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Availability:
This version is available at: 11583/2898892 since: 2021-05-10T10:24:03Z

Publisher:
Taylor & Francis

Published
DOI:10.1080/03081087.2021.1922337

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25 April 2024

The Largest Entry in the Inverse of a Vandermonde Matrix

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December 11, 2020

Abstract

We investigate the size of the largest entry (in absolute value) in the inverse of certain Vandermonde matrices. More precisely, for every real $b > 1$, let $M_b(n)$ be the maximum of the absolute values of the entries of the inverse of the $n \times n$ matrix $[b^{ij}]_{0 \leq i,j < n}$. We prove that $\lim_{n \to +\infty} M_b(n)$ exists, and we provide some formulas for it.

1 Introduction

Let $\mathbf{a} = (a_0, a_1, \ldots, a_{n-1})$ be a list of n real numbers. The classical *Vandermonde matrix* $V(\mathbf{a})$ is defined as follows:

$$
V(\mathbf{a}) := \begin{bmatrix}
1 & a_0 & a_0^2 & \cdots & a_0^{n-1} \\
1 & a_1 & a_1^2 & \cdots & a_1^{n-1} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & a_{n-1} & a_{n-1}^2 & \cdots & a_{n-1}^{n-1}
\end{bmatrix}.
$$

As is well-known, the Vandermonde matrix $V(\mathbf{a})$ is invertible if and only if the a_i are pairwise distinct [3]. Formulas for the inverse $V(\mathbf{a})$ (when it exists) have been known at least since 1958 [5].

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In what follows, n is a positive integer and $b > 1$ is a fixed real number. Let us define the entries $c_{i,j,n}$ by

$$[c_{i,j,n}]_{0 \le i,j < n} = V(b^0, b^1, b^2, \dots, b^{n-1})^{-1}, \tag{1}$$

and let $M_b(n) = \max_{0 \le i,j < n} |c_{i,j,n}|$, the maximum of the absolute values of the entries of $V(1, b, b^2, \dots, b^{n-1})^{-1}$. The size of the entries of inverses of Vandermonde matrices have been studied for a long time (e.g., [1]). Recently, in a paper by the first two authors and Daniel Kane [2], we needed to estimate $M_2(n)$, and we proved that $M_2(n) \le 34$. In fact, even more is true: the limit $\lim_{n \to \infty} M_2(n)$ exists and equals $3 \prod_{i \ge 2} \left(1 + \frac{1}{2^i - 1}\right) \doteq 5.19411992918 \cdots$. In this paper, we generalize this result, replacing 2 with any real number greater than 1.

Our main results are as follows:

Theorem 1. *Let $b > 1$ and $n_0 = \lceil \log_b(1 + \frac{1}{b}) \rceil$. Then $|c_{i,j,n}| \le |c_{n_0, n_0, n}|$ for $i, j \ge n_0$. Hence $M_b(n) \in \{ |c_{i,j,n}| : 0 \le i, j \le n_0 \}$.*

Theorem 2. *Let $b \ge \tau = (1 + \sqrt{5})/2$ and $n \ge 2$. Then $M_b(n) \in \{ |c_{0,0,n}|, |c_{1,1,n}| \}$.*

Theorem 3. *For all real $b > 1$ the limit $\lim_{n \to \infty} M_b(n)$ exists.*

2 Preliminaries

For every real number x, and for all integers $0 \le i, j < n$, let us define the power sum

$$\sigma_{i,j,n}(x) := \sum_{\substack{0 \le h_1 < \cdots < h_i < n \\ h_1, \dots, h_i \ne j}} x^{h_1 + \cdots + h_i}.$$

The following lemma will be useful in later arguments.

Lemma 4. *Let i, j, n be integers with $0 \le i < n$, $0 \le j < n - 1$, and let x be a positive real number.*

(a) If $x > 1$, then $\sigma_{i,j,n}(x) \ge \sigma_{i,j+1,n}(x)$.

(b) If $x < 1$, then $\sigma_{i,j,n}(x) \le \sigma_{i,j+1,n}(x)$.

Proof. We have

$$\sigma_{i,j+1,n}(x) - \sigma_{i,j,n}(x) = \sum_{(h_1, \dots, h_i) \in S_{i,j,n}} x^{h_1 + \cdots + h_i} - \sum_{(h_1, \dots, h_i) \in T_{i,j,n}} x^{h_1 + \cdots + h_i},$$

where

$$S_{i,j,n} := \{0 \le h_1 < \cdots < h_i < n : j \in \{h_1, \dots, h_i\}, j + 1 \notin \{h_1, \dots, h_i\}\}$$

and
$$T_{i,j,n} := \{0 \le h_1 < \cdots < h_i < n : j \notin \{h_1, \ldots, h_i\}, \ j+1 \in \{h_1, \ldots, h_i\}\}.$$

Now there is a bijection $S_{i,j,n} \to T_{i,j,n}$ given by

$$(h_1, \ldots, h_i) \mapsto (h_1, \ldots, h_{i_0-1}, h_{i_0} + 1, h_{i_0+1}, \ldots, h_i),$$

where i_0 is the unique integer such that $h_{i_0} = j$. Hence, it follows easily that $\sigma_{i,j,n}(x) \ge \sigma_{i,j+1,n}(x)$ for $x > 1$, and $\sigma_{i,j,n}(x) \le \sigma_{i,j+1,n}(x)$ for $x < 1$. □

Lemma 5. *Let i, j, n be integers with $0 \le i, j < n$, and let x be a nonzero real number. Then*

$$\frac{\sigma_{n-i-1,j,n}(x)}{x^{n(n-1)/2-j}} = \sigma_{i,j,n}(x^{-1}).$$

Proof. Note that the subsets $\mathcal{A}$ of $\{0, 1, \ldots, n-1\} - \{j\}$ of cardinality $n - i - 1$ are in one-to-one correspondence with the subsets $\mathcal{A}'$ of cardinality i. The correspondence is given by the complement $\mathcal{A} \mapsto \mathcal{A}' = \{0, 1, \ldots, n-1\} - \{j\} - \mathcal{A}$. In particular, we have

$$\sum_{a \in \mathcal{A}} a = \sum_{k \in \{0, 1, \ldots, n\} - \{j\}} k - \sum_{a \in \mathcal{A}'} a = \frac{n(n-1)}{2} - j - \sum_{a \in \mathcal{A}'} a.$$

As a consequence, we get that

$$\sigma_{n-i-1,j,n}(x) = \sum_{\mathcal{A}} x^{\sum_{a \in \mathcal{A}} a} = x^{n(n-1)/2-j} \sum_{\mathcal{A}'} x^{-\sum_{a \in \mathcal{A}'} a} = x^{n(n-1)/2-j} \sigma_{i,j,n}(x^{-1}),$$

as claimed. □

Recall the following formula for the entries of the inverse of a Vandermonde matrix (see, e.g., [4, §1.2.3, Exercise 40]).

Lemma 6. *Let $a_0, \ldots, a_{n-1}$ be pairwise distinct real numbers. If $V(a_0, a_1, \ldots, a_{n-1})^{-1} = [d_{i,j}]_{0 \le i, j < n}$ then*

$$d_{n-1,j} X^{n-1} + d_{n-2,j} X^{n-2} + \cdots + d_{0,j} X^0 = \prod_{\substack{0 \le i < n \\ i \ne j}} \frac{X - a_i}{a_j - a_i}.$$

For $0 \le i, j < n$ define
$$\pi_{j,n} := \prod_{\substack{0 \le h < n \\ h \ne j}} |b^j - b^h|.$$

We now obtain a relationship between the entries of $V(b^0, b^1, \ldots, b^{n-1})^{-1}$ and $\sigma_{i,j,n}$ and $\pi_{j,n}$.

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Lemma 7. *Let* $V(b^0, b^1, \ldots, b^{n-1})^{-1} = [c_{i,j,n}]_{0 \le i,j < n}$. *Then*

$$|c_{i,j,n}| = \frac{\sigma_{n-i-1,j,n}}{\pi_{j,n}} \tag{2}$$

for $0 \le i, j < n$.

Proof. By Lemma 6, we have

$$\prod_{\substack{0 \le h < n \\ h \ne j}} \frac{X - b^h}{b^j - b^h} = \sum_{0 \le i < n} c_{i,j,n} X^i.$$

which in turn, by Vieta's formulas, gives

$$c_{n-i-1,j,n} = (-1)^i \left(\prod_{\substack{0 \le h < n \\ h \ne j}} \frac{1}{b^j - b^h} \right) \sum_{\substack{0 \le h_1 < \cdots < h_i < n \\ h_1, \ldots, h_i \ne j}} b^{h_1 + \cdots + h_i} \tag{3}$$

for $0 \le i < n$. The result now follows by the definitions of σ and π. $\square$

Next, we obtain some inequalities for π.

Lemma 8. *Define* $n_0 = \lceil \log_b(1 + \frac{1}{b}) \rceil$. *Then*

$$\pi_{j,n} \le \pi_{j+1,n} \quad \text{for } n_0 \le j < n.$$

Proof. For $0 \le j < n - 1$, we have

$$\pi_{j+1,n} := \prod_{\substack{0 \le h < n \\ h \ne j+1}} |b^{j+1} - b^h| = b^{n-1} \prod_{\substack{0 \le h < n \\ h-1 \ne j}} |b^j - b^{h-1}| = \frac{b^{n+j-1} - b^{n-2}}{b^{n-1} - b^j} \pi_{j,n}.$$

A quick computation shows that the following inequalities are equivalent:

$$\frac{b^{n+j-1} - b^{n-2}}{b^{n-1} - b^j} \ge 1 \iff b^j \ge \frac{b^{n-1} + b^{n-2}}{b^{n-1} + 1}.$$

Let n_0 be the minimum positive integer such that $b^{n_0} \ge 1 + \frac{1}{b}$. Then $n_0 = \lceil \log_b(1 + \frac{1}{b}) \rceil$. Hence, for $n_0 \le j < n$, we have

$$b^j \ge 1 + \frac{1}{b} > \frac{b^{n-1} + b^{n-2}}{b^{n-1} + 1},$$

so that

$$\pi_{j,n} \le \pi_{j+1,n} \quad \text{for } n_0 \le j < n. \tag{4}$$

$\square$

Finally, we have the easy

Lemma 9. *For* $0 \le i, j < n$ *we have* $c_{i,j,n} = c_{j,i,n}$.

Proof. $V(b^0, b^1, \ldots, b^{n-1})$ is a symmetric matrix, so its inverse is also. $\square$

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3   Proof of Theorem 1

Proof. Suppose $i, j \geq n_0$. Then

$$
\begin{aligned}
|c_{i,j,n}| &= \frac{\sigma_{n-i-1,j,n}}{\pi_{j,n}} \quad \text{(by (2))} \\
&\leq \frac{\sigma_{n-i-1,n_0,n}}{\pi_{j,n}} \quad \text{(by Lemma 4 (a))} \\
&\leq \frac{\sigma_{n-i-1,n_0,n}}{\pi_{n_0,n}} \quad \text{(by Lemma 8)} \\
&= |c_{i,n_0,n}| \quad \text{(by (2))},
\end{aligned}
$$

and so we get

$$
|c_{i,j,n}| \leq |c_{i,n_0,n}|. \tag{5}
$$

But

$$
c_{i,n_0,n} = c_{n_0,i,n} \tag{6}
$$

by Lemma 9. Make the substitutions n_0 for i and i for j in (5) to get

$$
|c_{n_0,i,n}| \leq |c_{n_0,n_0,n}|. \tag{7}
$$

The result now follows by combining Eqs. (5), (6), and (7). □

4   Proof of Theorem 2

Proof. Since $b \geq \tau$, it follows that $b \geq 1 + 1/b$. Hence in Theorem 1 we can take $n_0 = 1$, and this gives $M_b(n) \in \{|c_{0,0,n}|, |c_{1,0,n}|, |c_{0,1,n}|, |c_{1,1,n}|\}$. However, by explicit calculation, we have

$$
\begin{aligned}
\sigma_{n-1,1,n} &= b^{n(n-1)/2-1} \\
\sigma_{n-2,1,n} &= b^{n(n-1)/2-1} + \sum_{(n-1)(n-2)/2-1 \leq i \leq n(n-1)/2-3} b^i,
\end{aligned}
$$

so that

$$
\sigma_{n-1,1,n} \leq \sigma_{n-2,1,n}. \tag{8}
$$

Hence

$$
\begin{aligned}
|c_{1,0,n}| &= |c_{0,1,n}| \quad \text{(by Lemma 9)} \\
&= \frac{\sigma_{n-1,1,n}}{\pi_{1,n}} \quad \text{(by (2))} \\
&\leq \frac{\sigma_{n-2,1,n}}{\pi_{1,n}} \quad \text{(by (8))} \\
&= |c_{1,1,n}| \quad \text{(by (2))},
\end{aligned}
$$

and the result follows. □

5 Proof of Theorem 3

Proof. We have

$$|c_{i,j,n}| = \frac{\sigma_{n-i-1,j,n}}{\pi_{j,n}}$$

$$= \frac{\sigma_{n-i-1,j,n}(b)}{\prod_{\substack{0 \le h < n \\ h \ne j}} |b^j - b^h|}$$

$$= \frac{\sigma_{n-i-1,j,n}(b)}{\prod_{\substack{0 \le h < n \\ h \ne j}} (b^h \cdot |b^{j-h} - 1|)}$$

$$= \frac{\sigma_{n-i-1,j,n}(b)}{b^{n(n-1)/2-j}} \cdot \frac{1}{\prod_{\substack{0 \le h < n \\ h \ne j}} |b^{j-h} - 1|}$$

$$= \sigma_{i,j,n}(b^{-1}) \frac{1}{\prod_{\substack{0 \le h < n \\ h \ne j}} |b^{j-h} - 1|},$$

where we used Lemma 5.

For $x < 1$ define

$$\sigma_{i,j,\infty}(x) = \sum_{\substack{0 \le h_1 < \cdots < h_i < \infty \\ h_1,\ldots,h_i \ne j}} \frac{1}{x^{h_1 + \cdots + h_i}},$$

with the convention $\sigma_{0,j,\infty}(x) := 1$.

Hence the limits

$$\ell_{i,j} := \lim_{n \to +\infty} |c_{i,j,n}|$$

$$= \lim_{n \to +\infty} \sigma_{i,j,n}(b^{-1}) \frac{1}{\prod_{\substack{0 \le h < n \\ h \ne j}} |b^{j-h} - 1|}$$

$$= \lim_{n \to +\infty} \sigma_{i,j,n}(b^{-1}) \prod_{0 \le h < j} \frac{1}{b^{j-h} - 1} \prod_{j < h < n} \frac{1}{1 - b^{j-h}}$$

$$= \lim_{n \to +\infty} \sigma_{i,j,n}(b^{-1}) \prod_{1 \le s \le j} \frac{1}{b^s - 1} \prod_{1 \le t < n-j} \frac{1}{1 - b^{-t}}$$

$$= \sigma_{i,j,\infty}(b^{-1}) \left(\prod_{1 \le s \le j} \frac{1}{b^s - 1} \right) \left(\prod_{t \ge 1} \frac{1}{1 - b^{-t}} \right) \tag{9}$$

exist and are finite.

From Theorem 1 we see that

$$\lim_{n \to +\infty} M_b(n) = \max_{0 \le i \le j < n_0} \lim_{n \to +\infty} |c_{i,j,n}| = \max_{0 \le i \le j \le n_0} \ell_{i,j},$$

and the proof is complete. $\square$

From this theorem we can explicitly compute $\lim_{n\to+\infty} M_b(n)$ for $b \geq \tau$.

Corollary 10. *Let* $\alpha \doteq 2.324717957$ *be the real zero of the polynomial* $X^3 - 3X^2 + 2X - 1$.

(a) *If* $b \geq \alpha$, *then* $\lim_{n\to\infty} M_b(n) = \prod_{t\geq 1}(1 - b^{-t})^{-1}$.

(b) *If* $\tau \leq b \leq \alpha$, *then* $\lim_{n\to\infty} M_b(n) = \frac{b^2 - b + 1}{b(b-1)^2} \prod_{t\geq 1}(1 - b^{-t})^{-1}$.

Proof. From Theorem 2 we know that for $b \geq \tau$ we have $\lim_{n\to\infty} M_b(n) \in \{\ell_{0,0}, \ell_{1,1}\}$. Now an easy calculation based on (9) shows that

$$\ell_{0,0} = \prod_{t\geq 1}(1 - b^{-t})^{-1}$$

$$\ell_{1,1} = \frac{b^2 - b + 1}{b(b-1)^2} \prod_{t\geq 1}(1 - b^{-t})^{-1}.$$

By solving the equation $\frac{b^2 - b + 1}{b(b-1)^2} = 1$, we see that for $b \geq \alpha$ we have $\ell_{0,0} \geq \ell_{1,1}$, while if $\tau \leq b \leq \alpha$ we have $\ell_{1,1} \geq \ell_{0,0}$. This proves both parts of the claim. $\square$

Remark 11. The quantity $M_b(n)$ converges rather slowly to its limit when b is close to 1. The following table gives some numerical estimates for $M_b(n)$.

b	$\lim_{n\to\infty} M_b(n)$
3	1.78531234199853419036786
$\alpha \doteq 2.3247$	2.4862447382651613433
2	5.194119929182595417
$\tau \doteq 1.61803$	26.788216012030303413
1.5	67.3672156
1.4	282.398
1.3	3069.44
1.2	422349.8

6 Final remarks

We close with a conjecture we have been unable to prove.

Conjecture 12. Let $b > 1$ and $n_0 = \lceil \log_b(1 + \frac{1}{b}) \rceil$. Then, for all sufficiently large n, we have $M_b(n) = |c_{i,i,n}|$ for some i, $0 \leq i \leq n_0$.

Acknowledgment

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