A Fast Quasi-Conformal Mapping Preconditioner for Electromagnetic Integral Equations

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Boundary Element Methods (BEMs) are efficient strategies to numerically solve electromagnetic radiation and scattering problems. Unfortunately, however, classical BEM formulations suffer from ill-conditioning when the frequency is low, or the discretization density is high. In the past, several remedies have been presented for these ill-conditioning problems including preconditioners based on Calderón identities, hierarchical bases, and current decompositions. While effective, these strategies however require ad-hoc procedures including mesh-refinements, new basis function definitions, and adapted fast methods that, if not implemented properly, can become computationally cumbersome.

In this work we adopted a different and new approach: we leveraged a spectral equivalence that can be established between integral operators defined on manifolds linked by conformal or quasi-conformal mappings. Moreover, a fast and sparse operator algebra is available for spheres (or circles in 2D). Thus, we derived equivalent matrix products discretizing respectively: a sphere mapping, an inversion on the sphere of the original operator, and a mapping back on the initial manifold. The steps are equipped with appropriate fast algorithms. We finally show that this sequence represents a valid, optimal, and multiplicative preconditioner for the original operator. For fixing ideas, consider for example the single layer operator in 2D for which the matrix elements are computed as

$$\left[\boldsymbol{S}_{\lambda,\lambda}\right]_{i,j} = \int_{\Gamma} \lambda_i(\boldsymbol{r}) \int_{\Gamma} G(\boldsymbol{r}'-\boldsymbol{r})\, \lambda_j(\boldsymbol{r}')d\boldsymbol{r}'d\boldsymbol{r}$$

where $G(\boldsymbol{r}'-\boldsymbol{r}) = -\frac{j}{4}H_0^{(2)}(k|\boldsymbol{r}'-\boldsymbol{r}|)$ is the Green function (with k the wavenumber and $H_0^{(2)}(x)$ the Hankel function of the second kind) and λ_n are the set of basis functions used to discretize the operator. Performing a change of variable, the same matrix elements can be computed using integrals on the circular boundary Γ_c

$$\left[\boldsymbol{S}_{\lambda,\lambda}\right]_{i,j} = \int_{\Gamma_c} \lambda_i\big(\boldsymbol{\Phi}(\tilde{\boldsymbol{r}})\big) \int_{\Gamma_c} G\left(\boldsymbol{\Phi}(\tilde{\boldsymbol{r}}')-\boldsymbol{\Phi}(\tilde{\boldsymbol{r}})\right) \lambda_j\big(\boldsymbol{\Phi}(\tilde{\boldsymbol{r}}')\big) J_{\boldsymbol{\Phi}}(\tilde{\boldsymbol{r}}') J_{\boldsymbol{\Phi}}(\tilde{\boldsymbol{r}}) d\tilde{\boldsymbol{r}}' d\tilde{\boldsymbol{r}}$$

where $\boldsymbol{\Phi}(\tilde{\boldsymbol{r}}')$ is a known mapping function that morphs a circular boundary Γ_c to the scatterer boundary Γ, $J_{\boldsymbol{\Phi}}(\tilde{\boldsymbol{r}})$ the determinant of the Jacobian. Letting $\boldsymbol{\Phi}(\tilde{\boldsymbol{r}}')$ be Lipschitz continuous it can be proved that the operator matrix $\boldsymbol{S}_{\lambda,\lambda}$ is spectrally equivalent to the mapped operator matrix $\tilde{\boldsymbol{S}}_{\tilde{\lambda},\tilde{\lambda}}$ for which the elements are computed as

$$\left[\tilde{\boldsymbol{S}}_{\tilde{\lambda},\tilde{\lambda}}\right]_{i,j} = \int_{\Gamma_c} \lambda_i\big(\boldsymbol{\Phi}(\tilde{\boldsymbol{r}})\big) \int_{\Gamma_c} G(\tilde{\boldsymbol{r}}'-\tilde{\boldsymbol{r}})\, \lambda_j\big(\boldsymbol{\Phi}(\tilde{\boldsymbol{r}}')\big) d\tilde{\boldsymbol{r}}' d\tilde{\boldsymbol{r}} = \int_{\Gamma_c} \tilde{\lambda}_i(\tilde{\boldsymbol{r}}) \int_{\Gamma_c} G(\tilde{\boldsymbol{r}}'-\tilde{\boldsymbol{r}})\, \tilde{\lambda}_j(\tilde{\boldsymbol{r}}) d\tilde{\boldsymbol{r}}' d\tilde{\boldsymbol{r}}$$

where $\tilde{\lambda}_n$ are a mapped version of the original basis functions. For operators computed on circular boundaries, parametrized with the angular variable θ, and discretized with the exponentials basis functions $\tilde{e}_m = e^{-\mathrm{j}m\theta}$ (eigenfunctions of the Laplace-Beltrami operator on the circle in 2D, we would use spherical harmonics in 3D), the obtained moment matrix, is diagonal. Hence, we discretize directly the inverse of the operator relative to $\tilde{\boldsymbol{S}}_{\tilde{\lambda},\tilde{\lambda}}$, getting the diagonal matrix $\widetilde{\boldsymbol{S}^{-1}}_{\tilde{e},\tilde{e}}$, and by using Gram matrices to perform a change of basis we compute the matrix $\widetilde{\boldsymbol{P}}_{\tilde{\lambda},\tilde{\lambda}} = \boldsymbol{G}_{\tilde{\lambda},\tilde{e}}\, \widetilde{\boldsymbol{S}^{-1}}_{\tilde{e},\tilde{e}} \boldsymbol{G}_{\tilde{e},\tilde{\lambda}}$ that for an increasing number of functions $\tilde{e}_m$ converges to the matrix $\widetilde{\boldsymbol{S}^{-1}}_{\tilde{\lambda},\tilde{\lambda}}$. This last matrix is spectrally equivalent to the inverse of the original operator matrix, that reads $\boldsymbol{G}_{\lambda,\lambda}(\boldsymbol{S}_{\lambda,\lambda})^{-1}\boldsymbol{G}_{\lambda,\lambda} \asymp \widetilde{\boldsymbol{S}^{-1}}_{\tilde{\lambda},\tilde{\lambda}}$, thus $\widetilde{\boldsymbol{P}}_{\tilde{\lambda},\tilde{\lambda}}$ can be used to precondition $\boldsymbol{S}_{\lambda,\lambda}$, obtaining an operator free from dense-discretization breakdown. Numerical results confirm theoretical predictions with a condition number robust to mesh refinement (Figure 1).

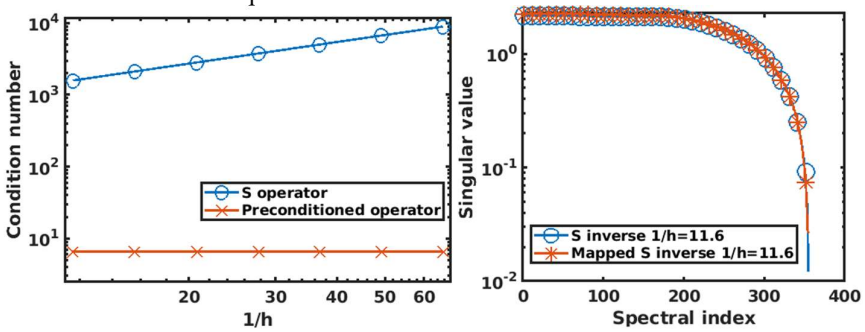

Figure 1. Left: condition numbers of $\boldsymbol{S}_{\lambda,\lambda}$ and $\widetilde{\boldsymbol{P}}_{\tilde{\lambda},\tilde{\lambda}}(\boldsymbol{G}_{\lambda,\lambda})^{-1}\boldsymbol{S}_{\lambda,\lambda}$. Right: spectra of $\widetilde{\boldsymbol{P}}_{\tilde{\lambda},\tilde{\lambda}}$ and $\boldsymbol{G}_{\lambda,\lambda}(\boldsymbol{S}_{\lambda,\lambda})^{-1}\boldsymbol{G}_{\lambda,\lambda}$.