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Binary operations inspired by generalized entropies applied to figurate numbers

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Abstract: The generalized entropies, which are coming from generalizations of the standard statistical mechanics, have binary composition operations that can be applied to the study of numbers. Here we will discuss these operations, inspired by entropies, and their use to study some of the figurate numbers.

Keywords: Generalized entropies, q-calculus, Abelian groups, Generalized sums, binary operations, OEIS, On-Line Encyclopedia of Integer Sequences, Figurate numbers, Triangular numbers, Pentagonal numbers, Pronic numbers, Hexagonal numbers.

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Introduction

The generalized entropies, which are coming from generalization of standard statistical mechanics, have composition operations that can be applied to the study of numbers. These entropies, such as the Havrda–Charvát and Tsallis entropy [1,2], are non-additive. It means that they are following rules of composition, which are different from the usual operation of addition. The Tsallis composition rule is defined in [3] as a pseudo-additivity. In order to consider the class of all the generalized entropies, not only the Tsallis one, let us call “generalized additivity” or "generalized sum", the rule concerning composition of two elements. I used the locution "generalized sum" in a discussion about the rules of binary composition that we can obtain from the transcendental functions [4].

The approach, given in [4], is using the method based on the generators of algebras as explained in [5-7]. What we can do for entropies, the same we can do for sequences of numbers [8]. Using this approach, we have studied the q-integers and the symmetric q-integers [9,10]. And we can study the generalized sums that are the binary operations of groupoids formed by Mersenne, Fermat, Cullen, Woodall, Thabit numbers, Repunits and many others [11-15]. Here, we apply the same approach to the category of the figurate numbers. These are numbers which can be represented by a regular geometrical arrangement of equally spaced points. The most common figurate numbers are given by https://mathworld.wolfram.com/FigurateNumber.html [16].

Before the study of numbers, let us start considering the binary operation and indicate it by $\bullet$. It is an operation combining two elements to obtain another element. In particular, this operation has a peculiar meaning when it acts on a set in a manner that its two domains and its codomain are the same set. This happens when it combines any two elements *a,b* of set *A t*o form another element of *A*, denoted *a•b*. *This property is the closure.*

To qualify (*A*,•) as a group, the set and operation must satisfy further requirements beside the c*losure.* We need *associativity* (for all *a*,*b* and *c* in *A,* it holds (*a*•*b*)•*c* = *a*•(*b*•*c*), the *identity element* (an element *e* exists in *A*, such that for all elements *a* in *A*, it is *e*•*a* = *a*•*e* = *a*), and the *inverse element* (for each *a* in *A*, there exists an element *b* in *A* such that *a*•*b* = *b*•*a* = *e*, where *e* is the identity). The notation used before is inherited from the multiplicative operation. If the group is an Abelian group, the operation is indicated by + and the *identity element* by 0 (*neutral element*) and the inverse element of a as $-a$ (*opposite element*). In this case, the group is called an additive group.

We can obtain the operation of a group by means of functions. Actually, if a function $G(x)$ exists, which is invertible $G^{-1}(G(x))=x$, we can use it as a *generator*, to generate an algebra [5,6]. In this manner, *G* can be used to define the *group law* $\Phi(x,y)$, such as:

$$\Phi(x,y)=G\left(G^{-1}(x)+G^{-1}(y)\right).$$

$\Phi(x,y)$is the operation $x\oplus y$.

How can we apply this approach to the sequence of integers? To propose a simple example, let us use the gnomonic numbers, which are defined as $g_n=2n-1$. They are figurate numbers. We have:

$$G(n)=2n-1 \quad , \quad G^{-1}(n)=(n+1)/2 \quad .$$

Therefore:

$$g_n \oplus g_m = G\left(\frac{g_n}{2}+\frac{1}{2}+\frac{g_m}{2}+\frac{1}{2}\right) = g_n + g_m + 1 = g_{n+m} = 2(n+m)-1$$

Then, the generalized sum of the groupoid of gnomonic numbers is:

$$g_n \oplus g_m = g_n + g_m + 1$$

This sum is giving another element of the set of gnomonic numbers. Actually, we can obtain all these numbers using the generalized sum:

$$g_{n+1} = g_n \oplus g_1 = g_n + g_1 + 1 \quad , \quad g_1 = 1$$

Numbers are: 1, 3, 5, 7, 9, 11, 13 ... The On-Line Encyclopedia of Integer Sequences, A005408, tells us that they are the odd numbers.

Therefore, in the case of the integer sequences, to find the generator means to find n as a function of g_n . In the case of the following figurate numbers, the approach is quite simple.

figurate number	formula
centered pentagonal number	$\frac{1}{2}(5n^2+5n+2)$
centered square number	$n^2+(n-1)^2$
centered triangular number	$\frac{1}{2}(3n^2-3n+2)$
decagonal number	$4n^2-3n$
gnomonic number	$2n-1$
heptagonal number	$\frac{1}{2}n(5n-3)$

hex number	$3n^2+3n+1$
hexagonal number	$n(2n-1)$
octagonal number	$n(3n-2)$
pentagonal number	$\frac{1}{2}n(3n-1)$
hexagonal number	$n(2n-1)$
pronic number	$n(n+1)$
square number	n^2
triangular number	$\frac{1}{2}n(n+1)$

Some numbers from the list of common figurate numbers given by mathworld.wolfram.com

Some of the above-mentioned numbers have been considered before, and their binary operations derived.

Triangular numbers T_n [17]:

$$T_n \oplus T_m = T_n + T_m + \frac{1}{4}\left[1 - (1+8T_n)^{1/2} - (1+8T_m)^{1/2} + (1+8T_n)^{1/2}(1+8T_m)^{1/2}\right]$$

$$T_{n+1} = T_n + 1 + \frac{1}{2}\left[-1 + (1+8T_n)^{1/2}\right]$$

Pronic (oblong) numbers O_n [18]:

$$O_m \oplus O_n = O_m + O_n + 1/2 + 2(O_m + 1/4)^{1/2}(O_n + 1/4)^{1/2} - (O_m + 1/4)^{1/2} - (O_n + 1/4)^{1/2}$$

Pentagonal numbers p_n , with $\frac{1}{9} + \frac{8}{3} p_n = A_n$ [19]:

$$p_m \oplus p_n = p_{m+n} = p_m + p_n + \frac{1}{12} + \frac{1}{4}\left(\sqrt{A_m} + \sqrt{A_n}\right) + \frac{3}{4}\sqrt{A_m A_n}$$

$$p_n \oplus p_1 = p_n + p_1 + \frac{1}{12} + \frac{1}{4}\left(\sqrt{A_n} + \sqrt{A_1}\right) + \frac{3}{4}\sqrt{A_n A_1}$$

Centered squares C_n [20]:

$$C_m \oplus C_n = C_m + C_n + 2(C_m - 1/2)^{1/2}(C_n - 1/2)^{1/2} - \sqrt{2}(C_m - 1/2)^{1/2} - \sqrt{2}(C_n - 1/2)^{1/2}$$

Let us add to the previous numbers, which are mentioned in [16], the **star numbers** (centered dodecagonal numbers) $S_n = 6n(n-1) - 1$ [21]:

$$S_n \oplus S_m = S_n + S_m + 2 + (3+6S_n)^{1/2} + (3+6S_m)^{1/2} + \frac{1}{3}(3+6S_n)^{1/2}(3+6S_m)^{1/2}$$

The figurate numbers F_n given in the table given above [16], and the star numbers too, have the general form:

$$F_n = a n^2 + b n + c$$

As a result, the generalized sum has the form:

$$F_n \oplus F_m = F_n + F_m + d + e\sqrt{g + h F_n} + e\sqrt{g + h F_m} + f\sqrt{g + h F_n}\sqrt{g + h F_m}$$

Let us consider the hexagonal numbers, for instance, which are $H_n = n(2n-1)$, $H_1 = 1$:

$$H_n \oplus H_m = H_n + H_m + \frac{1}{4} + \frac{1}{4}\sqrt{1+8H_n} + \frac{1}{4}\sqrt{1+8H_m} + \frac{1}{4}\sqrt{1+8H_n}\sqrt{1+8H_m}$$

$$H_n \oplus H_1 = H_n + H_1 + \frac{1}{4} + \frac{1}{4}\sqrt{1+8H_n} + \frac{1}{4}\sqrt{1+8H_1} + \frac{1}{4}\sqrt{1+8H_n}\sqrt{1+8H_1}$$
$$= \; H_n + 2 + \frac{1}{4}\sqrt{1+8H_n} + \frac{3}{4}\sqrt{1+8H_n} = H_n + 2 + \sqrt{1+8H_n}$$

Using this formula, we have: 1, 6, 15, 28, 45, 66, 91, 120, 153, 190, 231, 276, 325, 378, 435, 496, 561, 630, 703, 780, 861, 946, 1035, 1128, 1225, 1326, 1431, 1540, 1653, 1770, 1891, 2016, 2145, 2278, 2415, 2556, 2701, 2850, 3003, ... (On-Line Encyclopedia of Integer Sequences, A000384, Hexagonal numbers: a(n) = n*(2*n-1)).
If we analyse the square root, we can see that it contains numbers:

$$1+8H_n = 1+8n(2n-1) = 16n^2 - 8n + 1 = (4n+3)^2 = 16(n-1)^2 + 24(n-1) + 9$$

We have a link to OEIS A016838 a(n) = (4n + 3)^2.

Let us stress that the square roots contain interesting sequences of numbers, such as in the case of the pentagonal numbers [19]. Pentagonal numbers (OEIS A000326) possess a generalized sum involveing another integer sequence, OEIS A016969, and this sequence contains OEIS A007528, that is the primes of the 6n-1.

Here we have discussed the composition operations for the figurate numbers. These binary operations can be obtained from a law based on a function and its inverse. The sequence of a figurate number can be therefore seen as a groupoid, with a specific binary operation. In the case of figurate numbers containing n to powers greater than 2, the calculus is more complex, however we can imagine to find an approach to obtain a similar binary operation.

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