POINCARÉ AND HARDY INEQUALITIES ON HOMOGENEOUS TREES

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ABSTRACT. We study Hardy-type inequalities on infinite homogeneous trees. More precisely, we derive optimal Hardy weights for the combinatorial Laplacian in this setting and we obtain, as a consequence, optimal improvements for the Poincaré inequality.

1. INTRODUCTION

Given a linear, elliptic, second-order, symmetric nonnegative operator P on Ω, where Ω is a (e.g. Euclidean) domain, a *Hardy weight* is a nonnegative function W such that the following inequality holds

$$q(u) \geq \int_\Omega W u^2 \, \mathrm{d}x \quad \forall u \in C_c^\infty(\Omega), \tag{1.1}$$

where $q(u) = \langle u, Pu \rangle$ is the quadratic form associated to P. Clearly, the final (and most ambitious) goal is to get weights W such that inequality (1.1) is not valid for $V \geq W$, $V \neq W$, i.e. the operator $P - W$ is *critical* in the sense of [18, Definition 2.1]. When $P = -\Delta$ is the Laplace–Beltrami operator on a Riemannian manifold, the problem of the existence of Hardy weights has been widely studied in the literature, either in the Euclidean setting, see e.g [4, 9, 10, 20, 27, 28, 29] or on general manifolds, see e.g. [11, 17, 18, 24, 26, 30, 35]. Recently, the attention has also been devoted to the discrete setting, see e.g. [8, 21, 22, 23] and references therein.

The present paper is motivated by some recent results obtained in [5], see also [1] and [6], within the context of Cartan–Hadamard manifolds M. In particular, when M is the hyperbolic space $\mathbb{H}^N$, i.e. the simplest example of manifold with negative sectional curvature, the following Hardy weight has been determined when P being the Laplace–Beltrami operator $-\Delta_{\mathbb{H}^N}$ on $\mathbb{H}^N$ with $N \geq 3$:

$$W(r) = \frac{(N-1)^2}{4} + \frac{1}{4\,r^2} + \frac{(N-1)(N-3)}{4} \frac{1}{\sinh^2 r},$$

where $r = d(o, x) > 0$ denotes the geodesic distance of x from a fixed pole $o \in \mathbb{H}^N$. Besides, it is proved that the operator $-\Delta_{\mathbb{H}^N} - W$ is *critical* in $\mathbb{H}^N \setminus \{o\}$. It is worth noticing that the number $\frac{(N-1)^2}{4}$ in $W(r)$ coincides with the bottom of the L^2-spectrum of $-\Delta_{\mathbb{H}^N}$. Hence, the existence of the above weight yields the following improved Poincaré inequality:

$$\int_{\mathbb{H}^N} |\nabla_{\mathbb{H}^N} u|^2 \, \mathrm{d}v_{\mathbb{H}^N} - \frac{(N-1)^2}{4} \int_{\mathbb{H}^N} u^2 \, \mathrm{d}v_{\mathbb{H}^N} \geq \int_{\mathbb{H}^N} R\, u^2 \, \mathrm{d}v_{\mathbb{H}^N} \quad \forall u \in C_c^\infty(\mathbb{H}^N),$$

where the remainder term is

$$R(r) = \frac{1}{4\,r^2} + \frac{(N-1)(N-3)}{4} \frac{1}{\sinh^2 r} \sim \frac{1}{4\,r^2} \quad \text{as } r \to +\infty, \tag{1.2}$$

and, as a consequence of the criticality issue, all constants in (1.2) turn out to be sharp.

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Let $\Gamma = (V, E)$ denote a locally finite graph, where V and E denote a countably infinite set of vertices and the set of edges respectively. We recall that the combinatorial Laplacian Δ of a function f in the set $C(V)$ of real valued functions defined on V is defined by

$$\Delta f(x) := \sum_{y\sim x} \Big(f(x) - f(y) \Big) = m(x) f(x) - \sum_{y\sim x} f(y) \qquad \forall x \in V\,,$$

where $m(x)$ is the degree of x, i.e. the number of neighbors of x. The existence of Hardy weights for the combinatorial Laplacian or for more general operators on graphs has been recently studied in literature (see again [8, 21, 22, 23]).

We set our analysis on the case where the graph Γ is the homogeneous tree $\mathbb{T}_{q+1}$, i.e. a connected graph with no loops such that every vertex has $q+1$ neighbours, and we focus on the transient case, namely we always assume $q \geq 2$. $\mathbb{T}_{q+1}$ has been the object of investigation of many papers either in the field of harmonic analysis or of PDEs, see e.g. to [2, 3, 7, 12, 13, 14, 15, 16, 19, 31]. In particular, the homogeneous tree is in many respects a discrete analogue of the hyperbolic plane; we refer the reader to [7] for a discussion on this point. Therefore, since $\mathbb{T}_{q+1}$ is the basic example of graph of exponential growth, as $\mathbb{H}^N$ is the basic example of Riemannian manifold with exponential growth, it is natural to investigate whether the above mentioned results in $\mathbb{H}^N$ have a counterpart in $\mathbb{T}_{q+1}$: this will be the main goal of the paper.

In $\mathbb{T}_{q+1}$ the operator Δ is bounded on ℓ^2 and its ℓ^2-spectrum is given by $[(q^{1/2}-1)^2, (q^{1/2}+1)^2]$ (see [15]). Hence the following Poincaré inequality holds

$$\frac{1}{2} \sum_{\substack{x,y\in\mathbb{T}_{q+1}\\ x\sim y}} \Big(\varphi(x) - \varphi(y) \Big)^2 \geq \Lambda_q \sum_{x\in\mathbb{T}_{q+1}} \varphi^2(x) \qquad \forall \varphi \in C_0(\mathbb{T}_{q+1}),$$

with $\Lambda_q := (q^{1/2}-1)^2$.

By [23, Theorem 0.2] a Hardy weight for Δ on a transient graph Γ, is given by $W_{opt} = \frac{\Delta G_o^{1/2}}{G_o^{1/2}}$, where $G_o(x) := G(x,o)$ is the positive minimal Green function and o is a fixed point. Furthermore, W_{opt} is optimal in the sense of Definition 2.2 below and this implies, in particular, that the operator $\Delta - W_{opt}$ is *critical*. If $\Gamma = \mathbb{T}_{q+1}$, then the function G_o can be written explicitly, see Proposition 2.3 below, and W_{opt} reads as follows:

$$W_{opt}(x) = \begin{cases} \Lambda_q + q^{1/2} - q^{-1/2} & \text{if } |x| = 0, \\ \Lambda_q & \text{if } |x| \geq 1\,, \end{cases} \tag{1.3}$$

where, for each vertex $x \in \Gamma$, $|x| = d(x,o)$ and d is the usual discrete metric. By exploiting the super-solutions technique, in the present paper we provide the following new family of Hardy weights for Δ on $\mathbb{T}_{q+1}$:

$$W_{\beta,\gamma}(x) = \begin{cases} q+1-q^{1/2}(\frac{1}{\gamma} + \frac{1}{\gamma\, q}) & \text{if } |x| = 0, \\ q+1-q^{1/2}(2^\beta + \gamma) & \text{if } |x| = 1, \\ q+1-q^{1/2}[(1+\frac{1}{|x|})^\beta + (1-\frac{1}{|x|})^\beta] & \text{if } |x| \geq 2, \end{cases}$$

where $0 \leq \beta \leq \log_2 q^{1/2}$ and $q^{-1/2} \leq \gamma \leq q^{-1/2} + q^{1/2} - 2^\beta$. Moreover, if $\beta = 1/2$ we prove that the weight $W_{1/2,\gamma}$ is optimal (see again Definition 2.2), hence the operator $\Delta - W_{1/2,\gamma}$ is *critical*. We notice that

$$W_{\beta,\gamma}(x) = \Lambda_q + q^{1/2}\frac{\beta(1-\beta)}{|x|^2} + o\Big(\frac{1}{|x|^2}\Big) \qquad \text{as } |x| \to \infty\,,$$

hence the slowest decay at infinity occurs exactly for $\beta = 1/2$.

It is readily seen that the quadratic form inequality associated to $\Delta - W_{opt}$ in (1.3) can be read as an (optimal) *local* improvement of the Poincaré inequality on $\mathbb{T}_{q+1}$ at o. A direct inspection reveals that the

weights $W_{\beta,\gamma}$ satisfy $W_{\beta,\gamma} > \Lambda_q$ on $\mathbb{T}_{q+1}$ for all $0 \le \beta \le \log_2\left(\frac{3}{2} - \frac{1}{2q}\right)$ and $\frac{1}{2} + \frac{1}{2q} \le \gamma \le 2 - 2^\beta$. Hence, for such values of β and γ, we derive the following family of *global* improved Poincaré inequalities:

$$\frac{1}{2} \sum_{\substack{x,y \in \mathbb{T}_{q+1} \\ x \sim y}} \Big(\varphi(x) - \varphi(y)\Big)^2 \ge \sum_{x \in \mathbb{T}_{q+1}} \Lambda_q \varphi^2(x) + \sum_{x \in \mathbb{T}_{q+1}} R_{\beta,\gamma}(x) \varphi^2(x) \qquad \forall \varphi \in C_0(\mathbb{T}_{q+1}), \tag{1.4}$$

where

$$0 \le R_{\beta,\gamma}(x) = \begin{cases} q^{1/2}(2 - \frac{1}{\gamma} - \frac{1}{\gamma\, q}) & \text{if } |x| = 0, \\ q^{1/2}(2 - 2^\beta - \gamma) & \text{if } |x| = 1, \\ q^{1/2}\Big(2 - (1 + \frac{1}{|x|})^\beta - (1 - \frac{1}{|x|})^\beta\Big) & \text{if } |x| \ge 2. \end{cases}$$

It is worth noticing that the maximum of $R_{\beta,\gamma}$ at o is reached by choosing γ as large as possible, namely by taking $\gamma = 2 - 2^\beta$. Since such value is maximum for $\beta = 0$, we conclude that, among the weights $W_{\beta,\gamma}$ improving the Poincaré inequality, the largest at o is $W_{0,1} \equiv W_{opt}$.

Even if (1.4) improves globally the Poincaré inequality, we do not know whether this improvement is sharp on the whole $\mathbb{T}_{q+1}$. Nevertheless, a *sharp* improvement is provided by the critical weight $W_{1/2,\gamma}$ outside the ball $B_2(o)$. More precisely, there holds

$$\frac{1}{2} \sum_{\substack{x,y \in \mathbb{T}_{q+1} \\ x \sim y}} \Big(\varphi(x) - \varphi(y)\Big)^2 \ge \sum_{x \in \mathbb{T}_{q+1}} \Lambda_q \varphi^2(x) + \sum_{x \in \mathbb{T}_{q+1}} \overline{R}(x) \varphi^2(x), \qquad \forall \varphi \in C_0(\mathbb{T}_{q+1} \setminus B_2(o)),$$

where

$$\overline{R}(x) = q^{1/2}\left[2 - \left(1 + \frac{1}{|x|}\right)^{1/2} - \left(1 - \frac{1}{|x|}\right)^{1/2}\right] \qquad \text{if } |x| \ge 2$$

and the constant $q^{1/2}$ is sharp. Notice that

$$\overline{R}(x) \sim q^{1/2}\, \frac{1}{4|x|^2} \quad \text{as } |x| \to +\infty,$$

namely the decay of the remainder term is of the same order of that provided by (1.2) in $\mathbb{H}^N$, thereby confirming the analogy between $\mathbb{T}_{q+1}$ and $\mathbb{H}^N$.

Following the arguments used in the particular case of a homogeneous tree, in the last part of the paper we find a class of Hardy weights for the combinatorial Laplacian on rapidly growing *radial* trees, i.e. trees where the number of neighbours of a vertex x only depends on the distance of x from a fixed vertex o. This is a first result which might shed light on future related investigations on more general graphs.

The paper is organized as follows. In Section 2 we introduce the notation and we state our main results, namely Theorem 2.7, where we provide a family of optimal weights for Δ on $\mathbb{T}_{q+1}$, and Theorem 2.11 where we state the related improved Poincaré inequality. Section 3 is devoted to the proof of the statements of Section 2. Finally, in Section 4 we present a generalization of our results in the context of radial trees.

2. Notation and main results

We consider a graph $\Gamma = (V, E)$, where V and E denote a countably infinite set of vertices and the set of edges respectively, with the usual discrete metric d. If $(x, y) \in E$ we say that x and y are neighbors and we write $x \sim y$. We assume that Γ is a connected graph, that is, for every $x, y \in V$ there exists a finite sequence of vertices $x_1, \dots, x_n$ such that $x_0 = x$, $x_n = y$ and $x_j \sim x_{j+1}$ for $j = 0, \dots, n-1$. We also require that $(x, y) \in E$ if and only if $(y, x) \in E$. We use the notation $m(x)$ to indicate the degree of x, that is the number of edges that are attached to x and we assume that Γ is locally finite, i.e. $m(x) < \infty$ for all $x \in V$. When a vertex $o \in V$ is fixed let $x \mapsto |x|$ be the function which associates to each vertex x the distance $d(x, o)$ and define $B_r(o) = \{x \text{ s.t. } |x| < r\}$. We denote by $C(V)$ the set of real valued function defined on

V and by $C_0(V)$ the subspace consisting on finitely supported functions. Finally, we introduce the space of square summable functions

$$\ell^2(V) = \{f \in C(V) \text{ s.t. } \sum_{x\in V} f^2(x) < +\infty\}.$$

This is a Hilbert space with the inner product

$$\langle f, g\rangle = \sum_{x\in V} f(x)g(x),$$

and the induced norm $\|f\| = \sqrt{\langle f, f\rangle}$. As shown in [33, 34]

$$\langle \Delta\varphi, \varphi\rangle_{\ell^2} = \frac{1}{2}\sum_{\substack{x,y\in V\\ x\sim y}} \Big(\varphi(x) - \varphi(y)\Big)^2 \qquad \forall \varphi \in C_0(V).$$

More generally, we consider Schrödinger operators $H = \Delta + Q$ where Q is any potential. A function f is called H-(super)harmonic in V if

$$Hf(x) = 0 \quad (Hf(x) \geq 0) \qquad \forall x \in V.$$

By Hardy-type inequality for a positive Schrödinger operator H we mean an inequality of the form

$$\langle H\varphi, \varphi\rangle \geq \langle W\varphi, \varphi\rangle \qquad \forall \varphi \in C_0(V),$$

where $W \not\equiv 0$ is a nonnegative function in $C(V)$. We write $h(\varphi)$ and $W(\varphi)$ in place of $\langle H\varphi, \varphi\rangle$ and $\langle W\varphi, \varphi\rangle$, respectively. In particular we denote $h_\Delta(\varphi) = \langle \Delta\varphi, \varphi\rangle$.
In [23] the authors introduce the notion of optimal weight for a Hardy-type inequality; we recall some fundamental definitions that we need in the later discussion.

Definition 2.1. *Let h be a quadratic form associated with a Schrödinger operator H, such that $h \geq 0$ on $C_0(V)$. The form h is called* **subcritical** *in V if there is a nonnegative $W \in C_0(V)$, $W \not\equiv 0$, such that $h - W \geq 0$ on $C_0(V)$. A positive form h which is not subcritical is called* **critical** *in V.*

In [22, Theorem 5.3] it is shown that the criticality of h is equivalent to the existence of a unique positive function which is H-harmonic. Such a function is called the ground state of h and we have the further definition:

Definition 2.2. *Let h be a quadratic form associated with a Schrödinger operator H. We say that a positive function $W : V \to [0, \infty)$ is an* **optimal** *Hardy weight for h in V if*

- *$h - W$ is critical in V (***criticality***);*
- *$h - W \geq \lambda W$ fails to hold on $C_0(V \setminus K)$ for all $\lambda > 0$ and all finite $K \subset V$ (***optimality near infinity***);*
- *the ground state of $h - W$, $\Psi \notin \ell^2_W$ (***null-criticality***), namely*

$$\sum_{x\in V} \Psi^2(x)W(x) = +\infty.$$

In the following, for shortness, we will say that the operator H is critical if and only if its associated quadratic form h is critical.

Finally, we recall that a function $u : V \to \mathbb{R}$ is named proper on V if $u^{-1}(K)$ is finite for all compact sets $K \subset u(V)$.

2.1. **Hardy-type inequalities on $\mathbb{T}_{q+1}$.** In this subsection we shall state various Hardy-type inequalities on the homogeneous tree $\mathbb{T}_{q+1}$ with $q \geq 2$. We start with an optimal inequality for Δ obtained by combining the explicit formula of the Green function and [23, Theorem 0.2].

Proposition 2.3. *For all $\varphi \in C_0(\mathbb{T}_{q+1})$ the following inequality holds:*

$$\frac{1}{2} \sum_{\substack{x,y \in \mathbb{T}_{q+1} \\ x \sim y}} \Big(\varphi(x) - \varphi(y)\Big)^2 \geq \sum_{x \in \mathbb{T}_{q+1}} W_{opt}(x)\varphi^2(x),$$

where

$$W_{opt}(x) = \begin{cases} \Lambda_q + q^{1/2} - q^{-1/2} & \text{if } |x| = 0, \\ \Lambda_q & \text{if } |x| \geq 1. \end{cases} \tag{2.1}$$

Furthermore, the weight W_{opt} is optimal for Δ.

Remark 2.4. *As a consequence of the results of* [23, Theorem 0.2] *it follows that $G^{1/2}$ is the ground state of $h_\Delta - W_{opt}$. Furthermore, it is readily checked that*

$$\sum_{x \in \mathbb{T}_{q+1}} G(x)W_{opt}(x) = +\infty\,,$$

namely $G^{1/2} \notin \ell^2_{W_{opt}}$.

In the next theorem we state a family of Hardy-type inequalities depending on two parameters β, γ. The weights $W_{\beta,\gamma}$ provided can be seen as a generalization of W_{opt}. Indeed, if we fix $\beta = 0$ and $\gamma = 1$ in the statement below, we obtain W_{opt}.

Theorem 2.5. *For all $0 \leq \beta \leq \log_2 q^{1/2}$ and $q^{-1/2} \leq \gamma \leq q^{1/2} + q^{-1/2} - 2^\beta$ the following inequality holds*

$$\frac{1}{2} \sum_{\substack{x,y \in \mathbb{T}_{q+1} \\ x \sim y}} \Big(\varphi(x) - \varphi(y)\Big)^2 \geq \sum_{x \in \mathbb{T}_{q+1}} W_{\beta,\gamma}(x)\varphi^2(x) \qquad \forall \varphi \in C_0(\mathbb{T}_{q+1})\,,$$

where $W_{\beta,\gamma} \geq 0$ is defined as follows:

$$W_{\beta,\gamma}(x) = \begin{cases} q + 1 - q^{1/2}(\frac{1}{\gamma} + \frac{1}{q\gamma}) & \text{if } |x| = 0, \\ q + 1 - q^{1/2}(2^\beta + \gamma) & \text{if } |x| = 1, \\ q + 1 - q^{1/2}[(1 + \frac{1}{|x|})^\beta + (1 - \frac{1}{|x|})^\beta] & \text{if } |x| \geq 2. \end{cases}$$

Remark 2.6. *Notice that*

$$W_{\beta,\gamma}(x) = \Lambda_q + q^{1/2}\frac{\beta(1-\beta)}{|x|^2} + o\Big(\frac{1}{|x|^2}\Big) \qquad \text{as } |x| \to \infty.$$

Since

$$\max_\beta \beta(1-\beta) = 1/4,$$

which is reached for $\beta = 1/2$, $W_{1/2,\gamma}$ is the largest among the $W_{\beta,\gamma}$ at infinity.

On the other hand, in order to maximize the value of $W_{\beta,\gamma}$ at o, γ has to be taken as large as possible, namely $\gamma = q^{-1/2} + q^{1/2} - 2^\beta$. Since this quantity is maximum for $\beta = 0$, the largest weight at o is $W_{0,\overline{\gamma}}$ with $\overline{\gamma} = q^{-1/2} + q^{1/2} - 1$. Notice that: $W_{0,\overline{\gamma}} \equiv W_{opt}$ for $|x| \geq 2$, while $W_{0,\overline{\gamma}}(o) > W_{opt}(o)$ and $W_{opt}(|x| = 1) > W_{0,\overline{\gamma}}(|x| = 1)$, hence the two weights are not globally comparable.

The previous remark suggests that, in order to have the largest weight at infinity, one has to fix $\beta = 1/2$ in Theorem 2.5. This intuition is somehow confirmed by the statement below.

Theorem 2.7. *For all $q^{-1/2} \le \gamma \le q^{-1/2} + q^{1/2} - 2^{1/2}$ the following inequality holds*

$$\frac{1}{2} \sum_{\substack{x,y \in \mathbb{T}_{q+1} \\ x \sim y}} \Big(\varphi(x) - \varphi(y)\Big)^2 \ge \sum_{x \in \mathbb{T}_{q+1}} W_{1/2,\gamma}(x) \varphi^2(x) \qquad \forall \varphi \in C_0(\mathbb{T}_{q+1}),$$

where

$$W_{1/2,\gamma}(x) = \begin{cases} q + 1 - q^{1/2}(\frac{1}{\gamma} + \frac{1}{q\gamma}) & \text{if } |x| = 0, \\ q + 1 - q^{1/2}(2^{1/2} + \gamma) & \text{if } |x| = 1, \\ q + 1 - q^{1/2}[(1 + \frac{1}{|x|})^{1/2} + (1 - \frac{1}{|x|})^{1/2}] & \text{if } |x| \ge 2. \end{cases}$$

Furthermore, the weights $W_{1/2,\gamma}$ are optimal Hardy weights for Δ in the sense of Definition 2.2.

Using the same argument it is also possible to show that the weights we obtained in Theorem 2.5 are optimal near infinity, i.e. the constant is sharp in $\mathbb{T}_{q+1} \setminus K$ for every compact set K.

Corollary 2.8. *For all $0 \le \beta < \min\{\log_2 q^{1/2}, 1\}$ and $q^{-1/2} \le \gamma \le q^{-1/2} + q^{1/2} - 2^{\beta}$ the following inequality holds*

$$\frac{1}{2} \sum_{\substack{x,y \in \mathbb{T}_{q+1} \\ x \sim y}} \Big(\varphi(x) - \varphi(y)\Big)^2 \ge \sum_{x \in \mathbb{T}_{q+1}} W_{\beta,\gamma}(x) \varphi^2(x) \qquad \forall \varphi \in C_0(\mathbb{T}_{q+1}). \tag{2.2}$$

Moreover, the constant 1 in front of the r.h.s. term is sharp at infinity, in the sense that inequality (2.2) fails on $C_0(\mathbb{T}_{q+1} \setminus K)$ if we replace $W_{\beta,\gamma}$ with $CW_{\beta,\gamma}$, for all $C > 1$ and all compact set K.

2.2. **Improved Poincaré inequalities.** We shall provide three examples of improved Poincaré inequalities derived by the Hardy-type inequalities stated in the previous subsection. We recall that the Poincaré inequality on $\mathbb{T}_{q+1}$ reads

$$\frac{1}{2} \sum_{\substack{x,y \in \mathbb{T}_{q+1} \\ x \sim y}} \Big(\varphi(x) - \varphi(y)\Big)^2 \ge \Lambda_q \sum_{x \in \mathbb{T}_{q+1}} \varphi^2(x) \qquad \forall \varphi \in C_0(\mathbb{T}_{q+1}), \tag{2.3}$$

and the constant Λ_q is sharp in the sense that the above inequality cannot hold with a constant $\Lambda > \Lambda_q$.

The following improved Poincaré inequality is an immediate consequence of Theorem 2.3.

Proposition 2.9. *The following inequality holds*

$$\frac{1}{2} \sum_{\substack{x,y \in \mathbb{T}_{q+1} \\ x \sim y}} \Big(\varphi(x) - \varphi(y)\Big)^2 \ge \Lambda_q \sum_{x \in \mathbb{T}_{q+1}} \varphi^2(x) + \sum_{x \in \mathbb{T}_{q+1}} R_q(x) \varphi^2(x) \qquad \forall \varphi \in C_0(\mathbb{T}_{q+1}), \tag{2.4}$$

where

$$R_q(x) = \begin{cases} q^{1/2} - q^{-1/2} & \text{if } |x| = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Furthermore, the operator $\Delta - \Lambda_q - R_q$ is critical, hence the inequality does not hold with any $R > R_q$.

Notice that (2.4) improves (2.3) only locally, namely at o. The next statement provides a global improvement of (2.3).

Theorem 2.10. *For all $0 \le \beta \le \log_2 \left(\frac{3}{2} - \frac{1}{2q}\right)$ and $\frac{1}{2} + \frac{1}{2q} \le \gamma \le 2 - 2^{\beta}$, it holds*

$$\frac{1}{2} \sum_{\substack{x,y \in \mathbb{T}_{q+1} \\ x \sim y}} \Big(\varphi(x) - \varphi(y)\Big)^2 \ge \Lambda_q \sum_{x \in \mathbb{T}_{q+1}} \varphi^2(x) + \sum_{x \in \mathbb{T}_{q+1}} R_{\beta,\gamma} \varphi^2(x) \qquad \forall \varphi \in C_0(\mathbb{T}_{q+1}), \tag{2.5}$$

where

$$0 \le R_{\beta,\gamma}(x) = \begin{cases} q^{1/2}(2 - \frac{1}{\gamma} - \frac{1}{q\gamma}) & \text{if } |x| = 0, \\ q^{1/2}(2 - 2^{\beta} - \gamma) & \text{if } |x| = 1, \\ q^{1/2}\Big(2 - (1 + \frac{1}{|x|})^{\beta} - (1 - \frac{1}{|x|})^{\beta}\Big) & \text{if } |x| \ge 2. \end{cases}$$

Notice that (2.5) improves globally (2.3) but it gives no information about the sharpness of $R_{\beta,\gamma}$. A sharp improvement is instead provided by the next theorem which holds for functions supported outside the ball $B_2(o)$.

Theorem 2.11. *The following inequality holds*

$$\frac{1}{2} \sum_{\substack{x,y \in \mathbb{T}_{q+1} \\ x \sim y}} \Big(\varphi(x) - \varphi(y)\Big)^2 \ge \sum_{x \in \mathbb{T}_{q+1}} \Lambda_q \varphi^2(x) + \sum_{x \in \mathbb{T}_{q+1}} \overline{R}(x)\varphi^2(x) \qquad \forall \varphi \in C_0(\mathbb{T}_{q+1} \setminus B_2(o)), \tag{2.6}$$

where

$$\overline{R}(x) = q^{1/2}\left[2 - \left(1 + \frac{1}{|x|}\right)^{1/2} - \left(1 - \frac{1}{|x|}\right)^{1/2}\right] \qquad \text{if } |x| \ge 2.$$

Moreover, the constant $q^{1/2}$ is sharp in the sense that inequality (2.6) cannot hold if we replace the remainder term $\overline{R}$ with $C\Big[2 - (1 + \frac{1}{|x|})^{1/2} - (1 - \frac{1}{|x|})^{1/2}\Big]$ and $C > q^{1/2}$.

3. Proofs of the results

We collect here the proofs of the results stated in Section 2.

3.1. Proofs of Hardy-type inequalities.

Proof of Proposition 2.3. Consider the function $\tilde{u}(x) = \sqrt{G(x,o)}$, where G is the Green function on $\mathbb{T}_{q+1}$. By [23, Theorem 0.2] we only need to show that

$$\frac{\Delta\tilde{u}(x)}{\tilde{u}(x)} = W_{opt}(x).$$

By the explicit formula for the Green function on $\mathbb{T}_{q+1}$ given in [32, Lemma 1.24] we have

$$\tilde{u}(x) = \sqrt{\frac{q}{q-1}\left(\frac{1}{q}\right)^{|x|}}.$$

For $x \neq o$, we obtain that

$$\frac{\Delta\tilde{u}(x)}{\tilde{u}(x)} = \left(q + 1 - \left(\frac{1}{q}\right)^{1/2} - q\left(\frac{1}{q}\right)^{-1/2}\right) = \left(q^{1/2} - 1\right)^2 = \Lambda_q.$$

For $x = o$ we get

$$\begin{aligned} \frac{\Delta\tilde{u}(o)}{\tilde{u}(o)} &= \left[(q+1)\left(\frac{q}{q-1}\right)^{1/2} - (q+1)\left(\frac{1}{(q-1)^{1/2}}\right)\right]\left(\frac{q-1}{q}\right)^{1/2} \\ &= q + 1 - \frac{q+1}{q^{1/2}} = \Lambda_q + q^{1/2} - q^{-1/2} > \Lambda_q. \end{aligned}$$

□

Proof of Theorem 2.5. The statement follows from [8, Proposition 3.1] by providing a suitable positive supersolution to the equation $\Delta u = W_{\beta,\gamma} u$ in $\mathbb{T}_{q+1}$. To this aim, we define the function:

$$u_{\beta,\gamma}(x) = \begin{cases} q^{-|x|/2}|x|^\beta & \text{if } |x| \geq 1, \\ \gamma & \text{if } |x| = 0. \end{cases} \tag{3.1}$$

Now, by writing $u = u_{\beta,\gamma}$, we have

$$\frac{\Delta u(o)}{u(o)} = q + 1 - (q+1)\frac{q^{-1/2}}{\gamma} = q + 1 - q^{1/2}\Big(\frac{1}{\gamma} + \frac{1}{q\gamma}\Big),$$

which is nonnegative if $\gamma \geq q^{-1/2}$.
Next, for every x such that $|x| = 1$, we have

$$\frac{\Delta u(x)}{u(x)} = q + 1 - q\frac{q^{-1}2^\beta}{q^{-1/2}} - \frac{\gamma}{q^{-1/2}} = q + 1 - q^{1/2}(2^\beta + \gamma),$$

which is nonnegative if $\gamma \leq q^{1/2} + q^{-1/2} - 2^\beta$. The restriction $\beta \leq 1/2 \log_2 q$ comes out to make the inequality consistent $q^{-1/2} \leq \gamma \leq q^{-1/2} + q^{1/2} - 2^\beta$.
Finally, for every x such that $|x| \geq 2$, we have

$$\begin{aligned} \frac{\Delta u(x)}{u(x)} &= q + 1 - q\frac{q^{-(|x|+1)/2}(|x|+1)^\beta}{q^{-|x|/2}|x|^\beta} - \frac{q^{-(|x|-1)/2}(|x|-1)^\beta}{q^{-|x|/2}|x|^\beta} \\ &= q + 1 - q^{1/2}\left[\left(1 + \frac{1}{|x|}\right)^\beta + \left(1 - \frac{1}{|x|}\right)^\beta\right] \geq 0. \end{aligned} \tag{3.2}$$

If $\beta \leq 1$, then the function $f : \mathbb{R}^+ \to \mathbb{R}$ defined by $f(x) = x^\beta$ is concave. It follows that

$$f\left(\frac{1}{2}\left(1 + \frac{1}{|x|}\right) + \frac{1}{2}\left(1 - \frac{1}{|x|}\right)\right) = f(1) \geq \frac{1}{2}f\left(1 + \frac{1}{|x|}\right) + \frac{1}{2}f\left(1 - \frac{1}{|x|}\right),$$

that is equivalent to

$$2 \geq \left(1 + \frac{1}{|x|}\right)^\beta + \left(1 - \frac{1}{|x|}\right)^\beta.$$

Then,

$$\frac{\Delta u(x)}{u(x)} = q + 1 - q^{1/2}\left[\left(1 + \frac{1}{|x|}\right)^\beta + \left(1 - \frac{1}{|x|}\right)^\beta\right] \geq q + 1 - 2q^{1/2} = \Lambda_q > 0 \qquad \forall |x| \geq 2,$$

which proves (3.2).

If $\log_2 q^{1/2} \geq \beta > 1$, notice that the function $h : [2, +\infty) \to \mathbb{R}$ defined by $h(x) = (1 + \frac{1}{x})^\beta + (1 - \frac{1}{x})^\beta$ is decreasing. Then h reaches its maximum at 2. Thus to show the validity of (3.2) it suffices to prove that

$$h(x) \leq h(2) = \left(\frac{3}{2}\right)^\beta + \left(\frac{1}{2}\right)^\beta \leq q^{1/2} + q^{-1/2}. \tag{3.3}$$

Notice that for every $\beta \geq 1$

$$\frac{\mathrm{d}}{\mathrm{d}\beta}\left[\left(\frac{3}{2}\right)^\beta + \left(\frac{1}{2}\right)^\beta\right] = 2^{-\beta}(3^\beta \log(3/2) - \log(2)) \geq 0.$$

Hence

$$\left(\frac{3}{2}\right)^\beta + \left(\frac{1}{2}\right)^\beta \leq \left(\frac{3}{2}\right)^{\log_2 q^{1/2}} + \left(\frac{1}{2}\right)^{\log_2 q^{1/2}} \leq 2^{\log_2 q^{1/2}} + 2^{-\log_2 q^{1/2}} = q^{1/2} + q^{-1/2},$$

so that (3.3) holds and the proof is concluded. □

Remark 3.1. *Note that the statement of Theorem 2.5 can be enriched by considering the family of radial functions*

$$u_{\alpha,\beta,\gamma}(x) = \begin{cases} q^{\alpha|x|}|x|^{\beta} & \text{if } |x| \geq 1, \\ \gamma & \text{if } |x| = 0, \end{cases}$$

with $\alpha \in \mathbb{R}$ and β and γ as in Theorem 2.5. Indeed, a straightforward computation shows that for $|x| \geq 2$

$$W_{\alpha,\beta,\gamma}(x) = \frac{\Delta u_{\alpha,\beta,\gamma}(x)}{u_{\alpha,\beta,\gamma}(x)} = q + 1 - q^{\alpha+1}\left(1 + \frac{1}{|x|}\right)^{\beta} - q^{-\alpha}\left(1 - \frac{1}{|x|}\right)^{\beta}.$$

Nevertheless,

$$W_{\alpha,\beta,\gamma}(x) = q + 1 - q^{1+\alpha} - q^{-\alpha} + o(1) \qquad \text{as } |x| \to +\infty,$$

which is maximum for $\alpha = -1/2$. Therefore, the choice $\alpha = -1/2$ turns out to be the best to get a weight as large as possible at ∞.

We shall now prove our main result, i.e. Theorem 2.7.

Proof of Theorem 2.7. Consider the Schrödinger operator $H := \Delta + Q$, with

$$Q(x) = \begin{cases} 0 & \text{if } |x| = 0, \\ q^{1/2} & \text{if } |x| = 1, \\ -\Lambda_q & \text{if } |x| \geq 2. \end{cases}$$

Step 1. We construct an optimal Hardy weight for H. To this aim, we exploit [23, Theorem 1.1] that provides an optimal Hardy weight for a Schrödinger operator H by using H-harmonic functions.
For the sake of completeness we start by briefly recalling the statement of [23, Theorem 1.1]:
given two positive H-superharmonic functions u, v which are H-harmonic outside a finite set, if the function $u_0 := u/v$ is proper and $\sup_{x\sim y} u_0(x)/u_0(y) < +\infty$, then $\widetilde{W} := \frac{H[(uv)^{1/2}]}{(uv)^{1/2}}$ is an optimal weight for H.
Next we define

$$u(x) := \begin{cases} \gamma & \text{if } |x| = 0, \\ q^{-|x|/2} & \text{if } |x| \geq 1, \end{cases}$$

$$v(x) := \begin{cases} \gamma & \text{if } |x| = 0, \\ |x|q^{-|x|/2} & \text{if } |x| \geq 1. \end{cases}$$

Now we show that u, v satisfy the hypothesis of the above-mentioned theorem.
Indeed,

$$Hu(o) = (q+1)(\gamma - q^{-1/2}) + Q(o)\gamma \geq 0,$$
$$Hv(o) = (q+1)(\gamma - q^{-1/2}) + Q(o)\gamma \geq 0.$$

If $|x| = 1$, then

$$\begin{aligned} Hu(x) &= (q+1)q^{-1/2} - qq^{-1} - \gamma + q^{-1/2}q^{1/2} = q^{1/2} + q^{-1/2} - \gamma \geq 2^{1/2}, \\ Hv(x) &= (q+1)q^{-1/2} - 2q^{-1}q - \gamma + Q(x)q^{-1/2} \\ &\geq q^{1/2} + q^{-1/2} - 2 - q^{-1/2} - q^{1/2} + 2^{1/2} + 1 = 2^{1/2} + 1 - 2 > 0. \end{aligned}$$

If $|x| \geq 2$, then

$$\begin{aligned} Hu(x) &= (q+1)q^{-|x|/2} - qq^{-(|x|+1)/2} - q^{-(|x|-1)/2} - \Lambda_q q^{-|x|/2} \\ &= q^{-|x|/2}(q+1-2q^{1/2}-\Lambda_q) = 0; \\ Hv(x) &= (q+1)|x|q^{-|x|/2} - (|x|+1)qq^{-(|x|+1)/2} - (|x|-1)q^{-(|x|-1)/2} - \Lambda_q q^{-|x|/2} \\ &= |x|q^{-|x|/2}(q+1-2q^{1/2}-\Lambda_q) = 0. \end{aligned}$$

Define now

$$u_0(x) := \frac{u(x)}{v(x)} = \begin{cases} 1 & \text{if } |x| = 0, \\ \frac{1}{|x|} & \text{otherwise.} \end{cases}$$

The function u_0 is proper because $\lim_{|x|\to\infty} u_0(|x|) = 0$ and $u_0(|x|) > u_0(|x|+1) > 0$ for all $|x| \geq 1$, thus $u_0^{-1}(K)$ is finite for all compact set $K \subset (0,\infty)$.
Now consider $x \sim y$ and compute

$$\frac{u_0(x)}{u_0(y)} = \begin{cases} 1 & \text{if } |x| = 0, \\ 1/\gamma & \text{if } |y| = 0 \text{ and } |x| = 1, \\ 1+\frac{1}{|x|} & \text{if } |y| = |x|+1 \text{ and } |x| \geq 1, \\ 1-\frac{1}{|x|} & \text{if } |y| = |x|-1 \text{ and } |x| \geq 2. \end{cases}$$

Thus $\sup_{\substack{x,y\in\mathbb{T}_{q+1} \\ x\sim y}} \frac{u_0(x)}{u_0(y)} < +\infty$. Hence, from [23, Theorem 1.1] we conclude that the weight

$$\begin{aligned} \widetilde{W}(x) :&= \frac{H[(uv)^{1/2}](x)}{(uv)^{1/2}(x)} = \frac{\Delta(uv)^{1/2}(x)}{(uv)^{1/2}(x)} + Q(x) \\ &= \begin{cases} (q+1)(1-\frac{q^{-1/2}}{\gamma}) & \text{if } |x| = 0, \\ (q+1) - q^{1/2}(2^{1/2}+\gamma) + q^{1/2} & \text{if } |x| = 1, \\ (q+1) - q^{1/2}[(1+\frac{1}{|x|})^{1/2} + (1-\frac{1}{|x|})^{1/2}] - \Lambda_q & \text{if } |x| \geq 2 \end{cases} \end{aligned}$$

is an optimal weight for H.

Step 2. We derive an optimal Hardy weight for Δ. To this aim we prove that the three conditions of Definition 2.2 are satisfied by the operator $\Delta - W_{1/2,\gamma}$, where $W_{1/2,\gamma} := \widetilde{W} - Q$.

• *Criticality:* the optimal Hardy inequality, obtained considering the quadratic form h associated with H, namely

$$\frac{1}{2} \sum_{\substack{x,y\in\mathbb{T}_{q+1} \\ x\sim y}} \Big(\varphi(x) - \varphi(y)\Big)^2 + \sum_{x\in\mathbb{T}_{q+1}} Q(x)\varphi^2(x) \geq \sum_{x\in\mathbb{T}_{q+1}} \left(\frac{\Delta(uv)^{1/2}(x)}{(uv)^{1/2}(x)} + Q(x)\right)\varphi^2(x)$$

is equivalent to the Hardy inequality associated to Δ

$$\frac{1}{2} \sum_{\substack{x,y\in\mathbb{T}_{q+1} \\ x\sim y}} \Big(\varphi(x) - \varphi(y)\Big)^2 \geq \sum_{x\in\mathbb{T}_{q+1}} \frac{\Delta(uv)^{1/2}(x)}{(uv)^{1/2}(x)}\varphi^2(x) \qquad \forall \varphi \in C_0(\mathbb{T}_{q+1}).$$

Moreover,

$$W_{1/2,\gamma}(x) := \frac{\Delta(uv)^{1/2}(x)}{(uv)^{1/2}(x)} = \begin{cases} q+1-q^{1/2}(\frac{1}{\gamma}+\frac{1}{q\gamma}) & \text{if } |x| = 0, \\ q+1-q^{1/2}(2^{1/2}+\gamma) & \text{if } |x| = 1, \\ q+1-q^{1/2}[(1+\frac{1}{|x|})^{1/2} + (1-\frac{1}{|x|})^{1/2}] & \text{if } |x| \geq 2, \end{cases}$$

is nonnegative. The optimality of $\widetilde{W}$ for H implies that it does not exist a nonnegative function $f \not\equiv 0$ such that

$$\frac{1}{2}\sum_{\substack{x,y\in\mathbb{T}_{q+1}\\ x\sim y}}\Big(\varphi(x)-\varphi(y)\Big)^2-\sum_{x\in\mathbb{T}_{q+1}}W_{1/2,\gamma}(x)\varphi^2(x)\geq\sum_{x\in\mathbb{T}_{q+1}}f(x)\varphi^2(x),$$

or, equivalently, $\Delta - W_{1/2,\gamma}$ is critical.

• *Null-criticality of $\Delta - W_{1/2,\gamma}$ with respect to $W_{1/2,\gamma}$* : the function $z=(uv)^{1/2}$ is the ground state of $h_\Delta - W_{1/2,\gamma}$. Notice that

$$\begin{aligned} W_{1/2,\gamma}(x) &> W_{opt}(x) \qquad \text{if } |x|\geq 2,\\ z(x) &> G^{1/2}(x) \qquad \text{if } |x|\geq 2, \end{aligned}$$

where W_{opt} is defined by (2.1) and G is the Green function. Then by Remark 2.4

$$\sum_{x\in\mathbb{T}_{q+1}} z^2(x)W_{1/2,\gamma}(x)=+\infty.$$

• *Optimality near infinity* : suppose by contradiction that there exist $\overline{\lambda}>0$ and a compact set $K\subset\mathbb{T}_{q+1}$ such that

$$\frac{1}{2}\sum_{\substack{x,y\in\mathbb{T}_{q+1}\\ x\sim y}}\Big(\varphi(x)-\varphi(y)\Big)^2-\sum_{x\in\mathbb{T}_{q+1}}W_{1/2,\gamma}(x)\varphi^2(x)\geq\overline{\lambda}\sum_{x\in\mathbb{T}_{q+1}}W_{1/2,\gamma}(x)\varphi^2(x), \tag{3.4}$$

for all $\varphi\in C_0(\mathbb{T}_{q+1}\setminus K)$. Then, (3.4) holds true on $C_0(\mathbb{T}_{q+1}\setminus(K\cup B_2(o)))$. Notice that $W_{opt}\varphi^2\leq W_{1/2,\gamma}\varphi^2$ for all $\varphi\in C_0(\mathbb{T}_{q+1}\setminus(K\cup B_2(o)))$. It follows that

$$\begin{aligned}\frac{1}{2}\sum_{\substack{x,y\in\mathbb{T}_{q+1}\\ x\sim y}}\Big(\varphi(x)-\varphi(y)\Big)^2-\sum_{x\in\mathbb{T}_{q+1}}W_{opt}(x)\varphi^2(x)&\geq\frac{1}{2}\sum_{\substack{x,y\in\mathbb{T}_{q+1}\\ x\sim y}}\Big(\varphi(x)-\varphi(y)\Big)^2-\sum_{x\in\mathbb{T}_{q+1}}W_{1/2,\gamma}(x)\varphi^2(x)\\ &\geq\overline{\lambda}\sum_{x\in\mathbb{T}_{q+1}}W_{1/2,\gamma}(x)\varphi^2(x)\geq\overline{\lambda}\sum_{x\in\mathbb{T}_{q+1}}W_{opt}(x)\varphi^2(x),\end{aligned}$$

for all $\varphi\in C_0(\mathbb{T}_{q+1}\setminus(K\cup B_2(o)))$. This is a contradiction because W_{opt} is optimal for Δ. We checked the three conditions given in Definition 2.2. Hence $W_{1/2,\gamma}$ is optimal for Δ. □

Proof of Corollary 2.8. For $\beta<\min\{1/2\log q,1\}$ we have that $W_{\beta,\gamma}>W_{opt}$ on $B_2(o)^c$. Then, the thesis follows by repeating the same argument used for proving (3.4). □

3.2. **Proof of improved Poincaré inequalities.**

Proof of Theorem 2.10. Given $W_{\beta,\gamma}=\frac{\Delta u_{\beta,\gamma}}{u_{\beta,\gamma}}$, where $u_{\beta,\gamma}$ is defined by (3.1), it is easy to check that $W_{\beta,\gamma}$ is larger than Λ_q on $B_2(o)$ choosing the parameters $0\leq\beta\leq\log_2\left(\frac{3}{2}-\frac{1}{2q}\right)$ and $\frac{1}{2}+\frac{1}{2q}\leq\gamma\leq 2-2^\beta$.
Indeed,

$$q+1-(q+1)q^{-1/2}/\gamma\geq q+1-2q^{1/2}$$

is equivalent to $\frac{1}{2}+\frac{1}{2q}\leq\gamma$, and

$$q+1-q^{1/2}(2^\beta+\gamma)\geq q+1-2q^{1/2}$$

is equivalent to $\gamma\leq 2-2^\beta$. Notice that for this choice of γ and β it follows that $\beta\leq\log_2(\frac{3}{2})<1$, and we already proved in Theorem 2.5 that $W_{\beta,\gamma}\geq\Lambda_q$ on $B_2(o)^c$ for all $0\leq\beta<1$. □

Proof of Theorem 2.11. We know from Theorem 2.7 that the optimal weight $W_{1/2,\gamma}$ is larger than Λ_q for $|x| \geq 2$. Then we can define

$$\overline{R}(x) = W_{1/2,\gamma}(x) - \Lambda_q \qquad \forall x \in \mathbb{T}_{q+1} \setminus B_2(o),$$

and (2.6) follows. The sharpness of $q^{1/2}$ is consequence of the optimality of $\widetilde{W}$ for H where $\widetilde{W}$ and H are chosen such as in the proof of Theorem 2.7. □

4. Hardy-type inequalities on rapidly growing radial trees

In view of the results obtained on the homogeneous tree, here we attempt to generalise the family of Hardy inequalities given in Theorem 2.7 on a more general context, namely on radial trees. This could pave the way to future investigations on more general nonradial trees, by means of suitable comparison theorems, as in the Riemannian setting.

Let $T = (V, E)$ be an infinite tree. We call T a **radial** tree if the degree m depends only on $|x|$ (see e.g. [8, 34]). In the following we set $\overline{m} = m - 1$ to lighten the notation. For future purposes, we also note that the volume of the ball $B_n(o)$ is given by

$$\begin{aligned}
&\#B_1(o) = 1,\\
&\#B_2(o) = 2 + \overline{m}(0),\\
&\#B_3(o) = 2 + \overline{m}(0) + (\overline{m}(0)+1)\overline{m}(1),\\
&\qquad\vdots\\
&\#B_n(o) = 1 + (\overline{m}(0)+1)[1 + \overline{m}(1) + \overline{m}(1)\overline{m}(2) + \ldots + \overline{m}(1)\overline{m}(2)\overline{m}(3)\ldots\overline{m}(n-2)].
\end{aligned}$$

If particular, if $T = \mathbb{T}_{q+1}$, then $\overline{m} \equiv q$ and we have that $\#B_n(o) \sim q^{n-1}$ as $n \to +\infty$.

Next, recalling that the proof of Theorem 2.5 relies on the exploitation of the superharmonic functions $u_{\alpha,\beta}$ and that $u_{\alpha,\beta}(x) = |x|^\beta q^{\alpha|x|}$ for all $|x| \geq 1$, by analogy, we consider on T the family of positive and radial functions:

$$u_{\alpha,\beta}(x) := |x|^\beta \Psi^\alpha(|x|) \qquad \text{if } |x| \geq 1\,. \tag{4.1}$$

Regarding the choice of the function Ψ, since in $\mathbb{T}_{q+1}$ the function $q^{|x|}$ is related to $\#B_{|x|+1}(o)$ and since $\frac{\#B_{|x|+1}(o)}{\#B_{|x|}(o)} \sim q = \overline{m}$ as $|x| \to +\infty$, we assume that it satisfies the following condition

$$\Psi(|x|+1) = \overline{m}(|x|)\Psi(|x|) \quad \text{for all } |x| \geq 1. \tag{4.2}$$

Clearly, if $T = \mathbb{T}_{q+1}$, then (4.2) holds by taking $\Psi(|x|) = q^{|x|}$. We note that, conversely, for a given positive Ψ, condition (4.2) characterizes the tree we are dealing with through its degree, see Remark 4.2 below.

By showing that the function $u_{-1/2,\beta}$ is superharmonic on T, we obtain the following result.

Proposition 4.1. *Let $\Psi : (0,+\infty) \to \mathbb{R}$ be a positive function such that the map $(0,+\infty) \ni s \mapsto \frac{\Psi(s+1)}{\Psi(s)}$ is nondecreasing and let T be a radial tree with degree $\overline{m}+1$ satisfying condition (4.2). Then, for all $\beta < 1$ and $\frac{1}{\Psi^{1/2}(1)} \leq \gamma \leq \frac{1}{\Psi^{1/2}(1)}\left(\overline{m}(1) + 1 - \overline{m}^{1/2}(1)2^\beta\right)$ the following inequality holds*

$$\frac{1}{2}\sum_{\substack{x,y\in T\\ x\sim y}}\Big(\varphi(x)-\varphi(y)\Big)^2 \geq \sum_{x\in T} W_{\beta,\gamma}\varphi^2(x) \qquad \forall \varphi \in C_0(T)\,,$$

where $W_{\beta,\gamma}$ is the positive weight

$$W_{\beta,\gamma}(x) := \begin{cases} \overline{m}(0) + 1 - \frac{\overline{m}(0)+1}{\gamma\,\Psi^{1/2}(1)} & \text{if } |x| = 0, \\ \overline{m}(1) + 1 - \overline{m}^{1/2}(1)2^{\beta} - \Psi^{1/2}(1)\gamma & \text{if } |x| = 1, \\ \overline{m}(|x|) + 1 - \overline{m}^{1/2}(|x|)\Big(1 + \frac{1}{|x|}\Big)^{\beta} - \overline{m}^{1/2}(|x|-1)\Big(1 - \frac{1}{|x|}\Big)^{\beta} & \text{if } |x| \geq 2. \end{cases}$$

Remark 4.2. *It is readily seen that, by taking $\Psi(s) = q^s$ in Proposition 4.1, we get $T = \mathbb{T}_{q+1}$ and we re-obtain Theorem 2.7; however, Proposition 4.1 gives no information about the criticality of the operator $\Delta - W_{\beta,\gamma}$ on T. We also note that condition (4.2) yields rapidly growing trees, such as those generated, for instance, by the maps $\Psi_a(s) = e^{s^a}$ with $a > 1$.*

Proof. The proof follows the same lines of the proof of Theorem 2.5, namely we show that the function $u_{\alpha,\beta}$ in (4.1), with $\alpha = -1/2$ and $\beta < 1$, is superharmonic in $T \setminus B_2(0)$ and that it can be properly extended to o in order to get a superharmonic function on the whole T. Hence the statement follows by invoking [8, Proposition 3.1].

If $\beta < 1$ and $|x| \geq 2$ we have

$$\begin{aligned} \Delta u_{-1/2,\beta}(x) &= \Big(\overline{m}(|x|) + 1\Big)|x|^{\beta}\Psi^{-1/2}(|x|) - \overline{m}^{1/2}(|x|)(|x|+1)^{\beta}\Psi^{-1/2}(|x|) + \\ &\quad - (|x|-1)^{\beta}\overline{m}^{1/2}(|x|-1)\Psi^{-1/2}(|x|) \\ &= u_{-1/2,\beta}(x)\bigg(\overline{m}(|x|) + 1 - \overline{m}^{1/2}(|x|)\Big(1 + \frac{1}{|x|}\Big)^{\beta} - \overline{m}^{1/2}(|x|-1)\Big(1 - \frac{1}{|x|}\Big)^{\beta}\bigg). \end{aligned}$$

Since by hypothesis the function $\overline{m}$ is nondecreasing, we get

$$\begin{aligned} \Delta u_{-1/2,\beta}(x) = u_{-1/2,\beta}(x)\bigg(&\big(\overline{m}^{1/2}(|x|)) - 1\big)^2 + \overline{m}^{1/2}(|x|)\Big(2 - \Big(1 + \frac{1}{|x|}\Big)^{\beta} - \Big(1 - \frac{1}{|x|}\Big)^{\beta}\Big) \\ &+ \Big(\overline{m}^{1/2}(|x|) - \overline{m}^{1/2}(|x|-1)\Big)\Big(1 - \frac{1}{|x|}\Big)^{\beta}\bigg) > 0, \end{aligned}$$

for all $|x| \geq 2$.

Then we choose $\gamma := u_{-1/2,\beta}(o)$ such that $\Delta u_{-1/2,\beta}$ is nonnegative in $B_2(o)$. By a direct computation we have

$$\Delta u_{-1/2,\beta}(o) = (\overline{m}(0) + 1)(\gamma - \Psi^{-1/2}(1)) \geq 0,$$

for $\gamma \geq \Psi^{-1/2}(1)$. Furthermore, for $|x| = 1$ we get

$$\Delta u_{-1/2,\beta}(x) = (\overline{m}(1) + 1)\Psi^{-1/2}(1) - \overline{m}(1)2^{\beta}\Psi^{-1/2}(2) - \gamma \geq 0,$$

for $\gamma \leq \Psi^{-1/2}(1)\Big(\overline{m}(1) + 1 - \overline{m}^{1/2}(1)2^{\beta}\Big)$. This concludes the proof. □

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