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The complex world of oscillator noise

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***Abstract*—We review the modern oscillator noise analysis techniques based on phase-amplitude noise decompositions. While avoiding the extensive use of mathematical derivations, we aim at defining the common ground that forms the basis for modern oscillator noise analysis in order to provide an essential presentation of the Floquet-based approaches, clarifying their connections and differences.**

***Index Terms*—Oscillators, phase noise, amplitude noise, circuit simulation, Floquet theory**

I. Introduction

The study of fluctuations in oscillators has been a classical research topic in mathematics, physics and engineering since the first half of the XX century [1]–[4]. Besides the intellectual fascination for mathematically difficult problems, the importance of the topic is deeply rooted in practical applications, mainly in the fields of RF and microwave electronics, and of telecommunications. In fact, defining a precise frequency reference is fundamental for many applications, both electrical (e.g., transmitters and receivers) and optical (e.g., LASERs): the broadening of the generated spectral line is mainly due to the *phase noise* component of oscillator fluctuations, that as a consequence is the most commonly studied feature of oscillator noise (see [5] for a recent and exhaustive review). In a dual perspective, the definition of a precise time reference is also extremely important for digital applications, thus implying the necessity to keep under control the *time jitter* in clocked and in sampled systems. From the theoretical standpoint, phase noise and time jitter are simply the two sides of the same coin, a manifestation of the oscillator noisy behavior. As the microwave engineer is more often interested in the phase noise characterization, we will discuss the latter only. The time jitter estimation is discussed, for instance, in [6].

Despite this long history, oscillator noise has recently received a significant rejuvenation when a mathematically sound approach has been proposed in [6], [7] that takes care of some inconsistencies showed by classical approaches at vanishing offset frequency. Mathematical consistency is however attained at the cost of a significant complexity of the corresponding noise theory, that makes it impossible to provide a direct link to simplified circuit analysis and, thus, to simple yet sufficiently accurate closed form expressions that would make a direct connection to low-noise oscillator design rules such as the celebrated Leeson's formula [8], [9]. In other words, the use of advanced and mathematically sound theories is often confined to electronic design automation (EDA) tools for the computer-aided design (CAD) of oscillators.

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The aim of this review is to introduce such modern approaches to oscillator (phase and amplitude) noise analysis, and to discuss the relationship among them (avoiding as much as possible the corresponding mathematical subtleties) with the ultimate goal of clarifying their differences and connections.

II. Basics

The starting point of any oscillator noise theory is the set of equations that governs the state space evolution of the circuit in the absence of any fluctuations, which in the simplest case is a set of ordinary differential equations (ODEs) such as

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{f}(\mathbf{x}(t)) \tag{1}$$

where $\mathbf{x}(t)$ is the set of n variables describing the oscillator working point (WP), and $\mathbf{f}$ is an n size nonlinear function. Actually, circuit equations as implemented in circuit simulators take in general the form of a differential-algebraic equation (DAE) system [10]: the treatment is mathematically more involved [11], however since the basic results are the same, we discuss here the simpler ODE case only. We consider here purely analog systems, in which $\mathbf{f}$ is a smooth function. The case of mixed analog-digital circuits requires a more complex analysis due to the presence of jumps in the solution that prevent the direct exploitation of Floquet theory [12]–[14].

The oscillator is identified by a nonzero solution $\mathbf{x}_\mathrm{S}(t)$ of (1) characterized by the property of being periodic, i.e. there is a period $T > 0$ such that $\mathbf{x}_\mathrm{S}(t+T) = \mathbf{x}_\mathrm{S}(t)$. Clearly, a well designed oscillator should have a strongly stable WP, meaning that a limited perturbation of the circuit should be rapidly absorbed by the oscillator, whose state should therefore plunge back on the limit cycle (orbit) $\mathbf{x}_\mathrm{S}(t)$. Mathematically, this is guaranteed in the following way: the linear periodically time-varying (LPTV) system obtained by linearizing (1) around $\mathbf{x}_\mathrm{S}(t)$ should be characterized, besides by the unique structural Floquet exponent (FE) $\mu_1 = 0$ (see Appendix A for a brief introduction to Floquet theory), by the other $n-1$ FEs all satisfying $\mathrm{Re}\{\mu_i\} \ll 0$.

The presence of noise is translated into a dependency of the nonlinear function $\mathbf{f}$ on a proper set of $m_\mathrm{w} + m_\mathrm{c}$ noise sources, represented by a stochastic vector $\boldsymbol{\xi}_\mathrm{w}(t)$ of size m_w usually characterized as a set of uncorrelated, white Gaussian noise processes [15], and by the low-frequency (typically, flicker) fluctuations characterized by m_c scalar, independent and colored Gaussian noise sources $\xi_{\mathrm{c}m}(t)$: in this way the ODE system (1) is transformed into a stochastic ODE (S-ODE). As the noise sources are usually of limited magnitude, the customary procedure amounts to linearize the perturbed

S-ODE with respect to the noise sources thus leading to a Langevin equation

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{f}(\mathbf{x}(t)) + \mathbf{B}_{\mathrm{w}}(\mathbf{x}(t))\boldsymbol{\xi}_{\mathrm{w}}(t) + \sum_{m=1}^{m_{\mathrm{c}}} \mathbf{B}_{\mathrm{w}m}(\mathbf{x}(t))\xi_{\mathrm{c}m}(t) \quad (2)$$

where $\mathbf{x}(t)$ is the set of perturbed circuit variables, now stochastic processes. Matrix $\mathbf{B}_{\mathrm{w}}$, of size $n \times m_{\mathrm{w}}$, represents the possible noise source modulation of the white sources, while the m_{c} vectors $\mathbf{B}_{\mathrm{w}m}$ (of size n) are introduced to take into account the possible modulation of the colored sources.

At this point the approaches available in the literature are different ways of tackling the solution of (2): the most obvious one amounts to solve directly the nonlinear S-ODE (most probably numerically) and find the corresponding second-order statistical characterization of the noisy circuit variables $\mathbf{x}(t)$, namely the two-time correlation matrix $\mathbf{R}_{\mathbf{x},\mathbf{x}}(t_1, t_2) = \langle \mathbf{x}(t_1)\mathbf{x}^{\mathrm{T}}(t_2)\rangle$, where $\langle\cdot\rangle$ represents the expectation operator and T denotes the transpose. In many practical cases, noise is stationary and thus $\mathbf{R}_{\mathbf{x},\mathbf{x}}(t_1, t_2) = \mathbf{R}_{\mathbf{x},\mathbf{x}}(t)$ where $t = t_2 - t_1$. Thus, according to the Wiener-Kinchin theorem [16], the same information can be more effectively represented by a frequency domain function (the noise spectrum $\mathbf{S}_{\mathbf{x},\mathbf{x}}(\omega)$, or the power spectral density – PSD) that is the Fourier transform of $\mathbf{R}_{\mathbf{x},\mathbf{x}}(t)$. However, the direct numerical solution of a S-ODE is a tough task, in particular if the size n of the circuit is large and if an accurate determination of the statistical properties is sought for. Therefore, this approach is used mainly as a reference solution for validating manipulations of (2), and it is mostly applied to low dimensional test cases ($n = 2 \div 3$ or slightly more).

In most of the cases the problem is tackled by decomposing the fluctuating solution into phase and amplitude noise components, and by deriving (and solving) the corresponding S-ODEs: see Section V.

Finally, the noise sources are typically characterized as Gaussian, stationary processes. The Gaussian assumption allows to fully describe the statistical properties exploiting the average and the variance, i.e. the first two moments of the random process. In particular, the white components $\boldsymbol{\xi}_{\mathrm{w}}$ are here assumed uncorrelated and of unit amplitude, as the source strength (and the possible correlation, if present) can be included in the modulating matrix $\mathbf{B}_{\mathrm{w}}(t)$: $\mathbf{R}_{\boldsymbol{\xi}_{\mathrm{w}},\boldsymbol{\xi}_{\mathrm{w}}}(t_1, t_2) = \mathbf{I}\delta(t_1 - t_2)$, so that the corresponding PSD becomes $\mathbf{S}_{\boldsymbol{\xi}_{\mathrm{w}},\boldsymbol{\xi}_{\mathrm{w}}}(\omega) = \mathbf{I}$, where $\mathbf{I}$ is the identity matrix of size m_{w}. The uncorrelated, colored noise sources represent important physical processes, such as flicker noise, and are characterized by the corresponding (here, scalar) PSD $S_{\xi_{\mathrm{c}m},\xi_{\mathrm{c}m}}(\omega)$.

III. Phase definition and phase noise

The nature of autonomous systems makes their operation rather involved. Practical oscillators are characterized by a time-periodic working point that is a stable periodic orbit $\mathbf{x}_{\mathrm{S}}(t)$ (the limit cycle), i.e. a closed path in the state space continuously covered by the oscillator variables. Each point of the orbit is reached every T seconds, i.e. once per period of oscillation, and the actual operation of the circuit is characterized by the lack of a *fixed time reference,* meaning that even if the oscillator correctly operates on the designed periodic solution, the starting point of the orbit (i.e., the value $\mathbf{x}_{\mathrm{S}}(0)$) is randomly chosen by the peculiar initial conditions that are present at the time $t = 0$ of oscillator switch on. More mathematically, given the WP $\mathbf{x}_{\mathrm{S}}(t)$ the translated variables $\mathbf{x}_{\mathrm{S}}(t + t_0)$ are also a solution of (1) for any t_0. This suggests to consider a decomposition of the perturbed oscillator solution $\mathbf{x}(t)$ by separating the variation *along* the orbit by that taking place in the $(n-1)$-dimensional space that is linearly independent from the first one.

The behavior along the limit cycle is characterized by the concept of orbit *phase.* The exact mathematical definition of phase in the case of a noisy oscillator is a rather complex task, especially if $n > 2$: it involves the concept of orbit *isochron* [17]–[19], and it is beyond the scope of this review. We simply state here that the oscillator phase is defined as the function such that in the noiseless limit

$$\Phi(t) = \Phi(\mathbf{x}_{\mathrm{S}}(t)) = \omega_0 t \quad (3)$$

where $\omega_0 = 2\pi f_0 = 1/T$ is the WP (angular) frequency, and it is generalized for the noisy oscillator to a stochastic process whose average is equal to $\Phi(t)$. The corresponding second order statistical properties, i.e. the correlation function $R_{\Phi,\Phi}(t_1, t_2) = \langle \Phi(t_1)\Phi(t_2)\rangle$, defines the concept of *phase noise*. The remaining $n-1$ degrees of freedom required to fully characterize $\mathbf{x}(t)$ constitute the oscillator *orbital* – or *amplitude* – *noise.* As discussed in [20], for electronic oscillators the WP is normally a strongly stable orbit. This implies that the orbital perturbations will eventually decay, and the instantaneous WP is attracted back towards the noiseless orbit. Amplitude noise is therefore negligible with respect to phase noise, thus explaining the focus of the literature (and the designer's efforts) on phase fluctuations. Nevertheless, there are examples of autonomous systems, such as e.g. some models of biological systems [21], [22], for which orbital fluctuations are not negligible [23]. Furthermore, even in the electronic circuit case orbital contributions might become important far away from the oscillation harmonics, which in turn may impact the dynamic range of receivers operated in presence of strong adjacent channels [24], [25].

IV. Solution approaches: Linearization

The solution of (2) is almost always found by leveraging on the assumed small amplitude of the fluctuations induced by the noise sources, thus exploiting some degree of linearization. The simplest approach amounts to assume that the effect of noise is a perturbation of the oscillator orbit:

$$\mathbf{x}(t) = \mathbf{x}_{\mathrm{S}}(t) + \mathbf{x}_{\mathrm{n}}(t) \quad (4)$$

where $\mathbf{x}_{\mathrm{n}}(t)$ is a zero average vector stochastic process of size n.

The standard approach proceeds deriving a stochastic equation for $\mathbf{x}_{\mathrm{n}}(t)$ based on the linearization of (2) either around a DC value $\mathbf{x}_0$ that approximates $\mathbf{x}_{\mathrm{S}}(t)$, or directly around the oscillator limit cycle $\mathbf{x}_{\mathrm{S}}(t)$. In the first case the resulting system is linear time-invariant (LTI), while in the second it becomes linear periodically time varying (LPTV) [24], [26].

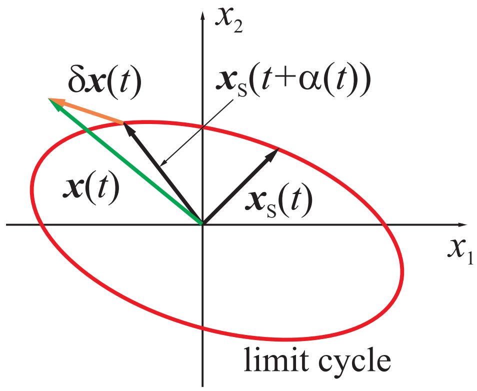

Fig. 1. Graphical representation of the decomposition of the noisy oscillator variables into phase and amplitude fluctuations for a generic 2D system.

The LTI analysis is extremely simple, but with a very limited accuracy and has been recognized as too crude for many applications, although in some specific cases, namely oscillators where little noise modulation takes place, the results might be in a reasonably good agreement with experiments [5].

On the other hand, the LPTV approach is much more popular, thanks to a combination of simplicity of the mathematical machinery and of an often quite good accuracy in the results, at least not asymptotically close to the harmonics of the oscillation frequency ω_0 where all linearized approaches yield a divergent spectrum [6]. The LPTV description was adopted in the classical work by Kurokawa [27], and has been generalized more recently within the framework of harmonic balance EDA tools as described in [28], where the Kurokawa results are derived as a special case. The methodology derived in [28] decomposes the noise description exploiting two formulations, one used to estimate noise far away from the harmonics of ω_0, the other very close to the harmonics where the *carrier modulation noise* is defined. This sophisticated decomposition allows for accurate results even quite close to the nominal oscillation frequency, thus improving the applicability of the LPTV description and making it a common tool among designers.

V. Solution approaches: Phase-amplitude decomposition

The path to the development of a well founded, and ultimately more accurate, theory of noise in oscillators was initiated by the seminal work of F. Kaertner who proposed in [29], [30] to decompose the noisy variables into a fluctuating term *along the limit cycle* (phase noise) and into *amplitude noise,* exploiting the decomposition shown in Fig. 1:

$$\mathbf{x}(t) = \mathbf{x}_S(t + \alpha(t)) + \delta\mathbf{x}(t). \tag{5}$$

The time reference fluctuation $\alpha(t)$ is a zero average (in the limit of negligible higher order terms of the α equation [15]) stochastic process that corresponds to phase noise

$$\Phi(t) = \omega_0(t + \alpha(t)), \tag{6}$$

while the amplitude fluctuations are represented by $\delta\mathbf{x}(t)$.

Of course the decomposition in (5) is not uniquely defined, however on the basis of the stability of the noiseless oscillator WP the basic idea is to choose the definition so that $\delta\mathbf{x}(t)$ *remains small* irrespective of time t. Notice that, on the contrary, the time fluctuation $\alpha(t)$ may be large *without forcing the oscillator instantaneous WP to wander far away from the orbit* $\mathbf{x}_S(t)$, in fact as shown in [6] the time perturbation $\alpha(t)$ has a variance that grows unbounded linearly with time.

Taking for granted the decomposition (5), the available theories amount to define a stochastic equation that enables the evaluation of the statistical properties of $\alpha(t)$ and of $\delta\mathbf{x}(t)$. Focusing on phase noise, we follow the most direct path: amplitude noise is simply neglected, by setting $\delta\mathbf{x}(t) = \mathbf{0}$. However, this choice has to be made wisely, in the sense that the S-ODE that defines the time evolution of $\alpha(t)$ should be determined by guaranteeing that the corresponding orbital perturbation remains arbitrarily small. As discussed in [6], [15] this implies that (2) has to be projected along the noiseless WP tangent, i.e. $\mathrm{d}\mathbf{x}_S/\mathrm{d}t$ whose versor is the Floquet eigenvector $\mathbf{u}_1(t)$ associated to the Floquet exponent $\mu_1 = 0$ (see Appendix A). As discussed in [30], this projection uniquely defines the phase and amplitude perturbations, as the corresponding equations are invariant with respect to linear changes of the state variables.

Although the projection along $\mathbf{u}_1(t)$ is mandatory to define the fluctuations along the orbit, the choice of the other Floquet eigenvectors as the remaining $n-1$ base elements used to define the amplitude noise is not strictly necessary. However, as shown in [31], this choice guarantees that even including the small amplitude noise, the defining equation for $\alpha(t)$ is left unchanged, thus preserving the results derived in [6], [7]. Since the Floquet basis is not in general orthogonal, the projection operation requires the use of the adjoint Floquet eigenvector $\mathbf{v}_1(t)$, also called the *perturbation projection vector* (PPV). The resulting nonlinear S-ODE for the time fluctuation $\alpha(t)$ is [6]

$$\begin{aligned}\frac{\mathrm{d}\alpha}{\mathrm{d}t} &= \mathbf{v}_1^{\mathrm{T}}(t+\alpha(t))\mathbf{B}_{\mathrm{w}}(t+\alpha(t))\boldsymbol{\xi}_{\mathrm{w}}(t)\\ &\quad + \sum_{m=1}^{m_{\mathrm{c}}} \mathbf{v}_1^{\mathrm{T}}(t+\alpha(t))\mathbf{B}_{\mathrm{w}m}(t+\alpha(t))\xi_{\mathrm{c}m}(t).\end{aligned} \tag{7}$$

Despite its nonlinear nature, this equation can be analyzed in detail (see [6] for white noise sources, [7] for flicker noise sources) and a general formulation for the resulting oscillator phase noise can be found whose characterization ultimately depends on the determination of the PPV, that becomes therefore the main quantity to be determined for phase noise assessment. The detailed analysis in [6], [7] shows that $\alpha(t)$ becomes asymptotically a Gaussian stationary stochastic process with variance fully defined by the harmonic components of the PPV and of the modulating functions

$$\sigma^2(t) = c_{\mathrm{w}}t + \frac{1}{\pi}\sum_{m=1}^{m_{\mathrm{c}}} |V_{0m}|^2 \int_{-\infty}^{+\infty} S_{\xi_{\mathrm{c}m},\xi_{\mathrm{c}m}}(\omega)\frac{1-\mathrm{e}^{\mathrm{j}\omega t}}{\omega^2}\,\mathrm{d}\omega \tag{8}$$

where

$$c_{\mathrm{w}}=\frac{1}{T}\int_0^T \mathbf{v}_1^{\mathrm{T}}(t)\mathbf{B}_{\mathrm{w}}(t)\mathbf{B}_{\mathrm{w}}^{\mathrm{T}}(t)\mathbf{v}_1(t)\ \mathrm{d}t \tag{9}$$

represents the contribution of white noise sources, while colored noise is weighted by the magnitude of the DC component of $\mathbf{v}_1^{\mathrm{T}}(t)\mathbf{B}_{\mathrm{c}m}(t)$

$$V_{0m}=\frac{1}{T}\int_0^T \mathbf{v}_1^{\mathrm{T}}(t)\mathbf{B}_{\mathrm{c}m}(t)\ \mathrm{d}t \tag{10}$$

The PSD of phase noise can also be expressed as a function of the same parameters, in fact [7] shows that the spectrum of the asymptotic value of the autocorrelation function for $\mathbf{x}_{\mathrm{S}}(t+\alpha(t))$ reads

$$\mathbf{S}_{\mathbf{x},\mathbf{x}}(\omega)=\sum_k \mathbf{X}_{\mathrm{S},k}\mathbf{X}_{\mathrm{S},k}^{\dagger}\mathbf{S}_k(\omega+k\omega_0) \tag{11}$$

where $\mathbf{X}_{\mathrm{S},k}$ is the amplitude of the k-th harmonic of $\mathbf{x}_{\mathrm{S}}(t)$ (assuming an exponential Fourier series), and

$$\mathbf{S}_k(\omega)=k^2\frac{\omega_0^2}{\omega^2}\left[c_{\mathrm{w}}+\sum_{m=1}^{m_{\mathrm{c}}}|V_{0m}|^2 S_{\xi_{\mathrm{c}m},\xi_{\mathrm{c}m}}(\omega)\right] \tag{12}$$

for $\omega \gg 0$, i.e. far away from the harmonics of ω_0, while close to the harmonics (i.e., for $\omega \approx 0$) a Lorentzian shape contribution is recovered, so as to avoid the nonphysical divergence for $\omega = 0$

$$\mathbf{S}_k(\omega)=\frac{\omega_0^2 k^2\left[c_{\mathrm{w}}+\sum_{m=1}^{m_{\mathrm{c}}}|V_{0m}|^2 S_{\xi_{\mathrm{c}m},\xi_{\mathrm{c}m}}(\omega)\right]}{\frac{\omega_0^4 k^4}{4}\left[c_{\mathrm{w}}+\sum_{m=1}^{m_{\mathrm{c}}}|V_{0m}|^2 S_{\xi_{\mathrm{c}m},\xi_{\mathrm{c}m}}(\omega)\right]^2+\omega^2} \tag{13}$$

Equation (12) is particularly interesting, as it is consistent with well known results concerning the scaling of the noise sources by ω^2 on the phase noise spectrum. The expression, however, is confined to a non-negligible frequency offset from the ω_0 harmonics. Close to these harmonics, the spectrum (13) becomes Lorentzian, which is again an expected result [20]. Notice also that the magnitude of the phase noise spectrum depends on the harmonic content of the PPV multiplied times the source modulation functions (see (9) and (10)).

A. Comparison among [6], [30] and [32]

We discuss here briefly the differences among the approaches presented in references [6], [30], [32]. A detailed comparison can be found in [33], where also the PSDs are presented. Notice that the same nonlinear S-ODE (7) was found in [30], however in order to find the phase noise characterization Kaertner made a zero order approximation of the S-ODE reducing it to the linear case:

$$\frac{\mathrm{d}\alpha}{\mathrm{d}t}=\mathbf{v}_1^{\mathrm{T}}(t)\mathbf{B}_{\mathrm{w}}(t)\boldsymbol{\xi}_{\mathrm{w}}(t)+\sum_{m=1}^{m_{\mathrm{c}}}\mathbf{v}_1^{\mathrm{T}}(t)\mathbf{B}_{\mathrm{w}m}(t)\xi_{\mathrm{c}m}(t). \tag{14}$$

Therefore, the results in [30] derive from an approximation of the correct phase equation (7) neglecting the α dependence on the right hand side. Although this does not impair the general shape of the output spectrum, that retains both the exclusive dependence on the spectral components of the PPV and of the modulating functions and the Lorentzian shape, the approximation becomes significant for the accurate description of specific behaviors such as injection locking or power/ground interference analysis: see [34] and references therein for a discussion.

The comparison with the Impulse Sensitivity Function (ISF) theory proposed in [32] (and summarized here in Appendix B) is made more complex by the several ISF definitions that can be exploited:

1) the *numerical ISF* defined in [32, Appendix A] as the phase fluctuation induced by a delta function perturbation in the oscillator variables calculated through time-domain simulations corresponds to the PPV [34], and therefore since in [32] the phase fluctuation is obtained through a linear response theory the time perturbation satisfies (14) (although in the original paper this relation is expressed in integral form), with the same limitations;
2) the *closed-form ISF* [32, Appendix B] corresponds to the projection along the direct Floquet eigenvector $\mathbf{u}_1(t)$, as opposite to the correct use of the PPV, and thus may severely undermine the accuracy of the calculated phase variation.

B. Amplitude noise

Projection of the full S-ODE along the other elements of the chosen basis yields a vector S-ODE having $\delta\mathbf{x}(t)$ as an unknown, whose solution characterizes the amplitude noise of the oscillator. Details on the projection procedure, and of the intricacies related to the use of Itô calculus, can be found in [15], [35]. In the simplest case, i.e. treating $\delta\mathbf{x}(t)$ as a linear perturbation of the limit cycle affected by phase noise, some of the present authors were able to prove that $\mathbf{x}_{\mathrm{S}}(t+\alpha(t))$ and $\delta\mathbf{x}(t)$ become asymptotically uncorrelated stochastic processes, while the corresponding amplitude noise PSD depends on the remaining $n-1$ FEs and Floquet eigenvectors [25]. The formulae are very complex, but nevertheless easily implementable into EDA tools provided that the relevant Floquet quantities have been accurately determined [36]–[38].

Notice that the same S-ODE for amplitude noise discussed in [25], [31] was already derived in [30]. Although the solution outlined in [30] is based on the linear phase equation (14) as opposed to the nonlinear equation (7), the resulting spectra show, as in [25], that only the Floquet exponents characterized by a magnitude of the real part much lower than ω_0 provide a significant contribution.

Amplitude noise was also tackled in [24], where the *amplitude* ISF is defined properly extending the concept of ISF used for phase noise characterization. The amplitude ISF basically amounts to select the Floquet subspace that mostly influences the amplitude fluctuation, thus corresponding to an approximation of the full theory in [25].

Finally, we remark that a careful treatment of the amplitude noise elimination leads to the presence of higher order terms in the phase noise equation (see [15], [39] and the references therein) that also influence the noiseless oscillation frequency.

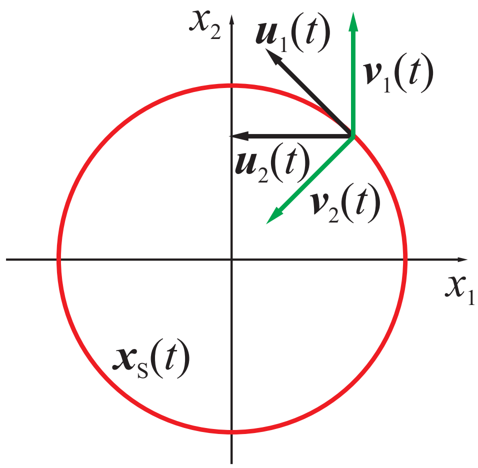

Fig. 2. Representation of the limit cycle for the simple 2D oscillator (15), along with the direct and adjoint Floquet eigenvectors (here shown for $t = \pi/8$).

VI. A simple 2D example

As an example we consider an extremely simple autonomous system, the 2D oscillator proposed in [40] written here in cartesian coordinates:

$$\dot{x}_1 = x_1 - x_2 - (x_1 + x_2)\sqrt{x_1^2 + x_2^2} + \epsilon\xi_1 \tag{15a}$$

$$\dot{x}_2 = x_1 + x_2 + (x_1 - x_2)\sqrt{x_1^2 + x_2^2} + \epsilon\xi_2 \tag{15b}$$

where ϵ is a parameter introduced to modulate the magnitude of the unit white Gaussian noise sources ξ (in other words, we set the colored noise sources to zero). The WP is defined by the solution of (15) for $\epsilon = 0$:

$$\mathbf{x}_\mathrm{S}(t) = \begin{bmatrix} x_{\mathrm{S}1}(t) \\ x_{\mathrm{S}2}(t) \end{bmatrix} = \begin{bmatrix} \cos(2t) \\ \sin(2t) \end{bmatrix}. \tag{16}$$

This very simple example allows to evaluate analytically the Floquet exponents and eigenvectors. The structural FE $\mu_1 = 0$ is characterized by the direct eigenvector $\mathbf{u}_1(t)$ and by the PPV $\mathbf{v}_1(t)$:

$$\mathbf{u}_1(t) = \begin{bmatrix} -\sin(2t) \\ \cos(2t) \end{bmatrix} \quad \mathbf{v}_1(t) = \begin{bmatrix} \cos(2t) - \sin(2t) \\ \cos(2t) + \sin(2t) \end{bmatrix}, \tag{17}$$

while for the second FE we find $\mu_2 = -1$ and

$$\mathbf{u}_2(t) = \frac{1}{2}\begin{bmatrix} -\cos(2t) - \sin(2t) \\ \cos(2t) - \sin(2t) \end{bmatrix} \quad \mathbf{v}_2(t) = 2\begin{bmatrix} -\cos(2t) \\ -\sin(2t) \end{bmatrix}. \tag{18}$$

For both eigenspaces, we have chosen to normalize to 1 the direct eigenvector, while for the adjoint $\mathbf{v}(t)$ the bi-orthogonality condition $\mathbf{u}_j^\mathrm{T}(t)\mathbf{v}_j(t) = 1$ was imposed. The four Floquet eigenvectors are shown in Fig. 2: notice that $\mathbf{u}_1$ and $\mathbf{u}_2$ are not orthogonal (though linearly independent), as well as the PPV and $\mathbf{v}_2$. On the other hand, the couples $(\mathbf{u}_1, \mathbf{v}_2)$ and $(\mathbf{u}_2, \mathbf{v}_1)$ are orthogonal.

Therefore, the correct time perturbation S-ODE (7) reads

$$\frac{\mathrm{d}\alpha}{\mathrm{d}t} = [\cos(2(t+\alpha(t))) - \sin(2(t+\alpha(t)))]\,\epsilon\xi_1(t) + [\cos(2(t+\alpha(t))) + \sin(2(t+\alpha(t)))]\,\epsilon\xi_2(t) \tag{19}$$

both for the rigorous theory in [6] and for the numerical ISF [32], while the approximated theory in [30] (and the original implementation of [32] with the numerical ISF) amounts to solve (14)

$$\frac{\mathrm{d}\alpha}{\mathrm{d}t} = [\cos(2t) - \sin(2t)]\,\epsilon\xi_1(t) + [\cos(2t) + \sin(2t)]\,\epsilon\xi_2(t). \tag{20}$$

Finally, the use of the closed form ISF leads to the Langevin equation

$$\frac{\mathrm{d}\alpha}{\mathrm{d}t} = -\sin(2t)\epsilon\xi_1(t) + \cos(2t)\epsilon\xi_2(t). \tag{21}$$

The classical approach to study S-ODEs such as the time perturbation equations above amounts to convert them into the corresponding Fokker-Planck equation [41] that defines the evolution of the probability density function $p(\alpha, t)$ for process $\alpha(t)$, the advantage being that such equation is entirely in the standard functions domain (i.e., no stochastic processes are involved). The derivation of the Fokker-Planck equation equivalent to (19), (20) and (21) leads to the same diffusion type equation:

$$\frac{\partial p}{\partial t} = D\frac{\partial^2 p}{\partial \alpha^2} \tag{22}$$

where however $D = \epsilon^2$ for (19) and (20), while $D = \epsilon^2/2$ for (21). This very simple behavior is due to the extreme symmetry of system (15): as clearly visible in (17) and (18), the components of the Floquet vectors exhibit a constant phase shift as a consequence of the rotational invariance of (15). The same argument justifies also the unexpected equivalence of the two approaches from [6] and [30], which is clearly peculiar to this specific case.

The solution of (22) is a Gaussian random process

$$p(\alpha, t) = \frac{1}{\sqrt{4\pi D t}}\,\mathrm{e}^{-\dfrac{\alpha^2}{4Dt}} \tag{23}$$

yielding a variance for the time perturbation equal to $\sigma^2(t) = 2Dt$ (linearly increasing with time, as expected). Therefore, the approaches (19) and (20) are characterized by a phase noise $\sigma^2(t) = 2\epsilon^2 t$, while the closed form ISF model (21) leads to a Gaussian process with variance $\sigma^2(t) = \epsilon^2 t$, i.e. half of the correct result. A comparison between the models is shown in Fig. 3.

VII. Conclusions

We have reviewed the available approaches to oscillator noise analysis that are currently implemented in modern EDA tools for low noise oscillator design. Starting from the common ground of Floquet analysis for the linearized system that is obtained as a result of the perturbation of the autonomous system equations around the noiseless working point, the approaches are presented in a unified way and they are then discussed with the aim of pointing out the common elements and the major differences.

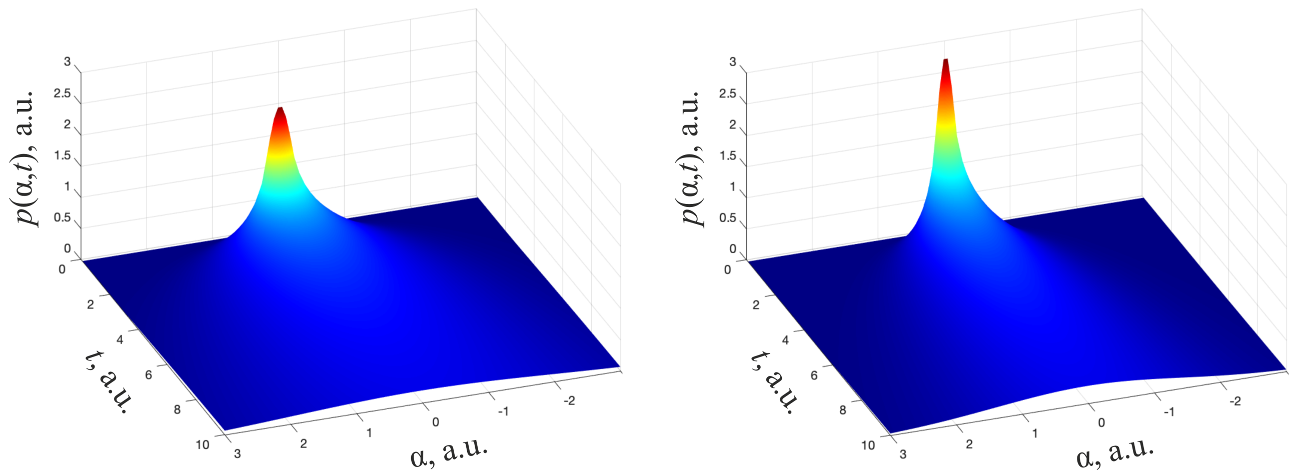

Fig. 3. Representation of the probability density function $p(\alpha, t)$ for the oscillator (15) according to the noise model in (19) and (20) (left), and to the closed form ISF (21) (right). In both cases $\epsilon^2 = 0.2$ was assumed.

Appendix A
A Floquet theory primer

Floquet theory [11], [42] is the basis for the most advanced oscillator noise theories, since it describes the input-output relationship of a *linear periodically time-varying* (LPTV) system of size n, such as

$$\frac{\mathrm{d}\mathbf{y}}{\mathrm{d}t} = \mathbf{A}(t)\mathbf{y}(t) \tag{24}$$

where $\mathbf{A}(t) = \mathbf{A}(t + T)$ is a T-periodic matrix of size n. Floquet theorem writes the solution of (24) with initial condition $\mathbf{y}(0) = \mathbf{y}_0$ as

$$\mathbf{y}(t) = \mathbf{U}(t)\mathbf{D}(t)\mathbf{V}(0)\mathbf{y}_0 \tag{25}$$

where $\mathbf{U}(t)$ and $\mathbf{V}(t)$ are two T-periodic invertible square matrices of size n such that $\mathbf{U}(t) = \mathbf{V}^{-1}(t)$, while matrix $\mathbf{D}(t)$ is diagonal:

$$\mathbf{D}(t) = \mathrm{diag}\left\{\exp(\mu_1 t), \ldots, \exp(\mu_n t)\right\}. \tag{26}$$

The set of the n complex numbers μ_i defines the Floquet exponents (FEs) of (24), while $\lambda_i = \exp(\mu_i T)$ are the corresponding Floquet multipliers (FMs).

Since $\mathbf{V}(t)\mathbf{U}(t) = \mathbf{I}_n$ (the identity matrix of size n), the columns $\mathbf{u}_i(t)$ of $\mathbf{U}(t)$ and the rows $\mathbf{v}_i^{\mathrm{T}}(t)$ of $\mathbf{V}(t)$ form a bi-orthogonal basis of $\mathbb{R}^n$. Function $\mathbf{u}_i(t)\exp(\mu_i t)$ is a solution of (24) with initial condition $\mathbf{u}_i(0)$. On the other hand, $\mathbf{v}_i(t)\exp(-\mu_i t)$ is a solution of the adjoint system associated to (24), i.e.

$$\frac{\mathrm{d}\mathbf{z}}{\mathrm{d}t} = -\mathbf{A}^{\mathrm{T}}(t)\mathbf{z}(t), \tag{27}$$

with initial condition $\mathbf{v}_i(0)$. Therefore, given the FE μ_i, $\mathbf{u}_i(t)$ is the associated *direct Floquet eigenvector,* while $\mathbf{v}_i(t)$ is the *adjoint Floquet eigenvector.* A geometrical interpretation can be found in [18]. The exponential dependence on μ_i implies that an oscillator has an asymptotically stable orbit if and only if all the FEs μ_i ($i = 2, \ldots, n$) have negative real part, or equivalently all the FMs λ_i ($i = 2, \ldots, n$) are found inside the unit circle of the complex plane.

A simple calculation [15] shows that the LPTV system associated to the linearization of an autonomous system around the oscillation noiseless working point $\mathbf{x}_\mathrm{S}(t)$ has always $\mu_1 = 0$ as a FE, with the associated direct (normalized) Floquet eigenvector being the tangent to the oscillator limit cycle $\mathbf{u}_1(t) = \dot{\mathbf{x}}_\mathrm{S}(t)/\|\dot{\mathbf{x}}_\mathrm{S}(t)\|$ ($\dot{}$ denotes time derivative). The corresponding adjoint Floquet eigenvector $\mathbf{v}_1(t)$ is the so-called *perturbation projection vector* (PPV) that plays the leading role in the assessment of phase noise [6], [18], [19]. However, also the other FEs and eigenvectors are of importance, both because they assess the stability of the circuit working point [43], and because they are required to express the oscillator amplitude noise [25]. The corresponding computation can be performed both in the time- and frequency-domains, see e.g. [36]–[38], [44]–[46].

Appendix B
The ISF theory [32]

According to the definition given in [32], the ISF $\Gamma(\omega_0\tau)$ "is a dimensionless, frequency- and amplitude-independent periodic function with period 2π which describes how much phase shift results from applying a unit impulse at time $t = \tau$". This means that the ISF defined in this way corresponds to the impulse response of the linearized equations defining the phase perturbation. However the operative definitions described in the original paper lead to different relations with reference to the quantities as used in this review (we consider here, for the sake of simplicity, the white noise sources) only:

1) the closed from ISF of [32, Appendix B] corresponds to the impulse response of the linearized equations defining the phase perturbation projected along the orbit tangent versor $\mathbf{u}_1(t)$:

$$\alpha(t) = \int_{-\infty}^{t} \mathbf{u}_1^{\mathrm{T}}(t)\mathbf{B}_\mathrm{w}(t)\boldsymbol{\xi}_\mathrm{w}(t)\ \mathrm{d}t. \tag{28}$$

Notice that the normalization of the unit tangent, as discussed in Appendix A, is necessary for establishing a direct relationship with the closed form ISF;

2) the numerical ISF of [32, Appendix A] is defined in an incremental way by introducing an impulse perturbation into the circuit equations, and determining the corresponding time evolution of the phase variation. As a consequence, it corresponds to the propagation of a deterministic source into a phase variation, that in turn was shown in [6] to be determined by the PPV:

$$\alpha(t) = \int_{-\infty}^{t} \mathbf{v}_1^{\mathrm{T}}(t)\mathbf{B}_{\mathrm{w}}(t)\boldsymbol{\xi}_{\mathrm{w}}(t)\ \mathrm{d}t. \tag{29}$$

As a final remark, we point out that the correct use of the ISF should be within the nonlinear S-ODE (7), and not by exploiting the Kaertner approximation (14). The latter corresponds directly to (29) while (28) leads to a different S-ODE:

$$\frac{\mathrm{d}\alpha}{\mathrm{d}t} = \mathbf{u}_1^{\mathrm{T}}(t)\mathbf{B}_{\mathrm{w}}(t)\boldsymbol{\xi}_{\mathrm{w}}(t). \tag{30}$$

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