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# The Largest Entry in the Inverse of a Vandermonde Matrix

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## Abstract

We investigate the size of the largest entry (in absolute value) in the inverse of certain Vandermonde matrices. More precisely, for every real  $b > 1$ , let  $M_b(n)$  be the maximum of the absolute values of the entries of the inverse of the  $n \times n$  matrix  $[b^{ij}]_{0 \leq i, j < n}$ . We prove that  $\lim_{n \rightarrow +\infty} M_b(n)$  exists, and we provide some formulas for it.

## 1 Introduction

Let  $\mathbf{a} = (a_0, a_1, \dots, a_{n-1})$  be a list of  $n$  real numbers. The classical *Vandermonde matrix*  $V(\mathbf{a})$  is defined as follows:

$$V(\mathbf{a}) := \begin{bmatrix} 1 & a_0 & a_0^2 & \cdots & a_0^{n-1} \\ 1 & a_1 & a_1^2 & \cdots & a_1^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & a_{n-1} & a_{n-1}^2 & \cdots & a_{n-1}^{n-1} \end{bmatrix}.$$

As is well-known, the Vandermonde matrix  $V(\mathbf{a})$  is invertible if and only if the  $a_i$  are pairwise distinct [3]. Formulas for the inverse  $V(\mathbf{a})$  (when it exists) have been known at least since 1958 [5].

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In what follows,  $n$  is a positive integer and  $b > 1$  is a fixed real number. Let us define the entries  $c_{i,j,n}$  by

$$[c_{i,j,n}]_{0 \leq i,j < n} = V(b^0, b^1, b^2, \dots, b^{n-1})^{-1}, \quad (1)$$

and let  $M_b(n) = \max_{0 \leq i,j < n} |c_{i,j,n}|$ , the maximum of the absolute values of the entries of  $V(1, b, b^2, \dots, b^{n-1})^{-1}$ . The size of the entries of inverses of Vandermonde matrices have been studied for a long time (e.g., [1]). Recently, in a paper by the first two authors and Daniel Kane [2], we needed to estimate  $M_2(n)$ , and we proved that  $M_2(n) \leq 34$ . In fact, even more is true: the limit  $\lim_{n \rightarrow \infty} M_2(n)$  exists and equals  $3 \prod_{i \geq 2} (1 + \frac{1}{2^i - 1}) \doteq 5.19411992918 \dots$ . In this paper, we generalize this result, replacing 2 with any real number greater than 1.

Our main results are as follows:

**Theorem 1.** *Let  $b > 1$  and  $n_0 = \lceil \log_b(1 + \frac{1}{b}) \rceil$ . Then  $|c_{i,j,n}| \leq |c_{n_0, n_0, n}|$  for  $i, j \geq n_0$ . Hence  $M_b(n) \in \{|c_{i,j,n}| : 0 \leq i, j \leq n_0\}$ .*

**Theorem 2.** *Let  $b \geq \tau = (1 + \sqrt{5})/2$  and  $n \geq 2$ . Then  $M_b(n) \in \{|c_{0,0,n}|, |c_{1,1,n}|\}$ .*

**Theorem 3.** *For all real  $b > 1$  the limit  $\lim_{n \rightarrow \infty} M_b(n)$  exists.*

## 2 Preliminaries

For every real number  $x$ , and for all integers  $0 \leq i, j < n$ , let us define the power sum

$$\sigma_{i,j,n}(x) := \sum_{\substack{0 \leq h_1 < \dots < h_i < n \\ h_1, \dots, h_i \neq j}} x^{h_1 + \dots + h_i}.$$

The following lemma will be useful in later arguments.

**Lemma 4.** *Let  $i, j, n$  be integers with  $0 \leq i < n$ ,  $0 \leq j < n - 1$ , and let  $x$  be a positive real number.*

(a) *If  $x > 1$ , then  $\sigma_{i,j,n}(x) \geq \sigma_{i,j+1,n}(x)$ .*

(b) *If  $x < 1$ , then  $\sigma_{i,j,n}(x) \leq \sigma_{i,j+1,n}(x)$ .*

*Proof.* We have

$$\sigma_{i,j+1,n}(x) - \sigma_{i,j,n}(x) = \sum_{(h_1, \dots, h_i) \in S_{i,j,n}} x^{h_1 + \dots + h_i} - \sum_{(h_1, \dots, h_i) \in T_{i,j,n}} x^{h_1 + \dots + h_i},$$

where

$$S_{i,j,n} := \{0 \leq h_1 < \dots < h_i < n : j \in \{h_1, \dots, h_i\}, j+1 \notin \{h_1, \dots, h_i\}\}$$

and

$$T_{i,j,n} := \{0 \leq h_1 < \dots < h_i < n : j \notin \{h_1, \dots, h_i\}, j+1 \in \{h_1, \dots, h_i\}\}.$$

Now there is a bijection  $S_{i,j,n} \rightarrow T_{i,j,n}$  given by

$$(h_1, \dots, h_i) \mapsto (h_1, \dots, h_{i_0-1}, h_{i_0} + 1, h_{i_0+1}, \dots, h_i),$$

where  $i_0$  is the unique integer such that  $h_{i_0} = j$ . Hence, it follows easily that  $\sigma_{i,j,n}(x) \geq \sigma_{i,j+1,n}(x)$  for  $x > 1$ , and  $\sigma_{i,j,n}(x) \leq \sigma_{i,j+1,n}(x)$  for  $x < 1$ .  $\square$

**Lemma 5.** *Let  $i, j, n$  be integers with  $0 \leq i, j < n$ , and let  $x$  be a nonzero real number. Then*

$$\frac{\sigma_{n-i-1,j,n}(x)}{x^{n(n-1)/2-j}} = \sigma_{i,j,n}(x^{-1}).$$

*Proof.* Note that the subsets  $\mathcal{A}$  of  $\{0, 1, \dots, n-1\} - \{j\}$  of cardinality  $n-i-1$  are in one-to-one correspondence with the subsets  $\mathcal{A}'$  of cardinality  $i$ . The correspondence is given by the complement  $\mathcal{A} \mapsto \mathcal{A}' = \{0, 1, \dots, n-1\} - \{j\} - \mathcal{A}$ . In particular, we have

$$\sum_{a \in \mathcal{A}} a = \sum_{k \in \{0, 1, \dots, n\} - \{j\}} k - \sum_{a \in \mathcal{A}'} a = \frac{n(n-1)}{2} - j - \sum_{a \in \mathcal{A}'} a.$$

As a consequence, we get that

$$\sigma_{n-i-1,j,n}(x) = \sum_{\mathcal{A}} x^{\sum_{a \in \mathcal{A}} a} = x^{n(n-1)/2-j} \sum_{\mathcal{A}'} x^{-\sum_{a \in \mathcal{A}'} a} = x^{n(n-1)/2-j} \sigma_{i,j,n}(x^{-1}),$$

as claimed.  $\square$

Recall the following formula for the entries of the inverse of a Vandermonde matrix (see, e.g., [4, §1.2.3, Exercise 40]).

**Lemma 6.** *Let  $a_0, \dots, a_{n-1}$  be pairwise distinct real numbers. If  $V(a_0, a_1, \dots, a_{n-1})^{-1} = [d_{i,j}]_{0 \leq i, j < n}$  then*

$$d_{n-1,j} X^{n-1} + d_{n-2,j} X^{n-2} + \dots + d_{0,j} X^0 = \prod_{\substack{0 \leq i < n \\ i \neq j}} \frac{X - a_i}{a_j - a_i}.$$

For  $0 \leq i, j < n$  define

$$\pi_{j,n} := \prod_{\substack{0 \leq h < n \\ h \neq j}} |b^j - b^h|.$$

We now obtain a relationship between the entries of  $V(b^0, b^1, \dots, b^{n-1})^{-1}$  and  $\sigma_{i,j,n}$  and  $\pi_{j,n}$ .

**Lemma 7.** Let  $V(b^0, b^1, \dots, b^{n-1})^{-1} = [c_{i,j,n}]_{0 \leq i, j < n}$ . Then

$$|c_{i,j,n}| = \frac{\sigma_{n-i-1,j,n}}{\pi_{j,n}} \quad (2)$$

for  $0 \leq i, j < n$ .

*Proof.* By Lemma 6, we have

$$\prod_{\substack{0 \leq h < n \\ h \neq j}} \frac{X - b^h}{b^j - b^h} = \sum_{0 \leq i < n} c_{i,j,n} X^i.$$

which in turn, by Vieta's formulas, gives

$$c_{n-i-1,j,n} = (-1)^i \left( \prod_{\substack{0 \leq h < n \\ h \neq j}} \frac{1}{b^j - b^h} \right) \sum_{\substack{0 \leq h_1 < \dots < h_i < n \\ h_1, \dots, h_i \neq j}} b^{h_1 + \dots + h_i} \quad (3)$$

for  $0 \leq i < n$ . The result now follows by the definitions of  $\sigma$  and  $\pi$ .  $\square$

Next, we obtain some inequalities for  $\pi$ .

**Lemma 8.** Define  $n_0 = \lceil \log_b(1 + \frac{1}{b}) \rceil$ . Then

$$\pi_{j,n} \leq \pi_{j+1,n} \quad \text{for } n_0 \leq j < n.$$

*Proof.* For  $0 \leq j < n - 1$ , we have

$$\pi_{j+1,n} := \prod_{\substack{0 \leq h < n \\ h \neq j+1}} |b^{j+1} - b^h| = b^{n-1} \prod_{\substack{0 \leq h < n \\ h-1 \neq j}} |b^j - b^{h-1}| = \frac{b^{n+j-1} - b^{n-2}}{b^{n-1} - b^j} \pi_{j,n}.$$

A quick computation shows that the following inequalities are equivalent:

$$\frac{b^{n+j-1} - b^{n-2}}{b^{n-1} - b^j} \geq 1 \quad \iff \quad b^j \geq \frac{b^{n-1} + b^{n-2}}{b^{n-1} + 1}.$$

Let  $n_0$  be the minimum positive integer such that  $b^{n_0} \geq 1 + \frac{1}{b}$ . Then  $n_0 = \lceil \log_b(1 + \frac{1}{b}) \rceil$ . Hence, for  $n_0 \leq j < n$ , we have

$$b^j \geq 1 + \frac{1}{b} > \frac{b^{n-1} + b^{n-2}}{b^{n-1} + 1},$$

so that

$$\pi_{j,n} \leq \pi_{j+1,n} \quad \text{for } n_0 \leq j < n. \quad (4)$$

$\square$

Finally, we have the easy

**Lemma 9.** For  $0 \leq i, j < n$  we have  $c_{i,j,n} = c_{j,i,n}$ .

*Proof.*  $V(b^0, b^1, \dots, b^{n-1})$  is a symmetric matrix, so its inverse is also.  $\square$

### 3 Proof of Theorem 1

*Proof.* Suppose  $i, j \geq n_0$ . Then

$$\begin{aligned}
 |c_{i,j,n}| &= \frac{\sigma_{n-i-1,j,n}}{\pi_{j,n}} \quad (\text{by (2)}) \\
 &\leq \frac{\sigma_{n-i-1,n_0,n}}{\pi_{j,n}} \quad (\text{by Lemma 4 (a)}) \\
 &\leq \frac{\sigma_{n-i-1,n_0,n}}{\pi_{n_0,n}} \quad (\text{by Lemma 8}) \\
 &= |c_{i,n_0,n}| \quad (\text{by (2)}),
 \end{aligned}$$

and so we get

$$|c_{i,j,n}| \leq |c_{i,n_0,n}|. \quad (5)$$

But

$$c_{i,n_0,n} = c_{n_0,i,n} \quad (6)$$

by Lemma 9. Make the substitutions  $n_0$  for  $i$  and  $i$  for  $j$  in (5) to get

$$|c_{n_0,i,n}| \leq |c_{n_0,n_0,n}|. \quad (7)$$

The result now follows by combining Eqs. (5), (6), and (7).  $\square$

### 4 Proof of Theorem 2

*Proof.* Since  $b \geq \tau$ , it follows that  $b \geq 1 + 1/b$ . Hence in Theorem 1 we can take  $n_0 = 1$ , and this gives  $M_b(n) \in \{|c_{0,0,n}|, |c_{1,0,n}|, |c_{0,1,n}|, |c_{1,1,n}|\}$ . However, by explicit calculation, we have

$$\begin{aligned}
 \sigma_{n-1,1,n} &= b^{n(n-1)/2-1} \\
 \sigma_{n-2,1,n} &= b^{n(n-1)/2-1} + \sum_{(n-1)(n-2)/2-1 \leq i \leq n(n-1)/2-3} b^i,
 \end{aligned}$$

so that

$$\sigma_{n-1,1,n} \leq \sigma_{n-2,1,n}. \quad (8)$$

Hence

$$\begin{aligned}
 |c_{1,0,n}| &= |c_{0,1,n}| \quad (\text{by Lemma 9}) \\
 &= \frac{\sigma_{n-1,1,n}}{\pi_{1,n}} \quad (\text{by (2)}) \\
 &\leq \frac{\sigma_{n-2,1,n}}{\pi_{1,n}} \quad (\text{by (8)}) \\
 &= |c_{1,1,n}| \quad (\text{by (2)}),
 \end{aligned}$$

and the result follows.  $\square$

## 5 Proof of Theorem 3

*Proof.* We have

$$\begin{aligned}
|c_{i,j,n}| &= \frac{\sigma_{n-i-1,j,n}}{\pi_{j,n}} \\
&= \frac{\sigma_{n-i-1,j,n}(b)}{\prod_{\substack{0 \leq h < n \\ h \neq j}} |b^j - b^h|} \\
&= \frac{\sigma_{n-i-1,j,n}(b)}{\prod_{\substack{0 \leq h < n \\ h \neq j}} (b^h \cdot |b^{j-h} - 1|)} \\
&= \frac{\sigma_{n-i-1,j,n}(b)}{b^{n(n-1)/2-j}} \cdot \frac{1}{\prod_{\substack{0 \leq h < n \\ h \neq j}} |b^{j-h} - 1|} \\
&= \sigma_{i,j,n}(b^{-1}) \frac{1}{\prod_{\substack{0 \leq h < n \\ h \neq j}} |b^{j-h} - 1|},
\end{aligned}$$

where we used Lemma 5.

For  $x < 1$  define

$$\sigma_{i,j,\infty}(x) = \sum_{\substack{0 \leq h_1 < \dots < h_i < \infty \\ h_1, \dots, h_i \neq j}} \frac{1}{x^{h_1 + \dots + h_i}},$$

with the convention  $\sigma_{0,j,\infty}(x) := 1$ .

Hence the limits

$$\begin{aligned}
\ell_{i,j} &:= \lim_{n \rightarrow +\infty} |c_{i,j,n}| \\
&= \lim_{n \rightarrow +\infty} \sigma_{i,j,n}(b^{-1}) \frac{1}{\prod_{\substack{0 \leq h < n \\ h \neq j}} |b^{j-h} - 1|} \\
&= \lim_{n \rightarrow +\infty} \sigma_{i,j,n}(b^{-1}) \prod_{0 \leq h < j} \frac{1}{b^{j-h} - 1} \prod_{j < h < n} \frac{1}{1 - b^{j-h}} \\
&= \lim_{n \rightarrow +\infty} \sigma_{i,j,n}(b^{-1}) \prod_{1 \leq s \leq j} \frac{1}{b^s - 1} \prod_{1 \leq t < n-j} \frac{1}{1 - b^{-t}} \\
&= \sigma_{i,j,\infty}(b^{-1}) \left( \prod_{1 \leq s \leq j} \frac{1}{b^s - 1} \right) \left( \prod_{t \geq 1} \frac{1}{1 - b^{-t}} \right) \tag{9}
\end{aligned}$$

exist and are finite.

From Theorem 1 we see that

$$\lim_{n \rightarrow +\infty} M_b(n) = \max_{0 \leq i \leq j < n_0} \lim_{n \rightarrow +\infty} |c_{i,j,n}| = \max_{0 \leq i \leq j \leq n_0} \ell_{i,j},$$

and the proof is complete.  $\square$

From this theorem we can explicitly compute  $\lim_{n \rightarrow +\infty} M_b(n)$  for  $b \geq \tau$ .

**Corollary 10.** Let  $\alpha \doteq 2.324717957$  be the real zero of the polynomial  $X^3 - 3X^2 + 2X - 1$ .

(a) If  $b \geq \alpha$ , then  $\lim_{n \rightarrow \infty} M_b(n) = \prod_{t \geq 1} (1 - b^{-t})^{-1}$ .

(b) If  $\tau \leq b \leq \alpha$ , then  $\lim_{n \rightarrow \infty} M_b(n) = \frac{b^2 - b + 1}{b(b-1)^2} \prod_{t \geq 1} (1 - b^{-t})^{-1}$ .

*Proof.* From Theorem 2 we know that for  $b \geq \tau$  we have  $\lim_{n \rightarrow \infty} M_b(n) \in \{\ell_{0,0}, \ell_{1,1}\}$ . Now an easy calculation based on (9) shows that

$$\begin{aligned} \ell_{0,0} &= \prod_{t \geq 1} (1 - b^{-t})^{-1} \\ \ell_{1,1} &= \frac{b^2 - b + 1}{b(b-1)^2} \prod_{t \geq 1} (1 - b^{-t})^{-1}. \end{aligned}$$

By solving the equation  $\frac{b^2 - b + 1}{b(b-1)^2} = 1$ , we see that for  $b \geq \alpha$  we have  $\ell_{0,0} \geq \ell_{1,1}$ , while if  $\tau \leq b \leq \alpha$  we have  $\ell_{1,1} \geq \ell_{0,0}$ . This proves both parts of the claim.  $\square$

*Remark 11.* The quantity  $M_b(n)$  converges rather slowly to its limit when  $b$  is close to 1. The following table gives some numerical estimates for  $M_b(n)$ .

$b$	$\lim_{n \rightarrow \infty} M_b(n)$
3	1.785312341998534190367486
$\alpha \doteq 2.3247$	2.4862447382651613433
2	5.194119929182595417
$\tau \doteq 1.61803$	26.788216012030303413
1.5	67.3672156
1.4	282.398
1.3	3069.44
1.2	422349.8

## 6 Final remarks

We close with a conjecture we have been unable to prove.

**Conjecture 12.** Let  $b > 1$  and  $n_0 = \lceil \log_b(1 + \frac{1}{b}) \rceil$ . Then, for all sufficiently large  $n$ , we have  $M_b(n) = |c_{i,i,n}|$  for some  $i$ ,  $0 \leq i \leq n_0$ .

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