=(w_2^{-1},\ldots,w_l^{-1})$. Then

$$\begin{aligned}
&\boldsymbol{\beta}(w_1^{-1})\wedge\cdots\wedge\boldsymbol{\beta}(w_l^{-1})\lrcorner[\mathbf{b}]_{m+\boldsymbol{\lambda}}\\
=\;&\boldsymbol{\beta}(w_1^{-1})\lrcorner\left(\boldsymbol{\beta}(w_2^{-1})\cdots\wedge\boldsymbol{\beta}(w_l^{-1})\lrcorner[\mathbf{b}]_{m+\boldsymbol{\lambda}}\right) \qquad\qquad (\text{Associativity of "}\wedge\text{" })\\
=\;&\boldsymbol{\beta}(w_1^{-1})\lrcorner\left(\frac{\Delta_0(\mathbf{w}_l^{-1}\backslash w_1^{-1})}{\prod_{j=2}^{l}w_j^{m-l+2}}\right)\overline{\sigma}_+(\mathbf{w}_l\backslash w_1)\sigma_-(\mathbf{w}_l\backslash w_1)[\mathbf{b}]_{m-l+1+\boldsymbol{\lambda}}\\
=\;&w_1^{-m+l-1}\frac{\Delta_0(\mathbf{w}_l^{-1}\backslash w_1^{-1})}{\prod_{j=2}^{l}w_j^{m-l+2}}\overline{\sigma}_+(w_1)\sigma_-(w_1)\overline{\sigma}_+(\mathbf{w}_k\backslash w_1)\sigma_-(\mathbf{w}_k\backslash w_1)[\mathbf{b}]_{m-k+\boldsymbol{\lambda}}.
\end{aligned}$$

Using the commutation rule (2.8) one then obtains

$$\begin{aligned}
=\;&\frac{w_1^{-m+l-1}}{w_1^{l-1}}(w_1-w_2)\cdots(w_1-w_l)\frac{\Delta_0(\mathbf{w}_l^{-1}\backslash w_1^{-1})}{\prod w_j^{m+l-2}}\cdot\overline{\sigma}_+(\mathbf{w}_l)\sigma_-(\mathbf{w}_l)[\mathbf{b}]_{m-k+\boldsymbol{\lambda}}\\
=\;&\frac{w_1^{-m}}{w_1^{l-1}w_2\cdots w_l}\prod\left(\frac{1}{w_j}-\frac{1}{w_1}\right)\frac{\Delta_0(\mathbf{w}_l^{-1}\backslash w_1^{-1})}{\prod w_j^{m+l-2}}\overline{\sigma}_+(\mathbf{w}_l)\sigma_-(\mathbf{w}_l)[\mathbf{b}]_{m-k+\boldsymbol{\lambda}}\\
=\;&\frac{\Delta_0(\mathbf{w}_l^{-1})}{\prod_{j=1}^{l}w_j^{m-l+1}}\overline{\sigma}_+(\mathbf{w}_l)\sigma_-(\mathbf{w}_l)[\mathbf{b}]_{m-k+\boldsymbol{\lambda}},
\end{aligned}$$

as desired. ∎

2.10 Corollary. *The generating function* (28) *acts on* $\mathcal{F}$ *according to:*

$$\sum_{\boldsymbol{\nu}\in\overline{\mathcal{P}}_l}[\boldsymbol{\beta}]^l_{\boldsymbol{\nu}}\mathbf{s}_{\boldsymbol{\nu}}(\mathbf{w}_l^{-1})\lrcorner[\mathbf{b}]_{m+\boldsymbol{\lambda}}=\prod_{j=1}^{l}w_j^{-m+l-1}\overline{\sigma}_+(\mathbf{w}_l)\sigma_-(\mathbf{w}_l)[\mathbf{b}]_{m-l+\boldsymbol{\lambda}}. \tag{46}$$

Proof. It is a consequence of equality (29) and of Proposition 2.9 up to dividing by the Vandermonde determinant. ∎

2.11 Proposition. *For all $k \geq 1$:*

$$\mathbf{b}(z_k)\wedge\cdots\wedge\mathbf{b}(z_1)\wedge[\mathbf{b}]_{m+\boldsymbol{\lambda}}=\prod_{j=1}^{k} z_j^{m+1}\Delta_0(\mathbf{z}_k)\sigma_+(\mathbf{z}_k)\overline{\sigma}_-(\mathbf{z}_k^{-1})[\mathbf{b}]_{m+k+\boldsymbol{\lambda}}.$$

Proof. By induction on $k \geq 1$. If $k = 1$, the formula reads as

$$\mathbf{b}(z_1)\wedge[\mathbf{b}]_{m+\boldsymbol{\lambda}}=z_1^{m+1}\sigma_+(\mathbf{z}_1)\overline{\sigma}_-(\mathbf{z}_1)[\mathbf{b}]_{m+1+\boldsymbol{\lambda}}$$

and this is Proposition 6.9 in [10]. Assume the formula holds for $k-1 \geq 0$. Then,

$$\begin{aligned}
&\mathbf{b}(z_k)\wedge\cdots\wedge\mathbf{b}(z_1)\wedge[\mathbf{b}]_{m+\boldsymbol{\lambda}}\\
=\;&\mathbf{b}(z_k)\wedge(\mathbf{b}(z_{k-1})\wedge\cdots\wedge\mathbf{b}(z_1)\wedge[\mathbf{b}]_{m+\boldsymbol{\lambda}}) \qquad \text{(Associativity of "}\wedge\text{")}\\
=\;&\mathbf{b}(z_k)\wedge z_{k-1}^{m+1}\cdots z_1^{m+1}\sigma_+(\mathbf{z}_{k-1})\overline{\sigma}_-(\mathbf{z}_{k-1})[\mathbf{b}]_{m+k-1+\boldsymbol{\lambda}}\cdot\Delta_0(\mathbf{z}_{k-1})\\
=\;&z_k^{m+k}z_{k-1}^{m+1}\cdots z_1^{m+1}\Delta_0(\mathbf{z}_{k-1})\sigma_+(\mathbf{z}_k)\overline{\sigma}_-(\mathbf{z}_k)\sigma_+(\mathbf{z}_{k-1})\overline{\sigma}_-(\mathbf{z}_{k-1})[\mathbf{b}]_{m+k+\boldsymbol{\lambda}}\\
=\;&z_k^{m+k+1}\textstyle\prod_{j=1}^{k-1}z_j^{m+1}\prod_{j=1}^{k-1}\displaystyle\left(1-\frac{z_j}{z_k}\right)\Delta_0(\mathbf{z}_{k-1})\cdot\\
&\cdot\,\sigma_+(z_1)\sigma_+(\mathbf{z}_{k-1})\overline{\sigma}_-(z_1)\overline{\sigma}_-(\mathbf{z}_{k-1})[\mathbf{b}]_{m+k-1+\boldsymbol{\lambda}}\\
=\;&\frac{z_k^{m+k+1}}{z_k^{k-1}}\prod_{j=1}^{k-1}z_j^{m+1}\prod_{j=1}^{k-1}(z_k-z_j)\Delta_0(\mathbf{z}_{k-1})\cdot\sigma_+(\mathbf{z}_k)\overline{\sigma}_-(\mathbf{z}_k)[\mathbf{b}]_{m+k-1+\boldsymbol{\lambda}}\\
=\;&\prod_{j=1}^{k}z_j^{m+1}\Delta_0(\mathbf{z}_k)\sigma_+(\mathbf{z}_k)\overline{\sigma}_-(\mathbf{z}_k)[\mathbf{b}]_{m+k+\boldsymbol{\lambda}}
\end{aligned}$$

as desired. ∎

2.12 Corollary. *The generating function* (26) *acts on on the basis element* $[\mathbf{b}]_{m+\boldsymbol{\lambda}}\in\mathcal{F}$*according to:*

$$\sum_{\boldsymbol{\mu}\in\overline{\mathcal{P}}_k}[\mathbf{b}]^k_{\boldsymbol{\mu}}\mathbf{s}_{\boldsymbol{\mu}}(\mathbf{z}_k)\wedge[\mathbf{b}]_{m+\boldsymbol{\lambda}}=\prod_{j=1}^{k}z_j^{m+1}\sigma_+(\mathbf{z}_k)\overline{\sigma}_-(\mathbf{z}_k)[\mathbf{b}]_{m+k+\boldsymbol{\lambda}} \tag{47}$$

Proof. By Proposition 2.11, using expression (26), dividing by the Vandermonde $\Delta_0(\mathbf{z}_k)$. ∎

3 Fermionic and Bosonic Vertex Representation of $gl(\bigwedge\mathcal{V})$.

3.1 Lemma. *The following commutation rule holds in* $\mathrm{End}_{\mathbb{Q}}(\mathcal{F})[z^{-1},w]]$

$$\overline{\sigma}_-(z)\overline{\sigma}_+(w) \;=\; \left(1-\frac{w}{z}\right)^{-1}\overline{\sigma}_+(w)\overline{\sigma}_-(z), \tag{48}$$

$$= \exp\left(\sum_{n\geq 0}\frac{1}{n}\frac{w^n}{z^n}\right)\overline{\sigma}_+(w)\overline{\sigma}_-(z). \tag{49}$$

Proof. Formula (48) is [10, Proposition 8.4, Formula (54)] and (49) uses the equality of formal power series $(1-x)^{-1} = \exp(\sum_{n\geq 1} x^n/n)$. ∎

3.2 Proposition. *Let* $p_n(\mathbf{z}_k^{-1}) = \sum_{i=1}^{k} z_i^{-n}$ *and* $p_n(\mathbf{w}_l) = \sum_{j=1}^{l} w_j^n$ *(the symmetric power sums Newton polynomials). The following equalities holds on* $\mathrm{End}_{\mathbb{Q}(\xi)}B(\xi)$*:*

$$\overline{\sigma}_-(\mathbf{z}_k) = \prod_{j=1}^{k}\overline{\sigma}_-(z_j) = \exp\left(-\sum_{n\geq 1}\frac{1}{n}p_n(\mathbf{z}_k^{-1})\frac{\partial}{\partial x_n}\right). \tag{50}$$

and

$$\sigma_-(\mathbf{w}_l) = \prod_{j=1}^{l}\sigma_-(z_j) = \exp\left(\sum_{n\geq 1}\frac{1}{n}p_n(\mathbf{w}_l^{-1})\frac{\partial}{\partial x_n}\right) \tag{51}$$

Therefore

$$\overline{\sigma}_-(\mathbf{z}_k)\sigma_-(\mathbf{w}_l)[\mathbf{b}]_{m+\boldsymbol{\lambda}} = \left[\exp\left(-\sum_{n\geq 1}\frac{1}{n}(p_n(\mathbf{z}_k^{-1}) - p_n(\mathbf{w}_l^{-1}))\frac{\partial}{\partial x_n}\right)\xi^m S_{\boldsymbol{\lambda}}(\mathbf{x})\right][\mathbf{b}]_0. \tag{52}$$

Proof. The operators

$$\sum_{n\geq 1}\frac{1}{n}\frac{1}{z_i^n}\frac{\partial}{\partial x_n},\quad \sum_{n\geq 1}\frac{1}{n}\frac{1}{z_j^n}\frac{\partial}{\partial x_n},\quad \sum_{n\geq 1}\frac{1}{n}\frac{1}{w_p^n}\frac{\partial}{\partial x_n},\quad \sum_{n\geq 1}\frac{1}{n}\frac{1}{w_q^n}\frac{\partial}{\partial x_n}$$

commute for all choices of $1 \leq i,j \leq k$ and $1 \leq p,q \leq l$. Then the product of their exponential is the exponentials of their sum:

$$\begin{aligned}
\overline{\sigma}_-(\mathbf{z}_k) = \prod_{j=1}^{k}\overline{\sigma}_-(z_j) &= \prod_{j=1}^{k}\exp\left(-\sum_{n\geq}\frac{1}{nz_j^n}\frac{\partial}{\partial x_n}\right)\\
&= \exp\left(-\sum_{n\geq 1}\frac{1}{n}\left(\frac{1}{z_1^n}+\cdots+\frac{1}{z_k^n}\right)\frac{\partial}{\partial x_n}\right)\\
&= \exp\left(-\sum_{n\geq 1}\frac{1}{n}p_n(\mathbf{z}_k^{-1})\frac{\partial}{\partial x_n}\right),
\end{aligned}$$

which validates (50). Formula (51) is checked analogously. Formula (52) follows from (50) and (51) and using again the fact that the operators $\sum_{n\geq 1}\frac{1}{n}p_n(\mathbf{z}_k^{-1})\frac{\partial}{\partial x_n}$ and $\sum_{n\geq 1}\frac{1}{n}p_n(\mathbf{w}_l^{-1})\frac{\partial}{\partial x_n}$ commute. Thus:

$$\overline{\sigma}_-(\mathbf{z}_k)\sigma_-(\mathbf{w}_l) = \exp\left(-\sum_{n\geq 1}\frac{1}{n}p_n(\mathbf{z}_k^{-1})\frac{\partial}{\partial x_n}\right)\exp\left(\sum_{n\geq 1}\frac{1}{n}p_n(\mathbf{w}_l^{-1})\frac{\partial}{\partial x_n}\right)$$

$$= \exp\left(-\sum_{n\geq 1}\frac{1}{n}(p_n(\mathbf{z}_k^{-1}) - p_n(\mathbf{w}_l^{-1}))\frac{\partial}{\partial x_n}\right). \tag{53}$$

■

3.3 Proposition. *The following commutation rules holds:*

$$\overline{\sigma}_-(\mathbf{z}_k)\overline{\sigma}_+(\mathbf{w}_l) = \exp\left(\sum_{n\geq 1}\frac{1}{n}p_n(\mathbf{w}_l)p_n(\mathbf{z}_k^{-1})\right)\overline{\sigma}_+(\mathbf{w}_l)\overline{\sigma}_-(\mathbf{z}_k). \tag{54}$$

Proof. We first prove that

$$\overline{\sigma}_-(\mathbf{z}_k)\overline{\sigma}_+(\mathbf{w}_l) = \prod_{i=1}^{k}\prod_{j=1}^{l}\left(1-\frac{w_j}{z_i}\right)^{-1}\overline{\sigma}_+(\mathbf{w}_l)\overline{\sigma}_-(\mathbf{z}_k) \tag{55}$$

For $k = l = 1$ the formula is Proposition (3.1). Suppose it holds for $k-1 \geq 1$ and $l = 1$. Then

$$\begin{aligned}
\overline{\sigma}_-(\mathbf{z}_k)\overline{\sigma}_+(\mathbf{w}_1) &= \prod_{i=1}^{k}\overline{\sigma}_-(z_i)\cdot\overline{\sigma}_+(w_1) && \text{(definition of } \overline{\sigma}_+(\mathbf{z}_k)\text{)}\\
&= \left(1-\frac{w_1}{z_k}\right)^{-1}\prod_{i=1}^{k-1}\overline{\sigma}_-(z_i)\overline{\sigma}_+(w_1)\overline{\sigma}_-(z_k) && \text{(first step of induction on } l\text{)}\\
&= \left(1-\frac{w_1}{z_k}\right)^{-1}\prod_{i=1}^{k-1}\left(1-\frac{w_1}{z_i}\right)^{-1}\overline{\sigma}_+(w_1)\prod_{i=1}^{k-1}\overline{\sigma}_-(z_i)\overline{\sigma}_-(z_k) && \text{(inductive hypothesis on } k\text{)}\\
&= \prod_{i=1}^{k}\left(1-\frac{w_1}{z_i}\right)^{-1}\overline{\sigma}_+(w_1)\overline{\sigma}_-(\mathbf{z}_k) && \text{(definition of } \overline{\sigma}_-(\mathbf{z}_k)\text{).}
\end{aligned}$$

Suppose now that (55) holds for all $k \geq 1$ and $l-1 \geq 0$. Then

$$\begin{aligned}
\overline{\sigma}_-(\mathbf{z}_k)\overline{\sigma}_+(\mathbf{w}_l) &= \overline{\sigma}_-(\mathbf{z}_k)\cdot\overline{\sigma}_+(w_l)\overline{\sigma}_-(\mathbf{w}_{l-1})\\
&= \prod_{i=1}^{k}\left(1-\frac{w_l}{z_i}\right)^{-1}\overline{\sigma}_+(w_l)\overline{\sigma}_-(\mathbf{z}_k)\overline{\sigma}_+(\mathbf{w}_{l-1})\\
&= \prod_{j=1}^{l}\left(1-\frac{w_l}{z_i}\right)^{-1}\prod_{\substack{1\leq i\leq k\\ 1\leq j\leq l-1}}\left(1-\frac{w_j}{z_i}\right)^{-1}\overline{\sigma}_+(w_l)\overline{\sigma}_+(\mathbf{w}_{l-1})\overline{\sigma}_-(\mathbf{z}_k)\\
&= \prod_{\substack{1\leq i\leq k\\ 1\leq j\leq l}}\left(1-\frac{w_j}{z_i}\right)^{-1}\overline{\sigma}_+(\mathbf{w}_l)\overline{\sigma}_-(\mathbf{z}_k),
\end{aligned}$$

which is precisely (55). To phrase (55) in the form (54) one first notice that

$$\left(1-\frac{w_j}{z_i}\right)^{-1}=\exp\left(\sum_{n\geq 1}\frac{1}{n}\frac{w_j^n}{z_i^n}\right).$$

By a simple manipulation one sees that

$$\prod_{\substack{1\leq i\leq k\\ 1\leq j\leq l}}\left(1-\frac{w_j}{z_i}\right)^{-1}=\prod_{\substack{1\leq i\leq k\\ 1\leq j\leq l}}\exp\left(\sum_{n\geq 1}\frac{1}{n}\frac{w_j^n}{z_i^n}\right)=\exp\left(\sum_{n\geq 1}\frac{1}{n}p_n(\mathbf{w}_l)p_n(\mathbf{z}_k^{-1})\right)$$

as desired. ■

3.4 Let $R_f(\mathbf{z}_k,\mathbf{w}_l^{-1}):\mathcal{F}\to\mathcal{F}[\mathbf{z}_k^{\pm 1},\mathbf{w}_l^{\pm 1}]$ defined on homogeneous elements as:

$$R_f(\mathbf{z}_k,\mathbf{w}_l^{-1})[\mathbf{b}]_{m+\boldsymbol{\lambda}}=\frac{\prod_{i=1}^k z_i^{m-l+1}}{\prod_{j=1}^l w_j^{m-l+1}}\xi^{k-l}[\mathbf{b}]_{m+\boldsymbol{\lambda}}$$

and $R_b(\mathbf{z}_k,\mathbf{w}_l^{-1})\in\mathrm{Hom}_{\mathbb{Q}[\xi]}(B(\xi),[\mathbf{z}_k^{\pm 1},\mathbf{w}_l^{\pm 1}])$ defined by

$$(R_b(\mathbf{z}_k,\mathbf{w}_l^{-1})\xi^m S_{\boldsymbol{\lambda}}(\mathbf{x}))[\mathbf{b}]_0=R_f(\mathbf{z}_k,\mathbf{w}_l^{-1})[\mathbf{b}]_{m+\boldsymbol{\lambda}}$$

from which

$$R_b(\mathbf{z}_k,\mathbf{w}_l^{-1})\cdot 1=\frac{\prod_{i=1}^k z_i^{m-l+1}}{\prod_{j=1}^l w_j^{m-l+1}}\xi^{k-l}$$

3.5 Proposition. *The map $R_f(\mathbf{z}_k,\mathbf{w}_l^{-1})$ commutes with Schubert derivations, in the sense that*

$$\sigma_\pm(\mathbf{z}_k)R_f(\mathbf{z}_k,\mathbf{w}_l^{-1})=R_f(\mathbf{z}_k,\mathbf{w}_l^{-1})\sigma_\pm(\mathbf{z}_k)\qquad\text{and}\qquad\overline{\sigma}_\pm(\mathbf{z}_k)R_f(\mathbf{z}_k,\mathbf{w}_l^{-1})=R_f(\mathbf{z}_k,\mathbf{w}_l^{-1})\overline{\sigma}_\pm(\mathbf{z}_k).$$

Proof. It is enough to prove that it commutes with $\sigma_{\pm i}$ and $\overline{\sigma}_{\pm j}$, $i,j\geq 0$, which are by definition $\mathbb{Q}[\mathbf{x}_k,\mathbf{w}_l^{-1}]$-linear. First of all recall that the product $\sigma_{\pm i}[\mathbf{b}]_{m+\boldsymbol{\lambda}}$ ($\boldsymbol{\lambda}\in\mathcal{P}_r$) is ruled by some Pieri's-like formulas

$$\sigma_{\pm i}[\mathbf{b}]_{m+\boldsymbol{\lambda}}=\sum_{\boldsymbol{\mu}\in P_\pm}[\mathbf{b}]_{m+\boldsymbol{\mu}},$$

where P_+ (resp. P_-) is the set of all partitions $\mu_1\geq\mu_2\geq\cdots\geq\mu_r$ ($r\geq\ell(\boldsymbol{\lambda})$) such that $\mu_1\geq\lambda_1\geq\cdots\geq\mu_k\geq\lambda_k$ and $|\boldsymbol{\mu}|=|\boldsymbol{\lambda}|+i$ (resp. $\lambda_1\geq\mu_1\geq\lambda_2\geq\mu_2\geq\cdots\geq\lambda_r\geq\mu_r$ and $|\boldsymbol{\mu}|=|\boldsymbol{\lambda}|-i$). Then we have

$$\sigma_{\pm i}R_f(\mathbf{z}_k,\mathbf{w}_l^{-1})[\mathbf{b}]_{m+\boldsymbol{\lambda}}\;=\;\sigma_{\pm i}\prod_{\substack{1\leq i\leq k\\ 1\leq j\leq l}}\frac{z_i^{m+l-1}}{w_j^{m-l+1}}\,\xi^{k-l}[\mathbf{b}]_{m+\boldsymbol{\lambda}}=\prod_{\substack{1\leq i\leq k\\ 1\leq j\leq l}}\frac{z_i^{m+l-1}}{w_j^{m-l+1}}\xi^{k-l}\sigma_{\pm i}[\mathbf{b}]_{m+\boldsymbol{\lambda}}$$

$$
\begin{aligned}
&= \prod_{\substack{1\le i\le k\\ 1\le j\le l}} \frac{z_i^{m+l-1}}{w_j^{m-l+1}}\xi^{k-l}\sum_{\boldsymbol{\mu}\in P_\pm}[\mathbf{b}]_{m+\boldsymbol{\mu}} = R_f(\mathbf{z}_k,\mathbf{w}_l^{-1})\sum_{\boldsymbol{\mu}\in P_\pm}[\mathbf{b}]_{m+\boldsymbol{\mu}}\\
&= R_f(\mathbf{z}_k,\mathbf{w}_l^{-1})\sigma_{\pm i}[\mathbf{b}]_{m+\boldsymbol{\lambda}}
\end{aligned}
$$

Thus $\sigma_\pm(\mathbf{z}_k)$ commutes with $R_f(\mathbf{z}_k,\mathbf{z}_l^{-1})$ and so do $\overline{\sigma}_\pm(\mathbf{z}_k)$. Indeed:

$$
\begin{aligned}
\overline{\sigma}_\pm(\mathbf{z}_k)R_f(\mathbf{z}_k,\mathbf{z}_l^{-1}) &= \overline{\sigma}_\pm(\mathbf{z}_k)R(\mathbf{z}_k,\mathbf{z}_l^{-1})\sigma_\pm(\mathbf{z}_k)\overline{\sigma}_\pm(\mathbf{z}_k)\\
&= \overline{\sigma}_\pm(\mathbf{z}_k)\sigma_\pm(\mathbf{z}_k)R(\mathbf{z}_k,\mathbf{w}_l^{-1})\overline{\sigma}_\pm(\mathbf{z}_k)\\
&= R_f(\mathbf{z}_k,\mathbf{w}_l^{-1})\overline{\sigma}_\pm(\mathbf{z}_k).
\end{aligned}
$$
∎

3.6 Theorem. *Notation as in* (38) *and* (39)*. Then:*

$$
\mathcal{E}_f(\mathbf{z}_k,\mathbf{w}_l^{-1}) = \exp\left(\sum_{n\ge1}\frac{1}{n}p_n(\mathbf{w}_l)p_n(\mathbf{z}_k^{-1})\right)\Gamma_f(\mathbf{z}_k,\mathbf{w}_l) \tag{56}
$$

and

$$
\mathcal{E}_b(\mathbf{z}_k,\mathbf{w}_l^{-1}) = \exp\left(\sum_{n\ge1}\frac{1}{n}p_n(\mathbf{w}_l)p_n(\mathbf{z}_k^{-1})\right)\Gamma_b(\mathbf{z}_k,\mathbf{w}_l.) \tag{57}
$$

where the fermionic and bosonic vertex operators are, respectively

$$
\begin{aligned}
\Gamma_f(\mathbf{z}_k,\mathbf{w}_l) &= R_f(\mathbf{z}_k,\mathbf{w}_l^{-1})\sigma_+(\mathbf{z}_k)\overline{\sigma}_+(\mathbf{w}_l)\overline{\sigma}_-(\mathbf{z}_k)\sigma_-(\mathbf{w}_l)\\
&= R_f(\mathbf{z}_k,\mathbf{w}_l^{-1})\exp\left(\sum_{n\ge1}x_n(p_n(\mathbf{z}_k)-p_n(\mathbf{w}_l))\right)\overline{\sigma}_-(\mathbf{z}_k)\sigma_-(\mathbf{w}_l).
\end{aligned} \tag{58}
$$

and

$$
\Gamma_b(\mathbf{z}_k,\mathbf{w}_l)=R_b(\mathbf{z}_k,\mathbf{w}_l^{-1})\exp\left(\sum_{n\ge1}x_n(p_n(\mathbf{z}_k)-p_n(\mathbf{w}_l))\right)\exp\left(-\sum_{n\ge1}\frac{p_n(\mathbf{z}_k^{-1})-p_n(\mathbf{w}_l^{-1})}{n}\frac{\partial}{\partial x_n}\right) \tag{59}
$$

Proof. We have:

$$
\begin{aligned}
\mathcal{E}_f(\mathbf{z}_k,\mathbf{w}_l^{-1}))[\mathbf{b}]_{m+\boldsymbol{\lambda}} &= [\mathbf{b}]^k(\mathbf{z}_k)\wedge[\boldsymbol{\beta}]^l(\mathbf{w}_l^{-1})\lrcorner[\mathbf{b}]_{m+\boldsymbol{\lambda}} && (\text{definition of } \mathcal{E}_f(\mathbf{z}_k,\mathbf{w}_l^{-1}))\\
&= [\mathbf{b}]^k(\mathbf{z}_k)\wedge\prod_{j=1}^{l}w_j^{-m+l-1}\overline{\sigma}_+(\mathbf{w}_l)\sigma_-(\mathbf{w}_l^{-1})[\mathbf{b}]_{m-l+\boldsymbol{\lambda}} && (\text{Corollary } 2.10)\\
&= \frac{\prod_{i=1}^k z_i^{m-l+1}}{\prod_{j=1}^l w_j^{m-l+1}}\sigma_+(\mathbf{z}_k)\overline{\sigma}_-(\mathbf{z}_k)\overline{\sigma}_+(\mathbf{w}_l)\sigma_-(\mathbf{w}_l^{-1})[\mathbf{b}]_{m+k-l+\boldsymbol{\lambda}} && (\text{Corollary } 2.12)\\
&= R(\mathbf{z}_k,\mathbf{w}_l^{-1})\sigma_+(\mathbf{z}_k)\overline{\sigma}_-(\mathbf{z}_k)\overline{\sigma}_+(\mathbf{w}_l)\sigma_-(\mathbf{w}_l^{-1})[\mathbf{b}]_{m+\boldsymbol{\lambda}} && (\text{Definition of } R(\mathbf{z}_k,\mathbf{w}_l^{-1}))
\end{aligned}
$$

By invoking the commutation relation proven in Proposition 3.3, one obtains

$$\mathcal{E}_f(\mathbf{z}_k, \mathbf{w}_l^{-1})[\mathbf{b}]_{m+\boldsymbol{\lambda}} = \exp\left(\sum_{n\geq 1}\frac{1}{n}p_n(\mathbf{w}_l)p_n(\mathbf{z}_k^{-1})\right) R(\mathbf{z}_k, \mathbf{w}_l^{-1})\sigma_+(\mathbf{z}_k)\overline{\sigma}_+(\mathbf{w}_l)\overline{\sigma}_-(\mathbf{z}_k)\sigma_-(\mathbf{w}_l)[\mathbf{b}]_{m+\boldsymbol{\lambda}} \tag{60}$$

which already proves that the expression of $\mathcal{E}_f(\mathbf{z}_k, \mathbf{w}_l^{-1})$ is exactly (56). To continue with, the $B(\xi)$-module structure of $\mathcal{F}$ says that $\mathcal{F}_m$ is an eigenspace of $\sigma_+(\mathbf{z}_k)\overline{\sigma}_+(\mathbf{w}_l)$ with eigenvalue

$$\begin{aligned}\prod_{i=1}^{k}\exp\left(\sum_{n\geq 1}x_n z_i^n\right)\prod_{j=1}^{l}\exp\left(-\sum_{n\geq 1}x_n w_j^n\right) &= \exp\left(\sum_{n\geq 1}x_n p_n(\mathbf{z}_k)\right)\exp\left(-\sum_{n\geq 1}x_n p_n(\mathbf{w}_l)\right)\\ &= \exp\left(\sum_{n\geq 1}x_n(p_n(\mathbf{z}_k)-p_n(\mathbf{w}_l))\right). \end{aligned}\tag{61}$$

Thus formula (60), up to replacing $\sigma_+(\mathbf{z}_k)\overline{\sigma}_+(\mathbf{w}_l)$ by its eigenvalue (61) with respect to $\mathcal{F}_m$, is precisely (56), with $\Gamma_f(\mathbf{z}_k, \mathbf{w}_l)$ given by expression (58). To prove (57) we recall that

$$(\mathcal{E}_b(\mathbf{z}_k, \mathbf{w}_l^{-1})\xi^m S_{\boldsymbol{\lambda}}(\mathbf{x}))[\mathbf{b}]_0 = \mathcal{E}_f(\mathbf{z}_k, \mathbf{w}_l^{-1})[\mathbf{b}]_{m+\boldsymbol{\lambda}} = \exp\left(\sum_{n\geq 1}\frac{1}{n}p_n(\mathbf{w}_l)p_n(\mathbf{z}_k^{-1})\right)\Gamma_f(\mathbf{z}_k, \mathbf{w}_l)[\mathbf{b}]_{m+\boldsymbol{\lambda}}.$$

Now

$$\Gamma_f(\mathbf{z}_k, \mathbf{w}_l)[\mathbf{b}]_{m+\boldsymbol{\lambda}} = R_f(\mathbf{z}_k, \mathbf{w}_l^{-1})\exp\left(\sum_{n\geq 1}x_n(p_n(\mathbf{z}_k)-p_n(\mathbf{w}_l))\right)\overline{\sigma}_-(\mathbf{z}_k)\sigma_-(\mathbf{w}_l)[\mathbf{b}]_{m+\boldsymbol{\lambda}}.$$

However, by Proposition 3.2, formula (52),

$$\overline{\sigma}_-(\mathbf{z}_k)\sigma_-(\mathbf{w}_l)[\mathbf{b}]_{m+\boldsymbol{\lambda}} = \left[\exp\left(-\sum_{n\geq 1}\frac{p_n(\mathbf{z}_k^{-1})-p_n(\mathbf{w}_l^{-1})}{n}\frac{\partial}{\partial x_n}\right)\xi^m S_{\boldsymbol{\lambda}}(\mathbf{x})\right][\mathbf{b}]_0.$$

which shows that

$$\Gamma_f(\mathbf{z}_k, \mathbf{w}_l)[\mathbf{b}]_{m+\boldsymbol{\lambda}} = (\Gamma_b(\mathbf{z}_k, \mathbf{w}_l^{-1})\xi^m S_{\boldsymbol{\lambda}}(\mathbf{x}))[\mathbf{b}]_0,$$

proving the theorem. ∎

3.7 Remark. In formula (57) let us set $k=l=1$ and call $z=z_1$ and $w=w_1$. Then $p_n(z^{\pm 1})=z^{\pm n}$ and $p_n(w^{\pm 1})=w^{\pm n}$. Then

$$\mathcal{E}_b(z,w) = \exp\left(\sum_{n\geq 1}\frac{1}{n}\frac{w^n}{z^n}\right)\Gamma_b(z,w)$$

where

$$\Gamma_b(z,w) = R_b(z,w)\exp(\sum_{n\geq 1}x_n(z^n-w^n))\exp\left(-\sum_{n\geq 1}\frac{z^{-n}-w^{-n}}{n}\frac{\partial}{\partial x_n}\right)$$

Keeping into account that

$$\exp\left(\sum_{n\geq 1}\frac{1}{n}\frac{w^n}{z^n}\right)=\frac{1}{1-\dfrac{w}{z}}$$

and using the definition of $R_b(z, w^{-1})$ one sees that

$$\mathcal{E}_b(z,w)_{|B^{(m)}}=\frac{\dfrac{z^m}{w^m}}{1-\dfrac{w}{z}}\exp(\sum_{n\geq 1}x_n(z^n-w^n))\exp\left(-\sum_{n\geq 1}\frac{z^{-n}-w^{-n}}{n}\frac{\partial}{\partial x_n}\right)$$

which is the celebrated DJKM formula.

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