Dispersion, spreading and sparsity of Gabor wave packets for metaplectic and Schrödinger operators

DISPERSION, SPREADING AND SPARSITY OF GABOR WAVE PACKETS FOR METAPLECTIC AND SCHRÖDINGER OPERATORS

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Abstract. Sparsity properties for phase-space representations of several types of operators have been extensively studied in recent articles, including pseudodifferential, Fourier integral and metaplectic operators, with applications to the analysis of Schrödinger-type evolution equations. It has been proved that such operators are approximately diagonalized by Gabor wave packets. While the latter are expected to undergo some spreading phenomenon, there is no record of this issue in the aforementioned results. In this note we prove refined estimates for the Gabor matrix of metaplectic operators, also of generalized type, where sparsity, spreading and dispersive properties are all noticeable. We provide applications to the propagation of singularities for the Schrödinger equation.

1. Introduction and discussion of the results

The relevance of the notion of wave packet in harmonic analysis and mathematical physics can be hardly overestimated. Roughly speaking, we say that a function g on $\mathbb{R}^d$ is a wave packet if it does possess good localization in phase space. To be more concrete, recall that good energy concentration of a function $g \in \mathcal{S}(\mathbb{R}^d) \setminus \{0\}$ (the Schwartz class) on a measurable set $T \subset \mathbb{R}^d$ is achieved if there exists $0 \le \delta_T \le 1/2$ such that

$$\left(\int_{\mathbb{R}^d \setminus T} |g(y)|^2 dy \right)^{1/2} \le \delta_T \, \|g\|_{L^2} .$$

The spectral content of g on a set $\Omega \subset \mathbb{R}^d$ is well concentrated if the analogous estimate is satisfied by its Fourier transform $\widehat{g}$ for small δ_Ω. Therefore g is concentrated on the cell $T \times \Omega$ in the phase space and the Donoho-Stark uncertainty principle prescribes a lower bound for the measure of such cell in terms of δ_T and δ_Ω [17].

The essential phase-space support of g can be moved to $(x+T) \times (\xi + \Omega)$ for any choice of $(x, \xi) \in \mathbb{R}^{2d}$ by applying a *phase-space shift* $\pi(x, \xi) = M_\xi T_x$ to g, namely

2020 *Mathematics Subject Classification.* 43A65, 35S05, 42B35, 81S30.

Key words and phrases. phase space analysis, Gabor wave packets, metaplectic operators, Schrödinger equation, dispersive estimates, almost diagonalization, modulation spaces, Wigner distribution.

as a result of the joint action of the modulation operator M_ξ and the translation operator T_x, respectively defined as

$$M_\xi g(y) = e^{2\pi i y\cdot\xi} g(y), \qquad T_x g(y) = g(y-x), \quad y \in \mathbb{R}^d.$$

Functions of the type $\pi(z)g$ for some fixed $z \in \mathbb{R}^{2d}$ and $g \in \mathcal{S}(\mathbb{R}^d)$ are called *Gabor wave packets* or *atoms*. In the case where $g(y) = e^{-\pi|y|^2}$ we speak of Gaussian wave packets; the latter are well-known textbook examples in physics.

Analysis of functions and operators in terms of Gabor wave packets is one of the primary purposes of modern time-frequency analysis [2, 14, 26]. For instance, a phase-space representation of a signal $f \in L^2(\mathbb{R}^d)$ is provided by the *short-time Fourier transform*, which ultimately amounts to a decomposition of f along the uniform boxes in phase space occupied by the Gabor atoms $\pi(z)g$, $z \in \mathbb{R}^{2d}$, for some fixed window function $g \in \mathcal{S}(\mathbb{R}^d) \setminus \{0\}$. It is defined as

$$V_g f(x,\xi) \coloneqq \langle f, \pi(x,\xi)g\rangle = \int_{\mathbb{R}^d} e^{-2\pi i y\cdot\xi} f(y)\, \overline{g(y-x)}\, dy, \quad (x,\xi) \in \mathbb{R}^{2d}.$$

Phase-space analysis of operators can be conducted along the same lines by investigating how they act at the atomic level. Precisely, the (continuous) *Gabor matrix* of a linear continuous operator $A : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$ with respect to analysis and synthesis windows $g, \gamma \in \mathcal{S}(\mathbb{R}^d) \setminus \{0\}$ is defined by

$$K_A(w,z) \coloneqq \langle A\pi(z)g, \pi(w)\gamma\rangle, \quad w, z \in \mathbb{R}^{2d}.$$

It can be regarded as an infinite matrix encoding the phase-space features of A, since its action on phase space reads as an integral operator with kernel K_A: under the additional assumption $\|g\|_{L^2} = \|\gamma\|_{L^2} = 1$ we have indeed the identity

$$V_\gamma(Af)(w) = \int_{\mathbb{R}^{2d}} K_A(w,z) V_g f(z) dz, \quad w \in \mathbb{R}^{2d}.$$

It is clear that sparsity of K_A is a highly desirable property, for both theoretical and numerical purposes. Several results concerning the approximate diagonalization of operators at the Gabor matrix level have been appearing in the literature, in particular for pseudodifferential operators [12, 23, 25, 31], Fourier integral operators [6, 7, 10] and propagators associated with Cauchy problems for Schrödinger-type evolution equations [8, 11]. We stress that wave packets should be tailored in order to best fit the geometry of the problem. For instance, the Gabor matrix of Fourier integral operators arising as propagators for strictly hyperbolic equations does not display a sparse behaviour, while analogous representations involving curvelet atoms do enjoy super-polynomial decay, cf. [4, 16]. See also [27, 36, 38] for other applications of wave packet analysis.

For the sake of concreteness let us focus on the Schrödinger propagator for the free particle $U(t) = e^{i(t/2\pi)\Delta}$, $t \in \mathbb{R}$, and fix $g \in \mathcal{S}(\mathbb{R}^d) \setminus \{0\}$. For any $t \in \mathbb{R}$ and $N \in \mathbb{N}$ there exists a constant $C = C(t, N) > 0$ such that the following decay estimate for the Gabor matrix elements of $U(t)$ holds:

$$|\langle e^{i(t/2\pi)\Delta}\pi(z)g, \pi(w)g\rangle| \le C(1 + |w - S_t z|)^{-N}, \quad w, z \in \mathbb{R}^{2d}, \tag{1}$$

where $S_t \in \mathbb{R}^{2d\times 2d}$ is the block matrix

$$S_t = \begin{bmatrix} I & 2tI \\ O & I \end{bmatrix}, \tag{2}$$

$I \in \mathbb{R}^{d\times d}$ is the identity matrix and $O \in \mathbb{R}^{d\times d}$ is the null matrix. We remark that $t \mapsto S_t$ coincides with the Hamiltonian flow for the free particle in phase space; precisely, the classical equations of motion with Hamiltonian $H(x, \xi) = |\xi|^2$ and initial datum $(x_0, \xi_0) \in \mathbb{R}^{2d}$ are solved by $(x(t), \xi(t)) = S_t(x_0, \xi_0)$. Hence (1) shows that the time evolution of wave packets under $U(t)$ approximately follows the classical flow, in accordance with the correspondence principle of quantum mechanics.

Nevertheless, a distinctive feature of wave propagation dynamics is the unavoidable effect of diffraction. In the situation under our attention it does manifest itself as the well-known phenomenon of the spreading of wave packets. Moreover, a straightforward consequence of the dispersive estimates for the Schrödinger propagator [37] is that there exists $C = C(\|g\|_{L^1\cap L^2}) > 0$ such that

$$|\langle e^{i(t/2\pi)\Delta}\pi(z)g, \pi(w)g\rangle| \le C(1 + |t|)^{-d/2}, \quad w, z \in \mathbb{R}^{2d}. \tag{3}$$

It may therefore appear quite unsatisfactory that there is no trace of such issues in quasi-diagonalization estimates as (1). The purpose of this note is exactly to prove refined estimates for the Gabor matrix of $U(t)$ where sparsity, spreading and dispersive phenomena are fully represented. To the best of our knowledge, we are not aware of systematic studies in this spirit for pseudodifferential or evolution operators.

Our quest is in fact motivated by the more general situation where $U(t)$ is the Schrödinger propagator corresponding to the Hamiltonian $H = Q^{\mathrm{w}}$, where Q is a real homogeneous quadratic polynomial on $\mathbb{R}^{2d}$ and Q^{w} denotes its Weyl quantization, (formally) defined as

$$Q^{\mathrm{w}} f(x) = \int_{\mathbb{R}^{2d}} e^{2\pi i(x-y)\cdot\xi} Q\left(\frac{x+y}{2}, \xi\right) f(y) dy d\xi.$$

For example, $(2\pi\xi_j)^{\mathrm{w}} = -i\partial_{x_j}$, $j = 1, \dots, d$, and $Q^{\mathrm{w}} = -\Delta$ for the choice $Q(x, \xi) = 4\pi^2|\xi|^2$.

The propagator $U(t) = e^{-2\pi i t Q^{\mathrm{w}}}$, $t \in \mathbb{R}$, turns out to be a *metaplectic operator*. In short, the metaplectic representation is a machinery which associates a symplectic matrix $S \in \mathrm{Sp}(d, \mathbb{R})$ with a member of the metaplectic group $\mu(S) \in \mathrm{Mp}(d, \mathbb{R})$, that is a unitary operator on $L^2(\mathbb{R}^d)$ defined up to the sign. If $\mathbb{R} \ni t \mapsto S_t \in \mathrm{Sp}(d, \mathbb{R})$

denotes the classical flow on phase space associated with the quadratic Hamiltonian $H(x,\xi)=Q(x,\xi)$ then $\mu(S_t)=\pm e^{-2\pi i t Q^{\mathrm{w}}}$ - see (2) for the free particle case. We refer to Section 2.5 below for further details and [18, 22] for comprehensive discussions on the metaplectic representation.

It is therefore convenient to focus on metaplectic operators as primary objects of our investigation. The spreading of wave packets under $\mu(S)$ is now connected with the *singular values* of $S\in\mathrm{Sp}(d,\mathbb{R})$ [5], which occur in couples (σ,σ^{-1}) of positive real numbers. We fix the ordering by labelling the largest d singular values in such a way that $\sigma_1\geq\ldots\geq\sigma_d\geq 1$; moreover we set $\Sigma=\mathrm{diag}(\sigma_1,\ldots,\sigma_d)$ and introduce the matrices

$$D=\begin{bmatrix}\Sigma & O\\ O & \Sigma^{-1}\end{bmatrix},\quad D'=\begin{bmatrix}\Sigma^{-1} & O\\ O & I\end{bmatrix},\quad D''=\begin{bmatrix}I & O\\ O & \Sigma^{-1}\end{bmatrix}.$$

The singular value decomposition of $S\in\mathrm{Sp}(d,\mathbb{R})$ (also known as the *Euler decomposition* in this setting) has a peculiar form due to the symplectic condition, namely there exist (non-unique) orthogonal and symplectic matrices U,V such that $S=U^\top DV$, cf. Proposition 2.1 below. Such factorization is identified by the triple (U,V,Σ). In the following for a given $S\in\mathrm{Sp}(d,\mathbb{R})$ we will denote by (U,V,Σ) an Euler decomposition of S and by D,D',D'' the above defined related matrices.

We are now in the position to state our first result, concerning rapidly decaying Gabor wave packets.

Theorem 1.1. *For any $g,\gamma\in\mathcal{S}(\mathbb{R}^d)$ and $N>0$ there exists $C>0$ such that, for every $S\in\mathrm{Sp}(d,\mathbb{R})$ and every Euler decomposition (U,V,Σ) of S,*

$$|\langle\mu(S)\pi(z)g,\pi(w)\gamma\rangle|\leq C(\det\Sigma)^{-1/2}(1+|D'U(w-Sz)|)^{-N},\quad z,w\in\mathbb{R}^{2d}. \tag{4}$$

We see that the simultaneous occurrence of sparsity, spreading and dispersive phenomena are represented by the quasi-diagonal structure along S, the dilation by $D'U$ and the factor $(\det\Sigma)^{-1/2}$ respectively. An equivalent form of the previous estimate where the spreading phenomenon is somehow more distributed follows by noticing that $D'U(w-Sz)=D'Uw-D''Vz$.

The special case of the free particle propagator is treated in detail in Section 4 below. We just mention here that, for any fixed $t\in\mathbb{R}$ and any Euler decomposition (U_t,V_t,Σ_t) of S_t, the estimate (4) reads

$$\left|\langle e^{i(t/2\pi)\Delta}\pi(z)g,\pi(w)\gamma\rangle\right|\leq C(1+|t|)^{-d/2}(1+|D_t'U_t(w-S_tz)|)^{-N},\quad w,z\in\mathbb{R}^{2d}.$$

We see that the features of both (1) and (3) are now represented, whereas the spreading phenomenon manifests itself as a dilation by the matrix $D_t'U_t$, the nature of which is investigated in Section 4.

We provide results in the same spirit of Theorem 1.1 for wave packets associated with less regular atoms; in particular we assume that g and γ satisfy certain phase-space decay conditions. The function spaces arising by imposing some weighted

Lebesgue regularity on the short-time Fourier transform of a function are of primary concern in time-frequency analysis and are known as *modulation spaces* [20]. To be precise, let $1 \le p < \infty$ and $s \in \mathbb{R}$, and define the polynomial weight function $v_s(z) \coloneqq (1+|z|)^s$ on $\mathbb{R}^{2d}$. The modulation space $M^p_{v_s}(\mathbb{R}^d)$ is defined as the subset of temperate distributions $f \in \mathcal{S}'(\mathbb{R}^d)$ such that, for any $g \in \mathcal{S}(\mathbb{R}^d) \setminus \{0\}$,

$$\|f\|_{M^p_{v_s}} \coloneqq \left(\int_{\mathbb{R}^{2d}} |\langle f, \pi(z)g\rangle|^p v_s(z)^p dz \right)^{1/p} < \infty.$$

Similarly, we say that $f \in M^\infty_{v_s}(\mathbb{R}^d)$ if there exists $C > 0$ such that, for any $g \in \mathcal{S}(\mathbb{R}^d) \setminus \{0\}$,

$$|\langle f, \pi(z)g\rangle| \le C(1+|z|)^{-s}, \quad z \in \mathbb{R}^{2d}.$$

We write $M^p(\mathbb{R}^d)$ for the unweighted case ($s = 0$).

We collect some of the properties of modulation spaces in Proposition 2.2 below. We just recall that they provide a refined framework of (Banach) spaces which encompasses several classical spaces of real harmonic analysis. For instance, we have that $M^2_{v_s}(\mathbb{R}^d)$, $s \in \mathbb{R}$, coincides with the Shubin-Sobolev space of order s [33], namely

$$Q^s(\mathbb{R}^d) = L^2_{v_s}(\mathbb{R}^d) \cap H^s(\mathbb{R}^d) = \{f \in \mathcal{S}'(\mathbb{R}^d) : f, \widehat{f} \in L^2_{v_s}(\mathbb{R}^d)\}.$$

Note in particular that $M^2(\mathbb{R}^d) = L^2(\mathbb{R}^d)$. Moreover, the modulation spaces $M^p_{v_s}$ are related to the Schwartz class (and its dual space $\mathcal{S}'$) via the following characterizations, for every $1 \le p \le \infty$,

$$\mathcal{S}(\mathbb{R}^d) = \bigcap_{s \ge 0} M^p_{v_s}(\mathbb{R}^d), \quad \mathcal{S}'(\mathbb{R}^d) = \bigcup_{s \ge 0} M^p_{v_{-s}}(\mathbb{R}^d). \tag{5}$$

Modulation spaces also provide an optimal environment where to investigate the behaviour of the Gabor matrix of a metaplectic operator, as evidenced by the following result.

Theorem 1.2. *(i) Let $1 \le p, q, r \le \infty$ satisfy $1/p + 1/q = 1 + 1/r$. For any $g \in M^p(\mathbb{R}^d)$ and $\gamma \in M^q(\mathbb{R}^d)$, $S \in \mathrm{Sp}(d, \mathbb{R})$ and any Euler decomposition (U, V, Σ) of S, there exists $H \in L^r(\mathbb{R}^{2d})$ such that*

$$|\langle \mu(S)\pi(z)g, \pi(w)\gamma\rangle| \le H(D'U(w - Sz)), \quad w, z, \in \mathbb{R}^{2d}, \tag{6}$$

with

$$\|H\|_{L^r} \le (\det \Sigma)^{1/2-1/r} \|g\|_{M^p} \|\gamma\|_{M^q}. \tag{7}$$

(ii) Let $s > 2d$. For any $g, \gamma \in M^\infty_{v_s}(\mathbb{R}^d)$, $S \in \mathrm{Sp}(d, \mathbb{R})$ and any Euler decomposition (U, V, Σ) of S, there exists $H \in L^\infty_{v_{s-2d}}(\mathbb{R}^{2d})$ such that (6) holds, with

$$\|H\|_{L^\infty_{v_{s-2d}}} \le (\det \Sigma)^{-1/2} \|g\|_{M^\infty_{v_s}} \|\gamma\|_{M^\infty_{v_s}}. \tag{8}$$

Here we used the notation $\|H\|_{L^\infty_{v_s}} = \|Hv_s\|_{L^\infty}$. We remark that the best decay in (7) is achieved in the case where $p = q = r = 1$, namely for Gabor atoms belonging to the modulation space $M^1(\mathbb{R}^d)$ - the so-called *Feichtinger algebra* [21, 29]. We also highlight the inclusion $M^\infty_{v_s}(\mathbb{R}^d) \subset M^1(\mathbb{R}^d)$ for $s > 2d$, which follows directly from the definition and the integrability of v_{-s} over $\mathbb{R}^{2d}$ for $s > 2d$.

In Theorem 3.3 we prove an estimate in the same spirit of Theorem 1.2 for the Gabor matrix of the so-called *generalized metaplectic operators*. This family of operators characterized by the sparsity of their phase-space representation has been introduced and studied in [6, 7] in connection with inverse-closed algebras of Fourier integral operators. Their main properties are recalled in Section 2.5.

Finally, we provide an application of the enhanced estimates for the Gabor matrix to the propagation of singularities for the Schrödinger equation in terms of new notions of *global* wave front sets, after Hörmander [28]. In fact, several notions of global wave front set have been introduced to detect (lack of) regularity at the modulation space level; see [32] for a more detailed discussion and [30, 39] for further applications.

Given an open cone Γ in $\mathbb{R}^{2d}$ and $g \in \mathcal{S}(\mathbb{R}^d) \setminus \{0\}$ we define the space of $M^1_{(g)}(\Gamma)$ of M^1-regular distributions on the cone Γ with respect to g as the set of all $f \in \mathcal{S}'(\mathbb{R}^d)$ such that

$$\|f\|_{M^1_{(g)}(\Gamma)} := \int_\Gamma |V_g f(z)| dz < \infty. \tag{9}$$

The next result shows that M^1-regularity of a function f on a conic subset of the phase space is preserved by the action of $\mu(S)$ provided that the cone evolves under S. We set $\mathbb{S}^{2d-1}$ for the sphere in $\mathbb{R}^{2d}$.

Theorem 1.3. *Let $S \in \mathrm{Sp}(d, \mathbb{R})$, $g, \gamma \in \mathcal{S}(\mathbb{R}^d) \setminus \{0\}$ and $\Gamma, \Gamma' \subset \mathbb{R}^{2d}$ be open cones such that $\overline{\Gamma' \cap \mathbb{S}^{2d-1}} \subset \Gamma \cap \mathbb{S}^{2d-1}$. If $f \in \mathcal{S}'(\mathbb{R}^d)$ is M^1-regular on Γ with respect to g then $\mu(S)f$ is M^1-regular on $S(\Gamma')$ with respect to γ.*

Precisely, given $r \geq 0$ there exists $C > 0$ such that, for any $f \in M^1_{v_{-r}}(\mathbb{R}^d) \cap M^1_{(g)}(\Gamma)$ (cf. (5)), $S \in \mathrm{Sp}(d, \mathbb{R})$ and any Euler decomposition (U, V, Σ) of S, the following estimate holds:

$$\|\mu(S)f\|_{M^1_{(\gamma)}(S(\Gamma'))} \leq C(\det \Sigma)^{1/2} \left(\|f\|_{M^1_{(g)}(\Gamma)} + (\det \Sigma)^r \|f\|_{M^1_{v_{-r}}(\mathbb{R}^d)} \right).$$

If we specialize the previous result to the free particle propagator we get

$$\left\| e^{i(t/2\pi)\Delta} f \right\|_{M^1_{(\gamma)}(S_t(\Gamma'))} \leq C \left((1 + |t|)^{d/2} \|f\|_{M^1_{(g)}(\Gamma)} + (1 + |t|)^{d(1/2+r)} \|f\|_{M^1_{v_{-r}}(\mathbb{R}^d)} \right),$$

where S_t is the classical flow in (2). The latter can be regarded as a microlocal refinement of known estimates, cf. for instance [40, Prop. 6.6] and Corollary 3.5 below.

In short, this note is organized as follows. In Section 2 we collect some auxiliary results. Section 3 is devoted to the proof of the main results. Section 4, as already anticipated, provides the example of the Schrödinger free propagator in detail.

2. Preliminaries

2.1. **Notation.** We set $|t|^2 = t \cdot t$, $t \in \mathbb{R}^d$, where $x \cdot y$ is the scalar product on $\mathbb{R}^d$. The bracket $\langle f, g\rangle$ denotes the extension to $\mathcal{S}'(\mathbb{R}^d) \times \mathcal{S}(\mathbb{R}^d)$ of the inner product $\langle f, g\rangle = \int_{\mathbb{R}^d} f(t)\overline{g(t)}dt$ on $L^2(\mathbb{R}^d)$.

The conjugate exponent p' of $p \in [1,\infty]$ is defined by $1/p + 1/p' = 1$ if $1 \le p < \infty$ and as $p' = 1$ if $p = \infty$. The symbol $\lesssim$ means that the underlying inequality holds up to a universal positive constant factor; the latter may possibly depend on some "allowable" parameter λ, in which case we write

$$f \lesssim_\lambda g \quad \Rightarrow \quad \exists C = C(\lambda) > 0 \, : \, f \le Cg.$$

Moreover, $f \asymp g$ stands for the case where both $f \lesssim g$ and $g \lesssim f$ hold.

The characteristic function of a set $A \subset \mathbb{R}^d$ is denoted by 1_A. Recall that $\Gamma \subset \mathbb{R}^d$ is a conic subset of $\mathbb{R}^d$ if it is invariant under multiplication by positive real numbers, namely $x \in \Gamma \Rightarrow \lambda x \in \Gamma$ for any $\lambda > 0$.

We choose the following normalization for the Fourier transform:

$$\mathcal{F}(f)(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^d} e^{-2\pi i x\cdot\xi} f(x)dx, \quad \xi \in \mathbb{R}^d.$$

The reflection operator is defined as $f^\vee(t) = f(-t)$, $t \in \mathbb{R}^d$.

Given $A, B \in \mathbb{R}^{d\times d}$ the direct sum $A \oplus B \in \mathbb{R}^{2d\times 2d}$ is defined as

$$A \oplus B = \mathrm{diag}(A, B) = \begin{bmatrix} A & O \\ O & B \end{bmatrix}.$$

2.2. **Symplectic matrices.** The canonical symplectic matrix $J \in \mathbb{R}^{2d\times 2d}$ is defined as

$$J = \begin{bmatrix} O & I \\ -I & O \end{bmatrix}.$$

The symplectic group $\mathrm{Sp}(d,\mathbb{R})$ is defined by

$$\mathrm{Sp}(d,\mathbb{R}) = \{S \in \mathrm{GL}(2d,\mathbb{R}) \, : \, S^\top J S = J\}.$$

Recall that the complex unitary group $U(d,\mathbb{C})$ is isomorphic to the subgroup of *symplectic rotations* $\mathrm{U}(2d,\mathbb{R})$ of $\mathrm{Sp}(d,\mathbb{R})$ [18], namely

$$\mathrm{U}(2d,\mathbb{R}) = \mathrm{Sp}(d,\mathbb{R}) \cap \mathrm{O}(2d,\mathbb{R}).$$

An equivalent, more concrete representation of symplectic rotations is

$$\mathrm{U}(2d,\mathbb{R}) = \left\{ \begin{bmatrix} A & -B \\ B & A \end{bmatrix} \in \mathbb{R}^{2d\times 2d} \, : \, AA^\top + BB^\top = I, \, AB^\top = B^\top A \right\}.$$

We recall a result on a SVD-like decomposition of symplectic matrices, also known as the *Euler decomposition* in the literature; see [34, Appendix B.2] for details and proofs.

Proposition 2.1. *For any $S \in \mathrm{Sp}(d,\mathbb{R})$ there exist $U, V \in \mathrm{U}(2d,\mathbb{R})$ such that*

$$S = U^\top D V, \quad D = \Sigma \oplus \Sigma^{-1},$$

where $\Sigma = \mathrm{diag}(\sigma_1, \ldots, \sigma_d)$ and $\sigma_1 \geq \ldots \geq \sigma_d \geq \sigma_d^{-1} \geq \ldots \geq \sigma_1^{-1}$ are the singular values of S.

We stress that while Σ is uniquely determined for given S once the order of the singular values is fixed, the matrices U and V appearing in such factorization are not unique in general due to possible occurrence of degenerate singular values.

We identify any Euler decomposition of S as $U^\top DV$ with the triple (U, V, Σ).

Recall from the Introduction that other useful related matrices are

$$D' = \Sigma^{-1} \oplus I, \quad D'' = I \oplus \Sigma^{-1}. \tag{10}$$

2.3. **Modulation spaces.** We provide a collection of time-frequency analysis tools that are needed below. The reader may consult [26] for further details and proofs of the mentioned results.

Recall that the short-time Fourier transform (STFT) of a temperate distribution $f \in \mathcal{S}'(\mathbb{R}^d)$ with respect to the window function $g \in \mathcal{S}(\mathbb{R}^d) \setminus \{0\}$ is defined as

$$V_g f(x,\xi) \coloneqq \langle f, \pi(x,\xi) g \rangle = \int_{\mathbb{R}^d} e^{-2\pi i y \cdot \xi} f(y) \overline{g(y-x)}\, dy. \tag{11}$$

The STFT is intimately connected with other well-known phase-space transforms such as the *Wigner distribution*

$$W(f,g)(x,\xi) = \int_{\mathbb{R}^d} e^{-2\pi i y\cdot\xi} f\left(x + \frac{y}{2}\right) \overline{g\left(x - \frac{y}{2}\right)}\, dy. \tag{12}$$

We write Wf when $f = g$. In particular, we have

$$W(f,g) = 2^d e^{4\pi i x\cdot\xi} V_{g^\vee} f(2x, 2\xi). \tag{13}$$

We also recall the orthogonality identity (also known as *Moyal formula* for the Wigner distribution):

$$\langle V_{g_1} f_1, V_{g_2} f_2 \rangle_{L^2} = \langle W(f_1, g_1), W(f_1, g_1) \rangle_{L^2} = \langle f_1, f_2 \rangle_{L^2} \overline{\langle g_1, g_2 \rangle_{L^2}}, \tag{14}$$

for any $f_1, g_1, f_2, g_2 \in L^2(\mathbb{R}^d)$. The behaviour of the Wigner distribution under time-frequency shifts is given by

$$W(\pi(w)f, \pi(z)g)(u) = c(w,z) M_{J(w-z)} T_{\frac{w+z}{2}} W(f,g)(u), \quad u, w, z \in \mathbb{R}^{2d}, \tag{15}$$

where $c(w,z)=e^{\pi i(w_1+z_1)\cdot(z_2-w_2)}$. The identity $W(\pi(z)f)(u)=Wf(u-z)$ is often referred to as the *covariance property* of Wf.

Recall from the Introduction that, for $1\leq p\leq\infty$ and $s\in\mathbb{R}$, the modulation space $M^p_{v_s}(\mathbb{R}^d)$ is the set of $f\in\mathcal{S}'(\mathbb{R}^d)$ such that, for any $g\in\mathcal{S}(\mathbb{R}^d)\setminus\{0\}$,

$$\|f\|_{M^p_{v_s}}\coloneqq\|V_gf\|_{L^p_{v_s}}=\left(\int_{\mathbb{R}^{2d}}|V_gf(z)|^pv_s(z)^pdz\right)^{1/p}<\infty, \tag{16}$$

with trivial modification in the case $p=\infty$. We collect below the relevant properties of modulation spaces that will be repeatedly used throughout this note, see [18, 26] for proofs and generalizations.

Proposition 2.2. *Consider $1\leq p\leq\infty$ and $s,r\in\mathbb{R}$ such that $|s|\leq r$.*

(i) $M^p_{v_s}(\mathbb{R}^d)$ is a Banach space with the norm (16), which is independent of the window function g (in the sense that different windows yield equivalent norms). Moreover, the class of admissible non-zero windows can be extended from $\mathcal{S}(\mathbb{R}^d)$ to $M^1_{v_r}(\mathbb{R}^d)$.

(ii) If $p<\infty$ the Schwartz class $\mathcal{S}(\mathbb{R}^d)$ is dense in $M^p_{v_s}(\mathbb{R}^d)$. Moreover, for any $g\in\mathcal{S}(\mathbb{R}^d)\setminus\{0\}$,

$$f\in\mathcal{S}(\mathbb{R}^d)\Longrightarrow V_gf,\, Wf\in\mathcal{S}(\mathbb{R}^{2d}).$$

(iii) If $p_1\leq p_2$ and $s_2\leq s_1$, then $M^{p_1}_{v_{s_1}}(\mathbb{R}^d)\subseteq M^{p_2}_{v_{s_2}}(\mathbb{R}^d)$. In particular, for $|s|\leq r$,

$$M^1_{v_r}(\mathbb{R}^d)\subseteq M^p_{v_s}(\mathbb{R}^d)\subseteq M^\infty_{v_{-r}}(\mathbb{R}^d).$$

(iv) If $1\leq p<\infty$ then $\left(M^p_{v_s}(\mathbb{R}^d)\right)'\simeq M^{p'}_{v_{-s}}(\mathbb{R}^d)$ and the duality is concretely given by

$$\langle f,h\rangle=\int_{\mathbb{R}^{2d}}V_gf(z)\overline{V_gh(z)}dz,$$

for $f\in M^p_{v_s}(\mathbb{R}^d)$, $h\in M^{p'}_{v_{-s}}(\mathbb{R}^d)$ and $g\in M^1_{v_r}(\mathbb{R}^d)$ with $\|g\|_{L^2}=1$.

It turns out that the STFT is injective in $\mathcal{S}'(\mathbb{R}^d)$, as a result of the following *inversion formula*: for $f\in\mathcal{S}'(\mathbb{R}^d)$ and $g,\gamma\in\mathcal{S}(\mathbb{R}^d)\setminus\{0\}$ such that $\langle g,\gamma\rangle\neq0$ we have

$$f=\frac{1}{\langle\gamma,g\rangle}\int_{\mathbb{R}^{2d}}V_gf(z)\pi(z)\gamma dz, \tag{17}$$

in the sense of temperate distributions. The same result extends to the case where $f\in M^p_{v_s}(\mathbb{R}^d)$ and $g,\gamma\in M^1_{v_r}(\mathbb{R}^d)\setminus\{0\}$ with s and r as in Proposition 2.2. The particular choice $\gamma=g$ yields

$$\mathrm{Id}_{M^p_{v_s}}=\frac{1}{\|g\|^2_{L^2}}V_g^*V_g, \tag{18}$$

where V_g^* is the adjoint STFT defined as a vector-valued integral by

$$V_g^* F = \int_{\mathbb{R}^{2d}} F(z)\pi(z) g dz, \quad F \in \mathcal{S}(\mathbb{R}^{2d}).$$

Moreover, for $g \in M^1_{v_r}(\mathbb{R}^d) \setminus \{0\}$ we have that $V_g : M^p_{v_s}(\mathbb{R}^d) \to L^p_{v_s}(\mathbb{R}^{2d})$ and $V_g^* : L^p_{v_s}(\mathbb{R}^{2d}) \to M^p_{v_s}(\mathbb{R}^d)$ are continuous maps.

The inversion formula enables an efficient phase-space analysis of operators as already mentioned in the Introduction. Consider a bounded linear operator $A : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$ and $g, \gamma \in \mathcal{S}(\mathbb{R}^d) \setminus \{0\}$; it is not restrictive to assume $\|g\|_{L^2} = \|\gamma\|_{L^2} = 1$. Using (18) we have that

$$A = V_\gamma^* V_\gamma A V_g^* V_g = V_\gamma^* \widetilde{A} V_g,$$

where $\widetilde{A} \coloneqq V_\gamma A V_g^*$ is an integral operator in $\mathbb{R}^{2d}$ with integral kernel given by the *Gabor matrix* K_A, that is

$$\widetilde{A}F(w) = \int_{\mathbb{R}^{2d}} K_A(w,z) F(z) dz, \quad K_A(w,z) = \langle A\pi(z)g, \pi(w)\gamma\rangle, \quad w \in \mathbb{R}^{2d}. \tag{19}$$

2.4. **Weyl operators.** Given $a \in \mathcal{S}'(\mathbb{R}^{2d})$ (*symbol*) the corresponding Weyl operator $a^{\mathrm{w}} : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$ is defined by duality as

$$\langle a^{\mathrm{w}} f, g\rangle = \langle a, W(g,f)\rangle, \quad f, g \in \mathcal{S}(\mathbb{R}^d),$$

where $W(g,f)$ is the Wigner distribution introduced in (12). Modulation spaces have been extensively employed as symbol classes as well as background spaces where to study boundedness of Weyl operators. Among the several results in this respect we highlight the special properties of Weyl operators with symbols in $M^{\infty,1}(\mathbb{R}^{2d})$ - the so-called *Sjöstrand class* after [35]. It is a modulation space of more general form than above, since its norm involves a mixed Lebesgue regularity condition on the short-time Fourier transform of a distribution: for any $g \in \mathcal{S}(\mathbb{R}^d) \setminus \{0\}$,

$$\|f\|_{M^{\infty,1}} \coloneqq \|V_g f\|_{L^{\infty,1}} = \int_{\mathbb{R}^d} \sup_{x \in \mathbb{R}^d} |V_g f(x,\xi)| d\xi < \infty.$$

We list below some results first proved in [25], see also [1, 12, 15] for generalizations and further results on almost diagonalization of operators.

Theorem 2.3. *Fix $g, \gamma \in M^1(\mathbb{R}^d)$ and consider $a \in \mathcal{S}'(\mathbb{R}^{2d})$. We have that $a \in M^{\infty,1}(\mathbb{R}^{2d})$ if and only if there exists a function $H \in L^1(\mathbb{R}^{2d})$ such that*

$$|\langle a^{\mathrm{w}} \pi(z) g, \pi(w)\gamma\rangle| \le H(w-z), \quad z, w \in \mathbb{R}^{2d}.$$

The controlling function H can be chosen as

$$H(w) = \sup_{z \in \mathbb{R}^{2d}} |V_\Phi a(z,w)|, \quad \Phi = W(\gamma, g),$$

hence $\|H\|_{L^1} \asymp \|a\|_{M^{\infty,1}}$. *Moreover,* a^{w} *is bounded on any modulation space* $M^p(\mathbb{R}^d)$, $1 \le p \le \infty$.

2.5. **Metaplectic operators.** Recall that the *metaplectic group* $\mathrm{Mp}(d,\mathbb{R})$ is the universal double cover of the symplectic group $\mathrm{Sp}(d,\mathbb{R})$. The corresponding faithful, strongly continuous unitary representation in $L^2(\mathbb{R}^d)$ allows us to directly interpret $\mathrm{Mp}(d,\mathbb{R})$ as a subgroup of $\mathcal{U}(L^2(\mathbb{R}^d))$, hence consisting of *metaplectic operators.* We use the notation $\mu(S)$ to denote metaplectic operators defined up to sign, where $S = \rho^{\mathrm{Mp}}(\mu(S)) \in \mathrm{Sp}(d,\mathbb{R})$ and $\rho^{\mathrm{Mp}} : \mathrm{Mp}(d,\mathbb{R}) \to \mathrm{Sp}(d,\mathbb{R})$ is the group projection, hence

$$\mu(AB) = \pm\mu(A)\mu(B), \quad A,B \in \mathrm{Sp}(d,\mathbb{R}).$$

An operator $\mu(S)$ satisfies the intertwining relation

$$\pi(Sz) = \mu(S)\pi(z)\mu(S)^{-1}, \quad z \in \mathbb{R}^{2d}.$$

We provide some elementary examples of metaplectic operators which are associated with special elements of $\mathrm{Sp}(d,\mathbb{R})$. In fact, it turns out that the metaplectic group is in some sense generated by operators $\mu(J)$, $\mu(S)$ and $\mu(\mathcal{C})$ defined below, cf. [18] for a precise account.

(1) The Fourier transform is a metaplectic operator associated with the canonical symplectic matrix, that is $\mu(J)f = \pm\mathcal{F}(f)$. Notice in particular that $\mu(-J) = \pm\mathcal{F}^{-1}$.

(2) Consider $A \in \mathrm{GL}(d,\mathbb{R})$ and set

$$S = \begin{bmatrix} A & O \\ O & (A^{-1})^{\top} \end{bmatrix}.$$

The metaplectic operator $\mu(S)$ acts as a rescaling by A:

$$\mu(S)f(t) = \pm|\det A|^{-1/2} f(A^{-1}t).$$

(3) Let $C \in \mathbb{R}^{d\times d}$ be a real symmetric matrix and set

$$\mathcal{C} = \begin{bmatrix} I & O \\ C & I \end{bmatrix}.$$

The metaplectic operator $\mu(\mathcal{C})$ is a chirp multiplication:

$$\mu(\mathcal{C})f(t) = \pm e^{\pi i t\cdot Ct} f(t).$$

We already mentioned that an important example of metaplectic operator is provided by the Schrödinger propagator for the free particle $U(t) = e^{i(t/2\pi)\Delta}$, $t \in \mathbb{R}$. This can be easily derived from the examples above since $U(t)$ is a Fourier multiplier with chirp symbol $m_t(\xi) = e^{-2\pi i t|\xi|^2}$ on $\mathbb{R}^d$, hence

$$U(t) = \mathcal{F}^{-1} m_t \mathcal{F} = \pm\mu(S_t), \quad S_t = \begin{bmatrix} I & 2tI \\ O & I \end{bmatrix}, \quad t \in \mathbb{R}. \tag{20}$$

A distinctive property of the Weyl calculus is known as *symplectic covariance* [18, Thm. 215]: for any $S \in \mathrm{Sp}(d,\mathbb{R})$ and $a \in \mathcal{S}'(\mathbb{R}^{2d})$,

$$(a \circ S)^{\mathrm{w}} = \mu(S)^{-1} a^{\mathrm{w}} \mu(S). \tag{21}$$

Metaplectic operators have been thoroughly studied in the framework of phase-space analysis [18, 22] and also in connection with the Schrödinger equation with quadratic Hamiltonians [9, 10, 13]. We mention below two relevant results concerning the Gabor matrix of a metaplectic operator and the boundedness on modulation spaces.

Theorem 2.4. *Consider $\mu(S) \in \mathrm{Mp}(d,\mathbb{R})$ and $g,\gamma \in \mathcal{S}(\mathbb{R}^d)$. For any $N \geq 0$ we have*

$$|\langle \mu(S)\pi(z)g, \pi(w)\gamma\rangle| \lesssim_{N,S} (1+|w-Sz|)^{-N}, \quad w,z \in \mathbb{R}^{2d}.$$

As a consequence, for any $1 \leq p \leq \infty$ and $s \in \mathbb{R}$, the operator $\mu(S)$ is bounded from $M^p_{v_s}(\mathbb{R}^d)$ into itself.

General families of operators characterized by the sparsity of their phase-space representation were introduced in [6, 7]. Given $S \in \mathrm{Sp}(d,\mathbb{R})$ and $g \in \mathcal{S}(\mathbb{R}^d)$, we say that a linear operator $A : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$ is in the class $FIO(S)$ of *generalized metaplectic operators* if there exists $H \in L^1(\mathbb{R}^{2d})$ such that

$$|\langle A\pi(z)g, \pi(w)g\rangle| \leq H(w-Sz), \quad w,z \in \mathbb{R}^{2d}. \tag{22}$$

The definition of $FIO(S)$ does not depend on the choice of $g \in \mathcal{S}(\mathbb{R}^d) \setminus \{0\}$. In fact, careful inspection of the proof of [6, Prop. 3.1] reveals that the class of admissible windows may be extended to $M^1(\mathbb{R}^d)$, hence the estimate (22) is equivalent to its *polarized* version with two arbitrary windows $g,\gamma \in M^1(\mathbb{R}^d)$, that is,

$$|\langle A\pi(z)g, \pi(w)\gamma\rangle| \leq H(w-Sz), \quad w,z \in \mathbb{R}^{2d}. \tag{23}$$

Sparsity of the Gabor matrix of generalized metaplectic operators provides non-trivial algebraic properties for $FIO(S)$ in the spirit of Theorem 2.3, as detailed in the following result.

Theorem 2.5. *Let $S, S_1, S_2 \in \mathrm{Sp}(d,\mathbb{R})$.*

(1) An operator $T \in FIO(S)$ is bounded on $M^p(\mathbb{R}^d)$ for any $1 \leq p \leq \infty$.
(2) If $T_1 \in FIO(S_1)$ and $T_2 \in FIO(S_2)$, then $T_1T_2 \in FIO(S_1S_2)$.
(3) If $T \in FIO(S)$ is invertible on $L^2(\mathbb{R}^d)$ then $T^{-1} \in FIO(S^{-1})$.
(4) $T \in FIO(S)$ if and only if there exist $a_1, a_2 \in M^{\infty,1}(\mathbb{R}^{2d})$ such that

$$T = a_1^{\mathrm{w}}\mu(S) = \mu(S)a_2^{\mathrm{w}}.$$

In particular, $a_2 = a_1 \circ S$.

In view of Theorem 2.4 we observe that the Gabor matrix $\langle \mu(S)\pi(z)g, \pi(w)\gamma\rangle$ of a metaplectic operator $\mu(S) \in \mathrm{Mp}(d,\mathbb{R})$ is well defined in the case where

$$g \in M^p(\mathbb{R}^d),\ \gamma \in M^q(\mathbb{R}^d), \quad \frac{1}{p} + \frac{1}{q} \geq 1. \tag{24}$$

To be precise,

$$\|\mu(S)\pi(z)g\|_{M^p} \leq \|\mu(S)\|_{\mathrm{op}}\|\pi(z)g\|_{M^p} = \|\mu(S)\|_{\mathrm{op}}\|g\|_{M^p}, \quad z \in \mathbb{R}^d,$$

hence by Proposition 2.2 (iv)

$$\begin{aligned} |\langle \mu(S)\pi(z)g, \pi(w)\gamma\rangle| &\leq \|\mu(S)\|_{\mathrm{op}}\|g\|_{M^p}\|\pi(w)\gamma\|_{M^{p'}} \\ &\leq \|\mu(S)\|_{\mathrm{op}}\|g\|_{M^p}\|\gamma\|_{M^{p'}} \\ &\leq \|\mu(S)\|_{\mathrm{op}}\|g\|_{M^p}\|\gamma\|_{M^q}, \end{aligned}$$

since from (24) we infer $q \leq p'$ and the inclusion $M^q(\mathbb{R}^d) \subset M^{p'}(\mathbb{R}^d)$ (Proposition 2.2 (iii)) yields the last inequality.

The same arguments apply to the Gabor matrix of $T \in FIO(S)$ as a consequence of Theorem 2.5.

2.6. **Technical lemmas.** A key technical tool for the main results is the following set of estimates.

Lemma 2.6. *Let $s > 1$, $a, b, \sigma \geq 1$ and $v \in \mathbb{R}$. Then*

$$\int_{\mathbb{R}} (a + |\sigma^{-1}u + v|)^{-s}(b + |u|)^{-s} du \lesssim_s (a + |v|)^{-s} b^{-s+1} + a^{-s+1}(b + |v|)^{-s+1}, \tag{25}$$

$$\int_{\mathbb{R}} (a+|u-v|)^{-s}(b+\sigma^{-1}|u|)^{-s} du \lesssim_s (a+\sigma^{-1}|v|)^{-s+1}b^{-s+1} + a^{-s+1}(b+\sigma^{-1}|v|)^{-s}. \tag{26}$$

Proof. We prove (25) under the assumption $|v| \geq 1$, otherwise the estimate is trivial since $\int_{\mathbb{R}}(b+|u|)^{-s}du \lesssim b^{-s+1}$. If $|\sigma^{-1}u+v| \geq |v|/2$ then $(a+|\sigma^{-1}u+v|)^{-s} \lesssim (a+|v|)^{-s}$, hence

$$\int_{\mathbb{R}} (a + |\sigma^{-1}u + v|)^{-s}(b + |u|)^{-s} du \lesssim (a + |v|)^{-s} b^{-s+1}.$$

If $|\sigma^{-1}u + v| \leq |v|/2$ then $|u| \geq \sigma|v|/2$, hence

$$\begin{aligned} \int_{\mathbb{R}} (a + |\sigma^{-1}u + v|)^{-s}(b + |u|)^{-s} du &\lesssim a^{-s+1}\sigma(b + \sigma|v|)^{-s} \\ &\leq a^{-s+1}(b + |v|)^{-s+1}. \end{aligned}$$

The proof of (26) in the non-trivial case $\sigma^{-1}|v| \geq 1$ follows by similar arguments, by considering separately the cases $|u - v| \geq |v|/2$ and $|u - v| < |v|/2$. $\square$

Remark 2.7. *For $s > d$ and $v \in \mathbb{R}^d$, the convolution inequality* (26) *with $\sigma = 1$ and $a = b$ can be improved in $\mathbb{R}^d$ as follows:*

$$\int_{\mathbb{R}^d} (a + |u - v|)^{-s}(a + |u|)^{-s} du \lesssim_s a^{-s+d}(a + |v|)^{-s}. \tag{27}$$

Notice that for $a = 1$ we have $v_{-s} * v_{-s} \lesssim v_{-s}$, cf. [26, Lem. 11.1.1(c)].

Lemma 2.8. *Let $\sigma_1, \dots, \sigma_d \geq 1$ and define the matrices*

$$\Sigma = \mathrm{diag}(\sigma_1, \dots, \sigma_d), \quad D' = \Sigma^{-1} \oplus I, \quad D'' = I \oplus \Sigma^{-1}.$$

For any $s > 2d$

$$\int_{\mathbb{R}^{2d}} (1 + |v - D''u|)^{-s}(1 + |D'u|)^{-s} du \lesssim_s (1 + |D'v|)^{-s+2d}, \quad v \in \mathbb{R}^{2d}. \tag{28}$$

Proof. The integral under our attention is

$$\int_{\mathbb{R}^{2d}} \left(1 + \sum_{j=1}^{d} |v_j - u_j| + \sum_{j=d+1}^{2d} |v_j - \sigma_{j-d}^{-1} u_j|\right)^{-s} \left(1 + \sum_{j=1}^{d} |\sigma_j^{-1} u_j| + \sum_{j=d+1}^{2d} |u_j|\right)^{-s} du.$$

We look at the latter as an iterated integral and we repeatedly apply Lemma 2.6; precisely we estimate each of the integrals with respect to $u_1, \dots, u_d$ as in (26) and the each one with respect to $u_{d+1}, \dots, u_{2d}$ as in (25). Careful inspection of the involved quantities reveals that the result after $2d$ steps is dominated by a sum of products of the form $A^{-s+2d}B^{-s+2d}$ with $A, B \geq 1$ such that

$$A + B = 2 + \sum_{j=1}^{d} \sigma_j^{-1}|v_j| + \sum_{j=d+1}^{2d} |v_j| > 1 + |D'v|.$$

The claim follows after noticing that $A^{-s+2d}B^{-s+2d} \leq (A+B)^{-s+2d}$ since $s > 2d$. ☐

Remark 2.9. *The decay rate $-s + 2d$ in* (28) *could be judged quite unsatisfactory at a first glance, since we are ultimately concerned with subconvolutive weights like v_{-s}* [3, 24]*. In fact, it is easy to realize that subconvolutivity still holds for $v_{-s}(A \cdot +b)$, for an invertible matrix $A \in \mathbb{R}^{2d}$ and $b \in \mathbb{R}^d$. The crucial point here is that the latter property does hold indeed, but only up to constants that depend on A, while the key feature of the estimate proved in Lemma 2.8 is precisely the uniformity with respect to the involved matrices, which comes at the price of an unavoidable loss in decay rate. Moreover, an inspection of the proof shows that Lemma 28 is specific to $\mathbb{R}^{2d}$, since the fine structure of the involved weights plays a key role via Lemma 2.6.*

We also emphasize that the issue is not even a consequence of the proof technique: while we do not claim that the decay $-s + 2d$ is sharp, a simple argument shows that the estimate in (28) *can not hold with $-s$ as exponent on the right-hand side. Let $d = 1$ for simplicity so that Σ reduces to a scalar $\sigma \geq 1$, and assume that* (28) *holds* *with an exponent α in place of $-s+2d$. Set $u=(u_1,u_2)$ and $v=(v_1,v_2)$, assuming in addition $v_1=0$. By Fatou's lemma we have that for $\sigma\to\infty$ the estimate reads*

$$\int_{\mathbb{R}^2}(1+|u_1|+|v_2|)^{-s}(1+|u_2|)^{-s}du_1du_2\leq C_s(1+|v_2|)^{\alpha},$$

or equivalently

$$\int_{\mathbb{R}}(1+|u_1|+|v_2|)^{-s}du_1\leq C_s'(1+|v_2|)^{\alpha}$$

for some new constant $C_s'>0$ depending only on s. A straightforward computation shows that the integral on the left-hand side is equivalent to $(1+|v_2|)^{-s+1}$ (up to a constant depending only on s), therefore we conclude that $\alpha\geq -s+1$.

3. Proof of the main results

We start this section with the proof of Theorem 1.1, namely a pointwise inequality for the Gabor matrix with Gabor atoms in the Schwartz class.

Proof of Theorem 1.1. We use the Moyal formula (14), the covariance property of Wigner distribution (15) and the symplectic covariance of the Weyl calculus (21). Hence

$$\begin{aligned}|\langle\mu(S)\pi(z)g,\pi(w)\gamma\rangle|^2&=\int_{\mathbb{R}^{2d}}W(\mu(S)\pi(z)g)(u)W(\pi(w)\gamma)(u)du\\&=\int_{\mathbb{R}^{2d}}W(\pi(z)g)(S^{-1}u)W\gamma(u-w)du\\&=\int_{\mathbb{R}^{2d}}Wg(S^{-1}u-z)W\gamma(u-w)du\\&=\int_{\mathbb{R}^{2d}}Wg(S^{-1}u+S^{-1}w-z)W\gamma(u)du.\end{aligned}$$

Direct application of Proposition 2.2 (ii) yields, for any $s\geq 0$,

$$|\langle\mu(S)\pi(z)g,\pi(w)\gamma\rangle|^2\lesssim\int_{\mathbb{R}^{2d}}v_{-s}(S^{-1}u+S^{-1}w-z)v_{-s}(u)du.$$

Recall that $S=U^\top DV$, hence $S^{-1}=V^\top D^{-1}U$ and therefore

$$|\langle\mu(S)\pi(z)g,\pi(w)\gamma\rangle|^2\lesssim\int_{\mathbb{R}^{2d}}v_{-s}(D^{-1}u+V(S^{-1}w-z))v_{-s}(u)du.$$

Set $v\coloneqq V(S^{-1}w-z)$. The change of variable $u=D''u'$ leads to

$$|\langle\mu(S)\pi(z)g,\pi(w)\gamma\rangle|^2\lesssim(\det\Sigma)^{-1}\int_{\mathbb{R}^{2d}}(1+|D'u+v|)^{-s}(1+|D''u|)^{-s}du.$$

We fix $s > 2d$ and apply Lemma 2.8 with D' and D'' interchanged. The claim then follows after setting $N = (s-2d)/2$, since $s > 2d$ is arbitrarily chosen and $D''v = D'U(w - Sz)$. $\Box$

We now prove Theorem 1.2, where Gabor atoms in suitable modulation spaces are considered.

Proof of Theorem 1.2. Fix $\phi, \psi \in \mathcal{S}(\mathbb{R}^d) \setminus \{0\}$ with $\|\phi\|_{L^2} = \|\psi\|_{L^2} = 1$; the reconstruction formula (17) applied to $g \in M^p(\mathbb{R}^d)$, $\gamma \in M^q(\mathbb{R}^d)$ (resp. $g, \gamma \in M^\infty_{v_s}(\mathbb{R}^d)$) yields

$$g = \int_{\mathbb{R}^{2d}} F(u)\pi(u)\phi du, \quad F = V_\phi g \in L^p(\mathbb{R}^{2d}) \text{ (resp. } F = V_\phi g \in L^\infty_{v_s}(\mathbb{R}^{2d}))$$

$$\gamma = \int_{\mathbb{R}^{2d}} G(v)\pi(v)\psi dv, \quad G = V_\psi \gamma \in L^q(\mathbb{R}^{2d}) \text{ (resp. } G = V_\psi \gamma \in L^\infty_{v_s}(\mathbb{R}^{2d})).$$

Then we have

$$\begin{aligned}|\langle \mu(S)\pi(z)g, \pi(w)\gamma\rangle| &\le \int_{\mathbb{R}^{4d}} |F(u)||G(v)||\langle \mu(S)\pi(z+u)\phi, \pi(w+v)\psi\rangle| dudv \\ &= \int_{\mathbb{R}^{4d}} |F(u-z)||G(v-w)||\langle \mu(S)\pi(u)\phi, \pi(v)\psi\rangle| dudv.\end{aligned}$$

Direct application of Theorem 1.1 with $N > \max\{2d, s\}$ (the reason of this choice will be clear in a moment) yields

$$|\langle \mu(S)\pi(z)g, \pi(w)\gamma\rangle| \lesssim_N (\det \Sigma)^{-1/2} \int_{\mathbb{R}^{4d}} |F(u-z)||G(v-w)| v_{-N}(D'Uv - D''Vu) dudv.$$

Set $\widetilde{F} = F \circ (D''V)^{-1}$ and $\widetilde{G} = G \circ (D'U)^{-1}$. Then

$$\begin{aligned}|\langle \mu(S)\pi(z)g, \pi(w)\gamma\rangle| &\lesssim_N (\det \Sigma)^{3/2} \int_{\mathbb{R}^{4d}} |\widetilde{F}(u - D''Vz)||\widetilde{G}(v - D'Uw)| v_{-N}(v-u) dudv \\ &= (\det \Sigma)^{3/2} \int_{\mathbb{R}^{4d}} |\widetilde{F}(u)||\widetilde{G}(v + D''Vz - D'Uw)| v_{-N}(v-u) dudv \\ &= (\det \Sigma)^{3/2} (|\widetilde{F}| * v_{-N} * |\widetilde{G}|^\vee)(D'Uw - D''Vz) \\ &= H(D'U(w - Sz)),\end{aligned}$$

where we defined

$$H(u) = (\det \Sigma)^{3/2} (v_{-N} * |\widetilde{F}| * |\widetilde{G}|^\vee)(u), \quad u \in \mathbb{R}^{2d}. \tag{29}$$

For $g \in M^p(\mathbb{R}^d)$ and $\gamma \in M^q(\mathbb{R}^d)$ we apply Young's inequality to prove that $H \in L^r(\mathbb{R}^{2d})$ for $1/p + 1/q = 1 + 1/r$, cf. (24). In particular, since $N > 2d$,

$$\begin{aligned} \|H\|_{L^r} &\le (\det \Sigma)^{3/2} \|v_{-N}\|_{L^1} \||\widetilde{F}| * |\widetilde{G}|^\vee\|_{L^r} \\ &\lesssim (\det \Sigma)^{3/2-1/p-1/q} \|F\|_{L^p} \|G\|_{L^q} \\ &\lesssim (\det \Sigma)^{1/2-1/r} \|g\|_{M^p} \|\gamma\|_{M^q} . \end{aligned}$$

For $g, \gamma \in M^\infty_{v_s}(\mathbb{R}^d)$, $s > 2d$, we note that

$$|\widetilde{F}(u)| \le \|g\|_{M^\infty_{v_s}} (1 + |(D'')^{-1}u|)^{-s}, \quad |\widetilde{G}(u)| \le \|\gamma\|_{M^\infty_{v_s}} (1 + |(D')^{-1}u|)^{-s}.$$

Therefore, since $N > s$ and again by Young's inequality,

$$\begin{aligned} \|H\|_{L^\infty_{v_{s-2d}}} &\le (\det \Sigma)^{3/2} \|v_{-N}\|_{L^1_{v_{s-2d}}} \||\widetilde{F}| * |\widetilde{G}|^\vee\|_{L^\infty_{v_{s-2d}}} \\ &\lesssim (\det \Sigma)^{3/2} \|g\|_{M^\infty_{v_s}} \|\gamma\|_{M^\infty_{v_s}} \left\| v_{-s}((D')^{-1}\cdot) * v_{-s}((D'')^{-1}\cdot) \right\|_{L^\infty_{v_{s-2d}}} \\ &\lesssim (\det \Sigma)^{-1/2} \|g\|_{M^\infty_{v_s}} \|\gamma\|_{M^\infty_{v_s}} , \end{aligned}$$

where in the last step we used Lemma 2.8 with the substitutions $u \mapsto (D')^{-1}(D'')^{-1}u$ and $v \mapsto (D')^{-1}v$. ☐

Remark 3.1. *Notice that after setting $\widetilde{H} = H \circ D'U$ the estimate* (6) *reads*

$$|\langle \mu(S)\pi(z)g, \pi(w)\gamma \rangle| \le \widetilde{H}(w - Sz),$$

while (7) *becomes*

$$\|\widetilde{H}\|_{L^r} \lesssim (\det \Sigma)^{1/2} \|g\|_{M^p} \|\gamma\|_{M^q} .$$

It is then clear that there is a trade-off between phase-space concentration of $\mu(S)$ along the graph of S and the spreading of wave packets.

We also emphasize that a similar balance involves the Gabor matrix decay and dispersion: arguing as before it is easy to show that if $g, \gamma \in M^\infty_{v_s}(\mathbb{R}^d)$, $s > 2d$,

$$\|H\|_{L^\infty_{v_s}} \lesssim (\det \Sigma)^{3/2} \|g\|_{M^\infty_{v_s}} \|\gamma\|_{M^\infty_{v_s}}.$$

Remark 3.2. *More precise claims on the regularity of H can be made in view of its structure, cf.* (29)*. Precisely, since $N > 2d$ it is easy to realize that v_{-N} belongs to the Wiener algebra $W(C_0, L^1)(\mathbb{R}^{2d})$ of continuous functions $f : \mathbb{R}^{2d} \to \mathbb{C}$ that are globally in L^1, namely*

$$\|f\|_{W(C_0,L^1)} \coloneqq \int_{\mathbb{R}^d} \left(\max_{x \in [0,1]^d} |f(x + y)| \right) dy < \infty.$$

The Wiener algebra is an example of more general amalgam spaces $W(X, L^p)(\mathbb{R}^{2d})$ with local and global components X and L^p respectively, obtained by mixing conditions on local regularity (encoded by some norm $\|\cdot\|_X$) and global L^p decay - see the original *paper* [19] *and the recent monograph* [14] *for further details and proofs of the mentioned results.*

We recall that $L^p(\mathbb{R}^d) = W(L^p, L^p)(\mathbb{R}^d)$ *for any* $1 \leq p \leq \infty$, *and that Wiener amalgam spaces do preserve Banach convolution triples in local and global components. With reference to our case, since* (C_0, L^p, C_0) *and* (L^1, L^p, L^p), $1 \leq p \leq \infty$, *are Banach convolution triples, we have that*

$$W(C_0, L^p)(\mathbb{R}^{2d}) * W(L^1, L^p)(\mathbb{R}^{2d}) \subset W(C_0, L^p)(\mathbb{R}^{2d}).$$

We thus infer that $v_{-N} * |\widetilde{F}| \in W(C_0, L^p)(\mathbb{R}^{2d})$ *and again, since*

$$W(C_0, L^p)(\mathbb{R}^{2d}) * W(L^q, L^q)(\mathbb{R}^{2d}) \subset W(C_0, L^r)(\mathbb{R}^{2d}), \quad \frac{1}{p} + \frac{1}{q} = \frac{1}{r},$$

arguing as before we conclude that $H = (\det \Sigma)^{3/2} v_{-N} * |\widetilde{F}| * |\widetilde{G}|^{\vee} \in W(C_0, L^r)(\mathbb{R}^{2d})$, *with*

$$\|H\|_{W(C_0,L^r)} \lesssim C_N (\det \Sigma)^{1/2-1/r} \|g\|_{M^p} \|\gamma\|_{M^q},$$

where we set $C_N = \|v_{-N}\|_{W(C_0,L^1)}$.

Similar arguments show that if $g, \gamma \in M^\infty_{v_s}(\mathbb{R}^d)$, $s > 2d$, *then* $H \in W(C_0, L^\infty_{v_s})(\mathbb{R}^{2d})$ *but a deterioration of the dispersive factor occurs as expected:*

$$\|H\|_{W(C_0,L^\infty_{v_s})} \lesssim (\det \Sigma)^{3/2} \|g\|_{M^\infty_{v_s}} \|\gamma\|_{M^\infty_{v_s}}.$$

We conclude with a result in the same spirit for generalized metaplectic operators.

Theorem 3.3. *Let* $1 \leq p, q, r \leq \infty$ *satisfy* $1/p + 1/q = 1 + 1/r$. *Consider* $S \in \mathrm{Sp}(d, \mathbb{R})$ *with an Euler decomposition* (U, V, Σ), $a \in M^{\infty,1}(\mathbb{R}^{2d})$ *so that* $T := a^{\mathrm{w}} \mu(S) \in FIO(S)$, *cf. Theorem 2.5. For any* $g \in M^p(\mathbb{R}^d)$, $\gamma \in M^q(\mathbb{R}^d)$ *there exists* $H \in L^r(\mathbb{R}^{2d})$ *such that, for any* $z, w \in \mathbb{R}^{2d}$,

$$|\langle T\pi(z)g, \pi(w)\gamma\rangle| \leq H(D'U(w - Sz)), \tag{30}$$

with

$$\|H\|_{L^r} \leq (\det \Sigma)^{1/2-1/r} \|a\|_{M^{\infty,1}} \|g\|_{M^p} \|\gamma\|_{M^q}.$$

Proof. We assume $\|g\|_{L^2} = \|\gamma\|_{L^2} = 1$ without loss of generality. Denoting by $K_{\mu(S)}(w, z) = \langle \mu(S)\pi(z)g, \pi(w)\gamma\rangle$ the Gabor matrix of $\mu(S)$ and similarly for $K_{a^{\mathrm{w}}}(w, z) = \langle a^{\mathrm{w}}\pi(z)\gamma, \pi(w)\gamma\rangle$, in view of (19), by Theorems 2.3 and 2.5 we have

$$\begin{aligned}
|\langle T\pi(z)g, \pi(w)\gamma\rangle| &= |\langle a^{\mathrm{w}}\mu(S)\pi(z)g, \pi(w)\gamma\rangle| \\
&\leq \int_{\mathbb{R}^{2d}} |K_{a^{\mathrm{w}}}(w, u)||K_{\mu(S)}(u, z)| du \\
&= \int_{\mathbb{R}^{2d}} |\langle a^{\mathrm{w}}\pi(u)\gamma, \pi(w)\gamma\rangle||\langle \mu(S)\pi(z)g, \pi(u)\gamma\rangle| du \\
&\leq \int_{\mathbb{R}^{2d}} H_a(w - u) H_S(D'Uu - D''Vz) du,
\end{aligned}$$

where H_S is the controlling function in Theorem 1.2 (i) and H_a is the one appearing in Theorem 2.3 with $g = \gamma$; in particular $\|H_a\|_{L^1} \asymp \|a\|_{M^{\infty,1}}$. The substitution $y = D'U(w-u)$ yields

$$|\langle T\pi(z)g, \pi(w)\gamma\rangle| \leq (\det\Sigma)\left[(H_a\circ(D'U)^{-1}) * H_S\right](D'U(w-Sz)).$$

The claim follows by Young inequality and Theorem 1.2 (i) after setting $H = (\det\Sigma)(H_a\circ(D'U)^{-1}) * H_S$:

$$\begin{aligned}\|H\|_{L^r} &\leq (\det\Sigma)\left\|H_a\circ(D'U)^{-1}\right\|_{L^1}\|H_S\|_{L^r}\\ &= \|H_a\|_{L^1}\|H_S\|_{L^r}\\ &\lesssim (\det\Sigma)^{1/2-1/r}\|a\|_{M^{\infty,1}}\|g\|_{M^p}\|\gamma\|_{M^q}.\end{aligned}$$

□

We conclude with the proof of Theorem 1.3; namely we study how the modulation space regularity on a cone in the phase space behaves under the action of a metaplectic operator.

Proof of Theorem 1.3. Fix $g, \gamma \in \mathcal{S}(\mathbb{R}^d)\setminus\{0\}$ with $\|g\|_{L^2} = \|\gamma\|_{L^2} = 1$, and Γ and Γ' as in the statement. From (19) with $A = \mu(S)$ and Theorem 1.1, for any $N > 0$ we have

$$\begin{aligned}|V_\gamma(\mu(S)f)(w)| &\leq \int_{\mathbb{R}^{2d}} |K_{\mu(S)}(w,z)||V_g f(z)|dz\\ &\lesssim_N (\det\Sigma)^{-1/2}\int_{\mathbb{R}^{2d}} v_{-N}(D'U(w-Sz))|V_g f(z)|dz\\ &\lesssim_N (\det\Sigma)^{-1/2}\int_{\mathbb{R}^{2d}} H(w-Sz)|V_g f(z)|dz,\end{aligned}$$

where we set $H = v_{-N}\circ D'U$. After naming $G = H\circ S = v_{-N}\circ D''V$ we apply Hölder's inequality and get

$$\begin{aligned}I &:= \|\mu(S)f\|_{M^1_{(\gamma)}(S(\Gamma'))}\\ &= \int_{S(\Gamma')} |V_\gamma(\mu(S)f)(w)|dw\\ &= \int_{\Gamma'} |V_\gamma(\mu(S)f)(Sw)|dw\\ &\lesssim (\det\Sigma)^{-1/2}\int_{\Gamma'}\int_{\mathbb{R}^{2d}} G(w-z)|V_g f(z)|dzdw.\end{aligned}$$

We then have $I \lesssim I_1 + I_2$, where

$$I_1 := (\det\Sigma)^{-1/2}\int_{\Gamma'}\int_{\Gamma} G(w-z)|V_g f(z)|dzdw,$$

$$I_2 := (\det \Sigma)^{-1/2} \int_{\Gamma'} \int_{\Gamma^c} G(w-z)|V_g f(z)| dz dw.$$

Young's inequality yields

$$I_1 \le \|G\|_{L^1} \|V_g f \cdot 1_\Gamma\|_{L^1} \lesssim (\det \Sigma)^{1/2} \|f\|_{M^1_{(g)}(\Gamma)} .$$

After setting $F(z) = |V_g f(z)| v_{-r}(z)$, the remaining integral is

$$I_2 = (\det \Sigma)^{-1/2} \int_{\Gamma'} \int_{\Gamma^c} G(w-z) v_r(z) F(z) dz.$$

The key point is now that

$$1 + |w - z| \asymp \max\{1 + |w|, 1 + |z|\}, \quad w \in \Gamma',\, z \in \Gamma^c,$$

hence

$$\begin{aligned} I_2 &\lesssim (\det \Sigma)^{-1/2} \int_{\Gamma'} \int_{\Gamma^c} G(w-z) v_r(w-z) F(z) dz \\ &\le (\det \Sigma)^{-1/2} \|(G \cdot v_r) * F\|_{L^1} \\ &\lesssim (\det \Sigma)^{-1/2} \|G \cdot v_r\|_{L^1} \|f\|_{M^1_{v_{-r}}} . \end{aligned}$$

Therefore, the remaining integral to estimate is

$$\|G \cdot v_r\|_{L^1} = \int_{\mathbb{R}^{2d}} (1 + |D'' z|)^{-N} (1 + |z|)^r dz.$$

Recall that $D'' = I \oplus \Sigma^{-1}$, cf. (10), and consider the elementary estimates

$$v_{-N}(D''z) \le v_{-N/2d}(z_1) \cdots v_{-N/2d}(z_d) v_{-N/2d}(\sigma_1^{-1} z_{d+1}) \cdots v_{-N/2d}(\sigma_d^{-1} z_{2d}),$$

$$v_r(z) \le v_r(z_1) \cdots v_r(z_{2d}).$$

As a result, the integral is dominated by $A^d B_1 \cdots B_d$, where

$$A := \int_{\mathbb{R}} (1 + |x|)^{-N/2d + r} dx,$$

$$B_j := \int_{\mathbb{R}} (1 + \sigma_j^{-1}|x|)^{-N/2d} (1 + |x|)^r dx, \quad j = 1, \ldots, d.$$

If N is large enough then $A < \infty$ and $B_j \lesssim \sigma_j^{1+r}$, therefore

$$I_2 \lesssim (\det \Sigma)^{1/2 + r} \|f\|_{M^1_{v_{-r}}} ,$$

and the claim follows. $\square$

Remark 3.4. *(1) Condition* (9) *can be generalized to introduce the notion of M^p-regularity, $1 \le p \le \infty$, on the cone Γ with respect to $g \in \mathcal{S}(\mathbb{R}^d) \setminus \{0\}$. The latter is satisfied for $f \in \mathcal{S}'(\mathbb{R}^d)$ if*

$$\|f\|_{M^p_{(g)}(\Gamma)} := \|V_g f \cdot 1_\Gamma\|_{L^p} < \infty. \tag{31}$$

Weighted versions of such conditions can be defined similarly. The proof of Theorem 1.3 can be easily modified in order to prove the estimate

$$\|\mu(S)f\|_{M^p_{(\gamma)}(S(\Gamma'))} \lesssim (\det\Sigma)^{1/2}\left(\|f\|_{M^p_{(g)}(\Gamma)} + (\det\Sigma)^r\,\|f\|_{M^p_{v_{-r}}}\right), \tag{32}$$

which however is not sharp unless $p = 1$ or $p = \infty$. We postpone further investigations on the issue to a subsequent contribution.

(2) The notion of M^p-regularity does not depend on the window g used to compute $V_g f$ in (31) *provided that a slightly smaller cone is allowed when changing window. This is indeed a consequence of* (32) *in the case where $S = I$. The properties of $M^p_{(g)}(\Gamma)$ as a function space will be object of future studies.*

Corollary 3.5. *Consider $1 \le p \le \infty$. There exists $C > 0$ such that, for any $f \in M^p(\mathbb{R}^d)$, $S \in \mathrm{Sp}(d,\mathbb{R})$,*

$$\|\mu(S)f\|_{M^p} \le C(\det\Sigma)^{|1/2-1/p|}\|f\|_{M^p}.$$

Proof. By choosing $\Gamma = \Gamma' = \mathbb{R}^{2d} \setminus \{0\}$ and $r = 0$ in Theorem 1.3 we see that the desired estimate holds for $p = 1$. Since $\mu(S)$ is unitary on $L^2(\mathbb{R}^d)$, the operator $\mu(S^{-1})$, and therefore $\mu(S)$, satisfies the same estimate for $p = \infty$. Interpolating with the trivial L^2-estimate, we obtain the desired result (modulation spaces interpolate like the corresponding L^p spaces [20]). $\square$

4. Applications to the free particle propagator

Let us consider the free particle propagator $U(t) = e^{i(t/2\pi)\Delta}$ and the corresponding classical flow (20); a straightforward computation shows that the largest d singular values of S_t coincide:

$$\sigma_j = \sigma(t) = (1 + 2t^2 + 2(t^2 + t^4)^{1/2})^{1/2} = \sqrt{1+t^2} + |t|, \quad j = 1, \dots, d.$$

Note in particular that $\sigma(t)$ is comparable to $1 + |t|$, $t \in \mathbb{R}$. An example of Euler decomposition (U_t, V_t, Σ_t) of S_t for $t \ge 0$ is given by

$$U_t = (1+\sigma(t)^2)^{-1/2}\begin{bmatrix}\sigma(t)I & I\\ -I & \sigma(t)I\end{bmatrix}, \quad V_t = (1+\sigma(t)^2)^{-1/2}\begin{bmatrix}I & \sigma(t)I\\ -\sigma(t)I & I\end{bmatrix}.$$

Theorem 1.1 thus yields

$$\left|\langle e^{i(t/2\pi)\Delta}\pi(z)g, \pi(w)\gamma\rangle\right| \le C(1+|t|)^{-d/2}(1 + |D_t' U_t(w - S_t z)|)^{-N}, \quad z, w \in \mathbb{R}^{2d}.$$

The spreading phenomenon manifests itself as a dilation by

$$D_t' U_t = (1+\sigma(t)^2)^{-1/2} \begin{bmatrix} I & \sigma(t)^{-1} I \\ -I & \sigma(t) I \end{bmatrix}.$$

We attempt to shed some light on the apparently unintelligible structure of such matrix by means of a toy example in dimension $d = 1$. Let $z = 0$ for simplicity and assume that the atom g is concentrated on the box $Q = \{(x,\xi) \in \mathbb{R}^2 : |x| < 1, |\xi| < 1\}$ in phase space. In view of (4) we are lead to consider

$$(D_t' U_t)^{-1}(Q) = \{(x,\xi) : |x + \sigma^{-1}(t)\xi| < \sqrt{1+\sigma(t)^2},\ |x - \sigma(t)\xi| < \sqrt{1+\sigma(t)^2}\}.$$

Therefore, the effect of $D_t' U_t$ on Q ultimately amounts to a horizontal stretch by a factor of approximately $\sigma(t)$. While the set $(D_t' U_t)^{-1}(Q)$ only represents an envelope of the actual phase-space evolution of the initial wave packet, the findings are consistent with the result in (2); see also [11, Fig. 1-8] for illuminating graphic representations. We stress that the estimate (1) is completely blind to such spreading effect.

Acknowledgments

The authors wish to express their gratitude to the referees for suggesting improvements, especially with regard to the issues discussed in Remarks 2.9 and 3.2.

The authors are members of the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM). The present research was partially supported by MIUR grant "Dipartimenti di Eccellenza" 2018–2022, CUP: E11G18000350001, DISMA, Politecnico di Torino.

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