When did torques and angular velocities become vectors? A historical comedy of errors

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Who discovered the vector properties of moments of forces and angular velocities? Among the many scientists who did, there were some of the greatest mathematical physicists at the turn of the 18th and 19th centuries, like Euler, Lagrange, Laplace, Poinsot, Poisson and Cauchy. Surprisingly, due to scientific rivalries, differences in views and poor communications, it took around three quarters of a century, from 1759 to 1834, to figure out that moments of forces and angular velocities are best represented by directed line segments. The present article relates a cautionary tale about the meanders, the detours and the dead ends of the history of science.

We shall present this subject more historically than some of the others in order to give some idea of the development of a physical theory or physical idea.
R. P. Feynman, *The Feynman Lectures on Physics* (1963), v. 1, ch. 26.

1. Introduction

It is astonishing how poorly we know the history of everyday results in elementary physics. This is the more to be regretted on account of the highly interesting stories that lie behind many formulas in our textbooks. Sometimes the slow unfolding of a scientific theory has more twists and turns than a spy novel. As a case in point, let us look at the history of the discovery of the vector properties of moments of forces and angular velocities.[1]

A note of caution is needed here. Since the word *vector* was formally adopted into mathematical physics by Sir William Rowan Hamilton in 1844 to denote the imaginary part of a quaternion, purists may question its use in connection with works published in the eighteenth century. However, we note that, as early as the seventeenth century, the Latin locution *radius vector* made its way into astronomy. We can thus feel justified in calling *vector* any quantity which can be represented by directed line segments subject to the parallelogram rule. In other words, our vectors are those encountered in high school physics textbooks.

The introduction of vectors in mechanics followed the development of the general theory, from single mass points to more complicated systems. In the pseudo-Aristotelian *Mechanical Problems*, a work probably of the fourth century B.C., the parallelogram of displacements appeared. The composition of velocities was enunciated in Galileo Galilei's *Discourses and Mathematical Demonstrations Relating to Two New Sciences* (1638). The parallelogram of forces was stated in Simon Stevin's *The Principles of the Art of Weighing* (1586) and demonstrated by Isaac Newton, Pierre Varignon and Bernard Lamy (1686/87). Before the end of the seventeenth century, all of these results were well established.[2]

As for the vector representation of moments and angular velocities, nothing happened before the creation of the dynamics of rigid bodies. The two lines of development began independently of each other, but then merged in such a way that it becomes difficult to follow the course of each of them separately.[3]

II. Angular velocities

In the early 1740s the dynamics of rigid bodies was still essentially limited to two-dimensional special problems, but things changed with the publication of the *Recherches sur la précession des équinoxes et sur la nutation de l'axe de la terre* (1749) by the philosopher, essayist and mathematician Jean le Rond d'Alembert.[4] This was an extremely difficult book to understand, a common characteristic of all of d'Alembert's scientific works. It contained a number of fundamental results, for example, the discovery of the instantaneous axis of rotation,[5] but a modern reader would be hard-pressed to find them in this complex tapestry of partial theories punctuated by awkward notations and semi-geometric demonstrations. Though the *Recherches* did not become the nearly definitive treatment of rigid bodies it was intended to be, it remained a source of inspiration for two generations of mathematical physicists.

Among those who fell under the spell of d'Alembert was Paolo Frisi (1728-1784), a Barnabite friar and a professor of mathematics at the University of Pisa. Frisi wrote prolifically, chiefly on hydraulics and astronomy, and was prominent in the advancement of the Enlightenment in Italy. Yet, for all of his efforts, he can be credited with only one first-class discovery: the statement and proof that two angular velocities about concurrent axes can be composed according to the parallelogram law.

Frisi presented his theorem in a memoir on the precession of the equinoxes published in 1759.[6] The proof was achieved via a sequence of lemmas based on the repeated application of the composition and decomposition of velocities to a rigid body rotating simultaneously about two concurrent axes. Thus Frisi demonstrated, more simply and clearly, the existence of d'Alembert's instantaneous axis of rotation, also going beyond d'Alembert in expressing its position by means of the parallelogram rule:

> In every body two rotations can be composed into one exactly in the same way that two forces, represented by the two sides of a parallelogram, are composed into a third force represented by the diagonal.[7]

There you have it, the first reference in history to angular velocities as vectors. Frisi was so convinced of the importance of his theorem that he kept refining its demonstration up to the very end of his life. In all he published three proofs, differing only in some details.[8]

In Italy, the composition of rotations immediately attracted some attention. Upon seeing the new theorem, Tommaso Perelli (1704-1783), one of Frisi's colleagues at the University of Pisa, devised his own proof, together with the demonstrations of a number of theorems on the maxima and minima of angular velocities. By modern standards of scholarship, Perelli was a remarkable man. While he is now remembered primarily as a mathematician, he was also an astronomer, a hydraulic engineer, a botanist, a music historian and a classical scholar. Unfortunately, his many interests prevented him from seriously pursuing mathematics. That is exactly what happened in this case: according to Frisi, Perelli wrote an account of his discovery, but never got around to publishing it.[9]

Another Italian mathematician who became interested in the parallelogram of rotations was Giulio Mozzi (1730-1813). Like Frisi and Perelli, Mozzi was an eclectic scholar of unbounded curiosity. He came from a noble family of Lucca, studied literature and wrote a couple of didactic poems, then turned to mathematics and became a student of Frisi. He published only one scientific work, the *Mathematical Discourse on the Instantaneous Rotation of Bodies* (1763), a slim volume written to alleviate the boredom of a prolonged illness. After this single burst of creativity, Mozzi abandoned mathematics and spent the rest of his life in politics. His *Discourse* lay unread for some fifty years.[10]

Mozzi was able to pack a lot of content into less than one hundred pages. He exposed some crucial errors of Johann Bernoulli and d'Alembert, sketched a general theory of the three-dimensional instantaneous motion of a rigid body acted upon by impulsive forces, showed that every infinitesimal rigid displacement is a screw motion (i.e., a rotation about an axis followed by a translation along the same direction) and described some properties of couples of forces. In keeping with his theory, he gave a proof of Frisi's theorem for the case of two impulsive forces acting on a rigid body with a fixed point.[11]

It might be expected that Frisi's theorem would receive the same attention outside Italy. Oddly enough, the composition of angular velocities went unnoticed abroad. D'Alembert, a great innovator with a few weaknesses, in his "Nouvelles recherches sur la précession des équinoxes" (1754), wrote that it would be wrong to consider separately the diurnal rotation of the Earth and the precession of the terrestrial axis.[12] Even Leonhard Euler, a mathematician and physicist of the highest standing, in his numerous papers on rigid bodies did not make use of the parallelogram of angular velocities.[13] This silence is difficult to explain. The most likely explanation is that mathematical physicists were then focused on a purely analytical approach to mechanics, to the detriment of geometrical constructions.

Ironically, the resurgence of the geometric composition of angular velocities came in the wake of the most powerful attack ever on the use of synthetic geometry in mathematical physics. Frisi's theorem reappeared in Joseph Louis Lagrange's *Méchanique analitique*, a book that purported "to condense the theory of [mechanics] and the method of solving the related problems to general formulas whose simple application produces all the necessary equations for the solution of each problem."[14] Once again, the (re)discovery did not follow the shortest path.

In the first edition (1788) Lagrange demonstrated the formulas for the composition of infinitesimal rotations: given a system of rectangular axes, three rotations $d\theta \cdot cos\lambda$, $d\theta \cdot cos\mu$, $d\theta \cdot cos\,\nu$ about the x-, y-, and z-axis, respectively, are equivalent to a single rotation $d\theta$ about the axis $\frac{x}{cos\lambda} = \frac{y}{cos\mu} = \frac{z}{cos\,\nu}$, where λ, μ, ν are respectively the angles between the axis of total rotation and the x-, y-, and z-axis. It is easy to see how close he came to establishing the vectorial character of infinitesimal rotations, yet he failed to do so.

About a quarter of a century after the first appearance of the book, Lagrange published a much enlarged second edition (1811-15), now entitled *Mécanique analytique*, which tackled the ideas of the younger generation of mathematical physicists. Perhaps inspired by the then new vector theories of moments (see next section), he completed his previous analysis by demonstrating that, under a rotation of the axes, partial rotations behave like the components of a linear velocity. This is very much in the spirit of modern Cartesian tensors: three quantities represent a vector when they transform in a certain way under an orthogonal transformation. Thus Lagrange succeeded at last in formulating the vectorial composition of infinitesimal rotations:

> It is clear from this development that the composition and resolution of rotational motions are entirely analogous to rectilinear motions. Indeed, if on the three axes of rotation $d\psi$, $d\omega$, $d\varphi$ one takes from their point of intersection lines proportional respectively to $d\psi$, $d\omega$, $d\varphi$, and if one draws on these three lines a rectangular parallelepiped, it is easy to see that the diagonal of this parallelepiped will be the axis of composed rotation $d\theta$ and will be at the same time proportional to this rotation $d\theta$. From this result and because the rotations about the same axis can be added or subtracted depending on whether they are in the same or opposite directions as the motions which are in the same or opposite directions, in general, one must conclude that the composition and resolution of rotational motions is done in the same manner and by the same laws that the composition or resolution of rectilinear motions, by substituting for rotational motions rectilinear motions along the direction of the axes of rotation.[15]

We do not know if Lagrange had taken his cue from Frisi. It is possible that he did, for we know from his correspondence that he had read Frisi's works, but he might also have more fully developed his own theory. However it may be, his contemporaries attributed all the merits of the discovery to him. It is through the *Mécanique analytique* that the vectorial theory of angular velocity made its way into modern literature.

Almost contemporaneously with Lagrange, the angular velocity vector appeared in a small book by Jacques-Frédéric Français on the rotation of rigid bodies. Interestingly, in those same years Français was also elaborating on Argand's vectorial interpretation of complex numbers. One wonders if there was a connection.[16]

III. Moments of vectors

Although the first correct ideas on three-dimensional rigid motion had emerged in the work of d'Alembert, it was Euler who brought the general theory to near perfection. This is easily verified by spending a couple of hours with the original texts. While today d'Alembert's *Recherches* is merely a historical relic, even modern physicists and engineers have much to learn from Euler's works on rigid dynamics (provided they read Latin). So, we should not be surprised when we find out that the discovery of the vectorial representation of moments was made by Euler in 1780.[17]

Euler was led to his discovery by a (seemingly to us) trivial problem: given an applied force **F**, to find its moment about a given straight line *l*. Back then, the moment of a force about an axis was defined, in purely geometric fashion, as the product of the intensity of the force by the length of the common perpendicular to the axis and the line of action of the force. Euler referred the problem to rectangular coordinates and, after much algebra, arrived at the simple expression

$$fP + gQ + hR,$$

where P, Q, R are respectively the moments of **F** about Ox, Oy, Oz, and f, g, h are the cosines of the angles formed by l with the coordinate axes.[18] Today we recognize in this formula a scalar product, and in P, Q, R the components of the moment $\mathbf{M}_{\mathrm{O}}$ about the origin O. Euler did not have the advantage of vector calculus, but knew that this was the length of the projection of the segment (P, Q, R) along the direction (f, g, h). In a flash he realized that moments of forces are represented by line segments. At the close of the paper, in the paragraph immediately following the derivation of his formula, he wrote:

> Therefore, the moments about three orthogonal axes can be composed exactly as the simple forces. For if three forces P, Q, R were applied at point a, acting along the directions af, ag, ah, they would form a force equal to $fP + gQ + hR$ acting along the direction az. This marvelous harmony deserves to be considered with the greatest attention, for in general mechanics it can deliver no small development.[19]

From the vantage point of 21st-century science, we can fully appreciate the significance of this veritable prophecy. Sadly, Euler could not reap what he had sown, for he was already past seventy and almost completely blind.

Euler, however, holds another surprise for us. By searching through his *Opera omnia* (up to now 86 thick volumes, yet still in progress), we can find to our amazement that he had already demonstrated this very same formula in 1764 without realizing its vectorial interpretation.[20] We are thus forced to conclude that even this supreme mathematical physicist could not remember all the details of his 866-plus works, and that in 1764 he did not yet know the geometrical meaning of the scalar product.

If history progressed linearly, the discovery of the vector representation of moments should have exerted a significant impact on dynamics. Yet the demon of perversity intervened again. Euler's paper appeared as late as 1793 in the *Acta* of the St. Petersburg Academy of Science. By that date, the Revolution had cut off most scientific communications between the French school of mathematical physics and the rest of the world. This unfortunate chain of events probably delayed the development of mechanics by a number of years.

Unaware of Euler's result, in 1798 Pierre Simon Laplace stumbled unexpectedly on the vector properties of angular momentum. He had set himself the problem of finding a "natural" frame of reference for an isolated system of mass points. The solution he proposed was the *invariable plane*, that is, in modern terms, the plane passing through the center of mass and orthogonal to what we now call the angular momentum vector. This became an instant classic: thoughout the nineteenth century the invariable plane figured prominently in every book on mechanics and astronomy.[21]

Of course, scientists in those days had no idea of the existence of an angular momentum *vector*. Laplace in effect started from an early formulation of the conservation of angular momentum, the *principle of the conservation of areas*: in the motion of an isolated system of mass points, the sum of the projections on a fixed plane of the areas described in unit time by the radii vectores drawn from any fixed point to all the points of the system, multiplied by their masses, are constant in time.[22] By rotating the axes, he demonstrated that there is a plane of maximum projection, and that the sum of these projections on any plane at right angles to this one is zero. This indicated the existence of a privileged direction in space for isolated systems. Regrettably, Laplace buried intuitive geometry under rather menacing algebra. If he had employed a symmetric notation, he would have realized that mass-areas transform vectorially.[23]

In the following year Laplace published a two-page follow-up to this paper, where he demonstrated that the axis about which the moment of momentum of the whole system is the greatest possible is orthogonal to the invariable plane. Since he employed neither geometry nor algebra, resorting instead to a verbal description of the operations with vectors, the end result is somewhat difficult to follow.[24]

The connection between the theorems of Euler and Laplace was made explicite by Gaspard de Prony, one of the leaders of the newly founded Ecole Polytechnique, in his lectures to engineers (apparently, amid the turmoil of the Napoleonic wars, Euler's memoir had somehow reached Paris). While Prony seldom added anything new to the topic, he had the merit of clarifying and making widely known the first results in the vector theory of moments.[25]

Once those basic results had been achieved, things progressed rapidly. In 1803 a complete vector theory of moments entered mechanics thanks to the young French mathematician Louis Poinsot. Geometry was then enjoying a renaissance and Poinsot was one of those riding the crest of the new wave. His first publication, a rigorous treatise somewhat deceptively entitled *Éléments de statique*, went head-on against the analytical mechanics of Lagrange and Laplace by founding statics on synthetic geometry. The *Statique* was in many ways an innovative book, especially in the section on rigid bodies. Since a rigid body can both translate and rotate, Poinsot introduced *two* independent causes of motion, forces and *couples of forces*. As is well known, a couple is a system of two equal, parallel and oppositely directed forces, whose magnitude is measured by the product of the intensity of the forces by the distance between their lines of action. Poinsot demonstrated that if we represent a couple with a directed segment perpendicular to its plane, we can combine couples by the parallelogram rule. Statics was thus reduced to vector geometry. In a successive memoir, Poinsot gave vectorial proofs of the conservation of momentum and angular momentum in dynamics.[26]

A different geometric representation of moments was developed shortly afterwards by the mathematical physicist Siméon Denis Poisson. His motivations partly lay outside of science: Poisson, a *protégé* of Laplace, saw with mounting concern the rise of Poinsot and tried to undermine his theory of couples. Starting from Laplace's discussion of mass-areas, he remarked that the moment of a force about a point is numerically equal to the double of the area of a triangle having the vertex in the centre of moments and the force as its base. It was therefore natural to consider the triangle itself as the geometrical representation of the moment. These triangle-moments shared many properties with forces: they could be added by means of their projections and obeyed the familiar parallelogram rule. Just this once, academic politics resulted in something productive, for Poisson's idea marked a significant step towards the definition of the cross product of vectors.[27]

Yet another geometric representation of moments was proposed by Jacques Philippe Marie Binet in 1815. Binet, a professor of analysis, mechanics and astronomy at the Ecole Polytechnique and at the Collège de France for about forty years, was an able mathematician and an attentive reader, whose *forte* was the detailed development of promising concepts formulated by others. His theory of moments was based on the fact that the motion of a rigid body with a fixed point O is completely determined by moments alone. Binet substituted every applied force **F** with a parallel force **F**' whose line of action was at unit distance from O and whose moment about O was the same as that of **F**; he called **F**' the *momens* of **F** about O. This is equivalent, in current terminology, to taking the moments $\mathbf{F}'_1$, $\mathbf{F}'_2$, $\mathbf{F}'_3$, … etc. of the forces $\mathbf{F}_1$, $\mathbf{F}_2$, $\mathbf{F}_3$, … etc. and rotating them by ninety degrees. If Binet had pursued this line of thought, he could have achieved something worthwhile.[28]

After 1820 the time was ripe for someone to organize all the different views involved in the the theory of moments into a general formulation. As in other branches of mathematics and mechanics, the onus fell to Augustin Louis Cauchy to clear up the muddle of apparently conflicting ideas. In 1826 he published, in consecutive pages of his *Exercices de mathématiques*, five papers in which he brought the theory almost to its final formulation.[29] Essentially, Cauchy took the best parts from the theories of moments then in existence. His *moments linéaires* are vectors, like Poinsot's couples and Binet's *momens*, which represent Poisson's triangles. Except for the lack of a proper vector notation, this is the modern theory.

IV. Polemics and controversies

These results were quickly taken up by textbooks and treatises. The parallelogram rule for moments and angular velocities was discussed in the second editions of both Lagrange's *Mécanique analytique* and Poisson's *Traité de mécanique*, Poinsot's *Statique* was reprinted many times and translated into several languages, and Poisson's vector-areas appeared in every textbook of analytic geometry up to the end of the century. In a few years, early vector mechanics had moved from research to pedagogy.

The appearance of several different theories of moments obviously led to some priority controversies. There were two such polemics, both erupting in 1827. They contributed nothing to science, but allow us to understand how scientists viewed vector mechanics before the advent of vector calculus.

The first controversy arose after the appearance of Cauchy's linear moments. Poinsot accused Cauchy of having published results which were merely repetitions of his theorems on couples of forces disguised under a different notation. Cauchy replied that his theory was more general than Poinsot's, for it could be applied to every physical entity that can be represented by a directed line segment. While posterity has accepted Cauchy's judgement, it must be conceded that he should have better acknowledged the achievements of Poinsot.[30]

A second controversy began when Poisson published a short account of the recent history of the theory of moments in which he maintained that Poinsot's work was entirely derived from that of his predecessors. Poinsot replied with a long and detailed assessment of the theory of couples. The introduction of couples, he wrote, had entailed a geometrical composition of moments, whereas up to then there had only been the algebraic sum of certain mathematical expressions. Vector entities had definitively taken their place in mechanics.[31]

V. Angular velocities and moments of vectors

By 1815 it had become clear to researchers that a number of fundamental entities in mechanics could be represented geometrically by means of directed line segments and plane surfaces. From then on, it was mainly a question of formulating a unified treatment of the whole matter.

Binet was the earliest to consider the connection between the geometric representations of torque and angular velocity. In the previously cited paper on moments, he wrote the law of rotational dynamics (i.e. that the external torque is equal to the time-rate of angular momentum) in the form

$$\sum_i m_i \left(y_i \frac{d^2 z_i}{dt^2} - z_i \frac{d^2 y_i}{dt^2} \right) = \sum_i M_i cos\lambda_i,$$

with two other similar equations found by cyclic permutation of the letters, x, y, z and λ, μ, ν; here M_i is the moment acting on the ith particle, λ_i, μ_i, ν_i are respectively the angles between the plane of the moment M_i and the yz-, zx-, and xy-plane, and the sums are to be taken over the particles. This is not too far from the current vector formulation $\sum \frac{d\boldsymbol{H}_O}{dt} = \sum \boldsymbol{M}_O$ expressed in rectangular Cartesian components (it is left as an easy exercise for the reader to demonstrate that the orthogonal projection $Mcos\lambda$ of the surface M on the yz-plane is equal to the projection of the normal vector of length M along the x-axis). In a second paper, Binet introduced the *areal velocity* as a vector quantity; he did not supply a proof, instead simply remarking that areal velocities are the moments of velocities about a fixed point.[32]

The definitive unification of the geometric representations of moments and angular velocities was achieved by Poinsot in his *Théorie nouvelle de la rotation des corps*, first published in an abridged version in 1834. This is the work in which he considered the *dynamical* effects of couples. Of special interest to us is the first section, since here Poinsot provided a study of the vectorial properties of angular velocity which closely followed the corresponding study for couples of forces in the *Eléments de statique*. In particular, he introduced the *couple of small rotations* (which turned out to be a pure translation) and the *accelerating couple*. He remarked that any proposition concerning the composition of forces has its counterpart in the composition of small rotations; for example, the theory of the central axis is the same as that of the instantaneous axis of rotation. By presenting a general overview of vectors in mechanics, Poinsot's *Théorie nouvelle* paved the way for the invention of vector calculus.[33]

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[1] This article is a general survey. For more detail on these questions, see S. Caparrini, "The discovery of the vector representation of moments and angular velocity," *Arch. Hist. Exact Sci.* **56** (2002). Most of the historical works cited in the article can be freely downloaded from *Google Books*, *Archive.org* or *Gallica*.

[2] The history of the discovery of the vectorial properties of forces and velocities is long and varied. For a more detailed overview see, for example, R. Dugas, *A History of Mechanics*, (New York: Dover, 1988). During the eighteenth and nineteenth centuries, physicists and mathematicians spent much energy debating various proofs of the parallelogram of forces; see M. Lange, "Why do forces add vectorially? A forgotten controversy in the foundations of classical mechanics," *Am. J. Phys.* **79** (2011).

[3] Historians of science have paid little attention to the introduction of the parallelogram law in rigid body dynamics. For a history of mechanics where this question is placed in a general context see S. Caparrini and C. Fraser, "Mechanics in the Eighteenth Century," in *The Oxford Handbook of the History of Physics*, edited by J. Buchwald and R. Fox (Oxford University Press, 2013).

[4] *Recherches sur la précession des équinoxes et sur la nutation de l'axe de la terre dans le systême newtonien* (Paris, 1749). The *Recherches* have been republished in volume 7, series I, of the modern edition of d'Alembert's *Oeuvres.*
[5] *Recherches* …, art. 72.
[6] P. Frisi, "Problematum praecessionis aequinoctiorum nutationis axis, aliarumque vicissitudinum diurni motus geometrica solutio, cujus specimen a Regia Berolinensi Scientiarum Academiae anno MDCCLVI praemium obtinuit," in *Paulli Frisii Dissertationum variarum*, vol. I (Lucca, 1759). This remarkable result of Frisi was rediscovered by the Italian historians R. Marcolongo and F. Ricci only at the beginning of the twentieth century. See R. Marcolongo, "Sul teorema della composizione delle rotazioni istantanee. Appunti per la storia della Matematica nel secolo XVIII," *Bollettino di bibliografia e di storia delle scienze matematiche* **9** (1906); F. Ricci, "Paolo Frisi e la composizione de' moti rotatori," *Rivista geografica italiana* **13** (1906).
[7] "Problematum …," p. 24. The translation from Latin is mine.
[8] P. Frisi, "Problematum praecessionis aequinoctiorum nutationis axis, …," p. 18-24; "De inequalitatibus motus Terrae, et Lunae circa axem ex Astronomorum hypothesibus," *De Bononiensi scientiarum et artium Instituto atque Academia commentarii* **5**(2), (1767), p. 31; *De gravitate universali corporum* (Milan, 1768), p. 35; *Cosmographia physica et mathematica* (Milan, 1774/75), vol. 2, p. 33; *Mechanicam universam et mechanicae applicationem ad aquarum fluentium theoriam continens*, in *Paulli Frisii Operum*, vol. 3 (Milan, 1783), p. 136; "De rotatione corporum," *De Bononiensi scientiarum et artium Instituto atque Academia commentarii* **6**(2) (1783), p. 57.
[9] See P. Frisi, *Lettera del sig. abate Paolo Frisi intorno agli studi del sig. dottor Tommaso Perelli a monsig. Angelo Fabroni* (Pisa, 1784), p. 24.
[10] G. Mozzi, *Discorso matematico sopra il rotamento momentaneo dei corpi* (Naples, 1763). For details on Mozzi see M. Ceccarelli, "Giulio Mozzi (1730-1813)," in *Distinguished Figures in Mechanism and Machine Science: Their Contributions and Legacies. Part I*, edited by M. Ceccarelli (Dordrecht: Springer, 2007).
[11] *Discorso*…, art. 52.
[12] J. d'Alembert, "Nouvelles recherches sur la précession des équinoxes et sur la figure de la Terre et de la Lune," in *Recherches sur differens points importans du système du monde*, part. 2 (Paris, 1754), p. 179.
[13] Euler's works on rigid motion have been republished in volumes 3-4 and 8-9 of series II of his *Opera omnia*.
[14] J.L. Lagrange, *Méchanique analitique* (Paris, 1788), from the Introduction. All quotations from the *Méchanique analitique* come from the modern English translation: *Analytical Mechanics*, translated into English from the second edition by Auguste Boissonade and Victor N. Vagliente (Dordrecht: Kluwer, 1997).
[15] J.L. Lagrange, *Mécanique analytique*, vol. 1 (Paris, 1811), partie I, § III, art. 12. English translation by A. Boissonade and V. N. Vagliente.
[16] J-.F. Français, *Mémoire sur le mouvement de rotation d'un corps solide libre autour de son centre de masse* (Paris, 1813), p. 28; "Nouveaux principes de Géométrie de position, et interprétation géométrique des symboles imaginaires", *Annales de Mathématiques pures et appliquées* **4**, 61-71 (1813-14). Quite often in the histories of mathematics it is mentioned that the geometrical interpretation of complex numbers influenced the birth of vector calculus: see M. J. Crowe, *A history of vector analysis: the evolution of the idea of a vectorial system* (University of Notre Dame Press, Notre Dame, 1967). However, it can be argued the other way round. There is evidence to show that the parallelogram of forces was one of the factors which led to the geometric view of complex numbers: see S. Caparrini, "On the Common Origin of Some of the Works on the Geometrical Interpretation of Complex Numbers," in *Two Cultures: Essays in honour of David Speiser*, edited by K. Williams (Birkhäuser, Basel-Boston-Berlin, 2006).
[17] L. Euler, "De momentis virium respectu axis cuiuscunque inveniendis; ubi plura insignia symptomata circa binas rectas, non in eodem plano sitas, explicantur," *Novi commentarii Academiae Scientiarum Imperialis Petropolitanae* **7** (1793). See also L. Euler, "Methodus facilis omnium virium momenta respectu axis cuius cunque determinandi," *Novi commentarii Academiae Scientiarum Imperialis Petropolitanae* **7** (1793). Both texts are reprinted in volume 9 of series II of Euler's *Opera omnia*. They are discussed in S. Caparrini, "Euler's Influence on the Birth of Vector Mechanics," in *Leonhard Euler: Life, Work and Legacy*, edited by R. E. Bradley and E. C. Sandifer (Elsevier, Amsterdam, 2007).
[18] More precisely, Euler reduced the mechanical problem to a problem in analytic geometry; given two straight lines in space, to find their distance. His solution was ---HHHHHHH, where ----. At the time, this result was new. It was rediscovered by Gaspard Monge in 1793: see--HHHHHHH-----.
[19] "De momentis virium …," art. 35. Translation (from Latin) mine.
[20] L. Euler, "De aequilibrio et motu corporum flexuris elasticis iunctorum," *Novi commentarii Academiae Scientiarum Imperialis Petropolitanae* **13** (1769), art. 20. The paper was written in 1764 but was published only in 1769. It was reprinted in volume 11, series II, of Euler's *Opera omnia*.
[21] P. S. Laplace, "Mémoire sur la détermination d'un plan qui reste toujours parallèle a lui même, dans le mouvement d'un système de corps agissant d'une manière quelconque les unes sur les autres et libres de toute action étrangère", *Journal de l'Ecole Polytechnique*, t. II, cahier V (1798). For a general history of angular momentum see A. Borrelli, "Angular momentum between physics and mathematics," in *Mathematics Meets Physics*, edited by K.-H. Schlote and M. Schneider (Verlag Harri Deutsch, Frankfurt a. M., 2011).
[22] For a modern view of the principle of the conservation of areas see J. Casey, "Areal velocity and angular momentum for non-planar problems in particle mechanics," *Am. J. Phys.* **75** (2007).
[23] The term *mass-area* was originally coined by J.C. Maxwell; see his *Matter and Motion* --HHHHHH-----.

[24] P.S. Laplace, "Sur la mécanique", *Journal de l'Ecole Polytechnique*, t. II, cahier VI (1799). Both articles are reprinted in volume 14 of Laplace's *Oeuvres*.

[25] G.C.F.M. R. de Prony, "Mécanique philosophique ou analyse raisonnée des divers parties de la science de l'équilibre et du mouvement", *Journal de l'Ecole Polytechnique*, t. III, cahiers VII-VIII (1801), p. 110.

[26] L. Poinsot, *Eléments de statique* (Paris, 1803); "Mémoire sur la composition des moments et des aires," *Journal de l'Ecole Polytechnique*, t. VI, cahier XIII (1804). The *Statique* was translated into English and was republished several times during the nineteenth century. Poinsot has not been studied enough in a formal, scholarly manner; see I. Grattan.Guinness, "From anomaly to fundament: Louis Poinsot's theories of the couple in mechanics," *Historia Mathematica* **41** (2014).

[27] S.D. Poisson, "Note sur différentes propriétés des projections," *Correspondance sur l'Ecole Polytechnique* **1** (1808).

[28] J.P.M. Binet, "Mémoire sur la composition des forces et sur la composition des moments," *Journal de l'Ecole Polytechnique*, t. X, cahier XVII (1815).

[29] The five papers by Cauchy on linear moments are: "Sur les moments linéaires"; "Sur les moments linéaires de plusieurs forces appliquées à différents points"; "Usage des moments linéaires dans la recherche des équations d'équilibre d'un système invariable entièrement libre dans l'espace"; "Sur les conditions d'équivalence de deux systèmes de forces appliquées à des points liés invariablement les uns aux autres"; "Usage des moments linéaires dans la recherche des équations d'équilibre d'un système invariable assujetti à certaines conditions". They appeared in volume 1 of the *Exercices de Mathématiques* (Paris, 1826) and were reprinted in volume 6, series II, of Cauchy's *Oeuvres*.

[30] L. Poinsot, "Note de M. Poinsot sur l'article no. 6 du *Bulletin* dernier, relatif aux *Exercices de mathématiques* de M. Cauchy, 5e. et 6e. livraisons," *Bulletin des sciences mathématiques astronomiques, physiques et chimiques* **7** (1827); A.-L. Cauchy, "Note de M. Cauchy, sur l'article du Bulletin des Sciences du mois d'avril, no. 178," *Bulletin des sciences mathématiques astronomiques, physiques et chimiques* **7** (1827).

[31] S.D. Poisson, "Note sur la composition des momens," *Bulletin des sciences mathématiques astronomiques, physiques et chimiques* **7** (1827); L. Poinsot, "Mémoire sur la composition des moments en Mécanique," *Bulletin des sciences mathématiques astronomiques, physiques et chimiques* **8** (1827); S.D. Poisson, "Addition a la note sur la composition des momens" *Bulletin des sciences mathématiques astronomiques, physiques et chimiques* **8** (1828); L. Poinsot, "Note relatif à la composition des moments," *Bulletin des sciences mathématiques astronomiques, physiques et chimiques* **9** (1828). Also see P. Radelet-de Grave, "La composition des moments en mécanique, ou la querelle des couples," *Sciences et techniques en perspective* **4** (2000).

[32] J.P.M. Binet, "Mémoire sur la composition des forces…," (1815), p. 343; "Mémoire sur les principes généraux de Dynamique, et en particulier sur un nouveau principe de Mécanique générale," *Journal de l'Ecole Polytechnique*, t. XII, cahier XIX (1823), p. 164.

[33] L. Poinsot, *Théorie nouvelle de la rotation des corps. Extrait d'un Mémoire lu à l'Académie des Sciences de l'Institut, le 19 mai 1834* (Paris, 1834). For the influence of mechanics on the early history of vector calculus see S. Caparrini, "Early Theories of Vectors," in *Essays on the History of Mechanics: in Memory of Clifford Ambrose Truesdell and Edoardo Benvenuto*, edited by M. Corradi, A. Becchi, F. Foce and O. Pedemonte (Birkhäuser, Basel-Boston-Berlin, 2003). (In 2003 this paper was awarded the Slade Prize by the British Society for the History of Science.)