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Convergence Analysis of Weighted SPSA-based Consensus Algorithm in Distributed Parameter Estimation Problem

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Abstract: In this paper, we study a distributed parameter estimation problem in a large-scale network of communication sensors. The goal of the sensors is to find a global estimate of an unknown parameter minimizing, which minimizes some aggregate cost function. Each sensor can communicated to a few “neighbors”, furthermore, the communication channels have limited capacities. To solve the resulting optimization problem, we use a *weighted* modification of the distributed consensus-based SPSA algorithm whose main advantage over the alternative method is its ability to work in presence of arbitrary *unknown-but-bounded* noises whose statistical characteristics can be unknown. We provide a convergence analysis of the weighted SPSA-based consensus algorithm and show its efficiency via numerical simulations.

Keywords: Sensor network, randomized algorithms, consensus, distributed parameter estimation

1. INTRODUCTION

Multi-agent systems and technologies have found numerous applications in engineering, from mobile robotics to distributed computing (Bullo et al., 2009; D.Bertsekas and Tsitsiklis, 1989; Olfati-Saber et al., 2007; Ren and Cao, 2011; Shoham and Leyton-Brown, 2008). Coordination of simple and inter-replaceable agents enables them to solve complex problems more efficiently than centralized systems, enhancing also their reliability and resilience. Being a special class of multi-agent systems, *sensor networks* constituted by low-power miniature wireless sensor devices “promise to revolutionize sensing in a wide range of application domains” (Tubaishat and Madria, 2003) due to their reliability, ease of deployment and cost-efficiency.

Obviously, data fusion from numerous sensors leads to more accurate estimates of the unknown parameters than small sensor groups can provide. However, as a sensor network becomes large, data acquisition and processing at a single center become virtually impossible, and distributed algorithms are needed that require only local interactions among sensors. A number of problems arises in design of such networks (Tubaishat and Madria, 2003) caused by time-varying topology of the network, limited capabilities of individual sensors (low power, small memory and minor computational capacity) and communication constraints (a large amount of data can lead to traffic congestion).

To provide accurate data fusion in the face of uncertainties (e.g. measurement noises and other kinds of unknown signals), distributed stochastic optimization is commonly used: the desired estimate of an unknown parameter should deliver an optimum to a certain mean-risk functional. In distributed optimization, most studied are methods for convex optimization, e.g. the alternating direction method of multipliers (ADMM) (Boyd et al., 2011) and subgradient methods (Nedić and Olshevsky, 2016; Rabbat and Nowak, 2004). For non-convex optimization, methods of surrogate functions have been used (Di Lorenzo and Scutari, 2016). There are also algorithms that embed a dynamic average consensus protocol into optimization process (Falsone et al., 2020; Xie and Guo, 2018). Most of methods, however, assume that some statistical characteristics of the uncertain parameters are known, for instance, the noises are Gaussian or have zero expectation. In this paper, we are concerned with situation where the random signals are completely unknown yet supposed to be bounded (Granichin and Amelina, 2015), which makes many statistical methods inapplicable.

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Note that the proposed approach can be used for other types of measuring parameters (e.g., bearing/azimuth).

In general, the problem is to find an estimation $\hat{\theta}_t$ of an unknown parameter θ:

$$\hat{\theta}_t^\star = \arg\min_{\hat{\theta}_t} ||\hat{\theta}_t - \theta||^2. \quad (2)$$

In this paper, we consider a more difficult problem setting. First, the solution of the optimization problem (2) needs to be found in a distributed way. Second, we impose the following *communication constraints*: at time instant t, each sensor $i \in \mathcal{N}$ is able to measure the squared distance to not more than one target. In practice, due to hardware constraints, the number of communication channels that can be used is usually less than the dimension of space or equal to it. Without loss of generality, in this paper, we assume that each sensor is able to collect data only from d neighbors. In this case and if there is no noise, we can use standard triangular approaches to determine the target position. However, if positions of all m targets need to be computed, then we have to simultaneously collect $m(d-1)$ measurements, and it is often impossible in practice. Third, we assume that there is the *unknown-but-bounded* noise involved in the measuring process, which is considered in the next subsection.

Suppose sensor i estimates the state of target l at time instant t. The sensor is able to collect the distances to the same target measured by its neighbors $j \in \bar{\mathcal{N}}_t^i \subset \mathcal{N}^i$, $|\bar{\mathcal{N}}_t^i| = d$. Let $\mathbf{u}_t^i = [j_1, \dots, j_d, l]^{\mathrm{T}}$, $j_1, \dots, j_d \in \bar{\mathcal{N}}_t^i$, be a vector defining a set of neighbors used to collect measurements associated with target l at time instant t. Denote by

$$\bar{\rho}_t^j(\mathbf{u}^i) = \rho(\mathbf{s}^i, \mathbf{r}^{h(\mathbf{u}_t^i)}) - \rho(\mathbf{s}^j, \mathbf{r}^{h(\mathbf{u}_t^i)}) \ \forall j \in \bar{\mathcal{N}}_t^i, \quad (3)$$

a residual between a measurement of sensor i and its neighbors. Here and after, $h(\mathbf{u}_t^i) : \mathbb{R}^{d+1} \to \mathbb{R}$ gives the last element of $\mathbf{u}_t^i$.

In this case, using the square difference formula we get d equations

$$\bar{\rho}_t^j(\mathbf{u}_t^i) = (\mathbf{s}^j - \mathbf{s}^i)^{\mathrm{T}}(2\mathbf{r}^{h(\mathbf{u}_t^i)} - \mathbf{s}^j - \mathbf{s}^i), j \in \bar{\mathcal{N}}_t^i.$$

This allows us to derive

$$C^{\mathbf{u}_t^i}\mathbf{r}_t^{h(\mathbf{u}_t^i)} = D^{\mathbf{u}_t^i}, \quad \mathbf{r}_t^{h(\mathbf{u}_t^i)} = [C^{\mathbf{u}_t^i}]^{-1}D^{\mathbf{u}_t^i}, \quad (4)$$

where

$$C^{\mathbf{u}_t^i} = 2\begin{bmatrix}(\mathbf{s}^{j_1} - \mathbf{s}^i)^{\mathrm{T}} \\ \cdots \\ (\mathbf{s}^{j_d} - \mathbf{s}^i)^{\mathrm{T}}\end{bmatrix}, D^{\mathbf{u}_t^i} = \begin{bmatrix}\bar{\rho}_t^1(\mathbf{u}_t^i) + \|\mathbf{s}^{j_1}\|^2 - \|\mathbf{s}^i\|^2 \\ \cdots \\ \bar{\rho}_t^d(\mathbf{u}_t^i) + \|\mathbf{s}^{j_d}\|^2 - \|\mathbf{s}^i\|^2\end{bmatrix}.$$

Using the introduced notations, we define the measurements of sensor $i \in \mathcal{N}$ at time instant t as follows:

$$y_t^i = F_t^i(\mathbf{u}_t^i, \mathbf{x}_t^i) + v_t^i = \|\hat{\mathbf{r}}_t^{h(\mathbf{u}_t^i)} - [C^{\mathbf{u}_t^i}]^{-1}D^{\mathbf{u}_t^i}\|^2 + v_t^i, \quad (5)$$

where v_t^i is the unknown-but-bounded additive noise, $\mathbf{x}_t^i$ is the measurement point depending on currently available estimate $\hat{\mathbf{r}}_t^{h(\mathbf{u}_t^i)}$ at time instant t. For example, $\mathbf{x}_t^i = \hat{\mathbf{r}}_t^{h(\mathbf{u}_t^i)}$.

3.1 Distributed Optimization

Denote by $\mathcal{F}_{t-1}$ the σ-algebra of all probabilistic events, which happened up to time instant t. $\mathbb{E}_{\mathcal{F}_{t-1}}$ denotes the conditional expectation with respect to the σ-algebra $\mathcal{F}_{t-1}$. This σ-algebra is generated by the values of all random variables (i.e., position of targets, noise, changes in communication topology) at time instants $\tau = \{1, 2, \dots, t\}$.

Let $\mathbf{u}_t = [\mathbf{u}_t^1, \dots, \mathbf{u}_t^n]^{\mathbf{T}}$ be the common vector defining the sets of neighbors used to collect measurements from each sensor. The multi-sensor multi-target problem can be formulated as the following minimization problem: to find estimate $\hat{\theta}_t = \mathrm{col}\{\hat{\mathbf{r}}_t^{h(\mathbf{u}_t^1)}, \dots, \hat{\mathbf{r}}_t^{h(\mathbf{u}_t^m)}\}$ that minimizes the following loss function

$$\hat{\theta}_t^\star = \arg\min_{\hat{\theta}_t} \bar{F}_t(\mathbf{u}_t, \hat{\theta}_t),$$

$$\bar{F}_t(\mathbf{u}_t, \hat{\theta}_t) = \mathbb{E}_{\mathcal{F}_{t-1}} \sum_{i \in \mathcal{N}} F_t^i(\mathbf{u}_t^i, \hat{\mathbf{r}}_t^{h(\mathbf{u}_t^i)}). \quad (6)$$

Usually, during optimization, each sensor fuses the needed information from all available neighboring nodes. In our problem setting, we mentioned the communication constraints that prohibit such communication strategy of the sensors. These communication constraints arise due to hardware and physical limitations since the bandwidths of communication channels is not unlimited. When a large number of sensors send and receive messages at the same time, communication becomes a bottleneck. To deal with this, we propose to choose communication links between sensors randomly. More formally, for each sensor $i \in \mathcal{N}$, we randomize the communication topology described by graph $\mathcal{G}_A$ at each time instant t to satisfy topology constraints such as the maximum number of links equals to d. We use a randomly chosen subgraph $\mathcal{G}_{B_t} \subset \mathcal{G}_A$ associated with adjacency matrix $B_t = [b_t^{i,j}]$, where the rows contain no more than d nonzero entries. Afterwards, the observable target at time instant t contained in $\mathbf{u}_t^i$ is generated from a uniform distribution independently for each sensor $i \in \mathcal{N}$ as in gossip algorithm (Boyd et al., 2011). We randomize the communication topology described by graph $\mathcal{G}_A$ based on the strategy similar to one presented in Amelina et al. (2014).

4. WEIGHTED SPSA-BASED CONSENSUS ALGORITHM

Let $\mathbf{u}_k^i$ and $\mathbf{\Delta}_k^i \in \mathbb{R}^d$, $k = 1, 2, \dots,$ $i \in \mathcal{N}$, be independent random variables. We generate $\mathbf{\Delta}_k^i$ called the *simultaneous test perturbation* from Bernoulli distribution with each component independently taking values $\pm\frac{1}{\sqrt{d}}$ with probabilities $\frac{1}{2}$. Let $\mathbf{e}_{h(\mathbf{u}_k^i)} \in \mathbb{R}^m$ be the sparse vector corresponding to the current target that sensor i observes, then $\hat{\Delta}_k^i = \mathbf{e}_{h(\mathbf{u}_k^i)} \otimes \mathbf{\Delta}_k^i$. In this case, $\hat{\Delta}_k^i$ is the vector of all zeros except for the rows that corresponds to $h(\mathbf{u}_k^i)$.

Let $\mathbb{U}^{i,l}$ be a set containing all possible subsets $\bar{\mathcal{N}}_t^i$ for target l. The neighborhood of sensor i at time instant t is defined by the i-th row of matrix B_t associated with graph $\mathcal{G}_{B_t}$. This row is defined by subset $\bar{\mathcal{N}}_t^i$ generated from the uniform distribution on the set $\mathbb{U}^{i,l}$.

Next, we introduce a weighted version of SPSA-based consensus algorithm. We define diagonal matrix $\Psi = [\psi_{ij}]$, where $\psi_{ij} > 0$ if $i = j$ and $\psi_{ij} = 0$ otherwise. At initialization step, for each $i \in \mathcal{N}$, we choose initial vector $\hat{\theta}_0^i \in \mathbb{R}^{md}$, positive step-size α_k, matrix Ψ, gain coefficient γ, and the scale of perturbation $\beta > 0$.

In order to get estimates $\{\widehat{\theta}_t^i\}$ of overall state vectors $\{\theta_t^i\}$ based on measurement points $\{\mathbf{x}_t^i\}$, we propose to use the weighted algorithm with two measurements of distributed sub-functions $F_t^i(\mathbf{u}_t^i, \mathbf{x}_t^i)$:

$$\begin{cases} \mathbf{x}_{2k}^i = \widehat{\theta}_{2k-2}^i + \beta\hat{\Delta}_k^i,\ \mathbf{x}_{2k-1}^i = \widehat{\theta}_{2k-2}^i - \beta\hat{\Delta}_k^i, \\ \widehat{\theta}_{2k-1}^i = \widehat{\theta}_{2k-2}^i, \\ \widehat{\theta}_{2k}^i = \widehat{\theta}_{2k-1}^i - \alpha_k \Psi \Bigg[\hat{\Delta}_k^i \dfrac{y_{2k}^i - y_{2k-1}^i}{2\beta} + \\ \qquad \gamma \displaystyle\sum_{j \in \mathcal{N}_{2k-1}^i} b_{2k-1}^{i,j} (\widehat{\theta}_{2k-1}^i - \widehat{\theta}_{2k-1}^j) \Bigg]. \end{cases} \quad (7)$$

Consider the last equation of the algorithm (7): the first part is similar to SPSA-like algorithm from Granichin and Amelina (2015) and the second one coincides with Local Voting Protocol (LVP) from Amelina et al. (2015), where it was studied for stochastic networks in the context of load balancing problem. The SPSA part represents a stochastic gradient descent of sub-functions $F_t^i(\mathbf{u}_t^i, \mathbf{x}_t^i)$, and LVP part is determined for each agent i by the weighted sum of differences between the information about the current estimate $\widehat{\theta}_{2k-1}^i$ of agent i and available information about the estimates of its neighbors.

Further, we use notation $\bar{\theta}_t = \mathrm{col}\{\widehat{\theta}_t^1, \dots, \widehat{\theta}_t^n\}$ for the common vector of estimates of all agents at time instant t. Also, we introduce the following: $\bar{\mathbf{y}}_t = \mathrm{col}\{y_t^1, \dots, y_t^n\}$, $\bar{\Delta}_{t\div 2} = \mathrm{diag}_{nmd}(\mathrm{col}\{\hat{\Delta}_{t\div 2}^1, \dots, \hat{\Delta}_{t\div 2}^n\})$. Using these notations, the algorithm (7) can be rewritten in the following form

$$\bar{\theta}_{2k} = \bar{\theta}_{2k-1} - \alpha_k \bar{\Psi} \left[\bar{\Delta}_k \left(\frac{\bar{\mathbf{y}}_{2k} - \bar{\mathbf{y}}_{2k-1}}{2\beta} \otimes \mathbf{1}_{md} \right) + \gamma(\mathcal{L}(B_{2k-1}) \otimes I_{md})\bar{\theta}_{2k-1} \right]. \quad (8)$$

The algorithm (7) runs in parallel at each sensor to estimate $\widehat{\theta}_t$. In the next section, we show that all these n sequences converge to the neighborhood of true vector θ_t.

5. MAIN RESULT

In this section, we provide a convergence analysis of the proposed algorithm. First, let us formulate assumptions about the dynamics of the targets, noise, and network topology.

Assumption 1: $\forall i \in \mathcal{N}$, $k = 1, 2, \dots$, matrices $C_{2k}^{\mathbf{u}_k^i}$, $C_{2k-1}^{\mathbf{u}_k^i}$ are invertible.

Assumption 2: For $k = 1, 2, \dots$, the successive differences $\tilde{v}_k^i = v_{2k}^i - v_{2k-1}^i$ of measurement noise are bounded: $|\tilde{v}_k^i| \le c_v < \infty$, or $\mathbb{E}(\tilde{v}_k^i)^2 \le c_v^2$ if sequence $\{\tilde{v}_t^i\}$ is random.

Assumption 3: For all $k = 1, 2, \dots,\ i \in \mathcal{N}, l \in \mathcal{M}$:

a) vectors $\mathbf{u}_k^i$, $\mathbf{\Delta}_k^i$, are mutually independent;

b) if $\mathbf{u}_k^i$, $\mathbf{\Delta}_k^i$are random, they do not depend on the σ-algebra $\mathcal{F}_{2k-2}$;

c) if $\tilde{v}_k^i$ are random, then random vectors $\mathbf{u}_k^i$, $\mathbf{\Delta}_k^i$, and elements $\tilde{v}_k^i$ are independent;

d) $\mathbb{E}\|\mathbf{\Delta}_k^i\|^2 \le \sigma_\Delta^2$, $\mathbb{E}[\mathbf{\Delta}_k^i(\mathbf{\Delta}_k^i)^{\mathrm{T}}] \le \sigma_\Delta^2 I_{md}$.

Assumption 4: a) For all $i \in \mathcal{N},\ j \in \bar{\mathcal{N}}_t^i$ weights $b_t^{i,j}$ are independent random variables with mean $\mathbb{E}b_t^{i,j} = b_{av}^{i,j}$, and $\mathbb{E}[(\mathcal{L}(B_t) - \mathcal{L}(B_{av}))(\mathcal{L}(B_t) - \mathcal{L}(B_{av}))^{\mathrm{T}}] \le Q_B$, $B_{av} = [b_{av}^{i,j}]$. Denote $b_{\max}$ as the maximum element of Q_{B};

b) Graph $\mathcal{G}_{B_{av}}$ is strongly connected.

Our analysis of the proposed algorithm applied to the problem presented in subsection 3.1 relies on the following definition.

Definition 2. A covariance matrix of residual has *an asymptotically efficient upper bound* $S > 0$ if $\exists \bar{k}$ such that $\forall k > \bar{k}$

$$\mathbb{E}[(\bar{\theta}_{2k} - \mathbf{1}_n \otimes \theta)(\bar{\theta}_{2k} - \mathbf{1}_n \otimes \theta)^{\mathrm{T}}] \le \frac{1}{k} S + o(\frac{1}{k}).$$

The following theorem shows the asymptotically efficient upper bound of the covariance matrix of residual provided by the algorithm (7).

Theorem 1: If Assumptions *1–4* hold, $\alpha_k = \frac{1}{k}$ and $-(\gamma\bar{\lambda}_2 + \frac{2}{m})\bar{\Psi} + \frac{1}{2}I_{nmd}$ is stable (Gantmacher and Brenner (2005)), *then* the covariance matrix of residual provided by the algorithm (7) has asymptotically efficient upper bound S, which is the solution of the following equation

$$S\left((\gamma\bar{\lambda}_2 + \frac{2}{m})\bar{\Psi}^{\mathrm{T}} - \frac{1}{2}I_{nmd}\right) + \left((\gamma\bar{\lambda}_2 + \frac{2}{m})\bar{\Psi} - \frac{1}{2}I_{nmd}\right)S = 4nc_v^2\bar{\Psi}\bar{\Psi}^{\mathrm{T}}. \quad (9)$$

6. SIMULATION

In this section, we present a numerical experiment, which illustrates the performance of the suggested algorithm (7).

Given a distributed network of 5 sensors monitoring an area of interest. Let $\mathcal{N} = \{1, 2, 3, 4, 5\}$ be the set of sensors. Each sensor has no more than two active communication channels at each time instant, i.e., $|\bar{\mathcal{N}}_t^i| = 2$. The communication channels are used to collect data from the neighbors. Within the area of interest, there are 10 targets. The sensors have to estimate their states. At time instant t, $\mathbf{s}^i = [s^{i,1}, s^{i,2}]^{\mathrm{T}} \in \mathbb{R}^2$ is the current state of sensor $i \in \mathcal{N}$, $\mathbf{r}^l = [r^{l,1}, r^{l,2}]^{\mathrm{T}} \in \mathbb{R}^2$ is the state of target $l \in \mathcal{M} = \{1, 2, \dots, 10\}$, $\theta = \mathrm{col}\{\mathbf{r}^1, \dots, \mathbf{r}^{10}\}$ is the common state of all targets.

In this sumulation, we consider hybrid noise which is uniformly distributed around constants that change with time, e.g. $v_k^i = \pm 1 + 0.1 * sin(k)$, where the sign in front of 1 switches each 50-th iteration.

According to Theorem 1, the step-size parameter α has to be equal to $\frac{1}{k}$. However, the algorithm (7) working on each node with the parameter $\alpha_k = \frac{1}{k^{1-\rho}}, \forall \rho > 0$ has more consistent convergence. In this simulation, the following parameter were chosen: $\alpha_k = \frac{1}{k^{3/5}}$, $\beta = 0.1$, $\gamma = 1.0$ were chosen to satisfy the conditions of Theorem *1*. The targets are located in the interval $[0; 100]$. The targets and sensors coordinates are random values uniformly distributed in intervals $[0; 100]$ and $[100; 120]$ respectively.

Let us consider for every target l and sensor i at each time instant t the covariance matrix of residuals $\tilde{\Sigma}_t^{i,l} \in \mathbb{R}^{d \times d}$, which is represented as a part of the common covariance matrix. Fig. 1 shows how the average first diagonal element of the covariance matrix of residuals $\tilde{\Sigma}_t^{i,l}$ depending on different matrices Ψ evolves over time. It is well seen that the new algorithm converges. The algorithm for choosing optimal Ψ will be studied in future works.

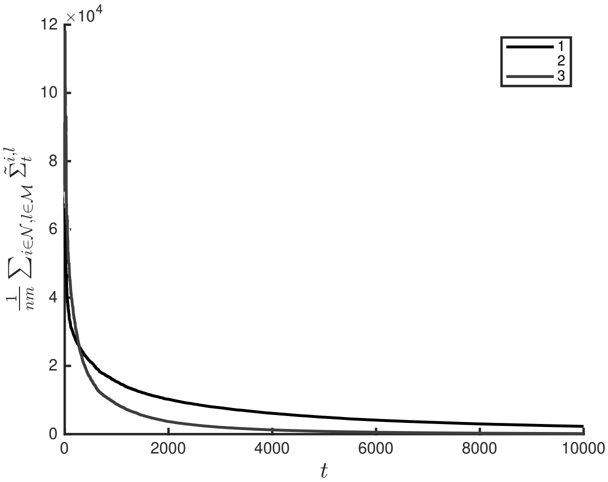

Fig. 1. The average value over all sensors and targets of the first entry of covariance matrix of residuals $\tilde{\Sigma}_t^{i,l}$, where (1): $\Psi_1 = I_{md}$, (2): $\Psi_1 = 0.5I_{md}$, (3): $\Psi_3 = 2I_{md}$

7. CONCLUSION

In this paper, we study the weighted SPSA-based consensus algorithm. We provide the convergence analysis of this algorithm in stationary case. We also determine a suitable step-size of the algorithm based on this analysis. The method is validated through simulation, where the parameters were chosen based on the convergence analysis.

APPENDIX

The proof of Theorem 1:

Denote $\mathbf{d}_t^i = \widehat{\theta}^i_{2\lceil \frac{t-1}{2} \rceil} - \theta$, $\bar{\mathbf{d}}_t = \mathrm{col}\{\mathbf{d}_t^1, \ldots, \mathbf{d}_t^n\}$, where $\lceil \cdot \rceil$ is a ceiling function, $\nu_k = \bar{\mathbf{d}}_{2k}$, $D_k = \nu_k \nu_k^{\mathrm{T}}$, $\Sigma_k = \mathbb{E}[D_k]$, $\bar{\mathbf{s}}_k = \frac{\alpha_k}{2\beta} \bar{\Delta}_k((\bar{\mathbf{y}}_{2k} - \bar{\mathbf{y}}_{2k-1}) \otimes \mathbf{1}_{md})$, $\bar{\mathbf{v}}_t = \mathrm{col}\{\tilde{v}_t^1, \ldots, \tilde{v}_t^n\}$, $\bar{\mathbf{u}}_k = \mathrm{col}\{\mathbf{u}_k^1, \ldots, \mathbf{u}_k^n\}$, $\bar{\Psi} = I_n \otimes \Psi$.

Let $\bar{\mathcal{F}}_{k-1} = \sigma\{\mathcal{F}_{k-1}, \bar{\mathbf{v}}_{2k-1}, \bar{\mathbf{v}}_{2k}, \bar{\mathbf{u}}_k, \bar{\Delta}_k\}$ be the σ-algebra of probabilistic events generated by $\mathcal{F}_{k-1}$, $\bar{\mathbf{v}}_{2k-1}$, $\bar{\mathbf{v}}_{2k}$, $\bar{\Delta}_k$, $\tilde{\mathcal{F}}_{k-1} = \sigma\{\mathcal{F}_{k-1}, \bar{\mathbf{v}}_{2k-1}, \bar{\mathbf{v}}_{2k}, \bar{\mathbf{u}}_k\}$, and $\hat{\mathcal{F}}_{k-1} = \sigma\{\mathcal{F}_{k-1}, \bar{\mathbf{v}}_{2k-1}, \bar{\mathbf{v}}_{2k}\}$: $\mathcal{F}_{k-1} \subset \hat{\mathcal{F}}_{k-1} \subset \tilde{\mathcal{F}}_{k-1} \subset \bar{\mathcal{F}}_{k-1} \subset \mathcal{F}_k$.

Using that $\bar{\theta}_{2k-1} = \bar{\theta}_{2k-2}$ and $\mathcal{L}(B_{2k-2})\mathbf{1}_n = 0$, we get

$$\nu_k = \bar{\theta}_{2k} - \mathbf{1}_n \otimes \theta =$$
$$= \bar{\mathbf{g}}_k - \bar{\Psi}\bar{\mathbf{s}}_k - \alpha_k \gamma \bar{\Psi}[(\mathcal{L}(B_{2k-2}) - \mathcal{L}(B_{av})) \otimes I_{md}]\nu_{k-1},$$

where $\bar{\mathbf{g}}_k = [I_{nmd} - \alpha_k \gamma \bar{\Psi}(\mathcal{L}(B_{av}) \otimes I_{md})]\nu_{k-1}$. Then,

$$D_k = \bar{\mathbf{g}}_k \bar{\mathbf{g}}_k^{\mathrm{T}} - \bar{\mathbf{g}}_k \bar{\mathbf{s}}_k^{\mathrm{T}} \bar{\Psi}^{\mathrm{T}} - \bar{\Psi}\bar{\mathbf{s}}_k \bar{\mathbf{g}}_k^{\mathrm{T}} + \bar{\Psi}\bar{\mathbf{s}}_k \bar{\mathbf{s}}_k^{\mathrm{T}} \bar{\Psi}^{\mathrm{T}} -$$
$$\alpha_k \gamma (\bar{\mathbf{g}}_k - \bar{\Psi}\bar{\mathbf{s}}_k)\nu_{k-1}^{\mathrm{T}}[(\mathcal{L}(B_{2k-2}) - \mathcal{L}(B_{av})) \otimes I_{md}]^{\mathrm{T}} \bar{\Psi}^{\mathrm{T}} -$$
$$\alpha_k \gamma \bar{\Psi}[(\mathcal{L}(B_{2k-2}) - \mathcal{L}(B_{av})) \otimes I_{md}]\nu_{k-1}(\bar{\mathbf{g}}_k^{\mathrm{T}} - \bar{\mathbf{s}}_k^{\mathrm{T}} \bar{\Psi}^{\mathrm{T}}) +$$
$$\alpha_k^2 \gamma^2 \bar{\Psi}[(\mathcal{L}(B_{2k-2}) - \mathcal{L}(B_{av})) \otimes I_{md}]D_{k-1}[(\mathcal{L}(B_{2k-2}) -$$
$$\mathcal{L}(B_{av})) \otimes I_{md}]^{\mathrm{T}} \bar{\Psi}^{\mathrm{T}}.$$

1. Consider σ-algebra $\bar{\mathcal{F}}_{k-1}$.

Now, we take the conditional expectation over σ-algebra $\bar{\mathcal{F}}_{k-1}$ and apply Assumption *4*:

$$\mathbb{E}_{\bar{\mathcal{F}}_{k-1}}[D_k] \leq \bar{\mathbf{g}}_k \bar{\mathbf{g}}_k^{\mathrm{T}} - \bar{\mathbf{g}}_k \bar{\mathbf{s}}_k^{\mathrm{T}} \bar{\Psi}^{\mathrm{T}} - \bar{\Psi}\bar{\mathbf{s}}_k \bar{\mathbf{g}}_k^{\mathrm{T}} + \bar{\Psi}\bar{\mathbf{s}}_k \bar{\mathbf{s}}_k^{\mathrm{T}} \bar{\Psi}^{\mathrm{T}} +$$
$$\alpha_k^2 \gamma^2 b_{\max} \|D_{k-1}\|^2 \bar{\Psi}\bar{\Psi}^{\mathrm{T}}, \qquad (10)$$

where $b_{\max}$ is the maximum element of Q_{B}.

2. Consider σ-algebra $\tilde{\mathcal{F}}_{k-1}$.

After we take the conditional expectation over σ-algebra $\tilde{\mathcal{F}}_{k-1}$ step by step:

$$\mathbb{E}_{\tilde{\mathcal{F}}_{k-1}}[D_k] \leq \bar{\mathbf{g}}_k \bar{\mathbf{g}}_k^{\mathrm{T}} - \bar{\mathbf{g}}_k \mathbb{E}_{\tilde{\mathcal{F}}_{k-1}}[\bar{\mathbf{s}}_k^{\mathrm{T}}] \bar{\Psi}^{\mathrm{T}} - \bar{\Psi}\mathbb{E}_{\tilde{\mathcal{F}}_{k-1}}[\bar{\mathbf{s}}_k] \bar{\mathbf{g}}_k^{\mathrm{T}} +$$
$$\bar{\Psi}\mathbb{E}_{\tilde{\mathcal{F}}_{k-1}}[\bar{\mathbf{s}}_k \bar{\mathbf{s}}_k^{\mathrm{T}}] \bar{\Psi}^{\mathrm{T}} + \alpha_k^2 \gamma^2 b_{\max} \|D_{k-1}\|^2 \bar{\Psi}\bar{\Psi}^{\mathrm{T}}. \qquad (11)$$

Under fulfilment of Assumption *4b*, we have $\bar{\lambda}_2 > 0$ (see Olfati-Saber and Murray (2004)). Hence, for the first term in (11) we derive

$$\bar{\mathbf{g}}_k \bar{\mathbf{g}}_k^{\mathrm{T}} \leq D_{k-1} - \alpha_k \gamma \bar{\lambda}_2 (\bar{\Psi} D_{k-1} + D_{k-1} \bar{\Psi}^{\mathrm{T}}) +$$
$$\alpha_k^2 \gamma^2 \bar{\lambda}_{\max}^2 \|D_{k-1}\|^2 \bar{\Psi}\bar{\Psi}^{\mathrm{T}}.$$

By virtue of Assumptions *1, 3* we can evaluate the second and the third term in (11) as following. Denote $\mathbf{r}^{h(\mathbf{u}_k^i)} = \mathbf{e}_{h(\mathbf{u}_k^i)} \otimes [C^{\mathbf{u}_k^i}]^{-1} D^{\mathbf{u}_k^i}$, $\widehat{\mathbf{r}}_t^{h(\mathbf{u}_k^i)} = \mathrm{diag}_{md}(\mathbf{e}_{h(\mathbf{u}_k^i)} \otimes I_d)\widehat{\theta}_t^i$, $\tilde{v}_k^i = v_{2k}^i - v_{2k-1}^i$, then $\forall i \in \{1, \ldots, n\}$:

$$y_{2k}^i - y_{2k-1}^i = 4\beta(\hat{\Delta}_k^i)^{\mathrm{T}}(\widehat{\mathbf{r}}_{2k-2}^{h(\mathbf{u}_k^i)} - \mathbf{r}^{h(\mathbf{u}_k^i)}) + \tilde{v}_k^i.$$

Under Assumption *3* we have $\mathbb{E}_{\tilde{\mathcal{F}}_{k-1}}[\tilde{v}_k^i \hat{\Delta}_k^i] = 0$. Denote $\bar{R}_t = \mathrm{diag}_{nmd}(\mathrm{col}\{\mathbf{e}_{h(\mathbf{u}^1_{t \div 2})} \otimes I_d, \ldots, \mathbf{e}_{h(\mathbf{u}^n_{t \div 2})} \otimes I_d\})$.By Assumption *3*, using that Δ_k^i is drawn from the symmetric distribution, for the fourth term in (11), we obtain

$$\mathbb{E}_{\tilde{\mathcal{F}}_{k-1}}[\bar{\mathbf{s}}_k \bar{\mathbf{s}}_k^{\mathrm{T}}] \leq 4\alpha_k^2 \mathbb{E}_{\tilde{\mathcal{F}}_{k-1}}[\bar{\Delta}_k (\bar{\Delta}_k)^{\mathrm{T}} \bar{R}_k D_{k-1} \bar{R}_k \bar{\Delta}_k^i (\bar{\Delta}_k)^{\mathrm{T}}] +$$
$$\frac{\alpha_k^2}{4\beta^2} \sum_{i \in \mathcal{N}} (\tilde{v}_k^i)^2 \mathbb{E}_{\tilde{\mathcal{F}}_{k-1}}[\bar{\Delta}_k^i (\bar{\Delta}_k)^{\mathrm{T}}].$$

3. Consider σ-algebra $\hat{\mathcal{F}}_{k-1}$.

Denote $Z_k = \mathbb{E}[\bar{\Delta}_k (\bar{\Delta}_k)^{\mathrm{T}}]$. Summing up the second and the third term from (11) and taking the conditional expectation over σ-algebra $\hat{\mathcal{F}}_{k-1}$,we derive the following:

$$-\mathbb{E}_{\hat{\mathcal{F}}_{k-1}}[\bar{\mathbf{g}}_k \bar{\mathbf{s}}_k^{\mathrm{T}}] \bar{\Psi}^{\mathrm{T}} - \bar{\Psi}\mathbb{E}_{\hat{\mathcal{F}}_{k-1}}[\bar{\mathbf{s}}_k \bar{\mathbf{g}}_k^{\mathrm{T}}] \leq$$
$$-2\frac{\alpha_k}{m}(D_{k-1} Z_k^{\mathrm{T}} \bar{\Psi}^{\mathrm{T}} + \bar{\Psi} Z_k D_{k-1}) +$$
$$2\alpha_k^2 \gamma \frac{1}{m} \bar{\lambda}_{\max} \bar{\Psi}(D_{k-1} Z_k^{\mathrm{T}} + Z_k D_{k-1}) \bar{\Psi}^{\mathrm{T}}.$$

4. Consider σ-algebra $\hat{\mathcal{F}}_{k-1}$.

After we take the conditional expectation over σ-algebra $\hat{\mathcal{F}}_{k-1}$:

$$\mathbb{E}_{\hat{\mathcal{F}}_{k-1}}[\bar{\mathbf{s}}_k \bar{\mathbf{s}}_k^{\mathrm{T}}] \leq 4\frac{\alpha_k^2}{m^2} \|D_{k-1}\| \mathbb{E}_{\hat{\mathcal{F}}_{k-1}}[\|\bar{\Delta}_k\|^4] +$$
$$\frac{\alpha_k^2}{4\beta^2} \sum_{i \in \mathcal{N}} (\tilde{v}_k^i)^2 Z_k.$$

5. Consider σ-algebra $\mathcal{F}_{k-1}$:

Finally, taking the conditional expectation over σ-algebra $\mathcal{F}_{k-1}$, by virtue of Assumption *2*, for the fourth term in (11) we get

$$\bar{\Psi}\mathbb{E}_{\mathcal{F}_{k-1}}[\bar{\mathbf{s}}_k\bar{\mathbf{s}}_k^{\mathrm{T}}]\bar{\Psi}^{\mathrm{T}} \leq 4\frac{\alpha_k^2}{m^2}\|D_{k-1}\|\bar{\Psi}\|\mathbb{E}_{\mathcal{F}_{k-1}}[\|\bar{\Delta}_k\|^4]\bar{\Psi}^{\mathrm{T}}+ \frac{\alpha_k^2}{4\beta^2}nc_v^2\bar{\Psi}Z_k.$$

Summing up the bounds, taking the unconditional expectation, and considering that $\|z_k^{-1}Z_k - I_{nmd}\| = \mathcal{O}(k^{-1})$, $\alpha_k z_k = k^{-1}$ we derive the following from (11)

$$\Sigma_k \leq \Sigma_{k-1} - (\alpha_k\gamma\bar{\lambda}_2 + \frac{1}{k}\frac{2}{m})(\Sigma_{k-1}\bar{\Psi}^{\mathrm{T}} + \bar{\Psi}\Sigma_{k-1})+ 4\alpha_k^2 z_k nc_v^2\bar{\Psi}\bar{\Psi}^{\mathrm{T}} + \frac{1}{k}\kappa_k\mathcal{O}(\|\Sigma_{k-1}\|) + o(k^{-2}),$$

where $\{\kappa_k\} : \kappa_k \to 0$ if $k \to \infty$.

Let $\alpha_k = \frac{1}{k}$, $W_k = \frac{1}{\frac{1}{k}\gamma\bar{\lambda}_2+\frac{1}{k}\frac{2}{m}}(\Sigma_k - \frac{1}{k}S)$. Consider the equation (9): if $-(\gamma\bar{\lambda}_2 + \frac{2}{m})\bar{\Psi} + \frac{1}{2}I_{nmd}$ is stable, then there exists a positive-definite matrix S which is a solution of this Lyapunov equation.

Then, according to Lemma 9 from Granichin and Polyak (2003), $W_k \xrightarrow[k\to\infty]{} 0$.

This completes the proof of Theorem *1*.

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