PRACTICAL CENTRAL BINOMIAL COEFFICIENTS

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Abstract. A *practical number* is a positive integer n such that all positive integers less than n can be written as a sum of distinct divisors of n. Leonetti and Sanna proved that, as $x \to +\infty$, the central binomial coefficient $\binom{2n}{n}$ is a practical number for all positive integers $n \leq x$ but at most $O(x^{0.88097})$ exceptions. We improve this result by reducing the number of exceptions to $\exp\big(C(\log x)^{4/5} \log\log x\big)$, where $C > 0$ is a constant.

1. Introduction

A *practical number* is a positive integer n such that all positive integers less than n can be written as a sum of distinct divisors of n. Practical numbers were defined by Srinivasan [15], althought they were already used by Fibonacci to decompose rational numbers as sums of unit fractions [12, pag. 121]. Estimates for the counting function of practical numbers were given by Hausman and Shapiro [3], Tenenbaum [16], Margenstern [8], Saias [13], and, lastly, Weingartner [17], who proved that the number of practical numbers up to x is asymptotic to $cx/\log x$, as $x \to +\infty$, where $c = 1.33607\ldots$ [18], settling a conjecture of Margenstern [8].

In analogy with Goldbach's conjecture and prime triplet conjecture, Melfi [10] proved that every positive even integer is the sum of two practical numbers, and that there are infinitely many triples $(n, n+2, n+4)$ of practical numbers. Moreover, Melfi [9] proved that every Lucas sequence $(U_n(P,Q))$ satisfying some mild conditions contains infinitely many practical numbers, and Sanna [14] showed that $U_n(P,Q)$ is practical for at least $\gg_{P,Q} x/\log x$ positive integers $n \leq x$, as $x \to +\infty$; and asked for a nontrivial upper bound.

Leonetti and Sanna [7] studied binomial coefficients that are practical numbers. They proved that, for fixed $\varepsilon > 0$ and as $x \to +\infty$, all binomial coefficients $\binom{n}{k}$, with $0 \leq k \leq n \leq x$, are practical numbers but at most $O_\varepsilon\big(x^{2-(2^{-1}\log 2-\varepsilon)/\log\log x}\big)$ exceptions. Furthermore, they showed that the central binomial coefficient $\binom{2n}{n}$ is a practical number for all positive integers $n \leq x$ but at most $O(x^{0.88097})$ exceptions. In this note, we give the following improvement of the last result.

Theorem 1.1. *For $x \geq 3$ the central binomial coefficient $\binom{2n}{n}$ is a practical number for all positive integers $n \leq x$ but at most $\exp\big(C(\log x)^{4/5}\log\log x\big)$ exceptions, where $C > 0$ is a constant.*

We remark that (as already pointed out in [7]), likely, there are only finitely many positive integers n such that $\binom{2n}{n}$ is not a practical number, but proving so could be out of reach. In fact, if n is a power of 2 whose base 3 representation does not contain the digit 2, then $\binom{2n}{n}$ is not a practical number [7, Proposition 2.1]. However, establishing whether there are finitely or infinitely many such powers of 2 is an open problem [2, 4, 6, 11].

2. Preliminaries

We need some preliminary results.

Lemma 2.1. *If d is a practical number and n is a positive integer divisible by d and having all prime factors not exceeding $2d$, then n is a practical number.*

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Proof. See [7, Lemma 2.2]. □

For every positive integer n, let $s_2(n)$ be the number of nonzero binary digits of n.

Lemma 2.2. *For every positive integer n, the exponent of 2 in the prime factorization of $\binom{2n}{n}$ is equal to $s_2(n)$.*

Proof. A result of Kummer [5] says that for every prime number p and for all positive integers m, n the exponent of p in the prime factorization of $\binom{m+n}{n}$ is equal to the number of carries in the addition $m + n$ done in base p. If $m = n$ and $p = 2$ then we get the desired claim. □

Lemma 2.3. *We have*

$$\#\{n \le x : s_2(n) \le \varepsilon(\log n/\log 2 + 1)\} \le x^{\left(\frac{1}{\log 2}+o(1)\right)\varepsilon \log(1/\varepsilon)},$$

uniformly as $\varepsilon \log x \to +\infty$ and $\varepsilon \to 0^+$.

Proof. Put $N := \lfloor \log x/\log 2 + 1\rfloor$ and $k := \lceil \varepsilon(\log n/\log 2 + 1)\rceil$. Then

$$C := \#\{n \le x : s_2(n) \le \varepsilon(\log n/\log 2 + 1)\} \le \#\{n < 2^N : s_2(n) \le k\},$$

where the right-hand side is the number of binary strings of length N having at most k nonzero bits (including $n = 0$ to the count). Therefore,

$$C \le \sum_{j=0}^{k} \binom{N}{j} \le \sum_{j=0}^{k} \frac{N^j}{j!} = \sum_{j=0}^{k} \frac{k^j}{j!}\left(\frac{N}{k}\right)^j < \left(\frac{eN}{k}\right)^k < e^{(1-\log\varepsilon)(\varepsilon(\log x/\log 2+1)+1)},$$

and the claim follows recalling that $\varepsilon \log x \to +\infty$ and $\varepsilon \to 0^+$. □

The following result of Erdős and Kolesnik is the key to the proof of Theorem 1.1.

Theorem 2.4. *There exist constants $c_1, c_2 > 0$ such that, for all integers m, n, r with*

$$2 \le m \le n/2 \quad \text{and} \quad 1 \le r \le c_1\left(\frac{(\log m)^3}{(\log n)^2 \log\log n}\right)^{1/4},$$

there exist at least $c_2 r m^{1/r}/(4^r \log m)$ prime numbers $p \in [m^{1/r}, n^{1/r}]$ such that $p^r \mid\mid \binom{n}{m}$.

Proof. See [1, Theorem 2]. □

Corollary 2.1. *There exists a constant $c_3 > 0$ such that, for all integers n, r with*

$$n \ge 3 \quad \text{and} \quad 1 \le r \le c_3\left(\frac{\log n}{\log\log n}\right)^{1/4},$$

there exists a prime number $p \in [n^{1/r}, (2n)^{1/r}]$ such that $p^r \mid\mid \binom{2n}{n}$.

Proof. The claim follows by replacing m and n with n and $2n$, respectively, in Theorem 2.4. □

3. Proof of Theorem 1.1

Fix $C > \max\big((5\log 2)^{-1}, (2/c_3)^4\big)$, where c_3 is the constant of Corollary 2.1. Assume that x is sufficiently large and put $E := \exp\big(C(\log x)^{4/5}\log\log x\big)$ and $\varepsilon := (\log x)^{-1/5}$. Let $n \le x$ be a positive integer and let v be the exponent of 2 in the prime factorization of $\binom{2n}{n}$. Since

$$\frac{1}{\log 2}\,\varepsilon\log(1/\varepsilon)\log x = \frac{1}{5\log 2}(\log x)^{4/5}\log\log x < C(\log x)^{4/5}\log\log x,$$

from Lemma 2.2 and Lemma 2.3 we get that $2^v \le n^\varepsilon$ for less than $\frac{1}{2}E$ choices of n. Hence, we can assume that $2^v > n^\varepsilon$ and $n > \frac{1}{2}E$, which excludes at most E positive integers not exceeding x. Then, since $n > \frac{1}{2}E$ and x is sufficiently large, we have

$$\frac{\log n}{\log\log n} > \frac{\log(\frac{1}{2}E)}{\log\log(\frac{1}{2}E)} > C(\log x)^{4/5} > \left(\frac{2(\log x)^{1/5}}{c_3}\right)^4.$$

Therefore,

$$r := \left\lfloor c_3 \left(\frac{\log n}{\log\log n}\right)^{1/4} \right\rfloor > \frac{1}{\varepsilon}.$$

Thanks to Corollary 2.1, there exists a prime number $p \in [n^{1/r}, (2n)^{1/r}]$ such that p^r divides $\binom{2n}{n}$. Now 2^v is a practical number, because all powers of 2 are practical numbers. Morever, since

$$p \leq (2n)^{1/r} < (2n)^{\varepsilon} < 2^{v+1},$$

from Lemma 2.1 it follows that $2^v p^r$ is a practical number. Finally, $2^v p^r$ divides $\binom{2n}{n}$, $2^v p^r \geq 2n$, and all prime factors of $\binom{2n}{n}$ are not exceeding $2n$, hence Lemma 2.1 yields that $\binom{2n}{n}$ is a practical number. The proof is complete.

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