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Generalized Sum of Stella Octangula Numbers

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A generalized sum is an operation that combines two elements to obtain another element, generalizing the ordinary addition. Here we discuss that concerning the Stella Octangula Numbers. We will also show that the sequence of these numbers, OEIS A007588, is linked to sequences OEIS A033431, OEIS A002378 (oblong or pronic numbers) and OEIS A003154 (star numbers). The Cardano formula is also discussed. In fact, the sequence of the positive integers can be obtained by means of Cardano formula from the sequence of Stella Octangula numbers.

Keywords: Groupoid Representations, Integer Sequences, Binary Operators, Generalized Sums, OEIS, On-Line Encyclopedia of Integer Sequences, Cardano formula.

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A generalized sum is a binary operation that combines two elements to obtain another element. In particular, this operation acts on a set in a manner that its two domains and its codomain are the same set. Some generalized sums have been previously proposed proposed for different sets of numbers (Fibonacci, Mersenne, Fermat, q-numbers, repunits and others). The approach was inspired by the generalized sums used for entropy [1,2]. The analyses of sequences of integers and q-numbers have been collected in [3].

Let us repeat here just one of these generalized sums, that concerning the Mersenne numbers [4]. These numbers are given by: $M_n = 2^n - 1$. The generalized sum is:

$$M_m \oplus M_n = M_{m+n} = M_m + M_n + M_m M_n$$

In particular:

$$M_n \oplus M_1 = M_{n+1} = M_n + M_1 + M_n M_1$$

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The generalized sum is the binary operation which is using two Mersenne numbers to have another Mersenne number.

Being $M_1=1$, from the formula of sum $\oplus$ given above we have:

$$M_{n+1}=2M_n+1$$

In [5], we have previosuly discussed some of the figurate numbers, those from https://mathworld.wolfram.com/FigurateNumber.html which are described by quadratic equations. Among the figurate numbers given by cubic equations we can find the Stella Octangula numbers. These numbers are defined in the following manner:

$$s_n=2n^3-n=n(2n^2-1)$$

(OEIS, On-Line Encyclopedia of Integer Sequences, A007588, 0, 1, 14, 51, 124, 245, 426, 679, 1016, 1449, 1990, 2651, 3444, 4381, 5474, 6735, 8176, 9809, 11646, 13699, 15980, ... [6].

Let us determine the generalised sum: $s_n\oplus s_m=s_{n+m}$. First, we can write:

$$n^3=\frac{n}{2}+\frac{s_n}{2} \quad , \quad m^3=\frac{m}{2}+\frac{s_m}{2} \quad , \quad (n+m)^3=\frac{n+m}{2}+\frac{s_{n+m}}{2}$$

$$\frac{n+m}{2}+\frac{s_{n+m}}{2}=\left(\sqrt[3]{\frac{n}{2}+\frac{s_n}{2}}+\sqrt[3]{\frac{m}{2}+\frac{s_m}{2}}\right)^3$$

$$\frac{n+m}{2}+\frac{s_{n+m}}{2}=\frac{n}{2}+\frac{s_n}{2}+\frac{m}{2}+\frac{s_m}{2}+3\left(\frac{n}{2}+\frac{s_n}{2}\right)^{2/3}\left(\frac{m}{2}+\frac{s_m}{2}\right)^{1/3}+3\left(\frac{n}{2}+\frac{s_n}{2}\right)^{1/3}\left(\frac{m}{2}+\frac{s_m}{2}\right)^{2/3}$$

$$s_{n+m}=s_n+s_m+3(n+s_n)^{2/3}(m+s_m)^{1/3}+3(n+s_n)^{1/3}(m+s_m)^{2/3}$$

The generalized sum is:

$$s_n \oplus s_m = s_n + s_m + 3(n+s_n)^{2/3}(m+s_m)^{1/3} + 3(n+s_n)^{1/3}(m+s_m)^{2/3}$$

In this formula, we have a sequence $(n+s_n) = n + 2n^3 - n = 2n^3$, that is : 2, 16, 54, 128, 250, 432, 686, 1024, 1458, 2000, 2662, 3456, 4394, 5488, 6750, 8192, 9826, 11664, 13718, 16000, ... (OEIS A033431) [7].

Then, we have:

$$s_{n+m} = s_n + s_m + 3(2n^3)^{2/3}(2m^3)^{1/3} + 3(2n^3)^{1/3}(2m^3)^{2/3}$$

$$s_{n+m} = s_n + s_m + 6n^2 m + 6nm^2 = s_n + s_m + 6(n^2 m + nm^2)$$

The generalized sum can be written as:

$$s_n \oplus s_m = s_n + s_m + 6(n^2 m + nm^2)$$

Moreover, being $s_1 = 1$:

$$s_{n+1} = s_n \oplus s_1 = s_n + 1 + 6(n^2 + n)$$

Numbers $O_n = n(n+1)$ are the oblong numbers, the groupoid of which we discussed in [8]. The sequence is OEIS A002378 (oblong or promic, pronic, or heteromecic numbers: a(n) = n*(n+1): 2, 6, 12, 20, 30, 42, 56, 72, 90, 110, 132, 156, 182, 210, 240, 272, 306, 342, 380, 420, ... [9].

The sequence of Stella Octangula numbers is linked to the sequence of the oblong (pronic) number.

However, we can also consider the numbers $a_n = 6n(n+1) + 1$. We have that:

$$a_n = 6n(n+1) + 1 = 6(n+1)n + 1 = S_{n+1}$$

where S_n is a star number $S_n = 6n(n-1) + 1$. The sequence of them, also known as centered dodecagonal numbers [10], is: 1, 13, 37, 73, 121, 181, 253, 337, 433, 541, 661, 793, 937, 1093, 1261, 1441, 1633, 1837, 2053, 2281, 2521, 2773, ... (OEIS A003154). The groupoid of star numbers was discussed in [11].

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Let us conclude observing that a Stella Octangula number is suitable for being solved immediately by means of the Cardano formula:

$$n^3 - \frac{n}{2} - \frac{S_n}{2} = 0 \quad , \quad n^3 + pn + q = 0$$

Cardano formula for root n has the form:

$$n = \sqrt[3]{-\frac{q}{2} + \sqrt[2]{\frac{q^2}{4} + \frac{p^3}{27}}} + \sqrt[3]{-\frac{q}{2} - \sqrt[2]{\frac{q^2}{4} + \frac{p^3}{27}}}$$

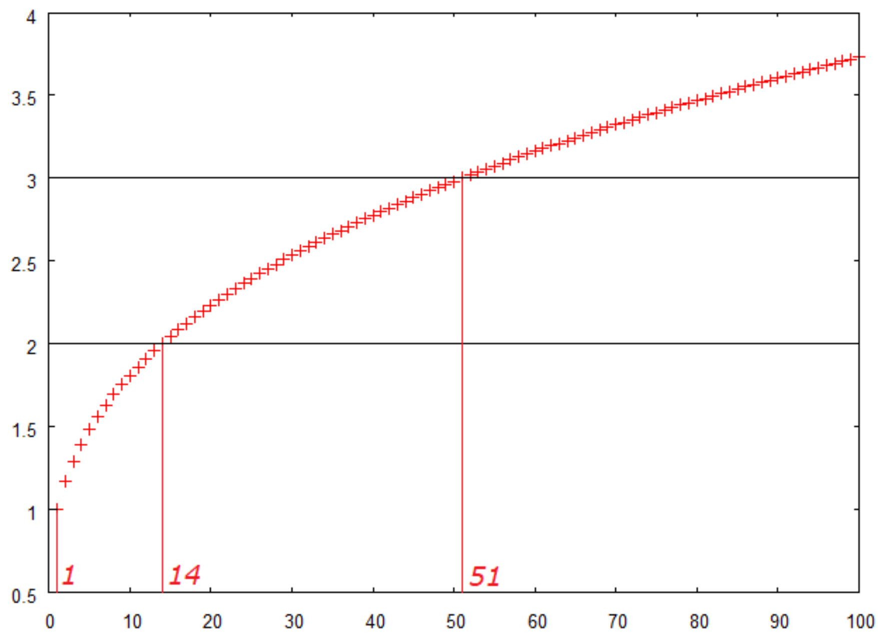

Fig.1 - Stella Octangula Numbers and Cardano formula.

The sequence of the positive integers can be obtained by means of Cardano formula from the sequence of Stella Octangula numbers.

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