One-dimensional high order finite elements embedding 3D state-based peridynamic subdomains

ONE-DIMENSIONAL HIGH ORDER FINITE ELEMENTS EMBEDDING 3D STATE-BASED PERIDYNAMIC SUBDOMAINS

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ABSTRACT

A technique to couple three-dimensional state-based peridynamics (PD) with one-dimensional high-order finite elements (FEs) is proposed in this work. The refined FEs employed are based on the well-established Carrera Unified Formulation (CUF) [1], whereas the coupling is realized using Lagrange multipliers satisfying the displacement continuity at the boundary interface with PD subdomains. This technique has already been introduced in a previous work [2], where a bond-based PD model was employed. In the present work, an extension to the more general state-based PD is proposed instead, see [3, 4]. Thanks to CUF, which can provide accurate solutions with reduced computational costs, it is demonstrated that complex structures of engineering interest and embedding detailed 3D PD subdomains can be solved with ease by using the proposed methodology. The resulting coupled models may be effectively used for describing crack propagation problems and whenever the failure is prescribed to small regions of the entire structural domain.

Keywords: State-Based Peridynamics, Carrera Unified Formulation, Higher-order finite elements

1 INTRODUCTION

Peridynamics (PD) is a non-local theory introduced by Silling in [5]. It assumes that a solid body is composed by material points and each pair of those interacts if their distance is shorter than a prescribed horizon radius δ. Peridynamics is based on integro-differential equations, which allow a description of discontinuous displacement fields, such as in the case of cracks. Nevertheless, PD can be affected by high computational costs, in particular when 3D structures are investigated. This is due to the non-local nature of the theory, that leads to numerical methods involving large and sparse matrices.

In this context, several researchers are working on models which are able to couple FEM and PD domains (i.e. [5]). In this work, the use of high order 1D finite elements to be coupled with 3D PD sub-domains is proposed. The 1D models are implemented in the Carrera Unified Formulation (CUF) framework [1]. The coupling is realized through the use of Lagrange multipliers, which do not modify the stiffness in both regions and guarantee displacement continuity at the interface.

2 STATE-BASED PERIDYNAMICS

In Figure 1, two material points x and x' are shown. They interact when they have a distance shorter than a prescribed horizon radius δ. These particles are shown in both initial and deformed configuration. In the latter, the force density vectors are defined at time t as **T**[**x**,t]($\boldsymbol{\xi}$) and **T**'[**x**',t]($\boldsymbol{\xi}$) where $\boldsymbol{\xi}$ is the relative position vector. The relative displacement vector is indicated with **η**.

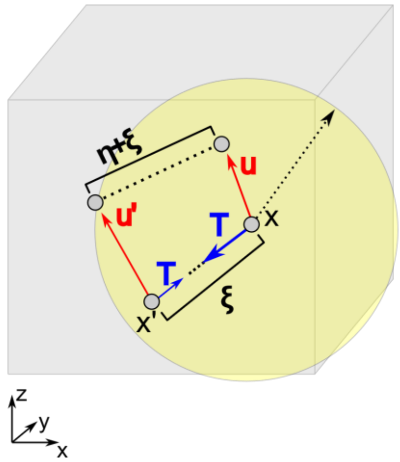


Figure 1: The positions of two material points in the initial and deformed configurations.

The equilibrium equation for the state-based peridynamics has an integral form and it reads as follow:

$$\rho\ddot{\boldsymbol{u}} = \int_{H_x} \{\mathbf{T}[\mathbf{x},\mathrm{t}](\boldsymbol{x}'-\boldsymbol{x}) - \mathbf{T}'[\mathbf{x},\mathrm{t}](\boldsymbol{x}-\boldsymbol{x}')\}dV_{x'} + \boldsymbol{b}(\boldsymbol{x},t)$$

where ρ is the material mass density, $\ddot{\boldsymbol{u}}$ is the acceleration vector, $dV_{x'}$ the infinitesimal volume associated to the particle x'. Finally, **b** is the force density vector. For further information about state-based peridynamics, interested readers can refer to [3].

3 HIGH ORDER 1D THEORIES BASED ON CLASSICAL ELASTICITY

In the framework of CUF, the 3D field of displacements can be expressed as a summation, of arbitrary expansion functions $F_\tau(x,z)$ and the vector of the generalized displacements $\boldsymbol{u}_\tau(y)$. In the case of one-dimensional beam theories, as in the case of this work, the displacements field is expressed as:

$$\boldsymbol{u}(x,y,z) = F_\tau(x,z)\boldsymbol{u}_\tau(y) \qquad \tau = 1,\dots,M \tag{1}$$

where M is the number of expansion terms. Different expansion functions can be utilized as $F_\tau(x,z)$. In this work, Taylor (TE) and Lagrange (LE) expansion are used. The Finite element method (FEM) is used to study the structures involved in this paper. Therefore, the generalized displacements can be expressed as:

$$\boldsymbol{u}_\tau(y) = N_i(y)\boldsymbol{u}_{\tau i} \qquad i = 1,\dots,n_{nodes} \tag{2}$$

in which n_{nodes} is the total number nodes per element, $N_i(y)$ are the one-dimensional shape functions and $\boldsymbol{u}_{\tau i}$ are the nodal unknowns.

In this work, the linear static problem is analyzed. The stiffness matrix is obtained via the application of the virtual displacement principle. After some manipulations, the virtual variation of the internal work reads:

$$\boldsymbol{\delta}(L_{int}) =< \delta\boldsymbol{\varepsilon}^T\boldsymbol{\sigma} >= \delta u_{sj}^T \boldsymbol{k}_{\mathbf{0}}^{ij\tau s}\boldsymbol{u}_{\tau i} \tag{3}$$

where $\boldsymbol{k}_{\mathbf{0}}^{ij\tau s}$ denotes the stiffness matrix in the form of fundamental nucleus, and the $<\cdot>$ operator represents the integral over the volume of the element.

It should be underlined that no assumption about the approximation order are made for the formulation of both matrices. Thus, any refined beam model can be obtained with this procedure. The assembled global stiffness matrix is obtained by looping through the indices i, j, τ, s. Then, the stiffness matrix is assembled over the entire FE domain, leading to the following linear system:

$$\boldsymbol{K}^{FE}\boldsymbol{U}^{FE} = \boldsymbol{F}^{FE} \tag{4}$$

where $\boldsymbol{U}^{FE}$ is the vector of FE nodal unknowns and $\boldsymbol{F}^{FE}$ the vector of external forces.

4 COUPLING TECHNIQUE

Let consider the beam shown in Figure 2 . The central region of the beam is modelled by 3D PD, while the rest of the domain is discretized with higher order 1D FEs. Given an interface zone I between PD domain and FEM region, Lagrange multipliers are employed in this work to satisfy displacement continuity at the interface.

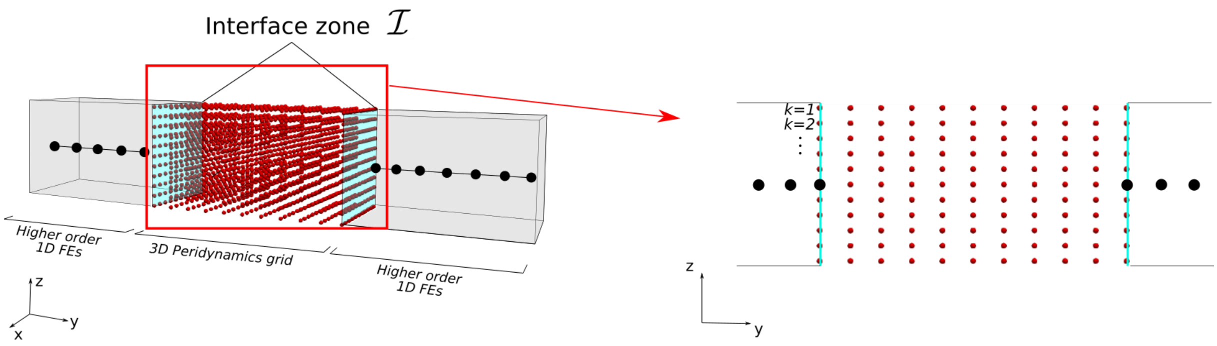


Figure 2: Refined 1D finite elements are coupled with PD particles through Lagrange multipliers. The Lagrangian of each particle k at the interface zone I is added to the system in order to satisfy displacement continuity.

To each particle k at the interface zone, the following Lagrangian is added to the linear system:

$$\boldsymbol{\pi}_k = \boldsymbol{\lambda}_k^T(\boldsymbol{u}_k^{PD} - \boldsymbol{u}^{FE}(x, y, z)) \tag{5}$$

where $\boldsymbol{\lambda}_k^T$ is the vector containing the Lagrange multipliers, $\boldsymbol{u}_k^{PD}$ is the displacement of the PD particle k, and $\boldsymbol{u}^{FE}$ is the displacement field of the counterpart.

The CUF equation are then introduced into Equation (5). The solution of the coupled problem is found by solving the following linear system:

$$\begin{bmatrix} \boldsymbol{K} & \boldsymbol{B}^T \\ \boldsymbol{B} & \boldsymbol{0} \end{bmatrix} \begin{pmatrix} \boldsymbol{U} \\ \boldsymbol{\lambda} \end{pmatrix} = \begin{pmatrix} \boldsymbol{F} \\ \boldsymbol{0} \end{pmatrix} \tag{6}$$

5 NUMERICAL RESULTS

5.1 I-section beam under bending and torsion

A cantilever I-section beam subjected to torsion and bending is investigated (Figure 3). The same problem has been studied in [6], where geometrical characteristics and material properties are also provided. Moreover, the same beam has been considered in [7], where Pagani and Carrera introduced the coupling technique between PD and FEM for the first time. Here, the bond-based PD formulation has been employed. Thus, a constrained value of Poisson ratio equal to 0.25 had to be considered. In this work, the implementation of state-based peridynamics allowed to remove this constraint. The Poisson ratio considered is equal to 0.2. The PD region is embedded into the FE domain, which is discretized with 10 four-node cubic (B4) elements. In order to do so, the cross-section of the 1D finite elements varies along the y-axis. Specifically, for $400 \leq y \leq 600$ mm, the cross-section is a disconnected domain where the PD region lays. The PD region has a grid spacing $\Delta x = 1$ mm and horizon radius $\delta = 2$ mm.

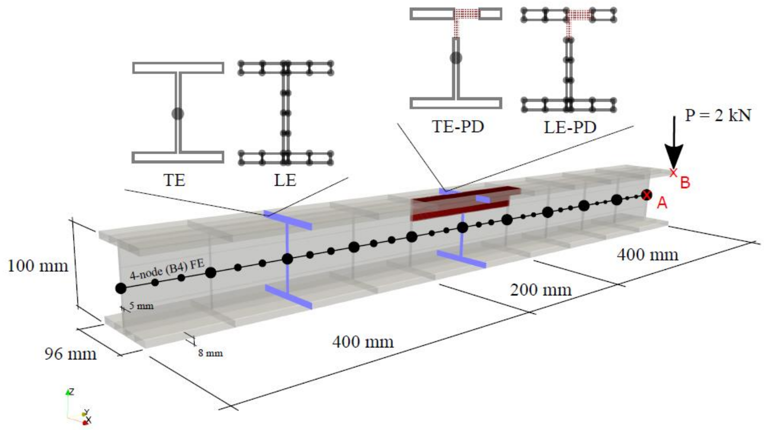
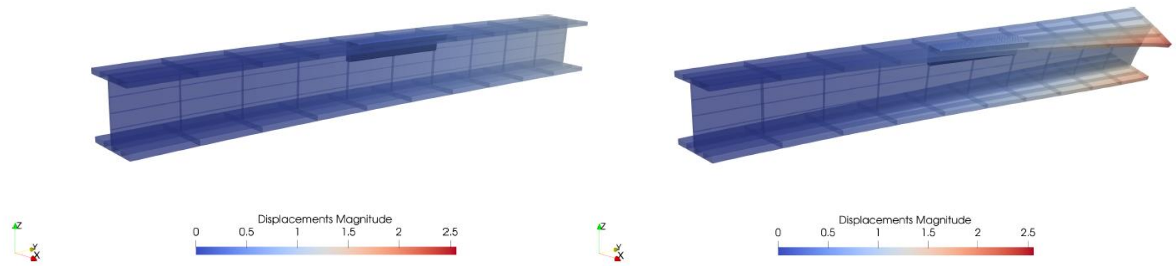


Figure 3: Cantilever I-section beam subjected to point load. The peridynamic zone coexists with the FE domain.

In Table 1 there is a comparison of the transverse displacements at points A and B from proposed coupled model and previous literature. Reference solutions were also provided by using a commercial FE software. The evaluation of displacements at these two points allows to appreciate the capability of the model in reproducing both bending and torsional behaviour. Note that low-order TE expansion are not able to find correct value of displacement in point B, due to the rigid behaviour of the beam in the cross-section plane. On the other hand, LE kinematics are able to provide accurate results. The difference in the beam behaviour is shown in Figure 4. Moreover, it should be underlined that both coupled models are coherent with the expected solutions.

Model	FE DOFs	PD DOFs	$-u_z$ [mm] (Point A)	$-u_z$ [mm] (Point B)
Reference solutions				
TE1	279	-	0.964	0.977
L9	8091	-	1.001	2.350
Nastran 2D	61000	-	1.006	2.437
Nastran 3D	355800	-	0.956	2.316
Present coupled model				
TE1 - PD	279	222507	0.954	0.967
L9 - PD	7911	222507	0.991	2.286

Table 2: Vertical displacement components at the center of the free-end I-section and at the loading point. The results from the present PD-FE coupled method are compared with those from the literature.



(a) TE1-PD (b) L9-PD

Figure 4: Deformed state of the I-section beam according to PD and low to high order FEs coupled models.

Finally, from Figure 5, it is clear that the stress state is not affected by the introduction of PD sub-domains into the model. In fact, the stress state remains congruent even at the interface zone.

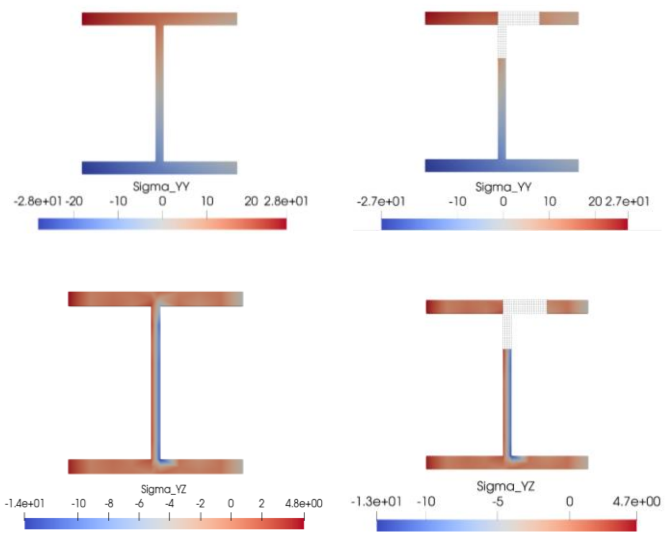


Figure 5: Distribution of axial stress and transverse shear stress across the FE-PD interface zone at y = 400 mm according to L9-PD model.

6 CONCLUSIONS

In this work, a coupling technique between 3D peridynamics (PD) and 1D high order finite elements is proposed. This method is based on the use of Lagrange mulitpliers, which have a clear physical meaning. PD regions are literally embedded into FE domains, and it is shown that the accuracy of the solution is comparable to the solution through full FEM discretization. Moreover, this approach has a low computational demand if compared to a situation where full 3D PD is used to modelled the solid body to investigate.

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