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Generalized sum of Pell Numbers

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Here we are proposing a generalized sum for Pell numbers. This sum contains four Pell numbers. By means of this generalized sum, the Pell number at position (n+m) in the sequence is given by the Pell numbers at positions n, m, (n-1) and (m-1).

Keywords: Groupoid Representations, Integer Sequences, Binary Operators, Generalized Sums, Fibonacci numbers, Lucas numbers, Pell numbers, OEIS A000129, OEIS A001333, OEIS, On-Line Encyclopedia of Integer Sequences.

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A generalized sum is an operation which combines some elements to obtain another element. This operation has a peculiar meaning when it acts on a set in a manner that its domains and its codomain are the same set. Of such a kind of operation we have proposed several examples for different sets of numbers (Mersenne, Fermat, q-numbers, repunits and others) [1]. The origin of the approach from the calculus of generalized entropies is exposed in [2,3]. In the examples given in [1-3], the generalized sum was a binary operation, that is, an operation based on two domains and a codomain which were the same set.

As an example of generalized sum, let us repeat here that concerning the Mersenne numbers [4]. These numbers are given by: $M_n = 2^n - 1$. The generalized sum is:

$$M_m \oplus M_n = M_{m+n} = M_m + M_n + M_m M_n$$

In particular:

$$M_n \oplus M_1 = M_{n+1} = M_n + M_1 + M_n M_1$$

Being $M_1 = 1$, we have $M_{n+1} = 2 M_n + 1$.

In fact, the generalized sum is based on two Mersenne numbers, to have a third Mersenne numbers. And in the cases discussed in [1-4], it is so. However, in the case of the Fibonacci and Lucas numbers for instance, the generalized sum is containing four Fibonacci and three Lucas numbers respectively [5]. Actually, a generalised sum can be imagined on more than two domains. However, for being useful, a simple rule is necessary to have an easy calculation.

Fibonacci numbers are given by the sequence A000045 in the OEIS. Let us consider $\varphi = (1+\sqrt{5})/2$ and $\psi = (1-\sqrt{5})/2$. Fibonacci and Lucas numbers are defined as:

$$F_n = (\varphi^n - \psi^n)/\sqrt{5} \quad ; \quad L_n = (\varphi^n + \psi^n)$$

Lucas numbers are given by sequence A000032 in the OEIS.

Let us imagine that we want to calculate F_{n+m} or L_{n+m} , that is the Fibonacci or the Lucas number at position (n+m) in the related sequences. We can used the generalized sums as shown in [5]:

$$F_n \oplus F_m = F_n F_m + F_n F_{m-1} + F_m F_{n-1}$$

$$L_n \oplus L_m = L_{n+m} = L_n L_m - (-1)^m L_{n-m}$$

The generalized sums for Fibonacci and Lucas numbers are not binary operations, because we have involved four or three numbers. However, as we can easily see, the generalized sum of the Fibonacci numbers $F_n \oplus F_m$ is involving, besides F_n, F_m , the Fibonacci F_{m-1}, F_{n-1} . So the rule is simply to use the preceding numbers. For the generalized sum of Lucas numbers, $L_n \oplus L_m = L_{n+m}$, the rule is to use $(-1)^m L_{n-m}$ besides L_n, L_m .

Here, we consider the Pell numbers P_n . As made for the Fibonacci numbers, let us use also the Lucas-like sequence H_n . For calculation, we define $\varphi = (1+\sqrt{2})$ and $\psi = (1-\sqrt{2})$, so that:

$$P_n = (\varphi^n - \psi^n)/(2\sqrt{2}) \quad , \quad H_n = (\varphi^n + \psi^n)/2$$

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Pell numbers are 0, 1, 2, 5, 12, 29, 70, 169, 408, 985, 2378, 5741, 13860, 33461, 80782, 195025, 470832, ... (A000129 in OEIS). $P_0=0$. H numbers are 1, 1, 3, 7, 17, 41, 99, 239, 577, 1393, 3363, 8119, 19601, 47321, 114243, 275807, 665857, ... (A001333 in OEIS). $H_0=1$.

We have:

$$\varphi^n=\sqrt{2}P_n+H_n \quad , \quad \psi^n=\sqrt{H_n}-P_n$$

Let us calculate $P_{n+m}=(\varphi^{n+m}-\psi^{n+m})/(2\sqrt{2})$. We find:

$$P_{n+m}=2\sqrt{2}\left(H_n P_m+H_m P_n\right)/(2\sqrt{2})=H_n P_m+H_m P_n$$

Numbers H_n, P_n are linked by the following relations:

$$H_n=1 \text{ if } n=0 \; ; \quad H_n=H_{n-1}+2P_{n-1} \text{ otherwise } (*)$$

$$P_n=0 \text{ if } n=0 \; ; \quad H_n=H_{n-1}+P_{n-1} \text{ otherwise } (**)$$

Then we have that: $H_n=P_n+P_{n-1}$. The generalized sum becomes:

$$P_m \oplus P_n=P_{n+m}=H_n P_m+H_m P_n=P_m\left(P_n+P_{n-1}\right)+P_n\left(P_m+P_{m-1}\right)$$

$$P_m \oplus P_n=P_m\left(P_n+P_{n-1}\right)+P_n\left(P_m+P_{m-1}\right)$$

It contains four Pell numbers, however two of them are determining the other two Pell numbers. That is, P_n, P_m determines P_{n-1}, P_{m-1} , as in the case of the Fibonacci numbers.

Of OEIS A000129, OEIS A001333 see please the detailed discussion and references given in the On-Line Encyclopedia of Integer Sequences. See please also https://en.wikipedia.org/wiki/Pell_number for (*) and (**).

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