Lowest order stabilization free virtual element method for the 2D Poisson equation

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Abstract

We analyze the first order Enlarged Enhancement Virtual Element Method (E^2VEM) for the Poisson problem. The method allows the definition of bilinear forms that do not require a stabilization term, thanks to the exploitation of higher order polynomial projections that are made computable by suitably enlarging the enhancement property (from which comes the prefix E^2) of local virtual spaces. We provide a sufficient condition for the well-posedness and optimal order a priori error estimates. We present numerical tests on convex and non-convex polygonal meshes that confirm the robustness of the method and the theoretical convergence rates.

Keywords: Virtual Element Methods, Poisson problem, polygonal meshes
65N12, 65N15, 65N30

1. Introduction

Virtual Element Methods (VEM) are polygonal methods for solving partial differential equations, that were first introduced in primal conforming form in [1] and were later on applied to most of the relevant problems of interest in applications, such as advection-diffusion-reaction equations [2], elastic and inelastic problems [3], parabolic and hyperbolic problems [4, 5], simulations in fractured media [6, 7]. Standard VEM discrete bilinear forms

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are characterized by the presence of an arbitrary non-polynomial stabilizing term that ensures the coercivity and that requires to be tuned depending on the problem analyzed. This arbitrarity of the discrete forms could be an issue, for instance, in the derivation of a posteriori error estimates [8, 9], where the stabilization term is always at the right-hand side when bounding the error in terms of the error estimator, both from above and from below. Moreover, the isotropic nature of the stabilization term becomes an issue when devising SUPG stabilizations [10, 11], in problems with anisotropic coefficients, or in the derivation of anisotropic a posteriori error estimates [12] or in complex non-linear problems [13]. Finally, we mention [14] where it has been shown the sensitivity of the solution of eigenvalue problems to variable parameters included in the discretization matrices.

Recently, the definition of VEM formulations that do not require an arbitrary non-polynomial stabilization term has received special interest. In particular, a preliminary version of this work has been made available to the scientific community as a preprint [15], and recent works developed and applied this approach to various problems such as linear and non-linear elasticity [16, 17, 18] and eigenproblems [19]. Moreover, in [20] a stabilization-free VEM formulation has been proposed for advection-diffusion problems in the advection-dominated regime and in [21] a comparison between the proposed method and standard Virtual Elements from [2] has been done, showing that the new formulation can induce smaller errors in the case of anisotropic diffusion tensors, due to the isotropic nature of the stabilization.

In this work, we analyze the Enlarged Enhancement Virtual Element Methods (E^2VEM), designed to allow the definition of a coercive bilinear form that involves only polynomial projections. In this framework, it is not required to add an arbitrary stabilizing bilinear form accounting for the non polynomial part of VEM functions. The method is based on the use of higher order polynomial projections in the discrete bilinear form with respect to the standard one [2] and on a modification of the VEM space to allow the computation of such projections. In particular, we extend the enhancement property that is used in the definition of the VEM space ([22], [2]), without changing the set of degrees of freedom. The degree of polynomial enrichment is chosen locally on each polygon, such that the discrete bilinear form is coercive.

The proof of well-posedness is quite elaborate, thus in this paper we deal only with the lowest order formulation and, for the sake of simplicity, we focus on the two dimensional Poisson's problem with homogenous Dirichlet boundary conditions, the extension to general boundary conditions being analogous to what is done for classical VEM. Moreover, the formulation and proofs presented in this work can also be easily extended to the case of a non constant anisotropic diffusion tensor.

The outline of the paper is as follows. In Section 2 we state our model problem. In Section 3 we introduce the approximation functional spaces and projection operators and we state the discrete problem. Section 4 contains the discussion about the well-posedness of the discrete problem under suitable sufficient conditions on the local projections. In Section 5 we prove optimal order a priori error estimates. Finally, Section 6 contains some numerical results assessing the rates of convergence of the method.

Throughout the work, $(\cdot,\cdot)_\omega$ denotes the standard L^2 scalar product defined on a generic $\omega \subset \mathbb{R}^2$, $\gamma^{\partial\omega}$ denotes the trace operator, that restricts on the boundary $\partial\omega$ an element of a space defined over $\omega \subset \mathbb{R}^2$. Inside the proofs, we decide to use a single character C for constants, independent of the mesh size, that appear in the inequalities, which means that we suppose to take at each step the maximum of the constants involved. Since the proofs require the definition of several auxiliary spaces and operators, we provide in Appendix A a table containing a summary of the relevant definitions.

2. Model Problem

Let $\Omega \subset \mathbb{R}^2$ be a bounded open set. We are interested in solving the following problem:

$$\begin{cases} -\Delta U = f & \text{in } \Omega, \\ U = 0 & \text{on } \partial\Omega. \end{cases} \tag{1}$$

Defining $a\colon \mathrm{H}^1_0(\Omega) \times \mathrm{H}^1_0(\Omega) \to \mathbb{R}$ such that,

$$a\left(U,W\right) := \left(\nabla U, \nabla W\right)_\Omega \quad \forall U,W \in \mathrm{H}^1_0(\Omega), \tag{2}$$

then, given $f \in \mathrm{L}^2(\Omega)$, the variational formulation of (1) is given by: find $U \in \mathrm{H}^1_0(\Omega)$ such that,

$$a\left(U,W\right) = \left(f, W\right)_\Omega \quad \forall W \in \mathrm{H}^1_0(\Omega)\,. \tag{3}$$

3. Discrete formulation

In order to define the discrete form of (3), $\mathcal{M}_h$ denotes a conforming polygonal tessellation of Ω and E denotes a generic polygon of $\mathcal{M}_h$. $\#\mathcal{M}_h$

denotes the number of polygons of $\mathcal{M}_h$ and the maximum diameter of all the polygons in $\mathcal{M}_h$ is denoted by h. Fixed $E \in \mathcal{M}_h$, let $\{x_i\}_{i=1}^{N_E^V}$ be its N_E^V vertices counter clockwise ordered, $\mathcal{E}_E$ the set of its edges and $\boldsymbol{n}^{\partial E}$ the outward-pointing unit normal vector to ∂E. We assume that $\mathcal{M}_h$ satifies the standard mesh assumptions for VEM (see for instance [23, 24]), i.e. $\exists \kappa > 0$ such that

1. for all $E \in \mathcal{M}_h$, E is star-shaped with respect to a ball of radius $\rho \geq \kappa h_E$, where h_E is the diameter of E;

2. for all edges $e \subset \partial E$, $|e| \geq \kappa h_E$.

Notice that the above conditions imply that, denoting by N_E^V the number of vertices of E, it holds

$$\exists N_{\max}^V > 0 \colon \forall E \in \mathcal{M}_h,\ N_E^V \leq N_{\max}^V \,. \tag{4}$$

For any given $E \in \mathcal{M}_h$, let $\mathbb{P}_k(E)$ be the space of polynomials of degree up to k defined on E. Let $\Pi_1^{\nabla,E} : \mathrm{H}^1(E) \to \mathbb{P}_1(E)$ be the $\mathrm{H}^1(E)$-orthogonal operator, defined up to a constant by the orthogonality condition: $\forall\, u \in \mathrm{H}^1(E)$,

$$\left(\nabla \left(\Pi_1^{\nabla,E} u - u\right), \nabla p\right)_E = 0 \ \ \forall\, p \in \mathbb{P}_1(E)\,. \tag{5}$$

In order to define $\Pi_1^{\nabla,E}$ uniquely, we choose any continuous and linear projection operator $\mathrm{P}_0 : \mathrm{H}^1(E) \to \mathbb{P}_0(E)$, whose continuity constant in H^1-norm is independent of h_E and continuous with respect to deformations of the geometry, and we impose $\forall\, u \in \mathrm{H}^1(E)$,

$$\mathrm{P}_0(\Pi_1^{\nabla,E} u - u) = 0. \tag{6}$$

Remark 1. *Under the current mesh assumptions, a suitable choice for* P_0 *is the integral mean on the boundary of* E*, i.e.*

$$\mathrm{P}_0(u) := \frac{1}{|\partial E|} \int_{\partial E} \gamma^{\partial E}(u)\, ds \quad \forall u \in \mathrm{H}^1(E).$$

Notice that this is a common choice, see for instance [2].

For any given $E \in \mathcal{M}_h$, let $l \in \mathbb{N}$ be given, as detailed in the next section, where we will choose l depending on N_E^V (see Theorem 1 and Section 4.3). Let $\mathcal{EN}_{1,l}^E$ be the set of functions $v \in \mathrm{H}^1(E)$ satisfying

$$(v, p)_E = \left(\Pi_1^{\nabla,E} v, p\right)_E \ \forall p \in \mathbb{P}_{l+1}(E) \ . \tag{7}$$

We define the Enlarged Enhancement Virtual Space of order 1 as

$$\mathcal{V}_{1,l}^E := \{v \in \mathcal{EN}_{1,l}^E : \Delta v \in \mathbb{P}_{l+1}(E)\,, \ \ \gamma^e(v) \in \mathbb{P}_1(e) \ \forall e \in \mathcal{E}_E, \ \ v \in C^0(\partial E)\}\,.$$

We define as degrees of freedom of this space the values of functions at the vertices of E (see [1, 2]).

Moreover, let $\boldsymbol{\ell} \in \mathbb{N}^{\#\mathcal{M}_h}$ be a vector and $\ell(E)$ denote the element corresponding to the polygon E, we define the global discrete space as

$$\mathcal{V}_{1,\boldsymbol{\ell}} := \{v \in \mathrm{H}_0^1(\Omega) \colon v_{|E} \in \mathcal{V}_{1,\ell(E)}^E\}\,.$$

Note that $v \in \mathcal{V}_{1,\boldsymbol{\ell}}$ is a continuous function that is a polynomial of degree 1 on each edge of the mesh.

To define our discrete bilinear form, let $\Pi_l^{0,E} \nabla : \mathrm{H}^1(E) \to [\mathbb{P}_l(E)]^2$ be the $\mathrm{L}^2(E)$-projection operator of the gradient of functions in $\mathrm{H}^1(E)$, defined, $\forall u \in \mathrm{H}^1(E)$, by the orthogonality condition

$$\left(\Pi_l^{0,E} \nabla u, \boldsymbol{p}\right)_E = (\nabla u, \boldsymbol{p})_E \ \ \forall \boldsymbol{p} \in [\mathbb{P}_l(E)]^2 \ . \tag{8}$$

Remark 2. *For each function $u \in \mathcal{V}_{1,l}^E$, the above projection is computable given the degrees of freedom of u, applying the Gauss-Green formula and exploiting* (7)*.*

Let $a_h^E \colon \mathrm{H}^1(E) \times \mathrm{H}^1(E) \to \mathbb{R}$ be defined as

$$a_h^E(u, v) := \left(\Pi_{\ell(E)}^{0,E} \nabla u, \Pi_{\ell(E)}^{0,E} \nabla v\right)_E \quad \forall u, v \in \mathrm{H}^1(E)\,, \tag{9}$$

and $a_h \colon \mathrm{H}^1(\Omega) \times \mathrm{H}^1(\Omega) \to \mathbb{R}$ as

$$a_h(u, v) := \sum_{E \in \mathcal{M}_h} a_h^E(u, v) \quad \forall u, v \in \mathcal{V}_{1,\boldsymbol{\ell}}\,. \tag{10}$$

We state the discrete problem as: find $u \in \mathcal{V}_{1,\boldsymbol{\ell}}$ such that

$$a_h(u, v) = \sum_{E \in \mathcal{M}_h} \left(f, \Pi_0^{0,E} v\right)_E \quad \forall v \in \mathcal{V}_{1,\boldsymbol{\ell}}\,, \tag{11}$$

where, $\forall E \in \mathcal{M}_h$, $\Pi_0^{0,E} \colon \mathrm{L}^2(E) \to \mathbb{R}$ is the $\mathrm{L}^2(E)$-projection, defined by

$$\Pi_0^{0,E} v := \frac{1}{|E|} (v, 1)_E \quad \forall v \in \mathrm{L}^2(E) . \tag{12}$$

The above projection is computable for any given $v \in \mathcal{V}_{1,l}^E$ exploiting (7).

4. Well-posedness

This section is devoted to prove the well-posedness of the discrete problem stated by (11), under suitable sufficient conditions on $\boldsymbol{\ell}$. The main result is given by Theorem 1, that implies the existence of an equivalent norm on $\mathcal{V}_{1,\boldsymbol{\ell}}$ and the well-posedness of (11).

First, we define, for any given $l \in \mathbb{N}$,

$$\mathcal{P}_l^{\ker}(E) = \left\{ \boldsymbol{p} \in [\mathbb{P}_l(E)]^2 : \int_{\partial E} \boldsymbol{p} \cdot \boldsymbol{n}^{\partial E} \gamma^{\partial E} (v - \mathrm{P}_0(v)) = 0 \quad \forall v \in \mathcal{V}_{1,l}^E \right\} . \tag{13}$$

Notice that the dimension of $\mathcal{P}_l^{\ker}(E)$ generally depends on the geometry of the polygon and the definition of P_0, but in Section 4.3 we provide an algorithm for enforcing the sufficient condition that is assumed in the following Theorem.

Theorem 1. *Let $E \in \mathcal{M}_h$, $u \in \mathcal{V}_{1,\ell(E)}^E$ and $\ell(E) \in \mathbb{N}$ such that the following condition is satisfied:*

$$(\ell(E) + 1)(\ell(E) + 2) - \dim \mathcal{P}_{\ell(E)}^{\ker}(E) \geq N_E^V - 1, \tag{14}$$

then

$$\Pi_{\ell(E)}^{0,E} \nabla u = 0 \implies \nabla u_{|E} = 0. \tag{15}$$

We omit in the following the proof of the case of triangles ($N_E^V = 3$ and $\ell(E) = 0$), indeed if E is a triangle, $\mathcal{V}_{1,\ell(E)}^E = \mathbb{P}_1(E)$ $\forall \ell(E) \geq 0$, and then $\Pi_{\ell(E)}^{0,E} \nabla u = \nabla u$ $\forall \ell(E) \geq 0$. Moreover, an explicit computation yields $\dim \mathcal{P}_0^{\ker}(E) = 0$ if E is a triangle. For technical reasons, the proof of Theorem 1 for a general polygon is split into two results, described in Sections 4.1 and 4.2, respectively. The proof relies on an auxiliary inf-sup condition that is proved by constructing a suitable Fortin operator, whose existence is guaranteed under condition (14).

4.1. Auxiliary inf-sup condition

In this section, after some auxiliary results, we prove through Proposition 1 that (15) is satisfied if the auxiliary inf-sup condition (26) holds true.

Lemma 1. *Let* $u \in \mathcal{V}_{1,l}^{E}$*, with* $l \geq 0$*. Then*

$$\Pi_l^{0,E} \nabla u = 0 \implies \Pi_1^{\nabla,E} u \in \mathbb{P}_0(E) \,.$$

Proof. Applying (8), we have

$$\Pi_l^{0,E} \nabla u = 0 \implies (\nabla u, \boldsymbol{p})_E = 0 \;\; \forall \boldsymbol{p} \in [\mathbb{P}_l(E)]^2 \,,$$

that implies

$$(\nabla u, \nabla p)_E = 0 \;\; \forall p \in \mathbb{P}_1(E) \,, \tag{16}$$

thanks to the relation $\nabla \mathbb{P}_1(E) \subseteq \nabla \mathbb{P}_{l+1}(E) \subseteq [\mathbb{P}_l(E)]^2$. Given (16) and (5),

$$\begin{aligned} \left(\nabla \Pi_1^{\nabla,E} u, \nabla p\right)_E = 0 \;\; \forall p \in \mathbb{P}_1(E) &\implies \nabla \Pi_1^{\nabla,E} u = 0 \\ &\implies \Pi_1^{\nabla,E} u \in \mathbb{P}_0(E) \,. \end{aligned}$$

□

Lemma 2. *Let* $u \in \mathcal{V}_{1,l}^{E}$*. If* $\Pi_l^{0,E} \nabla u = 0$*, then* (7) *can be rewritten as*

$$(u, p)_E = \mathrm{P}_0\left(u\right) \cdot (1, p)_E \;\; \forall p \in \mathbb{P}_{l+1}(E) \,, \tag{17}$$

where P_0 *is the projection operator defined in Section 3.*

Proof. Applying Lemma 1 and (6),

$$\Pi_l^{0,E} \nabla u = 0 \implies \Pi_1^{\nabla,E} u = \mathrm{P}_0\left(u\right) .$$

Then, (7) provides (17). □

We now need to introduce some notation and definitions. First, let $\mathcal{T}_E$ denote the sub-triangulation of E obtained linking each vertex of E to the centre of the ball with respect to which E is star-shaped, denoted by x_C. Let us define the set of internal edges of the triangulation $\mathcal{T}_E$ as $\mathcal{I}_E$. For any $i = 1, \ldots, N_E^V$, let $\tau_i \in \mathcal{T}_E$ be the triangle whose vertices are x_i, x_{i+1} (with $x_{N_E^V+1} \equiv x_1$) and x_C. Let e_i denote the edge $\overrightarrow{x_C x_i} \in \mathcal{I}_E$ and by $\boldsymbol{n}^{e_i}$ the outward-pointing unit normal vector to the edge e_i of τ_i. Then, for each polygon E, we can define the reference polygon $\hat{E}$, such that the mapping $F : \hat{E} \to E$ is given by

$$x = h_E \hat{x} + x_C. \tag{18}$$

Let Σ be the set of all admissible reference polygons, i.e. satisfying the mesh assumptions with the same regularity parameter as the polygons in the mesh.

Lemma 3 ([25, Proof of Lemma 4.9])**.** *Σ is compact.*

Definition 1. *Let $\mathrm{H}^1_{\mathcal{T}}(E)$ be the broken Sobolev space*

$$\mathrm{H}^1_{\mathcal{T}}(E) := \left\{ v \in \mathrm{L}^2(E) \colon v_{|\tau} \in \mathrm{H}^1(\tau)\ \forall \tau \in \mathcal{T}_E \right\}.$$

Let $u \in \mathrm{H}^1_{\mathcal{T}}(E)$, we define $\forall e_i \in \mathcal{I}_E$ the jump function $[\![\cdot]\!]_{e_i} : \mathrm{H}^1_{\mathcal{T}}(E) \to \mathrm{L}^2(e_i)$ such that

$$[\![u]\!]_{e_i} := \gamma^{e_i}\left(u_{|\tau_i}\right) - \gamma^{e_i}\left(u_{|\tau_{i-1}}\right).$$

Moreover, $[\![u]\!]_{\mathcal{I}_E}$ denotes the vector containing the jumps of u on each $e_i \in \mathcal{I}_E$. We endow $\mathrm{H}^1_{\mathcal{T}}(E)$ with the following seminorm and norm : $\forall u \in \mathrm{H}^1_{\mathcal{T}}(E)$,

$$|u|^2_{\mathrm{H}^1_{\mathcal{T}}(E)} := \sum_{\tau \in \mathcal{T}_E} \|\nabla u\|^2_{[\mathrm{L}^2(\tau)]^2} + \sum_{i=1}^{N_E^V} \left\| [\![u]\!]_{e_i} \right\|^2_{\mathrm{L}^2(e_i)}, \tag{19}$$

$$\|u\|^2_{\mathrm{H}^1_{\mathcal{T}}(E)} := |u|^2_{\mathrm{H}^1_{\mathcal{T}}(E)} + \|u\|^2_{\mathrm{L}^2(E)}. \tag{20}$$

Definition 2. *Let $V(E)$ be given by*

$$V(E) := \{ \boldsymbol{v} \in \left[\mathrm{L}^2(E)\right]^2 : \boldsymbol{v}_{|\tau} \in \mathrm{H}^{\mathrm{div}}(\tau)\ \forall \tau \in \mathcal{T}_E,\ [\![\boldsymbol{v}]\!]_{e_i} \in \mathrm{L}^\infty(e_i)\ \forall e_i \in \mathcal{I}_E \}. \tag{21}$$

Then $\forall \boldsymbol{v} \in V(E)$, we define its seminorm and its norm:

$$|\boldsymbol{v}|^2_{V(E)} := \sum_{\tau \in \mathcal{T}_E} \|\nabla \cdot \boldsymbol{v}\|^2_{\mathrm{L}^2(\tau)} + h_E^2 \left\| [\![\boldsymbol{v}]\!]_{\mathcal{I}_E} \right\|^2_{\mathrm{L}^\infty(\mathcal{I}_E)},$$

$$\|\boldsymbol{v}\|^2_{V(E)} := |\boldsymbol{v}|^2_{V(E)} + \|\boldsymbol{v}\|^2_{[\mathrm{L}^2(E)]^2}$$

where

$$\left\| [\![\boldsymbol{v}]\!]_{\mathcal{I}_E} \right\|_{\mathrm{L}^\infty(\mathcal{I}_E)} := \max_{i=1,\dots,N_E^V} \left\| [\![\boldsymbol{v}]\!]_{e_i} \right\|_{\mathrm{L}^\infty(e_i)}.$$

Remark 3. *Let us note that $[\mathbb{P}_l(E)]^2 \subset V(E)$. Hence, we can use $\|\cdot\|_{V(E)}$ as a norm for $[\mathbb{P}_l(E)]^2$. Notice that, since $[\mathbb{P}_l(E)]^2 \subset [C^0(E)]^2$, $\left\| [\![\boldsymbol{p}]\!]_{\mathcal{I}_E} \right\|_{\mathrm{L}^\infty(\mathcal{I}_E)} =$* 0, $\forall \boldsymbol{p} \in [\mathbb{P}_l(E)]^2$.

Definition 3. *Let* $\mathcal{V}_{1,l}^{E,\mathrm{P}_0}$ *be the space*

$$\mathcal{V}_{1,l}^{E,\mathrm{P}_0} := \left\{ v \in \mathcal{V}_{1,l}^{E} : \mathrm{P}_0(v) = 0 \right\}. \tag{22}$$

Definition 4. *Denoting by* $\{\psi_i\}_{i=1}^{N_E^V}$ *the set of Lagrangian basis functions of* $\mathcal{V}_{1,l}^{E}$*, let* $\mathcal{Q}(\partial E)$ *be the vector space*

$$\mathcal{Q}(\partial E) := \operatorname{span}\left\{ \gamma^{\partial E}(\psi_i - \mathrm{P}_0(\psi_i)) \right\}, \quad \forall i = 1, \dots, N_E^V - 1. \tag{23}$$

We remark that the above space is made up of continuous piecewise linear polynomials on each edge. Notice that $\forall q \in \mathcal{Q}(\partial E)$, $\exists! v \in \mathcal{V}_{1,l}^{E,\mathrm{P}_0}$ *such that* $q = \gamma^{\partial E}(v)$.

Definition 5. *Let* $\mathcal{R}_{\mathcal{Q}}(E)$ *be the vector space, lifting of* $\mathcal{Q}(\partial E)$ *on* E*, given by:*

$$\mathcal{R}_{\mathcal{Q}}(E) := \left\{ \bar{q} \in \mathrm{L}^2(E) \colon \bar{q}_{|\tau} \in \mathbb{P}_1(\tau)\ \forall \tau \in \mathcal{T}_E,\ \gamma^{\partial E}(\bar{q}) \in \mathcal{Q}(\partial E),\ \bar{q}(x_C) = 0 \right\}. \tag{24}$$

We note that $\mathcal{R}_{\mathcal{Q}}(E) \subset \mathrm{H}^1_{\mathcal{T}}(E) \cap C^0(E)$*. Hence, we use the norm* $\|\cdot\|_{\mathrm{H}^1_{\mathcal{T}}(E)}$ *defined in* (20) *as a norm for* $\mathcal{R}_{\mathcal{Q}}(E)$*. Notice that* $\sum_{i=1}^{N_E^V} \left\| [\![\bar{q}]\!]_{e_i} \right\|_{\mathrm{L}^2(e_i)} = 0$*. Let* $\{r_j\}_{j=1}^{N_E^V - 1}$ *denote a basis of* $\mathcal{R}_{\mathcal{Q}}(E)$.

Now, we can introduce the bilinear form b which is used in Proposition 1.

Definition 6. *Let* $b : \mathcal{R}_{\mathcal{Q}}(E) \times V(E) \to \mathbb{R}$*, such that* $\forall \bar{q} \in \mathcal{R}_{\mathcal{Q}}(E)$, $\forall \boldsymbol{v} \in V(E)$

$$b(\bar{q}, \boldsymbol{v}) = \sum_{\tau \in \mathcal{T}_E} \int_\tau \left[\nabla \bar{q}\, \boldsymbol{v} + \bar{q}\, \nabla \cdot \boldsymbol{v} \right] dx - \sum_{i=1}^{N_E^V} \int_{e_i} \gamma^{e_i}(\bar{q})\, [\![\boldsymbol{v}]\!]_{e_i} \cdot \boldsymbol{n}^{e_i} ds. \tag{25}$$

Remark 4. *In the following, for any given* $\bar{q} \in \mathcal{R}_{\mathcal{Q}}(E)$ *we use* $b(\bar{q}, \boldsymbol{v})$ *when* $\boldsymbol{v}$ *is a polynomial or a function of the* $\mathrm{H}(\mathrm{div}; E)$*-conforming VEM space [26]. In these cases, an application of the divergence theorem gives*

$$b(\bar{q}, \boldsymbol{v}) = \int_{\partial E} \bar{q}\, \boldsymbol{v} \cdot n^{\partial E}\, ds \quad \forall\, \bar{q} \in \mathcal{R}_{\mathcal{Q}}(E).$$

The following lemma gives the continuity of the bilinear form b.

Lemma 4. *Let b be given by* (25). *Then b is a bilinear form and $\exists C_b > 0$ independent of h_E such that*

$$b(\bar{q}, \boldsymbol{v}) \leq C_b \left\|\bar{q}\right\|_{\mathrm{H}^1_{\mathcal{T}}(E)} \left\|\boldsymbol{v}\right\|_{V(E)} \quad \forall \bar{q} \in \mathcal{R}_{\mathcal{Q}}(E), \forall \boldsymbol{v} \in V\left(E\right) .$$

Proof. The proof of this lemma can be found in Appendix B. □

The following proposition is the first step towards the proof of Theorem 1.

Proposition 1. *Let b denote the continuous bilinear form defined by* (25). *If $\exists \beta > 0$, independent of h_E, such that*

$$\forall \bar{q} \in \mathcal{R}_{\mathcal{Q}}(E), \qquad \sup_{\boldsymbol{p} \in [\mathbb{P}_l(E)]^2} \frac{b(\bar{q}, \boldsymbol{p})}{\left\|\boldsymbol{p}\right\|_{V(E)}} \geq \beta \left\|\bar{q}\right\|_{\mathrm{H}^1_{\mathcal{T}}(E)} , \tag{26}$$

then (15) *holds true.*

Proof. Let $u \in \mathcal{V}^E_{1,l}$, (8) yields,

$$\Pi^{0,E}_l \nabla u = 0 \implies \left(\nabla u, \boldsymbol{p}\right)_E = 0 \;\; \forall \boldsymbol{p} \in \left[\mathbb{P}_l(E)\right]^2 .$$

Applying Gauss-Green formula, the previous relation becomes

$$\left(\nabla u, \boldsymbol{p}\right)_E = \left(\gamma^{\partial E}(u)\, , \boldsymbol{p} \cdot n^{\partial E}\right)_{\partial E} - \left(u, \nabla \cdot \boldsymbol{p}\right)_E = 0 \;\; \forall \boldsymbol{p} \in \left[\mathbb{P}_l(E)\right]^2 .$$

Since $\nabla \cdot \boldsymbol{p} \in \mathbb{P}_{l-1}(E)$ we apply (17) and we obtain

$$\left(\gamma^{\partial E}(u)\, , \boldsymbol{p} \cdot n^{\partial E}\right)_{\partial E} - \mathrm{P}_0\left(u\right) \cdot \left(1, \nabla \cdot \boldsymbol{p}\right)_E = 0 \;\; \forall \boldsymbol{p} \in \left[\mathbb{P}_l(E)\right]^2 .$$

Then we can apply the divergence theorem and find the relation

$$\left(\gamma^{\partial E}(u - \mathrm{P}_0\left(u\right))\, , \boldsymbol{p} \cdot n^{\partial E}\right)_{\partial E} = 0 \;\; \forall \boldsymbol{p} \in \left[\mathbb{P}_l(E)\right]^2 . \tag{27}$$

We have $q = \gamma^{\partial E}(u - \mathrm{P}_0\left(u\right)) \in \mathcal{Q}(\partial E)$ ($\mathcal{Q}(\partial E)$ defined in (23)). Let $\bar{q} \in \mathcal{R}_{\mathcal{Q}}(E)$ be the lifting of q ($\mathcal{R}_{\mathcal{Q}}(E)$ defined in (24)), then the relation (27), applying the divergence theorem, is

$$b(\bar{q}, \boldsymbol{p}) = 0 \;\; \forall \boldsymbol{p} \in \left[\mathbb{P}_l(E)\right]^2 .$$

Then, since b is a continuous bilinear form, (26) implies $q \equiv 0$. Finally, since $u \in \mathcal{V}^E_{1,l}$, then $u = \mathrm{P}_0\left(u\right)$ and $\nabla u = \nabla \mathrm{P}_0\left(u\right) = 0$. □

4.2. Proof of the inf-sup condition

In this section we show that (26) holds with β independent of h_E. The proof relies on the technique known as Fortin trick [27], that consists in the following classical result.

Proposition 2 ([27, Proposition 5.4.2])**.** *Assume that there exists an operator* $\Pi_E : V(E) \to [\mathbb{P}_l(E)]^2$ *that satisfies,* $\forall \boldsymbol{v} \in V(E)$*,*

$$b(\bar{q}, \Pi_E \boldsymbol{v} - \boldsymbol{v}) = 0 \quad \forall \bar{q} \in \mathcal{R}_{\mathcal{Q}}(E)\,, \tag{28}$$

and assume that there exists a constant $C_\Pi > 0$*, independent of* h_E*, such that*

$$\|\Pi_E \boldsymbol{v}\|_{V(E)} \leq C_\Pi \|\boldsymbol{v}\|_{V(E)} \quad \forall \boldsymbol{v} \in V(E)\,. \tag{29}$$

Assume moreover that $\exists \eta > 0$*, independent of* h_E *such that*

$$\inf_{q \in \mathcal{R}_{\mathcal{Q}}(E)} \sup_{\boldsymbol{v} \in V(E)} \frac{b(q, \boldsymbol{v})}{\|q\|_{\mathrm{H}^1_{\mathcal{T}}(E)} \|\boldsymbol{v}\|_{V(E)}} \geq \eta\,. \tag{30}$$

Then the discrete inf-sup condition (26) *is satisfied, with* $\beta = \frac{\eta}{C_\Pi}$*.*

Remark 5. *The inf-sup constant* β *in* (26) *has to be independent of the mesh size in order to guarantee that the constant in* (46)*, involved in the coercivity of the bilinear form of* (11)*, is independent of the mesh size.*

Remark 6. *The operator* Π_E *defined in the following is such that the constant* C_Π *depends on* $N^V_{\max}$ *and on the continuity constant of* P_0*, both are bounded independently of* h_E *by assumption.*

Following the above results, we have to prove (30) and to show the existence of the operator Π_E satisfying (28) and (29). In the following proposition we achieve the first task.

Proposition 3. *Let* $b\colon \mathcal{R}_{\mathcal{Q}}(E) \times V(E) \to \mathbb{R}$ *be defined by* (25)*. Then the inf-sup condition* (30) *holds true.*

Proof. Let $q \in \mathcal{R}_{\mathcal{Q}}(E)$ be given arbitrarily. For any $\tau_i \in \mathcal{T}_E$, we recall that the vertices of τ_i are x_C, $x_i = \binom{x_{i,1}}{x_{i,2}}$ and $x_{i+1} = \binom{x_{i+1,1}}{x_{i+1,2}}$. Let e_i^∂ and $\boldsymbol{n}^{e_i^\partial}$ denote the edge $\overrightarrow{x_i x_{i+1}}$ and the outward-pointing unit normal vector to the edge e_i^∂, respectively. Let $\boldsymbol{\varphi}_i^1$, $\boldsymbol{\varphi}_i^2 \in [\mathbb{P}_1(\tau_i)]^2$ be given such that

$$\boldsymbol{\varphi}_i^1(x_1, x_2) = \begin{pmatrix} x_2 - x_{i,2} \\ -(x_1 - x_{i,1}) \end{pmatrix}, \boldsymbol{\varphi}_i^2(x_1, x_2) = \left|e_i^\partial\right| \boldsymbol{n}^{e_i^\partial} = \begin{pmatrix} x_{i+1,2} - x_{i,2} \\ -(x_{i+1,1} - x_{i,1}) \end{pmatrix}. \tag{31}$$

Let $\mathrm{B}^{\partial}(\tau_i) := \mathrm{span}\{\boldsymbol{\varphi}_i^1, \boldsymbol{\varphi}_i^2\} \subset [\mathbb{P}_1(\tau_i)]^2$. Notice that $\forall \boldsymbol{v} \in \mathrm{B}^{\partial}(\tau_i)$ we have $\nabla \cdot \boldsymbol{v} = 0$ and $\left\| \boldsymbol{v} \cdot \boldsymbol{n}^{e_i^{\partial}} \right\|_{\mathrm{L}^2(e_i^{\partial})}$ is a norm on $\mathrm{B}^{\partial}(\tau_i)$. Indeed, if $\boldsymbol{v} \in \mathrm{B}^{\partial}(\tau_i)$ and $\left\| \boldsymbol{v} \cdot \boldsymbol{n}^{e_i^{\partial}} \right\|_{\mathrm{L}^2(e_i^{\partial})} = 0$, then an explicit computation yields $\boldsymbol{v} \equiv \boldsymbol{0}$ on τ_i. Moreover, let $\mathrm{B}^{\partial}(\mathcal{T}_E) := \{\boldsymbol{v} : \boldsymbol{v}_{|\tau} \in \mathrm{B}^{\partial}(\tau)\ \forall \tau \in \mathcal{T}_E\} \subset V(E)$. Notice that $\forall \boldsymbol{v} \in \mathrm{B}^{\partial}(\mathcal{T}_E)$ we have that $\sum_{\tau \in \mathcal{T}_E} \|\nabla \cdot \boldsymbol{v}\|^2_{\mathrm{L}^2(\tau)} = 0$ and $\left\|\boldsymbol{v} \cdot \boldsymbol{n}^{\partial E}\right\|_{\mathrm{L}^2(\partial E)}$ is a norm on $\mathrm{B}^{\partial}(\mathcal{T}_E)$. We define $\boldsymbol{v}^{\star} \in \mathrm{B}^{\partial}(\mathcal{T}_E)$ such that $\boldsymbol{v}^{\star}{}_{|\partial E} \cdot \boldsymbol{n}^{\partial E} = \gamma^{\partial E}(q)$. In particular, $\forall \tau_i \in \mathcal{T}_E$ $\boldsymbol{v}^{\star}{}_{|\tau_i} = \frac{q(x_{i+1}) - q(x_i)}{\left|e_i^{\partial}\right|} \boldsymbol{\varphi}_i^1 + \frac{q(x_i)}{\left|e_i^{\partial}\right|} \boldsymbol{\varphi}_i^2$. Notice that $\|q\|_{\mathrm{L}^2(\partial E)} = \left\|\boldsymbol{v}^{\star} \cdot \boldsymbol{n}^{\partial E}\right\|_{\mathrm{L}^2(\partial E)}$ and $b(q, \boldsymbol{v}^{\star}) = \int_{\partial E} q \boldsymbol{v}^{\star} \cdot n^{\partial E}$. Then,

$$\begin{aligned}\sup_{\boldsymbol{v} \in V(E)} \frac{b(q, \boldsymbol{v})}{\|q\|_{\mathrm{H}^1_{\mathcal{T}}(E)} \|\boldsymbol{v}\|_{V(E)}} &\geq \frac{b(q, \boldsymbol{v}^{\star})}{\|q\|_{\mathrm{H}^1_{\mathcal{T}}(E)} \|\boldsymbol{v}^{\star}\|_{V(E)}} = \frac{\int_{\partial E} q\, \boldsymbol{v}^{\star} \cdot n^{\partial E}}{\|q\|_{\mathrm{H}^1_{\mathcal{T}}(E)} \|\boldsymbol{v}^{\star}\|_{V(E)}} \\ &= \frac{\|q\|_{\mathrm{L}^2(\partial E)}}{\|q\|_{\mathrm{H}^1_{\mathcal{T}}(E)}} \frac{\left\|\boldsymbol{v}^{\star} \cdot \boldsymbol{n}^{\partial E}\right\|_{\mathrm{L}^2(\partial E)}}{\|\boldsymbol{v}^{\star}\|_{V(E)}}\,. \end{aligned} \tag{32}$$

We have to estimate from below the last two factors. We notice that $\|q\|_{\mathrm{L}^2(\partial E)}$ is a norm on $\mathcal{R}_{\mathcal{Q}}(E)$, since $q \in \mathbb{P}_1(\mathcal{T}_E)$ and $q(x_C) = 0$. Thus, we can exploit the equivalence of norms on finite dimensional spaces. Hence, regarding the first norm, we get, by a scaling argument,

$$\begin{aligned}\|q\|^2_{\mathrm{L}^2(\partial E)} &= \sum_{e \in \partial E} \|q\|^2_{\mathrm{L}^2(e)} = h_E \sum_{\hat{e} \in \partial \hat{E}} \|\hat{q}\|^2_{\mathrm{L}^2(\hat{e})} \geq C h_E \left(\sum_{\hat{\tau} \in \mathcal{T}_{\hat{E}}} \|\hat{q}\|^2_{\mathrm{L}^2(\hat{\tau})} + \left\|\hat{\nabla} \hat{q}\right\|^2_{[\mathrm{L}^2(\hat{\tau})]^2} \right) \\ &= C h_E \left(\sum_{\tau \in \mathcal{T}_E} h_E^{-2} \|q\|^2_{\mathrm{L}^2(\tau)} + \|\nabla q\|^2_{[\mathrm{L}^2(\tau)]^2} \right) \geq C h_E \min\{1, h_E^{-2}\} \|q\|^2_{\mathrm{H}^1_{\mathcal{T}}(E)} \\ &\geq C h_E \|q\|^2_{\mathrm{H}^1_{\mathcal{T}}(E)}\ . \end{aligned} \tag{33}$$

Notice that the constant above is independent of the choice of reference element by Lemma 3. The second norm is estimated using the definition of dual norm and the trace inequality

$$\left\|\gamma^{\partial E}(w)\right\|_{\mathrm{L}^2(\partial E)} \leq C h_E^{\frac{1}{2}} \left(h_E^{-2} \|w\|^2_{\mathrm{L}^2(E)} + \|\nabla w\|^2_{[\mathrm{L}^2(E)]^2} \right)^{\frac{1}{2}} \quad \forall w \in \mathrm{H}^1(E)\,,$$

as follows:

$$\left\|\boldsymbol{v}^\star \cdot \boldsymbol{n}^{\partial E}\right\|_{\mathrm{L}^2(\partial E)} = \sup_{\chi \in \mathrm{L}^2(\partial E)} \frac{\left(\boldsymbol{v}^\star \cdot \boldsymbol{n}^{\partial E}, \chi\right)_{\partial E}}{\|\chi\|_{\mathrm{L}^2(\partial E)}} \geq \sup_{w \in \mathrm{H}^1(E)} \frac{\left(\boldsymbol{v}^\star \cdot \boldsymbol{n}^{\partial E}, \gamma^{\partial E}(w)\right)_{\partial E}}{\|\gamma^{\partial E}(w)\|_{\mathrm{L}^2(\partial E)}}$$
$$\geq C h_E^{-\frac{1}{2}} h_E \sup_{w \in \mathrm{H}^1(E)} \frac{h_E^{-1}\left(\boldsymbol{v}^\star \cdot \boldsymbol{n}^{\partial E}, \gamma^{\partial E}(w)\right)_{\partial E}}{\left(h_E^{-2}\|w\|^2_{\mathrm{L}^2(E)} + \|\nabla w\|_{[\mathrm{L}^2(E)]^2}\right)^{\frac{1}{2}}} . \tag{34}$$

Let $w^\star \in \mathrm{H}^1(E)$ be such that

$$(\nabla w^\star, \nabla \varphi)_E + h_E^{-2}(w^\star, \varphi)_E = h_E^{-1}\left(\boldsymbol{v}^\star \cdot \boldsymbol{n}^{\partial E}, \gamma^{\partial E}(w)\right)_{\partial E} .$$

Notice that $\hat{w}^\star = w^\star \circ F$ (F being the mapping defined by (18)) is the solution of

$$\left(\hat{\nabla}\hat{w}^\star, \hat{\nabla}\hat{\varphi}\right)_{\hat{E}} + (\hat{w}^\star, \hat{\varphi})_{\hat{E}} = \left(\hat{\boldsymbol{v}}^\star \cdot \boldsymbol{n}^{\partial \hat{E}}, \gamma^{\partial \hat{E}}(\hat{\varphi})\right)_{\partial \hat{E}} \quad \forall \hat{\varphi} \in \mathrm{H}^1(\hat{E}) .$$

Notice that

$$\sup_{\hat{w} \in \mathrm{H}^1(\hat{E})} \frac{\left(\hat{\boldsymbol{v}}^\star \cdot \boldsymbol{n}^{\partial \hat{E}}, \gamma^{\partial \hat{E}}(\hat{w})\right)_{\partial \hat{E}}}{\|\hat{w}\|_{\mathrm{H}^1(\hat{E})}} = \frac{\left(\hat{\boldsymbol{v}}^\star \cdot \boldsymbol{n}^{\partial \hat{E}}, \gamma^{\partial \hat{E}}(\hat{w}^\star)\right)_{\partial \hat{E}}}{\|\hat{w}^\star\|_{\mathrm{H}^1(\hat{E})}} .$$

This relation holds true since the *greater than inequality* is trivial using the definition of sup and the *less than inequality* can be proved applying the property of inner products $|(x, y)|^2 \leq (x, x)\,(y, y)$, indeed

$$\sup_{\hat{w} \in \mathrm{H}^1(\hat{E})} \frac{\left(\hat{\boldsymbol{v}}^\star \cdot \boldsymbol{n}^{\partial \hat{E}}, \gamma^{\partial \hat{E}}(\hat{w})\right)_{\partial \hat{E}}}{\|\hat{w}\|_{\mathrm{H}^1(\hat{E})}} = \sup_{\hat{w} \in \mathrm{H}^1(\hat{E})} \frac{\left(\hat{\nabla}\hat{w}^\star, \hat{\nabla}\hat{w}\right)_{\hat{E}} + (\hat{w}^\star, \hat{w})_{\hat{E}}}{\|\hat{w}\|_{\mathrm{H}^1(\hat{E})}}$$
$$\leq \sup_{\hat{w} \in \mathrm{H}^1(\hat{E})} \frac{\|\hat{w}^\star\|_{\mathrm{H}^1(\hat{E})}\|\hat{w}\|_{\mathrm{H}^1(\hat{E})}}{\|\hat{w}\|_{\mathrm{H}^1(\hat{E})}} = \frac{\|\hat{w}^\star\|^2_{\mathrm{H}^1(\hat{E})}}{\|\hat{w}^\star\|_{\mathrm{H}^1(\hat{E})}} = \frac{\left(\hat{\boldsymbol{v}}^\star \cdot \boldsymbol{n}^{\partial \hat{E}}, \gamma^{\partial \hat{E}}(\hat{w}^\star)\right)_{\partial \hat{E}}}{\|\hat{w}^\star\|_{\mathrm{H}^1(\hat{E})}} .$$

Then, by choosing $\varphi = w^\star$ and $\hat{\varphi} = \hat{w}^\star$ in the equations above we get

$$\sup_{w \in \mathrm{H}^1(E)} \frac{h_E^{-1} \left(\boldsymbol{v}^\star \cdot \boldsymbol{n}^{\partial E}, \gamma^{\partial E}(w)\right)_{\partial E}}{\left(h_E^{-2} \left\|w\right\|^2_{\mathrm{L}^2(E)} + \left\|\nabla w\right\|_{[\mathrm{L}^2(E)]^2}\right)^{\frac{1}{2}}} \geq \frac{h_E^{-2} \left\|w^\star\right\|^2_{\mathrm{L}^2(E)} + \left\|\nabla w^\star\right\|^2_{[\mathrm{L}^2(E)]^2}}{\left(h_E^{-2} \left\|w^\star\right\|^2_{\mathrm{L}^2(E)} + \left\|\nabla w^\star\right\|_{[\mathrm{L}^2(E)]^2}\right)^{\frac{1}{2}}}$$

$$= \left(h_E^{-2} \left\|w^\star\right\|^2_{\mathrm{L}^2(E)} + \left\|\nabla w^\star\right\|^2_{[\mathrm{L}^2(E)]^2}\right)^{\frac{1}{2}} = \left(\left\|\hat{w}^\star\right\|^2_{\mathrm{L}^2(\hat{E})} + \left\|\hat{\nabla} \hat{w}^\star\right\|^2_{\left[\mathrm{L}^2\left(\hat{E}\right)\right]^2}\right)^{\frac{1}{2}}$$

$$= \frac{\left(\hat{\boldsymbol{v}}^\star \cdot \boldsymbol{n}^{\partial \hat{E}}, \gamma^{\partial \hat{E}}(\hat{w}^\star)\right)_{\partial \hat{E}}}{\left\|\hat{w}^\star\right\|_{\mathrm{H}^1(\hat{E})}} = \sup_{\hat{w} \in \mathrm{H}^1(\hat{E})} \frac{\left(\hat{\boldsymbol{v}}^\star \cdot \boldsymbol{n}^{\partial \hat{E}}, \gamma^{\partial \hat{E}}(\hat{w})\right)_{\partial \hat{E}}}{\left\|\hat{w}\right\|_{\mathrm{H}^1(\hat{E})}} .$$

Moreover, notice that the term $\sup_{\hat{w} \in \mathrm{H}^1(\hat{E})} \frac{\left(\hat{\boldsymbol{v}} \cdot \boldsymbol{n}^{\partial \hat{E}}, \gamma^{\partial \hat{E}}(\hat{w})\right)_{\partial \hat{E}}}{\left\|\hat{w}\right\|_{\mathrm{H}^1(\hat{E})}}$ is a norm on $B^\partial(\mathcal{T}_E)$. Indeed, if $\sup_{\hat{w} \in \mathrm{H}^1(\hat{E})} \frac{\left(\hat{\boldsymbol{v}} \cdot \boldsymbol{n}^{\partial \hat{E}}, \gamma^{\partial \hat{E}}(\hat{w})\right)_{\partial \hat{E}}}{\left\|\hat{w}\right\|_{\mathrm{H}^1(\hat{E})}} = 0$ then $\hat{\boldsymbol{v}} \cdot \boldsymbol{n}^{\partial \hat{E}} = 0$ and $\hat{\boldsymbol{v}} = 0$. Then, applying the above results to (34), recalling that $\sum_{\tau \in \mathcal{T}_E} \left\|\nabla \cdot \boldsymbol{v}\right\|^2_{\mathrm{L}^2(\tau)} = 0$ $\forall \boldsymbol{v} \in \mathrm{B}^\partial(\mathcal{T}_E)$, using the equivalence of norms on finite dimensional spaces and a scaling argument, we get

$$\begin{aligned}
\left\|\boldsymbol{v}^\star \cdot \boldsymbol{n}^{\partial E}\right\|_{\mathrm{L}^2(\partial E)} &\geq C h_E^{-\frac{1}{2}} h_E \sup_{\hat{w} \in \mathrm{H}^1(\hat{E})} \frac{\left(\hat{\boldsymbol{v}}^\star \cdot \boldsymbol{n}^{\partial \hat{E}}, \gamma^{\partial \hat{E}}(\hat{w})\right)_{\partial \hat{E}}}{\left\|\hat{w}\right\|_{\mathrm{H}^1(\hat{E})}} \geq C h_E^{-\frac{1}{2}} h_E \left\|\hat{\boldsymbol{v}}^\star\right\|_{V\left(\hat{E}\right)} \\
&= C h_E^{-\frac{1}{2}} \left(\sum_{\tau \in \mathcal{T}_E} \left\|\boldsymbol{v}^\star\right\|^2_{[\mathrm{L}^2(\tau)]^2} + h_E^2 \left\|[\![\boldsymbol{v}^\star]\!]_{\mathcal{I}_E}\right\|^2_{\mathrm{L}^\infty(\mathcal{I}_E)}\right)^{\frac{1}{2}} \\
&= C h_E^{-\frac{1}{2}} \left\|\boldsymbol{v}^\star\right\|_{V(E)} ,
\end{aligned} \tag{35}$$

where C is independent of h_E and of the choice of reference element by Lemma 3. The proof is thus concluded by applying the estimates (33) and (35) to (32). □

In the following, assuming (14), we prove the existence of an operator Π_E satisfying (28) and (29). First, we need some auxiliary results.

Definition 7. *Let* $\{r_i\}_{i=1}^{N_E^V - 1}$ *be a basis of* $\mathcal{R}_{\mathcal{Q}}(E)$*. Let us define the set of linear operators* $D_i : V(E) \to \mathbb{R}$ *such that* $\forall \boldsymbol{v} \in V(E)$

$$D_i(\boldsymbol{v}) := b(r_i, \boldsymbol{v}) \quad \forall i = 1, \dots, N_E^V - 1 .$$

Lemma 5. *If* $(\ell(E)+1)(\ell(E)+2) - \dim \mathcal{P}^{\ker}_{\ell(E)}(E) \geq N_E^V - 1$*, there exists a set of functions* $\boldsymbol{\pi}_j \in \left[\mathbb{P}_{\ell(E)}(E)\right]^2$ *such that*

$$D_i(\boldsymbol{\pi}_j) = \delta_{ij} \quad \forall i, j = 1, \ldots, N_E^V - 1. \tag{36}$$

Proof. In the following, with a slight abuse of notation we use ℓ instead of $\ell(E)$. Let $V_\ell^M(E)$ be the local mixed virtual element space of order ℓ, defined in [26], i.e.

$$V_\ell^M(E) := \{\boldsymbol{v} \in \mathrm{H}(\mathrm{div}; E) \cap \mathrm{H}(\mathrm{rot}; E) : \gamma^e\big(\boldsymbol{v} \cdot \boldsymbol{n}^{\partial E}\big) \in \mathbb{P}_\ell(e)\, \forall e \in \boldsymbol{\mathcal{E}}_E,\\ \mathrm{div}\boldsymbol{v} \in \mathbb{P}_\ell(E) \text{ and } \mathrm{rot}\boldsymbol{v} \in \mathbb{P}_{\ell-1}(E)\}.$$

Notice that $[\mathbb{P}_\ell(E)]^2 \subset V_\ell^M(E)$. For each $\boldsymbol{v} \in V_\ell^M(E)$, the degrees of freedom of $\boldsymbol{v}$ are defined [26] by

1. $\int_e \boldsymbol{v} \cdot \boldsymbol{n}^{\partial E}\, q \,\mathrm{ds},\ \forall e \in \boldsymbol{\mathcal{E}}_E,\ \forall q \in \mathbb{P}_\ell(e)$,
2. $\int_E \boldsymbol{v} \cdot \nabla p_\ell \,\mathrm{dx},\ \forall p_\ell \in \mathbb{P}_\ell(E)$,
3. $\int_E \boldsymbol{v} \cdot \boldsymbol{p}_{\boldsymbol{\ell}}^\perp \mathrm{dx},\ \forall \boldsymbol{p}_{\boldsymbol{\ell}}^\perp \in \{\boldsymbol{p}_{\boldsymbol{\ell}}^\perp \in [\mathbb{P}_\ell(E)]^2 : \int_E \boldsymbol{p}_{\boldsymbol{\ell}}^\perp \cdot \nabla q \ \mathrm{dx} = 0\, \forall q \in \mathbb{P}_{\ell+1}(E)\}$.

The number of degrees of freedom defined by the first, the second and the third condition is, respectively, $(\ell+1)N_E^V$, $\frac{(\ell+1)(\ell+2)}{2} - 1$ and $\frac{(\ell-1)(\ell+2)}{2} + 1$. Globally, $\dim V_\ell^M(E) = (\ell+1)N_E^V + \ell(\ell+2)$.

Notice that a possible choice for the basis of $\mathcal{P}_\ell(\partial E) := \{p \in \mathbb{P}_\ell(e)\,, \forall e \in \boldsymbol{\mathcal{E}}_E\}$ is composed by the $N_E^V - 1$ basis functions $\{\gamma^{\partial E}(r_i)\}_{i=1}^{N_E^V-1} \subset \mathcal{Q}(\partial E) \subset \mathcal{P}_\ell(\partial E)$, completed by a choice of linearly independent functions $\{q_i^C\}_{i=N_E^V}^{(\ell+1)N_E^V} \subset \mathcal{P}_\ell(\partial E)$. Moreover, for any $\boldsymbol{v} \in V_\ell^M(E)$ we have

$$D_i(\boldsymbol{v}) = b(r_i, \boldsymbol{v}) = \int_{\partial E} \big(\boldsymbol{v} \cdot \boldsymbol{n}^{\partial E}\big)\, \gamma^{\partial E}(r_i)\ ds \quad \forall i = 1, \ldots, N_E^V - 1\,,$$

by an application of the divergence theorem. Hence, the first set of degrees of freedom can be split into two groups, i.e.

- $D_i(\boldsymbol{v})$, $\forall i = 1, \ldots, N_E^V - 1$,
- $\int_{\partial E} \boldsymbol{v} \cdot \boldsymbol{n}^{\partial E}\, q_i^C \,\mathrm{ds},\ \forall i = N_E^V, \ldots, (\ell+1)N_E^V$.

Let $j \in \{1, \ldots, N_E^V - 1\}$ and let $V^R(E; j) \subset V_\ell^M(E)$ be

$$V^R(E; j) := \{\boldsymbol{v} \in V_\ell^M(E) : D_i(\boldsymbol{v}) = 0 \ \forall i = 1, \ldots, N_E^V - 1, \ i \neq j\}. \quad (37)$$

Notice that $\dim V^R(E; j) = \dim V_\ell^M(E) - (N_E^V - 1) + 1$. Moreover, we define $V^{\perp \mathbb{P}_\ell}(E) \subset V_\ell^M(E)$, given by

$$V^{\perp \mathbb{P}_\ell}(E) := \{\boldsymbol{v} \in V_\ell^M(E) \colon \mathrm{dof}(\boldsymbol{v}) \cdot \mathrm{dof}(\boldsymbol{p}) = 0 \, \forall \boldsymbol{p} \in [\mathbb{P}_\ell(E)]^2 \setminus \mathcal{P}_\ell^{\ker}(E)\} \quad (38)$$

where $\mathrm{dof}(\boldsymbol{v})$ denotes the vector of degrees of freedom of $\boldsymbol{v} \in V_\ell^M(E)$. Notice that $\dim V^{\perp \mathbb{P}_\ell}(E) = \dim V_\ell^M(E) - \big((\ell+1)(\ell+2) - \dim \mathcal{P}_\ell^{\ker}(E)\big)$. Since $(\ell+1)(\ell+2) - \dim \mathcal{P}_\ell^{\ker}(E) \geq N_E^V - 1$, then $\dim V^R(E; j) > \dim V^{\perp \mathbb{P}_\ell}(E)$ and thus

$$\exists \boldsymbol{w}_j \in V^R(E; j) \cap \left([\mathbb{P}_\ell(E)]^2 \setminus \mathcal{P}_\ell^{\ker}(E)\right), \ \boldsymbol{w}_j \neq \boldsymbol{0}.$$

Then we can choose $\boldsymbol{\pi}_j = \boldsymbol{w}_j$ such that $D_j(\boldsymbol{w}_j) = 1$, this is possible since $D_j(\boldsymbol{w}_j)$ cannot be zero. Indeed, by contradiction let us suppose that $D_j(\boldsymbol{w}_j) = 0$, then, by definition of $\mathcal{P}_\ell^{\ker}(E)$ (13), $\boldsymbol{w}_j \in \mathcal{P}_\ell^{\ker}(E)$. This is a contradiction since $\boldsymbol{w}_j \in [\mathbb{P}_\ell(E)]^2 \setminus \mathcal{P}_\ell^{\ker}(E)$ and $\boldsymbol{w}_j \neq \boldsymbol{0}$. □

In the following proposition we provide a definition of Π_E and prove (28) and (29).

Proposition 4. *Under the hypothesis of Theorem 1, let us define* $\Pi_E : V(E) \to [\mathbb{P}_\ell(E)]^2$ *such that* $\forall \boldsymbol{v} \in V(E)$

$$\Pi_E \boldsymbol{v} := \sum_{i=1}^{N_E^V - 1} D_i(\boldsymbol{v}) \boldsymbol{\pi}_i \,,$$

where $\boldsymbol{\pi}_i$ *satisfies* (36)*. Then* Π_E *satisfies* (28) *and* (29)*.*

Proof. Since

$$\forall \boldsymbol{v} \in V(E), \quad D_i(\Pi_E \boldsymbol{v}) = D_i(\boldsymbol{v}) \ \ \forall i = 1, \ldots, N_E^V - 1, \quad (39)$$

let us check that Π_E satisfies (28), indeed by construction $\forall r_i \in \mathcal{R}_\mathcal{Q}(E)$, $i = 1, \ldots, N_E^V - 1$, $\forall \boldsymbol{v} \in V(E)$:

$$b(r_i, \Pi_E \boldsymbol{v} - \boldsymbol{v}) = D_i(\Pi_E \boldsymbol{v} - \boldsymbol{v}) = 0.$$

Furthermore, let us consider $\widehat{\Pi_E \boldsymbol{v}} = \Pi_E \boldsymbol{v} \circ F$ defined on the reference polygon $\hat{E}$. Applying the linearity of the definition of the mapping $F : \hat{E} \to E$, presented in (18), we have

$$\widehat{\Pi_E \boldsymbol{v}} = \left(\sum_{i=1}^{N_E^V - 1} D_i(\boldsymbol{v}) \boldsymbol{\pi}_i \right) \circ F = \sum_{i=1}^{N_E^V - 1} D_i(\boldsymbol{v}) (\boldsymbol{\pi}_i \circ F) = \sum_{i=1}^{N_E^V - 1} D_i(\boldsymbol{v}) \hat{\boldsymbol{\pi}}_i \,. \tag{40}$$

Then, applying Lemma 4, we have $\forall i = 1, \ldots, N_E^V - 1$

$$|D_i(\boldsymbol{v})| = h_E \, |b(\hat{r}_i, \hat{\boldsymbol{v}})| \leq C_b h_E \, \|\hat{r}_i\|_{\mathrm{H}^1_{\mathcal{T}}(\hat{E})} \, \|\hat{\boldsymbol{v}}\|_{V(\hat{E})} \,. \tag{41}$$

Then, we want to prove the continuity of $\Pi_E \boldsymbol{v}$. Since

$$\Pi_E \boldsymbol{v} \in [\mathbb{P}_\ell(E)]^2 \implies \Pi_E \boldsymbol{v} \in C^0(E) \implies \left\| [\![\Pi_E \boldsymbol{v}]\!]_{\mathcal{I}_E} \right\|_{\mathrm{L}^\infty(\mathcal{I}_E)} = 0,$$

applying (40) and (41), we have

$$\begin{aligned}
\|\Pi_E \boldsymbol{v}\|^2_{V(E)} &= \|\Pi_E \boldsymbol{v}\|^2_{[\mathrm{L}^2(E)]^2} + \|\nabla \cdot \Pi_E \boldsymbol{v}\|^2_{\mathrm{L}^2(E)} \\
&= h_E^2 \left\| \sum_{i=1}^{N_E^V - 1} D_i(\boldsymbol{v}) \hat{\boldsymbol{\pi}}_i \right\|^2_{\mathrm{L}^2(\hat{E})} + \left\| \hat{\nabla} \cdot \left(\sum_{i=1}^{N_E^V - 1} D_i(\boldsymbol{v}) \hat{\boldsymbol{\pi}}_i \right) \right\|^2_{\mathrm{L}^2(\hat{E})} \\
&\leq C \sum_{i=1}^{N_E^V - 1} |D_i(\boldsymbol{v})|^2 \left(h_E^2 \, \|\hat{\boldsymbol{\pi}}_i\|^2_{[\mathrm{L}^2(\hat{E})]^2} + \left\| \hat{\nabla} \cdot \hat{\boldsymbol{\pi}}_i \right\|^2_{\mathrm{L}^2(\hat{E})} \right) \\
&\leq C N^V_{\max} \max_{i=1,\ldots,N_E^V - 1} \left\{ \|\hat{r}_i\|^2_{\mathrm{H}^1_{\mathcal{T}}(\hat{E})} \, \|\hat{\boldsymbol{\pi}}_i\|^2_{V(\hat{E})} \right\} h_E^2 \, \|\hat{\boldsymbol{v}}\|_{V(\hat{E})} \,.
\end{aligned} \tag{42}$$

We set $C(\hat{E}) := \max_i \|\hat{r}_i\|_{\mathrm{H}^1_{\mathcal{T}}(\hat{E})} \max_i \|\widehat{\boldsymbol{\pi}}_i\|_{V(\hat{E})}$. This is a continuous function on the set of admissible reference elements Σ, which is a compact set by Lemma 3. Indeed, $\|\hat{r}_i\|_{\mathrm{H}^1_{\mathcal{T}}(\hat{E})}$ is a continuous function $\forall i = 1, \ldots, N_E^V - 1$ on Σ. Moreover, by definition, $\widehat{\boldsymbol{\pi}}_i$ depends continuously on the set $\{\hat{r}_i\}_{i=1}^{N_E^V - 1}$. Then there exists $M = \max_{\hat{E} \in \Sigma} C(\hat{E}) > 0$. Finally, starting from (42), it results that $\exists C > 0$ such that

$$
\begin{aligned}
\left\|\Pi_E \boldsymbol{v}\right\|_{V(E)}^2 &\leq C h_E^2 \left\|\hat{\boldsymbol{v}}\right\|_{V\left(\hat{E}\right)} \\
&\leq C \left(h_E^2 \sum_{\hat{\tau} \in \mathcal{T}_{\hat{E}}} \left\|\hat{\boldsymbol{v}}\right\|_{[\mathrm{L}^2(\hat{\tau})]^2}^2 + \sum_{\hat{\tau} \in \mathcal{T}_{\hat{E}}} \left\|\hat{\nabla} \cdot \hat{\boldsymbol{v}}\right\|_{\mathrm{L}^2(\hat{\tau})}^2 + h_E^2 \left\| [\![\hat{\boldsymbol{v}}]\!]_{\mathcal{I}_{\hat{E}}} \right\|_{\mathrm{L}^\infty\left(\mathcal{I}_{\hat{E}}\right)}^2 \right) \\
&= C \left\|\boldsymbol{v}\right\|_{V(E)}^2 .
\end{aligned}
\tag{43}
$$

□

4.3. Numerical evaluation of the sufficient degree of projection

Algorithm 1 Algorithm for the computation of $\ell(E)$ on a given polygon

Input: A polygon $E \in \mathcal{M}_h$

Let $\ell(E)$ be the smallest number satisfying $(\ell(E)+1)(\ell(E)+2) \geq N_E^V - 1$

Compute the matrix B such that $B_{ij} = \left(\nabla \varphi_j, \boldsymbol{m}_i\right)_E$ $\forall \boldsymbol{m}_i \in [\widehat{\mathsf{M}}_{\Sigma,\ell(E)}(E)]^2$

Perform a QR decomposition of $B^\intercal$:

$$B^\intercal = QR$$

with $Q \in \mathbb{R}^{N_E^V \times (\ell(E)+1)(\ell(E)+2)}$ and $R \in \mathbb{R}^{(\ell(E)+1)(\ell(E)+2) \times (\ell(E)+1)(\ell(E)+2)}$

$N \leftarrow$ number of diagonal elements of R whose absolute value is $\geq$ 1e$-$12

while $N < N_E^V - 1$ **do**

  $\ell(E) \leftarrow \ell(E) + 1$

  Compute $\hat{B}$ such that $\hat{B}_{ij} = \left(\nabla \varphi_j, \boldsymbol{m}_i\right)_E$ $\forall \boldsymbol{m}_i \in [\widehat{\mathsf{M}}_{\ell(E)}(E)]^2$

  Perform a QR decomposition of $\hat{B}^\intercal - QQ^\intercal \hat{B}^\intercal$:

$$\hat{B}^\intercal - QQ^\intercal \hat{B}^\intercal = \hat{Q}\hat{R}$$

  with $\hat{Q} \in \mathbb{R}^{N_E^V \times (\ell(E)+1)}$ and $\hat{R} \in \mathbb{R}^{(\ell(E)+1) \times (\ell(E)+1)}$

  $B^\intercal \leftarrow \begin{bmatrix} B^\intercal & \hat{B}^\intercal \end{bmatrix}$

  $R \leftarrow \begin{bmatrix} R & Q^\intercal \hat{B}^\intercal \\ 0 & \hat{R} \end{bmatrix}$

  $Q \leftarrow \begin{bmatrix} Q & \hat{Q} \end{bmatrix}$

  $N \leftarrow$ number of diagonal elements of R whose absolute value is $\geq$ 1e $-$ 12

end while

return $\ell(E)$, B

In this section, we describe a way to compute the minimum $\ell(E)$ that satisfies (14) for a generic polygon $E \in \mathcal{M}_h$. Let us start considering the construction of $\Pi^{0,E}_{\ell(E)}\nabla$. The computation of the matrix representing the gradient projection follows standard VEM practice (see [28]). Let $\{\varphi_j, \quad j = 1,\dots,N_E^V\}$ be a basis of $\mathcal{V}^E_{1,\ell(E)}$ and let

$$\widehat{\mathsf{M}}_p\left(E\right) := \left\{\frac{(x-x_E)^{\alpha_1}(y-y_E)^{\alpha_2}}{h_E^{\alpha_1+\alpha_2+1}},\ \text{with}\ p=\alpha_1+\alpha_2\right\}$$

be the set of homogeneus scaled monomials of degree p and $[\widehat{\mathsf{M}}_p\left(E\right)]^2 := \left\{\begin{pmatrix} m\\ 0\end{pmatrix},\ m \in \widehat{\mathsf{M}}_p\left(E\right)\right\} \cup \left\{\begin{pmatrix} 0\\ m\end{pmatrix},\ m \in \widehat{\mathsf{M}}_p\left(E\right)\right\}$. We consider the scaled monomial basis $[\widehat{\mathsf{M}}_{\Sigma,\ell(E)}\left(E\right)]^2 := \{\boldsymbol{m}_k, \quad k = 1,\dots,(\ell(E)+1)(\ell(E)+2)\}$ of $\left[\mathbb{P}_{\ell(E)}(E)\right]^2$ given by the direct sum of $[\widehat{\mathsf{M}}_p\left(E\right)]^2$ with $0 \le p \le \ell(E)$. Since $\Pi^{0,E}_{\ell(E)}\nabla\varphi_j \in \left[\mathbb{P}_{\ell(E)}(E)\right]^2$, we have

$$\Pi^{0,E}_{\ell(E)}\nabla\varphi_j = \sum_{k=1}^{(\ell(E)+1)(\ell(E)+2)} \pi_{kj}\boldsymbol{m}_k\,, \quad \forall i = 1,\dots,N_E^V\,.$$

It is then easy to check that the matrix $\hat{\Pi}$ collecting the coefficients π_{kj} is obtained by solving the matrix system

$$G\hat{\Pi} = B\,, \tag{44}$$

where $G_{ik} = \left(\boldsymbol{m}_i, \boldsymbol{m}_k\right)_E$ is symmetric and positive definite and $B_{ij} = \left(\nabla\varphi_j, \boldsymbol{m}_i\right)_E$. Since $\dim \mathcal{V}^E_{1,\ell(E)} = N_E^V$, and thus $\dim \nabla\mathcal{V}^E_{1,\ell(E)} = N_E^V - 1$, then $\Pi^{0,E}_{\ell(E)}\nabla\colon \nabla\mathcal{V}^E_{1,\ell(E)} \to \left[\mathbb{P}_{\ell(E)}(E)\right]^2$ is injective if and only if the dimension of its range is $N_E^V - 1$. This implies that the desired rank of $\hat{\Pi}$ is $N_E^V - 1$ and, since G is non-singular, this is guaranteed if the rank of B is also $N_E^V - 1$. In order to determine for each polygon E the minimum $\ell(E)$ providing numerically the coercivity, we apply Algorithm 1. We first set $\ell(E)$ equal to the necessary condition of the injectivity for the projector $\Pi^{0,E}_{\ell(E)}\nabla$, i.e. (14) with $\dim \mathcal{P}^{\ker}_{\ell(E)}(E)$ set to zero. Then, we start by computing the corresponding matrix B. We perform a QR decomposition of $B^\intercal$: $B^\intercal = QR$, with the matrix Q of dimension $N_E^V \times (\ell(E)+1)(\ell(E)+2)$ and the matrix R of dimension $(\ell(E)+1)(\ell(E)+2) \times (\ell(E)+1)(\ell(E)+2)$. We evaluate if the number of non-zero elements of the diagonal of the matrix R is equal to the dimension of the space of gradients of VEM functions, i.e. $N_E^V - 1$. If not, we increase $\ell(E)$ until we satisfy the condition. Notice that the QR decomposition is updated incrementally at each iteration and that the additional cost of performing Algorithm 1 with respect to knowing $\ell(E)$ in advance is the QR decomposition of a matrix of dimension $\dim \left[\mathbb{P}_{\ell(E)}(E)\right]^2 \times N_E^V$. Once we have the value of $\ell(E)$ and the corresponding matrix B, we compute the matrix G and solve (44). The numerical robustness of this procedure with respect to h_E is guaranteed by the choice of the polynomial basis, that is such that both G and B are invariant with respect to rescalings of the polygon.

Remark 7. *In the implementation of Algorithm 1, we suggest the Householder QR decomposition at line 3 and the application of Givens rotations or modified Gram-Schmidt with renormalization at line 8.*

4.4. Coercivity of the discrete bilinear form

In this section we prove the coercivity of the discrete problem defined by (11) with respect to the standard $\mathrm{H}_0^1(\Omega)$ norm, denoted by

$$\|V\|_{\mathrm{H}_0^1(\Omega)} = \|\nabla V\|_{[\mathrm{L}^2(\Omega)]^2} \quad \forall V \in \mathrm{H}_0^1(\Omega)\,.$$

Let

$$\|v\|_{\boldsymbol{\ell}} := \left(\sum_{E \in \mathcal{M}_h} \left\| \Pi^{0,E}_{\ell(E)} \nabla v \right\|^2_{[\mathrm{L}^2(E)]^2} \right)^{\frac{1}{2}} \quad \forall v \in \mathcal{V}_{1,\boldsymbol{\ell}}\,.$$

We have the following result.

Proposition 5. *Suppose $\ell(E)$ satisfies* (14) $\forall E \in \mathcal{M}_h$. *Then, $\|\cdot\|_{\boldsymbol{\ell}}$ is a norm on $\mathcal{V}_{1,\boldsymbol{\ell}}$.*

Proof. Let $v \in \mathcal{V}_{1,\boldsymbol{\ell}}$ be given. It is clear from its definition that $\|v\|_{\boldsymbol{\ell}}$ is a semi-norm. Applying Theorem 1 and since $v \in \mathrm{H}_0^1(\Omega)$, we have that

$$\|v\|_{\boldsymbol{\ell}} = 0 \implies \|v\|_{\mathrm{H}_0^1(\Omega)} = 0 \implies v = 0\,.$$

□

Lemma 6. *We have that*

$$\|v\|_{\boldsymbol{\ell}} \leq \|v\|_{\mathrm{H}_0^1(\Omega)} \quad \forall v \in \mathcal{V}_{1,\boldsymbol{\ell}}\,. \tag{45}$$

Moreover, if $\ell(E)$ satisfies (14) $\forall E \in \mathcal{M}_h$*, then*

$$\exists c_* > 0\colon\ \|v\|_{\boldsymbol{\ell}} \geq c_* \|v\|_{\mathrm{H}^1_0(\Omega)} \quad \forall v \in \mathcal{V}_{1,\boldsymbol{\ell}}\,, \tag{46}$$

where c_* does not depend on h.

Proof. Relation (45) follows immediately by the definition of $\Pi^{0,E}_{\ell(E)}$ and an application of the Cauchy-Schwarz inequality. Indeed, let $E \in \mathcal{M}_h$, then

$$\left\|\Pi^{0,E}_{\ell(E)} \nabla v\right\|^2_E = \left(\Pi^{0,E}_{\ell(E)} \nabla v, \Pi^{0,E}_{\ell(E)} \nabla v\right)_E = \left(\nabla v, \Pi^{0,E}_{\ell(E)} \nabla v\right)_E \leq \|\nabla v\|_{[\mathrm{L}^2(E)]^2} \left\|\Pi^{0,E}_{\ell(E)} \nabla v\right\|_{[\mathrm{L}^2(E)]^2}.$$

On the other hand, by standard scaling arguments we have

$$\|v\|^2_{\boldsymbol{\ell}} = \sum_{E \in \mathcal{M}_h} \left\|\Pi^{0,E}_{\ell(E)} \nabla v\right\|^2_{[\mathrm{L}^2(E)]^2} = \sum_{E \in \mathcal{M}_h} \left\|\hat{\Pi}^{0,\hat{E}}_{\ell(E)} \hat{\nabla}\left(\hat{v} - P_0(\hat{v})\right)\right\|^2_{\left[\mathrm{L}^2\left(\hat{E}\right)\right]^2}.$$

Notice that $\forall \hat{E} \in \Sigma$, where Σ is the set of admissible reference elements, $\hat{v} - \mathrm{P}_0(\hat{v}) \in \mathcal{V}^{\hat{E},\mathrm{P}_0}_{1,\ell(E)}$. Moreover, $\forall \hat{w} \in \mathcal{V}^{\hat{E},\mathrm{P}_0}_{1,\ell(E)}$ both $\left\|\hat{\Pi}^{0,\hat{E}}_{\ell(E)} \hat{\nabla} \hat{w}\right\|_{\left[\mathrm{L}^2\left(\hat{E}\right)\right]^2}$ and $\left\|\hat{\nabla} \hat{w}\right\|_{\left[\mathrm{L}^2\left(\hat{E}\right)\right]^2}$ are norms. Then, by standard arguments about the equivalence of norms on finite dimensional spaces, we obtain $\forall \hat{E} \in \Sigma$

$$\left\|\hat{\Pi}^{0,\hat{E}}_{\ell(E)} \hat{\nabla} \hat{w}\right\|_{\left[\mathrm{L}^2\left(\hat{E}\right)\right]^2} \geq C(\hat{E}) \left\|\hat{\nabla} \hat{w}\right\|_{\left[\mathrm{L}^2\left(\hat{E}\right)\right]^2} \tag{47}$$

where

$$C(\hat{E}) = \frac{\min_{\hat{z} \in \mathcal{V}^{\hat{E},\mathrm{P}_0}_{1,\ell(E)} : \|\mathrm{dof}\,(\hat{z})\|_{l^2} = 1} \left\|\hat{\Pi}^{0,\hat{E}}_{\ell(E)} \hat{\nabla} \hat{z}\right\|_{\left[\mathrm{L}^2\left(\hat{E}\right)\right]^2}}{\sqrt{N^V_E - 1}\, \max_{i=1,\ldots,N^V_E - 1} \left\|\hat{\nabla} \hat{\psi}_i\right\|_{\left[\mathrm{L}^2\left(\hat{E}\right)\right]^2}}\,. \tag{48}$$

$C(\hat{E})$ is a continuous function on Σ, which is a compact set by Lemma 3. Indeed, $\hat{\Pi}^{0,\hat{E}}_{\ell(E)}$ is continuous on Σ, as well as functions in $\mathcal{V}^{\hat{E},\mathrm{P}_0}_{1,\ell(E)}$ following proofs of [25, Lemma 4.9] and [29, Lemma 4.5]. Moreover, $C(\hat{E}) > 0$, $\forall \hat{E} \in \Sigma$. Indeed, applying Proposition 2, it holds that $\forall \hat{z} \in \mathcal{V}^{\hat{E},\mathrm{P}_0}_{1,\ell(E)} : \|\mathrm{dof}\,(\hat{z})\|_{l^2} = 1$,

$$\begin{aligned}\left\|\hat{\Pi}^{0,\hat{E}}_{\ell(E)} \hat{\nabla} \hat{z}\right\|^2_{\left[\mathrm{L}^2\left(\hat{E}\right)\right]^2} &= \left(\hat{\nabla} \hat{z}, \hat{\Pi}^{0,\hat{E}}_{\ell(E)} \hat{\nabla} \hat{z}\right)_{\hat{E}} = \left(\hat{z}, \hat{\Pi}^{0,\hat{E}}_{\ell(E)} \hat{\nabla} \hat{z} \cdot \boldsymbol{n}^{\partial \hat{E}}\right)_{\partial \hat{E}} = b(\hat{z}_R, \hat{\Pi}^{0,\hat{E}}_{\ell(E)} \hat{\nabla} \hat{z}) \\ &\geq \beta \left\|\hat{\Pi}^{0,\hat{E}}_{\ell(E)} \hat{\nabla} \hat{z}\right\|_{\left[\mathrm{L}^2\left(\hat{E}\right)\right]^2} \|\hat{z}_R\|_{\mathrm{H}^1_{\mathcal{T}}(\hat{E})} > 0\,,\end{aligned}$$

where $\hat{z}_R$ is the lifting of $\gamma^{\partial \hat{E}}(\hat{z})$ on $\mathcal{R}_{\mathcal{Q}}(\hat{E})$. Then, $\exists m > 0$ such that $m := \min_{\hat{E}\in\Sigma} C(\hat{E})$. Finally, by standard scaling argument we obtain

$$\|v\|_{\boldsymbol{\ell}}^2 \geq m^2 \sum_{E\in\mathcal{M}_h} \left\|\hat{\nabla}\left(\hat{v} - \mathrm{P}_0(\hat{v})\right)\right\|^2_{\left[\mathrm{L}^2\left(\hat{E}\right)\right]^2} = m^2 \|v\|_{\mathrm{H}^1_0(\Omega)} . \tag{49}$$

□

In the following theorem, we provide a proof of the continuity and the coercivity of the discrete bilinear form. The coercivity property follows from Lemma 6.

Theorem 2. *Let a_h be the bilinear form defined by* (10)*. Then,*

$$a_h(w, v) \leq \|w\|_{\mathrm{H}^1_0(\Omega)} \|v\|_{\mathrm{H}^1_0(\Omega)} \, \forall w, v \in \mathcal{V}_{1,\boldsymbol{\ell}} . \tag{50}$$

Moreover, suppose $\ell(E)$ satisfies (14) *$\forall E \in \mathcal{M}_h$. Then,*

$$\exists C > 0, \text{ independent of } h\colon a_h(w, w) \geq C \|w\|^2_{\mathrm{H}^1_0(\Omega)} \, \forall w \in \mathcal{V}_{1,\boldsymbol{\ell}} . \tag{51}$$

Proof. Let $w, v \in \mathcal{V}_{1,\boldsymbol{\ell}}$ be given. Applying the Cauchy-Schwarz inequality and (45) we get

$$\begin{aligned} a_h(w, v) &= \sum_{E\in\mathcal{M}_h} \left(\Pi^{0,E}_{\ell(E)} \nabla w, \Pi^{0,E}_{\ell(E)} \nabla v\right)_E \\ &\leq \sum_{E\in\mathcal{M}_h} \left\|\Pi^{0,E}_{\ell(E)} \nabla w\right\|_{[\mathrm{L}^2(E)]^2} \left\|\Pi^{0,E}_{\ell(E)} \nabla v\right\|_{[\mathrm{L}^2(E)]^2} \\ &\leq \|w\|_{\boldsymbol{\ell}} \|v\|_{\boldsymbol{\ell}} \leq \|w\|_{\mathrm{H}^1_0(\Omega)} \|v\|_{\mathrm{H}^1_0(\Omega)} . \end{aligned}$$

Moreover, assuming that $\ell(E)$ satisfies (14) $\forall E \in \mathcal{M}_h$, we can apply the lower bound in (46) and get

$$a_h(w, w) = \|w\|^2_{\boldsymbol{\ell}} \geq (c_*)^2 \|w\|^2_{\mathrm{H}^1_0(\Omega)} .$$

□

This theorem implies that the bilinear form a_h of the problem (11) satisfies the hypothesis of the Lax-Milgram theorem, hence the problem admits a unique solution.

5. A priori error estimates

In this section we derive error estimates for the proposed method, in H_0^1 norm and in the standard L^2 norm. First, we recall classical results for Virtual Element Methods concerning the interpolation error and the polynomial projection error (see [8, 2]).

Lemma 7. *Let $U \in \mathrm{H}^2(\Omega)$, then there exists $C > 0$ such that $\forall\, h$, $\exists U_\mathrm{I} \in \mathcal{V}_{1,\boldsymbol{\ell}}$ satisfying*

$$\left\|U - U_\mathrm{I}\right\|_{\mathrm{L}^2(\Omega)} + h \left\|U - U_\mathrm{I}\right\|_{\mathrm{H}_0^1(\Omega)} \leq C h^2 \left|U\right|_2 . \tag{52}$$

Proof. The proof of this result is detailed in [20], it follows a similar approach as the one in [8]. □

Lemma 8. *Let $U \in \mathrm{H}^2(\Omega)$, then there exist $C_1, C_2 > 0$ such that*

$$\left\|\Pi_{\boldsymbol{\ell}}^0 \nabla U - \nabla U\right\|_{\mathrm{L}^2(\Omega)} \leq C_1 h \left|U\right|_2 , \tag{53}$$

$$\left\|\Pi_0^0 U - U\right\|_{\mathrm{L}^2(\Omega)} \leq C_2 h \left\|U\right\|_{\mathrm{H}_0^1(\Omega)} . \tag{54}$$

Theorem 3. *Let $U \in \mathrm{H}^2(\Omega) \cap \mathrm{H}_0^1(\Omega)$ and $f \in \mathrm{L}^2(\Omega)$ be the solution and the right-hand side of (3), respectively. Then, $\exists C > 0$ such that the unique solution $u \in \mathcal{V}_{1,\boldsymbol{\ell}}$ to problem (11) satisfies the following error estimate:*

$$\left\|U - u\right\|_{\mathrm{H}_0^1(\Omega)} \leq C h \left(\left|U\right|_2 + \left\|f\right\|_{\mathrm{L}^2(\Omega)}\right) . \tag{55}$$

Proof. Let U_I be given by Lemma 7. Applying the triangle inequality, we have

$$\left\|U - u\right\|_{\mathrm{H}_0^1(\Omega)} \leq \left\|U - U_\mathrm{I}\right\|_{\mathrm{H}_0^1(\Omega)} + \left\|U_\mathrm{I} - u\right\|_{\mathrm{H}_0^1(\Omega)} . \tag{56}$$

We deal with the two terms separately. The first one can be bounded applying (52), i.e.

$$\left\|U - U_\mathrm{I}\right\|_{\mathrm{H}_0^1(\Omega)} \leq C h \left|U\right|_2 . \tag{57}$$

On the other hand, in order to deal with the second term of (56) let $\varepsilon = U_\mathrm{I} - u$. First, applying the coercivity of the bilinear form a_h (51) and the discrete problem (11), we have that $\exists C > 0$:

$$C \left\|\varepsilon\right\|_{\mathrm{H}_0^1(\Omega)}^2 \leq a_h\left(\varepsilon, \varepsilon\right) = a_h\left(U_\mathrm{I}, \varepsilon\right) - a_h\left(u, \varepsilon\right) = a_h\left(U_\mathrm{I}, \varepsilon\right) - \sum_{E \in \mathcal{M}_h} \left(f, \Pi_0^{0,E} \varepsilon\right)_E . \tag{58}$$

Applying the definition of the L^2 projectors and adding and subtracting terms, i.e. $\Pi_{\ell(E)}^{0,E}\nabla U$ and ∇U, we have

$$\begin{aligned}
a_h(\varepsilon,\varepsilon) &= a_h(U_\mathrm{I}-U,\varepsilon)+a_h(U,\varepsilon)-\sum_{E\in\mathcal{M}_h}\left(\Pi_0^{0,E}f,\varepsilon\right)_E\\
&= a_h(U_\mathrm{I}-U,\varepsilon)+\sum_{E\in\mathcal{M}_h}\left(\Pi_{\ell(E)}^{0,E}\nabla U-\nabla U,\nabla\varepsilon\right)_E+(\nabla U,\nabla\varepsilon)_E-\left(\Pi_0^{0,E}f,\varepsilon\right)_E\\
&= a_h(U_\mathrm{I}-U,\varepsilon)+\sum_{E\in\mathcal{M}_h}\left(\Pi_{\ell(E)}^{0,E}\nabla U-\nabla U,\nabla\varepsilon\right)_E+\left(f-\Pi_0^{0,E}f,\varepsilon\right)_E.
\end{aligned}$$

Let us consider the last three terms separately. The first one can be bounded applying (50) and (52), i.e.

$$a_h(U_\mathrm{I}-U,\varepsilon)\le C\left\|U_\mathrm{I}-U\right\|_{\mathrm{H}_0^1(\Omega)}\left\|\varepsilon\right\|_{\mathrm{H}_0^1(\Omega)}\le Ch\left|U\right|_2\left\|\varepsilon\right\|_{\mathrm{H}_0^1(\Omega)}. \tag{59}$$

Applying the Cauchy-Schwarz inequality and (53), the second term can be bounded as follows:

$$\begin{aligned}
\sum_{E\in\mathcal{M}_h}\left(\Pi_{\ell(E)}^{0,E}\nabla U-\nabla U,\nabla\varepsilon\right)_E &\le \sum_{E\in\mathcal{M}_h}\left\|\Pi_{\ell(E)}^{0,E}\nabla U-\nabla U\right\|_{\mathrm{L}^2(E)}\left\|\varepsilon\right\|_{\mathrm{H}_0^1(E)}\\
&\le Ch\left|U\right|_2\left\|\varepsilon\right\|_{\mathrm{H}_0^1(\Omega)}.
\end{aligned}\tag{60}$$

The last term can be bounded applying the definition of $\Pi_0^{0,E}$, the Cauchy-Schwarz inequality and (54), i.e.

$$\begin{aligned}
\sum_{E\in\mathcal{M}_h}\left(f-\Pi_0^{0,E}f,\varepsilon\right)_E &= \sum_{E\in\mathcal{M}_h}\left(f,\varepsilon-\Pi_0^{0,E}\varepsilon\right)_E\\
&\le \sum_{E\in\mathcal{M}_h}\left\|f\right\|_{\mathrm{L}^2(E)}\left\|\varepsilon-\Pi_0^{0,E}\varepsilon\right\|_{\mathrm{L}^2(E)}\le Ch\left\|f\right\|_{\mathrm{L}^2(\Omega)}\left\|\varepsilon\right\|_{\mathrm{H}_0^1(\Omega)}.
\end{aligned}\tag{61}$$

Finally, applying together (59), (60) and (61) into (58) and simplifying, we have

$$\left\|\varepsilon\right\|_{\mathrm{H}_0^1(\Omega)}\le Ch\left(\left|U\right|_2+\left\|f\right\|_{\mathrm{L}^2(\Omega)}\right). \tag{62}$$

Considering together (57) and (62) we prove (55). □

Theorem 4. *Let Ω be convex. Let $U\in\mathrm{H}^2(\Omega)\cap\mathrm{H}_0^1(\Omega)$ and $f\in\mathrm{H}^1(\Omega)$ be the solution and the right-hand side of* (3)*,respectively. Then, $\exists C>0$ such* *that the unique solution $u \in \mathcal{V}_{1,\ell}$ to problem* (11) *satisfies the following error estimate:*

$$\|U - u\|_{\mathrm{L}^2(\Omega)} \le C h^2 \left(|U|_2 + \|f\|_{\mathrm{H}^1_0(\Omega)} \right) . \tag{63}$$

Proof. Let us define the auxiliary problem: let $\Psi \in \mathrm{H}^2(\Omega) \cap \mathrm{H}^1_0(\Omega)$ the solution of $a\,(V, \Psi) = (U - u, V)_\Omega \; \forall V \in \mathrm{H}^1_0(\Omega)$. From the definition of Ψ, we get:

$$\exists C > 0 : \quad |\Psi|_2 \le C \, \|U - u\|_{\mathrm{L}^2(\Omega)} , \tag{64}$$

$$\exists C > 0 : \quad \|\Psi\|_{\mathrm{H}^1_0(\Omega)} \le C \, \|U - u\|_{\mathrm{L}^2(\Omega)} . \tag{65}$$

Let us denote by Ψ_I the interpolant of Ψ according to Lemma 7. Applying the auxiliary problem, the discrete problem (11) and the definition of the bilinear form a (2), we have

$$\begin{aligned}
\|U - u\|^2_{\mathrm{L}^2(\Omega)} &= (U - u, U - u)_\Omega = a\,(U - u, \Psi) \\
&= a\,(U, \Psi - \Psi_I) + a\,(U, \Psi_I) - a\,(u, \Psi) \\
&= a\,(U, \Psi - \Psi_I) + (f, \Psi_I)_\Omega - a\,(u, \Psi) \\
&= a\,(U, \Psi - \Psi_I) + (f, \Psi_I)_\Omega - \left(\sum_{E \in \mathcal{M}_h} \left(f, \Pi^{0,E}_0 \Psi_I \right)_E \right) \\
&\qquad + a_h\,(u, \Psi_I) - a\,(u, \Psi) + a\,(u, \Psi_I) - a\,(u, \Psi_I) \\
&= a\,(U - u, \Psi - \Psi_I) + \left(\sum_{E \in \mathcal{M}_h} \left(f, \Psi_I - \Pi^{0,E}_0 \Psi_I \right)_E \right) \\
&\qquad + a_h\,(u, \Psi_I) - a\,(u, \Psi_I) .
\end{aligned} \tag{66}$$

Let us consider the terms of the previous relation separately. First, applying the Cauchy-Schwarz inequality, (52), (54) and (64), we have, for the first term,

$$\begin{aligned}
a\,(U - u, \Psi - \Psi_I) &\le \|U - u\|_{\mathrm{H}^1_0(\Omega)} \|\Psi - \Psi_I\|_{\mathrm{H}^1_0(\Omega)} \\
&\le C h \, \|U - u\|_{\mathrm{H}^1_0(\Omega)} |\Psi|_2 \le C h \, \|U - u\|_{\mathrm{H}^1_0(\Omega)} \|U - u\|_{\mathrm{L}^2(\Omega)} ,
\end{aligned} \tag{67}$$

and, for the second one,

$$\begin{aligned}\sum_{E\in\mathcal{M}_h}\left(f,\Psi_I-\Pi_0^{0,E}\Psi_I\right)_E &= \sum_{E\in\mathcal{M}_h}\left(f-\Pi_0^{0,E}f,\Psi_I-\Pi_0^{0,E}\Psi_I\right)_E \\ &\le \sum_{E\in\mathcal{M}_h}\left\|f-\Pi_0^{0,E}f\right\|_{\mathrm{L}^2(E)}\left\|\Psi_I-\Pi_0^{0,E}\Psi_I\right\|_{\mathrm{L}^2(E)} \\ &\le Ch\left|f\right|_{\mathrm{H}^1(\Omega)}\sum_{E\in\mathcal{M}_h}\left\|\Psi_I-\Pi_0^{0,E}\Psi_I\right\|_{\mathrm{L}^2(E)}. \qquad (68)\end{aligned}$$

Applying the property

$$\forall E\in\mathcal{M}_h,\ \ \left\|\Psi_I-\Pi_0^{0,E}\Psi_I\right\|_{\mathrm{L}^2(E)}\le\left\|\Psi_I-\Pi_0^{0,E}\Psi\right\|_{\mathrm{L}^2(E)},$$

(52) and (54) to (68), we obtain

$$\begin{aligned}\sum_{E\in\mathcal{M}_h}\left(f,\Psi_I-\Pi_0^{0,E}\Psi_I\right)_E &\le Ch\left|f\right|_{\mathrm{H}^1(\Omega)}\sum_{E\in\mathcal{M}_h}\left\|\Psi_I-\Pi_0^{0,E}\Psi\right\|_{\mathrm{L}^2(E)} \\ &\le Ch\left|f\right|_{\mathrm{H}^1(\Omega)}\sum_{E\in\mathcal{M}_h}\left(\left\|\Psi_I-\Psi\right\|_{\mathrm{L}^2(E)}+\left\|\Psi-\Pi_0^{0,E}\Psi\right\|_{\mathrm{L}^2(E)}\right) \\ &\le Ch\left|f\right|_{\mathrm{H}^1(\Omega)}\left(h^2\left|\Psi\right|_2+h\left\|\Psi\right\|_{\mathrm{H}^1_0(\Omega)}\right). \qquad (69)\end{aligned}$$

We can omit higher order terms and apply (65), obtaining

$$\sum_{E\in\mathcal{M}_h}\left(f,\Psi_I-\Pi_0^{0,E}\Psi_I\right)_E\le Ch^2\left|f\right|_{\mathrm{H}^1(\Omega)}\left\|U-u\right\|_{\mathrm{L}^2(\Omega)}. \qquad (70)$$

Finally, we have to bound $a_h\left(u,\Psi_I\right)-a\left(u,\Psi_I\right)$. Then, applying the orthogonality property of $\Pi_{\ell(E)}^{0,E}$, adding and subtracting terms, we have

$$\begin{aligned}a_h\left(u,\Psi_I\right)-a\left(u,\Psi_I\right) &= \sum_{E\in\mathcal{M}_h}\left(\Pi_{\ell(E)}^{0,E}\nabla u,\nabla\Psi_I\right)_E-\left(\nabla u,\nabla\Psi_I\right)_E \\ &= \sum_{E\in\mathcal{M}_h}\left(\Pi_{\ell(E)}^{0,E}\nabla u-\nabla u,\nabla\Psi_I-\Pi_0^{0,E}\nabla\Psi_I\right)_E \\ &= \sum_{E\in\mathcal{M}_h}\left(\Pi_{\ell(E)}^{0,E}\nabla u-\Pi_{\ell(E)}^{0,E}\nabla U,\nabla\Psi_I-\Pi_0^{0,E}\nabla\Psi_I\right)_E \\ &\quad+\left(\Pi_{\ell(E)}^{0,E}\nabla U-\nabla U,\nabla\Psi_I-\Pi_0^{0,E}\nabla\Psi_I\right)_E \\ &\quad+\left(\nabla U-\nabla u,\nabla\Psi_I-\Pi_0^{0,E}\nabla\Psi_I\right)_E. \qquad (71)\end{aligned}$$

Notice that, applying (52) and (53), we have the property $\forall E \in \mathcal{M}_h$:

$$\left\| \nabla \Psi_I - \Pi_0^{0,E} \nabla \Psi_I \right\|_{\mathrm{L}^2(E)} \leq \left\| \nabla \Psi_I - \Pi_0^{0,E} \nabla \Psi \right\|_{\mathrm{L}^2(E)} \leq Ch \left| \Psi \right|_{2,E} .$$

Therefore, applying the continuity of the projection operator and (64), the first and the last term of (71) can be bounded as

$$\sum_{E \in \mathcal{M}_h} \left(\Pi_{\ell(E)}^{0,E} \nabla u - \Pi_{\ell(E)}^{0,E} \nabla U, \nabla \Psi_I - \Pi_0^{0,E} \nabla \Psi_I \right)_E + \left(\nabla U - \nabla u, \nabla \Psi_I - \Pi_0^{0,E} \nabla \Psi_I \right)_E \\ \leq Ch \left\| U - u \right\|_{\mathrm{H}_0^1(\Omega)} \left\| U - u \right\|_{\mathrm{L}^2(\Omega)} . \tag{72}$$

Similarly, the second term is bounded as

$$\sum_{E \in \mathcal{M}_h} \left(\Pi_{\ell(E)}^{0,E} \nabla U - \nabla U, \nabla \Psi_I - \Pi_0^{0,E} \nabla \Psi_I \right)_E \leq Ch^2 \left| U \right|_2 \left\| U - u \right\|_{\mathrm{L}^2(\Omega)} . \tag{73}$$

Finally, applying (67), (70), (72) and (73) to (66) and simplifying, we obtain

$$\left\| U - u \right\|_{\mathrm{L}^2(\Omega)} \leq C \left(h \left\| U - u \right\|_{\mathrm{H}_0^1(\Omega)} + h^2 \left| f \right|_{\mathrm{H}^1(\Omega)} + h^2 \left| U \right|_2 \right) .$$

Applying the H^1-estimate (Theorem 3) we obtain the relation (63). □

Remark 8. *Denoting by* $\Pi_1^{0,E}$ *the* L^2*-projector from* $\mathrm{L}^2(E)$ *to* $\mathbb{P}_1(E)$*, we can define the discrete problem* (11) *as*

$$a_h(u, v) = \sum_{E \in \mathcal{M}_h} \left(f, \Pi_1^{0,E} v \right)_E \quad \forall v \in \mathcal{V}_{1,\boldsymbol{\ell}} ,$$

and we can require $f \in \mathrm{L}^2(\Omega)$ *so* (63) *still holds as*

$$\left\| U - u \right\|_{\mathrm{L}^2(\Omega)} \leq Ch^2 \left(\left| U \right|_2 + \left\| f \right\|_{\mathrm{L}^2(\Omega)} \right) .$$

Remark 9 (Extension to more general elliptic problems)**.** *Consider the following diffusion-reaction model:*

$$\begin{cases} -\Delta U + U = f & \text{in } \Omega , \\ U = 0 & \text{on } \partial\Omega . \end{cases} \tag{74}$$

Table 1: Sufficient $\ell(E)$ for regular polygons up to 24 edges

N_E^V	3	4, 5	6, 7	8, 9	10, 11	12, 13	14, 15	16, 17	18, 19	20, 21	22, 23	24
$\ell(E)$	0	1	2	3	4	5	6	7	8	9	10	11
$\check{\ell}(N_E^V)$	0	1	1	2	2	2	3	3	3	3	4	4

Table 2: Sufficient $\ell(E)$ for non-regular convex polygons up to 24 edges

N_E^V	3	4, 5	6, 7	8, 9	10, 11	12, 13	14, 15	16, 17	18, 19	20, 21	22, 23	24
$\ell(E)$	0	1	1	2	2	2	3	3	3	3	4	4
$\check{\ell}(N_E^V)$	0	1	1	2	2	2	3	3	3	3	4	4

The coercivity of the bilinear form defined by (9) *and* (10) *allows us to discretize it as: find* $u \in \mathcal{V}_{1,\boldsymbol{\ell}}$ *such that*

$$a_h(u,v) + \sum_{E\in\mathcal{M}_h} \left(\Pi_0^{0,E} u, \Pi_0^{0,E} v\right)_E = \sum_{E\in\mathcal{M}_h} \left(f, \Pi_0^{0,E} v\right)_E \quad \forall v \in \mathcal{V}_{1,\boldsymbol{\ell}}\,. \tag{75}$$

If $\ell(E)$ *satisfies* (14) *locally on each polygon, we can prove the well-posedness of* (75) *following [2, Lemma 5.7]. Optimal order a priori error estimates can be proved as in [2, Theorem 5.1 and 5.2], using the interpolation result given by Lemma 7. In Section 6.2.3 we assess numerically the validity of such results.*

6. Numerical Results

This section is devoted to assess the theoretical results reported previously. First, we consider single polygons and investigate numerically which is the minimum degree $\ell(E)$ providing coercivity, then we carry out some convergence tests.

6.1. Coercivity tests

To test numerically the coercivity of the bilinear form a_h^E, we consider a set of polygons and we perform for each of them Algorithm 1 which returns

Table 3: Sufficient $\ell(E)$ for polygons with aligned edges up to 24 edges (built on the non-regular convex triangle)

N_E^V	3	4, 5	6, 7	8, 9	10, 11	12, 13	14, 15	16, 17	18, 19	20, 21	22, 23	24
$\ell(E)$	0	1	2	2	3	4	4	5	6	6	7	8
$\check{\ell}(N_E^V)$	0	1	1	2	2	2	3	3	3	3	4	4

Table 4: Sufficient $\ell(E)$ for polygons with aligned edges up to 24 edges (built on the non-regular convex hexagon)

N_E^V	7	8, 9	10, 11	12, 13	14, 15	16, 17	18, 19	20, 21	22, 23	24
$\ell(E)$	1	2	2	2	3	3	3	3	4	4
$\check{\ell}(N_E^V)$	1	2	2	2	3	3	3	3	4	4

the minimum $\ell(E)$ that ensures numerically the local coercivity. In view of Theorem 1, we define, for any $E \in \mathcal{M}_h$,

$$\check{\ell}(N_E^V) \text{ as the smallest } l \text{ such that } (l+1)(l+2) \geq N_E^V - 1\,.$$

Notice that Theorem 1 implies that the minimum $\ell(E)$ that is sufficient to obtain local coercivity on E satisfies $\ell(E) \geq \check{\ell}(N_E^V)$. In the following, we compute numerically the minimum $\ell(E)$ that induces the coercivity of the stiffness matrix for several sequences of polygons.

In Table 1 we display $\check{\ell}(N_E^V)$ and the minimum $\ell(E)$ computed by Algorithm 1 for regular polygons of n vertices having vertices $x_i = \left(\cos\left(\frac{(i-1)2\pi}{n}\right) \ \sin\left(\frac{(i-1)2\pi}{n}\right)\right)$, $i \in \{1, \ldots, n\}$. We can see that for these polygons the value of $\ell(E)$ provided by the algorithm corresponds to the one that we obtain if we use harmonic polynomials only (see [30]). This suggests that for regular polygons the proposed method seems to be stable if and only if the projection space contains the gradients of harmonic polynomials.

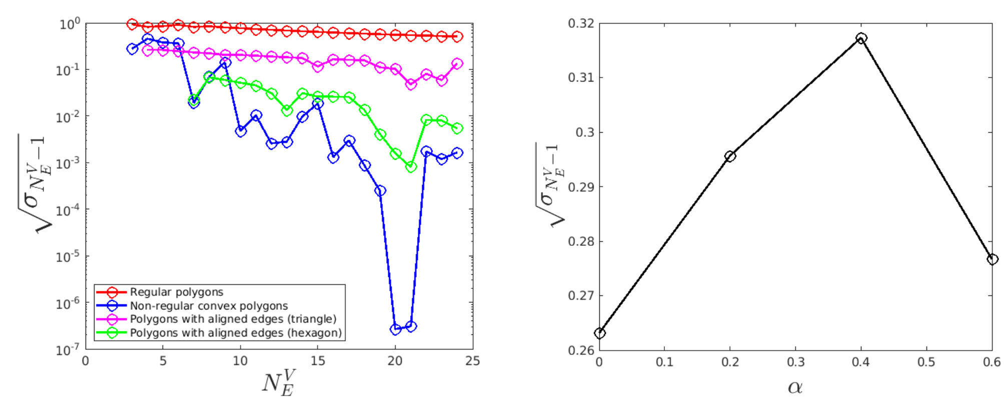
On the other hand, if we consider a sequence of non-regular convex polygons, the results in Table 2 suggest that we can take $\ell(E) = \check{\ell}(N_E^V)$. The vertices of such polygons are generated by sampling random points on a circle of radius 1 and imposing that the ratio of each edge and the diameter of the circle is ≥ 0.1.

A third test considers a sequence of polygons with aligned edges obtained starting from a non-equilateral triangle and then progressively splitting its edges into equal parts one at a time until all three edges are split into eight equal parts, thus generating a sequence of polygons up to 24 edges. In Table 3 we can see how the sufficient $\ell(E)$ that guarantees coercivity in this case is inside the range given by $\check{\ell}(N_E^V)$ and the sufficient $\ell(E)$ obtained for regular polygons.

A similar test is reported in Table 4, where the same procedure has been applied to a non-regular hexagon. We can see that in this case $\check{\ell}(N_E^V)$ is sufficient.

Lastly, we consider a sequence of polygons that are non convex. To generate this sequence, we start from the quadrilateral considered in the second test (Table 2), add the edge midpoints as vertices and move them towards its barycenter x_C with the transformation $S(x) = (1-\alpha)x + \alpha x_C$, thus obtaining a sequence of non-convex octagons. We select four polygons by choosing $\alpha \in \{0, 0.2, 0.4, 0.6\}$, larger is α smaller is the radius of the inscribed ball. In all these cases, the sufficient $\ell(E)$ that guarantees coercivity is $\check{\ell}(8) = 2$.

Finally, for each polygon we compute the $(N_E^V - 1)$-th from largest to smallest eigenvalue of the local stiffness matrix A^E, denoted by $\sigma_{N_E^V-1}$, using the value of $\ell(E)$ provided by Algorithm 1; $\sigma_{N_E^V-1} \neq 0$ ensures the rank of the stiffness matrix be equal to $N_E^V - 1$. In Figure 1, we depicted the



(a) Convex polygons (b) Concave polygons

Figure 1: Values of the $\sqrt{\sigma_{N_E^V-1}}$ for polygons analyzed in Section 6.1.

square root of $\sigma_{N_E^V-1}$ for all polygons considered in this section, these values are a numerical approximation of the local coercivity constant. Notice that the value of $\ell(E)$ can be different for polygons with the same N_E^V, for each polygon $\ell(E)$ is written in Tables 1, 2, 3 and 4.

The coordinates of all polygons considered in this section, except for the regular ones, are provided as supplementary materials to the paper.

6.2. Convergence tests

Let us consider problem (1) on the unit square with homogeneous Dirichlet boundary conditions and the right-hand side defined such that the exact solution is

$$U_{ex} = \sin(2\pi x)\sin(2\pi y).$$

In the following, we show, in log-log scale plots, the convergence curves of the L^2 and H^1 errors that we measure respectively as follows,

$$\mathrm{L}^2 \text{ error } = \sqrt{\sum_{E\in\mathcal{M}_h} \left\|\Pi_1^{\nabla,E} u - U_{ex}\right\|^2_{\mathrm{L}^2(E)}},$$

$$\mathrm{H}^1 \text{ error } = \sqrt{\sum_{E\in\mathcal{M}_h} \left\|\nabla\Pi_1^{\nabla,E} u - \nabla U_{ex}\right\|^2_{\mathrm{L}^2(E)}},$$

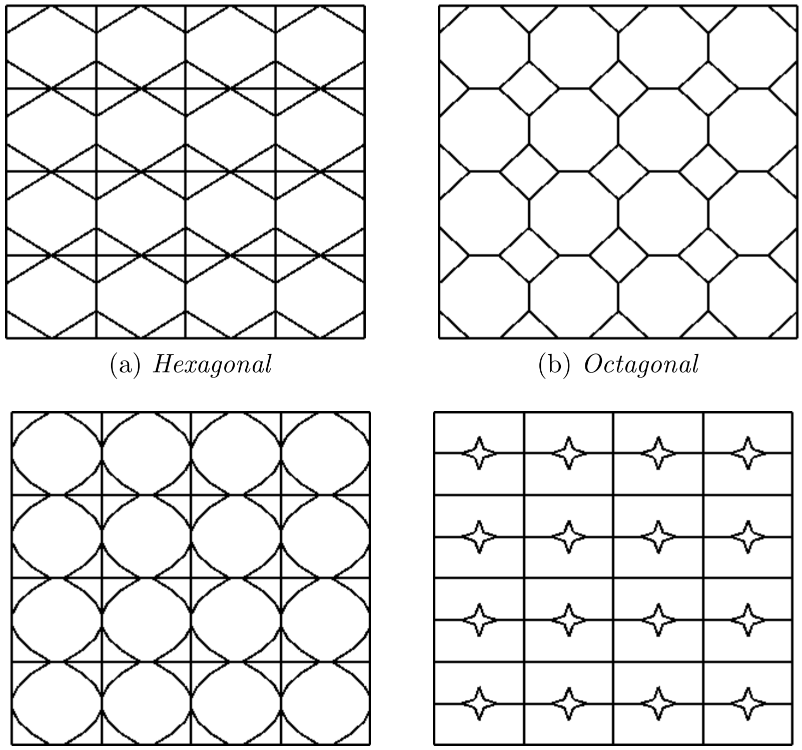
where u is the discrete solution of (11). Then, for each polygon $E \in \mathcal{M}_h$ we choose $\ell(E)$ such that the sufficient condition (14) is satisfied, as detailed below.

6.2.1. Meshes

We consider four sequences of meshes for the convergence test. The first sequence, labeled *Hexagonal*, is a tesselation made by hexagons and triangles, as it is shown in Figure 2a. For this mesh, Algorithm 1 $\ell(E) = 0$ on triangles and $\ell(E) = 2$ on hexagons. The second sequence, shown in Figure 2b and labeled *Octagonal*, is made by octagons, squares and triangles. It results $\ell(E) = 0$ on triangles, $\ell(E) = 1$ on squares, $\ell(E) = 2$ on octagons. Then, the third sequence, labeled *Hexadecagonal*, is made by hexadecagons and concave pentagons, as it is shown in Figure 2c. It results $\ell(E) = 1$ on the concave pentagons and $\ell(E) = 3$ on hexadecagons. Finally, the last sequence, labeled *Star Concave*, is a non-convex tessellation made by octagons and nonagons, as it is shown in Figure 2d. By Algorithm 1, $\ell(E) = 3$ on octagons and $\ell(E) = 2$ on nonagons. In each case we start from a mesh of $\#\mathcal{M}_h$ polygons then we refine it, obtaining meshes made by $4\#\mathcal{M}_h$, $16\#\mathcal{M}_h$ and $64\#\mathcal{M}_h$ polygons. The first and the third sequence start with $\#\mathcal{M}_h$ equal to 320, the second and the fourth with $\#\mathcal{M}_h$ equal to 164 and 192 respectively.

6.2.2. Convergence results

For the four mesh sequences, we report the trend of the H^1 and the L^2 errors in Figures 3a and 3b, respectively, decreasing the maximum diameter of the polygons. In the legends, we report the computed convergence rates with respect to h, denoted by α. We see that we get the expected values for all the meshes, as obtained in (55) and (63).



(a) *Hexagonal* (b) *Octagonal*

(c) *Hexadecagonal* (d) *Star Concave*

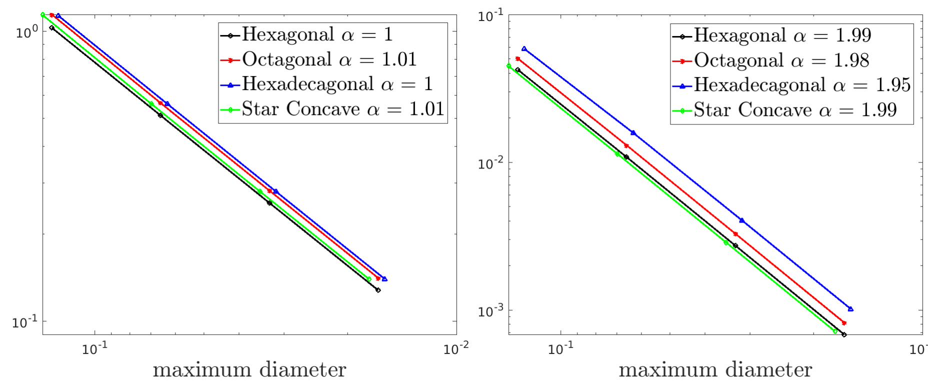
Figure 2: Meshes

6.2.3. Convergence of diffusion-reaction discrete problem

We finally report, in Figure 4, the H^1 and L^2 errors obtained for the four mesh sequences when solving (74) using the discrete formulation (75). We can see that the convergence rates α reported in the legends are optimal.

7. Conclusions

In this work, we present a structure-preserving Virtual Element formulation, where the bilinear forms involve only polynomial projections in the definition. We discuss a general proof of well-posedness of the lowest order method applied to the Poisson problem, identifying a sufficient condition. Then, we propose an algorithm to numerically ensure the stability of the proposed scheme, that exploits an incremental QR factorization, and we derive optimal a-priori error estimates. Numerical tests on convex and non-convex



(a) H^1 *error* (b) L^2 *error*

Figure 3: Logarithmic convergence plots

polygons show the robustness of the method and assess the expected rate of convergence.

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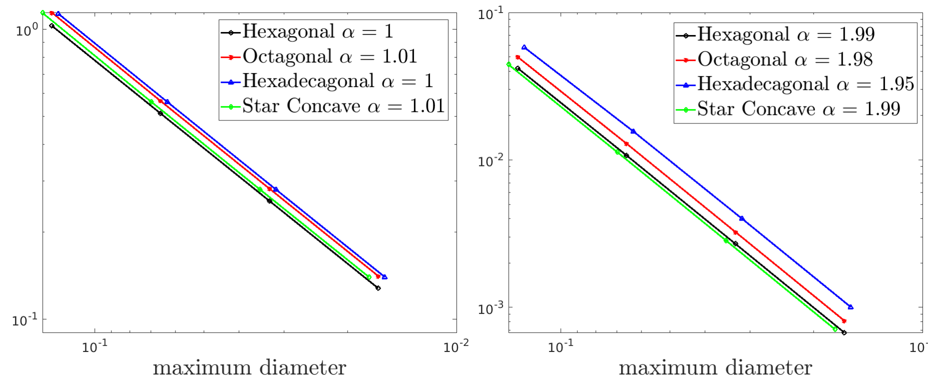

(a) H^1 *error*

(b) L^2 *error*

Figure 4: Logarithmic convergence plots for diffusion-reaction model

Appendix A. Notation Table

Geometry	
E	generic polygon
x_C	centre of the ball with respect to which E is star-shaped
$\mathcal{T}_E$	triangulation of E obtained linking each vertex of E to x_C
$\mathcal{I}_E$	edges of $\mathcal{T}_E$ internal to E

Operators	
$\gamma^{\partial E}$	trace operator on ∂E
$[\![\cdot]\!]_e$	jump operator over an edge e

Polynomial projectors	
$\mathbb{P}_k(E)$	space of polynomials defined on E up to degree k
$\Pi_1^{\nabla,E}$	H^1 orthogonal projector on $\mathbb{P}_1(E)$
P_0	projection operator onto the space of constants
$\Pi_\ell^{0,E}\nabla$	$\mathrm{L}^2(E)$ orthogonal projector of gradients on $[\mathbb{P}_\ell(E)]^2$

Local spaces	
$\mathcal{V}_{1,l}^E$	$\left\{\begin{array}{l} v \in \mathrm{H}^1(E) : \Delta v \in \mathbb{P}_{l+1}(E)\,, \quad \gamma^e(v) \in \mathbb{P}_1(e)\ \forall e \in \mathcal{E}_E, \\ v \in C^0(\partial E)\,, \quad (v,p)_E = \left(\Pi_1^{\nabla,E} v, p\right)_E\ \forall p \in \mathbb{P}_{l+1}(E) \end{array}\right\}$
$\mathcal{P}_l^{\ker}(E)$	$\left\{\boldsymbol{p} \in [\mathbb{P}_l(E)]^2 :\ \int_{\partial E} \boldsymbol{p}\cdot\boldsymbol{n}^{\partial E}\gamma^{\partial E}(v - \mathrm{P}_0(v)) = 0 \quad \forall v \in \mathcal{V}_{1,l}^E\right\}$
$\mathrm{H}^1_{\mathcal{T}}(E)$	$\left\{v \in \mathrm{L}^2(E) : v_{\lvert\tau} \in \mathrm{H}^1(\tau)\ \forall \tau \in \mathcal{T}_E\right\}$
$V(E)$	$\{\boldsymbol{v} \in [\mathrm{L}^2(E)]^2 : \boldsymbol{v}_{\lvert\tau} \in \mathrm{H}^{\mathrm{div}}(\tau)\ \forall\tau \in \mathcal{T}_E,\ [\![\boldsymbol{v}]\!]_{e_i} \in \mathrm{L}^\infty(e_i)\ \forall e_i \in \mathcal{I}_E\}$
$\mathcal{Q}(\partial E)$	span $\left\{\gamma^{\partial E}(\psi_i - \mathrm{P}_0(\psi_i))\ \ \forall i = 1,\dots,N_E^V - 1\right\}$ where ψ_i are basis functions of $V(E)$
$\mathcal{R}_{\mathcal{Q}}(E)$	$\left\{\bar{q} \in \mathrm{L}^2(E) : \bar{q}_{\lvert\tau} \in \mathbb{P}_1(\tau)\ \forall\tau \in \mathcal{T}_E,\ \gamma^{\partial E}(\bar{q}) \in \mathcal{Q}(\partial E),\ \bar{q}(x_C) = 0\right\}$

Norms	
$\lVert\bar{q}\rVert^2_{\mathrm{H}^1_{\mathcal{T}}(E)}$	$\lVert\bar{q}\rVert^2_{\mathrm{L}^2(E)} + \sum_{\tau\in\mathcal{T}_E} \lVert\nabla\bar{q}\rVert^2_{[\mathrm{L}^2(\tau)]^2} + \sum_{i=1}^{N_E^V} \left\lVert[\![\bar{q}]\!]_{e_i}\right\rVert^2_{\mathrm{L}^2(e_i)}$
$\lVert\boldsymbol{v}\rVert^2_{V(E)}$	$\lVert\boldsymbol{v}\rVert^2_{[\mathrm{L}^2(E)]^2} + \sum_{\tau\in\mathcal{T}_E} \lVert\nabla\cdot\boldsymbol{v}\rVert^2_{\mathrm{L}^2(\tau)} + \left\lVert[\![\boldsymbol{v}]\!]_{\mathcal{I}_E}\right\rVert^2_{\mathrm{L}^\infty(\mathcal{I}_E)}$

Appendix B. Proof of Lemma 4

In order to show the proof, we have to present a preliminary result.

Lemma 9. *Let $\bar{q} \in \mathcal{R}_{\mathcal{Q}}(E)$. Then $\exists C > 0$, independent of h_E, such that*

$$\sum_{i=1}^{N_E^V} |\bar{q}(x_i)| \leq C \sqrt{\sum_{\tau \in \mathcal{T}_E} \|\nabla \bar{q}\|^2_{\mathrm{L}^2(\tau)}} \,. \tag{B.1}$$

Proof. We notice that

$$\sum_{i=1}^{N_E^V} |\bar{q}(x_i)| = \frac{1}{2} \sum_{\tau \in \mathcal{T}_E} \left(|\bar{q}(x_{\tau,1})| + |\bar{q}(x_{\tau,2})| \right) \,, \tag{B.2}$$

where $x_{\tau,1}$ and $x_{\tau,2}$ are the vertices of τ that are on ∂E. We have that

$$\bar{q}_{|\tau} \in \tilde{\mathbb{P}}_1(\tau) = \{ p \in \mathbb{P}_1(\tau) : p(x_C) = 0 \} \,,$$

and

$$|\bar{q}(x_{\tau,1})| + |\bar{q}(x_{\tau,2})| = \left\| \mathrm{dof}_{\tilde{\mathbb{P}}_1(\tau)} \left(\bar{q}_{|\tau} \right) \right\|_{l^1} \,,$$

having chosen the values at $x_{\tau,1}$ and $x_{\tau,2}$ as set of degrees of freedom on $\tilde{\mathbb{P}}_1(\tau)$ and denoting by $\mathrm{dof}_{\tilde{\mathbb{P}}_1(\tau)}(\cdot)$ the operator returning the vector of such values. Using the mapping (18) we get

$$\left\| \mathrm{dof}_{\tilde{\mathbb{P}}_1(\tau)} \left(\bar{q}_{|\tau} \right) \right\|_{l^1} = \left\| \mathrm{dof}_{\tilde{\mathbb{P}}_1(\hat{\tau})} \left(\hat{\bar{q}}_{|\hat{\tau}} \right) \right\|_{l^1} \,.$$

The right-hand side of the above equation is a norm on $\tilde{\mathbb{P}}_1(\hat{\tau})$, as well as $\left\| \hat{\nabla} \hat{\bar{q}} \right\|_{\mathrm{L}^2(\hat{\tau})}$. Then, by standard arguments about the equivalence of norms in finite dimensional spaces, we have

$$\left\| \mathrm{dof}_{\tilde{\mathbb{P}}_1(\hat{\tau})} \left(\hat{\bar{q}}_{|\hat{\tau}} \right) \right\|_{l^1} \leq \frac{\sqrt{2} \max_{i=1,2} \left\| \mathrm{dof}_{\tilde{\mathbb{P}}_1(\hat{\tau})} \left(\hat{\chi}_i \right) \right\|_{l^1}}{\min_{\hat{w} \in \tilde{\mathbb{P}}_1(\hat{\tau}) :\, \hat{w}(\hat{x}_{\hat{\tau},1})^2 + \hat{w}(\hat{x}_{\hat{\tau},2})^2 = 1} \left\| \hat{\nabla} \hat{w} \right\|_{\mathrm{L}^2(\hat{\tau})}} \left\| \hat{\nabla} \hat{\bar{q}} \right\|_{\mathrm{L}^2(\hat{\tau})} \,,$$

where the $\hat{\chi}_i$ are Lagrangian in the degrees of freedom. Then, $\left\| \mathrm{dof}_{\tilde{\mathbb{P}}_1(\hat{\tau})} \left(\hat{\chi}_1 \right) \right\|_{l^1} = \left\| \mathrm{dof}_{\tilde{\mathbb{P}}_1(\hat{\tau})} \left(\hat{\chi}_2 \right) \right\|_{l^1} = 1$ and

$$\left\| \mathrm{dof}_{\tilde{\mathbb{P}}_1(\hat{\tau})} \left(\hat{\bar{q}}_{|\hat{\tau}} \right) \right\|_{l^1} \leq \frac{\sqrt{2}}{\min_{\hat{w} \in \tilde{\mathbb{P}}_1(\hat{\tau}) :\, \hat{w}(\hat{x}_{\hat{\tau},1})^2 + \hat{w}(\hat{x}_{\hat{\tau},2})^2 = 1} \left\| \hat{\nabla} \hat{w} \right\|_{\mathrm{L}^2(\hat{\tau})}} \left\| \hat{\nabla} \hat{\bar{q}} \right\|_{\mathrm{L}^2(\hat{\tau})} \,.$$

It can be proved by standard arguments that the constant in the above inequality is continuous with respect to $\hat{\tau}$, since it depends continuously on the deformation of the domain (see the proofs of [25, Lemma 4.9] and [29, Lemma 4.5]). It follows by compactness of the set of admissible reference elements, denoted by Σ, (Lemma 3) that there exists $M > 0$ such that

$$M = \max_{\hat{\tau}\in\Sigma} \frac{\sqrt{2}}{\min_{\hat{w}\in\tilde{\mathbb{P}}_1(\hat{\tau}):\, \hat{w}(\hat{x}_{\hat{\tau},1})^2+\hat{w}(\hat{x}_{\hat{\tau},2})^2=1} \left\|\hat{\nabla}\hat{w}\right\|_{\mathrm{L}^2(\hat{\tau})}},$$

and thus, starting again from (B.2) and applying the mapping (18), we get

$$\begin{aligned}\sum_{i=1}^{N_E^V} |\bar{q}(x_i)| &= \frac{1}{2}\sum_{\tau\in\mathcal{T}_E} \left\|\mathrm{dof}_{\tilde{\mathbb{P}}_1(\tau)}\left(\bar{q}_{|\tau}\right)\right\|_{l^1} = \frac{1}{2}\sum_{\hat{\tau}\in\mathcal{T}_{\hat{E}}} \left\|\mathrm{dof}_{\tilde{\mathbb{P}}_1(\hat{\tau})}\left(\hat{\bar{q}}_{|\hat{\tau}}\right)\right\|_{l^1} \\ &\leq \frac{M}{2}\sum_{\hat{\tau}\in\mathcal{T}_{\hat{E}}} \left\|\hat{\nabla}\hat{\bar{q}}\right\|_{\mathrm{L}^2(\hat{\tau})} = \frac{M}{2}\sum_{\tau\in\mathcal{T}_E} \left\|\nabla\bar{q}\right\|_{\mathrm{L}^2(\tau)} \leq \frac{M\sqrt{N_E^V}}{2}\sqrt{\sum_{\tau\in\mathcal{T}_E}\left\|\nabla\bar{q}\right\|^2_{\mathrm{L}^2(\tau)}},\end{aligned}$$

and we obtain (B.1) since N_E^V is uniformly bounded by (4). □

Now, we can present the proof of Lemma 4.

Proof. Let $\bar{q} \in \mathcal{R}_{\mathcal{Q}}(E)$ and $\boldsymbol{v} \in V(E)$ be given. Starting from (25) and applying the triangular inequality, we have

$$|b(\bar{q},\boldsymbol{v})| \leq \left|\sum_{\tau\in\mathcal{T}_E}\int_\tau \left[\nabla\bar{q}\,\boldsymbol{v} + \bar{q}\,\nabla\cdot\boldsymbol{v}\right]\,dx\right| + \left|\sum_{i=1}^{N_E^V}\int_{e_i} \gamma^{e_i}(\bar{q})\,[\![\boldsymbol{v}]\!]_{e_i}\cdot\boldsymbol{n}^{e_i}ds\right|. \quad \text{(B.3)}$$

Let us consider separately the two terms involved in the inequality. The first part can be analyzed applying the property,

$$\forall\bar{q}\in\mathcal{R}_{\mathcal{Q}}(E), \quad \sum_{\tau\in\mathcal{T}_E}\left(\|\bar{q}\|_{\mathrm{L}^2(\tau)} + \|\nabla\bar{q}\|_{[\mathrm{L}^2(\tau)]^2}\right) \leq \sqrt{2N_E^V}\,\|\bar{q}\|_{\mathrm{H}^1_{\mathcal{T}}(E)}$$

and the mesh assumption (4), as follows

$$
\begin{aligned}
\left|\sum_{\tau\in\mathcal{T}_E}\int_\tau \left[\nabla\bar{q}\,\boldsymbol{v}+\bar{q}\,\nabla\cdot\boldsymbol{v}\right]\,dx\right| &\le \sum_{\tau\in\mathcal{T}_E}\left(\|\nabla\bar{q}\|_{[\mathrm{L}^2(\tau)]^2}\|\boldsymbol{v}\|_{[\mathrm{L}^2(\tau)]^2}+\|\bar{q}\|_{\mathrm{L}^2(\tau)}\|\nabla\cdot\boldsymbol{v}\|_{\mathrm{L}^2(\tau)}\right)\\
&\le C\sum_{\tau\in\mathcal{T}_E}\left(\|\boldsymbol{v}\|_{[\mathrm{L}^2(\tau)]^2}+\|\nabla\cdot\boldsymbol{v}\|_{\mathrm{L}^2(\tau)}\right)\times\left(\|\nabla\bar{q}\|_{[\mathrm{L}^2(\tau)]^2}+\|\bar{q}\|_{\mathrm{L}^2(\tau)}\right)\\
&\le C\,\|\bar{q}\|_{\mathrm{H}^1_{\mathcal{T}}(E)}\sum_{\tau\in\mathcal{T}_E}\left(\|\boldsymbol{v}\|_{[\mathrm{L}^2(\tau)]^2}+\|\nabla\cdot\boldsymbol{v}\|_{\mathrm{L}^2(\tau)}\right).
\end{aligned}
$$

Moreover, let us consider the second term of (B.3), computing exactly the term $\|\gamma^{e_i}(\bar{q})\|_{\mathrm{L}^2(e_i)}$ and applying the properties $\forall\boldsymbol{v}\in V(E)$

$$
\begin{aligned}
&\sum_{i=1}^{N_E^V}\left\|[\![\boldsymbol{v}]\!]_{e_i}\right\|_{\mathrm{L}^2(e_i)}\le\sqrt{2N_E^V}\sqrt{\sum_{i=1}^{N_E^V}\left\|[\![\boldsymbol{v}]\!]_{e_i}\right\|^2_{\mathrm{L}^2(e_i)}}\,,\\
&\left\|[\![\boldsymbol{v}]\!]_{e_i}\right\|^2_{\mathrm{L}^2(e_i)}\le h_E\left\|[\![\boldsymbol{v}]\!]_{\mathcal{I}_E}\right\|^2_{\mathrm{L}^\infty(\mathcal{I}_E)}\,,\ \forall e_i\in\mathcal{I}_E\,,
\end{aligned}
$$

we have

$$
\begin{aligned}
\left|\sum_{i=1}^{N_E^V}\int_{e_i}\gamma^{e_i}(\bar{q})\,[\![\boldsymbol{v}]\!]_{e_i}\cdot\boldsymbol{n}^{e_i}ds\right| &\le\sum_{i=1}^{N_E^V}\|\gamma^{e_i}(\bar{q})\|_{\mathrm{L}^2(e_i)}\left\|[\![\boldsymbol{v}]\!]_{e_i}\cdot\boldsymbol{n}^{e_i}\right\|_{\mathrm{L}^2(e_i)}\\
&\le\sum_{i=1}^{N_E^V}\frac{\sqrt{h_{e_i}}}{\sqrt{3}}\,|\bar{q}(x_i)|\left\|[\![\boldsymbol{v}]\!]_{e_i}\right\|_{[\mathrm{L}^2(e_i)]^2}\le\frac{h_E}{\sqrt{3}}\left\|[\![\boldsymbol{v}]\!]_{\mathcal{I}_E}\right\|_{\mathrm{L}^\infty(\mathcal{I}_E)}\sum_{i=1}^{N_E^V}|\bar{q}(x_i)|\\
&\le Ch_E\left\|[\![\boldsymbol{v}]\!]_{\mathcal{I}_E}\right\|_{\mathrm{L}^\infty(\mathcal{I}_E)}\|\bar{q}\|_{\mathrm{H}^1_{\mathcal{T}}(E)}\,,
\end{aligned}
$$

where we apply Lemma 9 in the last step. Finally, substituting into (B.3), we obtain

$$
\begin{aligned}
|b(\bar{q},\boldsymbol{v})| &\le C\,\|\bar{q}\|_{\mathrm{H}^1_{\mathcal{T}}(E)}\left(\sum_{\tau\in\mathcal{T}_E}\left(\|\boldsymbol{v}\|_{[\mathrm{L}^2(\tau)]^2}+\|\nabla\cdot\boldsymbol{v}\|_{\mathrm{L}^2(\tau)}\right)+h_E\left\|[\![\boldsymbol{v}]\!]_{\mathcal{I}_E}\right\|_{\mathrm{L}^\infty(\mathcal{I}_E)}\right)\\
&\le C\,\|\bar{q}\|_{\mathrm{H}^1_{\mathcal{T}}(E)}\|\boldsymbol{v}\|_{V(E)}\,.
\end{aligned}
$$

□

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