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(Article begins on next page)

# Non-relativistic three-dimensional supergravity theories and semigroup expansion method 

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AbSTRACT: In this work we present an alternative method to construct diverse nonrelativistic Chern-Simons supergravity theories in three spacetime dimensions. To this end, we apply the Lie algebra expansion method based on semigroups to a supersymmetric extension of the Nappi-Witten algebra. Two different families of non-relativistic superalgebras are obtained, corresponding to generalizations of the extended Bargmann superalgebra and extended Newton-Hooke superalgebra, respectively. The expansion method considered here allows to obtain known and new non-relativistic supergravity models in a systematic way. In particular, it immediately provides an invariant tensor for the expanded superalgebra, which is essential to construct the corresponding Chern-Simons supergravity action. We show that the extended Bargmann supergravity and its Maxwellian generalization appear as particular subcases of a generalized extended Bargmann supergravity theory. In addition, we demonstrate that the generalized extended Bargmann and generalized extended Newton-Hooke supergravity families are related through a contraction process.

Keywords: Chern-Simons Theories, Supergravity Models, Gauge Symmetry, Classical Theories of Gravity

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## 1 Introduction

The formulation of a non-relativistic (NR) three-dimensional supergravity theory has recently been approached in [1] and subsequently developed in $[2,3]$. These last two years, the construction of NR supergravity actions has received a growing interest [4-8] considering different procedures. ${ }^{1}$ Such supergravity models correspond to supersymmetric extensions of diverse NR gravity theories. Unlike their bosonic counterparts, the construction of NR supergravity theories remains as a challenging task mainly motivated by the diverse applications of these models in the context of holography and relativistic field theory.

The first proposed NR supergravity theory corresponds to a supersymmetric extension of the Newton-Cartan gravity which was obtained by gauging a Bargmann superalgebra [1]. At the bosonic level, the Newton-Cartan formalism allows to formulate in a geometric way a Newtonian gravity model that resembles General Relativity [10, 11]. Newton-Cartan gravity theories have been largely studied and extended with diverse purposes [12-27]. Remarkably, Newton-Cartan geometry has been useful to approach strongly coupled condensed matter systems [28-38] and NR effective field theories [39-43]. Nevertheless, an action principle for a NR supergravity theory requires to consider an approach different from the Newton-Cartan supergravity one. To this end, a Chern-Simons (CS) formalism

[^0]was considered in [3] to construct a three-dimensional NR supergravity action invariant under an extended Bargmann superalgebra. Such superalgebra can be seen as the supersymmetric extension of the extended Bargmann algebra [44-48]. The extended Bargmann superalgebra admits an invariant bilinear form which ensures the proper construction of a well-defined CS action. Furthermore, the extended Bargmann gravity differs from the Newton-Cartan gravity at the matter coupling level, allowing all components of the Ricci tensor to be non-vanishing. On the other hand, the CS formalism has the advantage of offering a gauge-invariant action, this being an interesting three-dimensional toy model [49-51].

To go beyond Poincaré supergravity is a natural step to explore more general supergravity theories. Analogously, at the NR level, it is possible to extend the extended Bargmann supergravity to approach other features. The inclusion of a cosmological constant in a NR supergravity theory was presented in [6] which, in the flat limit, reproduces the extended Bargmann supergravity. The NR action is based on the extended Newton-Hooke superalgebra which appears as a Lie algebra expansion [52] of the $\mathcal{N}=2 \operatorname{AdS}$ superalgebra. More recently, a Maxwellian generalization of the extended Bargmann supergravity and its extension to an enlarged extended Bargmann supergravity were studied in [7] and [8], respectively. They can be seen as supersymmetric extensions of the Maxwellian extended Bargmann (MEB) [19] and enlarged extended Bargmann (EEB) gravity theories [22]. While the MEB supergravity action has been obtained by hand, the EEB supergravity theory has been found from the relativistic $\mathcal{N}=2$ AdS-Lorentz superalgebra through the semigroup expansion method [53].

The Lie algebra expansion procedure has been introduced in [54] and subsequently developed by expanding Maurer-Cartan forms [52, 55]. An expansion method based on semigroups ( $S$-expansion) has been then introduced in [53] and subsequently studied in [56-61]. Within the $S$-expansion procedure the expanded (super)algebra is obtained by combining the structure constants of a Lie (super)algebra with the multiplication law of a semigroup $S$. In addition, the $S$-expansion method provides us with the non-vanishing components of the invariant tensor for the expanded (super)algebra, which are crucial to construct CS actions. The $S$-expansion mechanism not only has been useful at the NR level ${ }^{2}$ [22, 27, 65-68] but also to obtain novel relativistic symmetries [69-73], superalgebras [74-79], and asymptotic symmetries [80-82], among others.

In this work, we present an alternative procedure to construct various NR supergravity theories by considering the $S$-expansion method. We extend the results obtained in [22, 65] in which diverse NR symmetries are obtained by expanding the Nappi-Witten algebra. The Nappi-Witten symmetry was introduced in [83, 84] and can be seen as a central extension of the homogeneous part of the Galilei algebra. Here, we apply the $S$-expansion procedure to the Nappi-Witten superalgebra introduced in [8] to obtain known and new NR superalgebras. We get two families of NR superalgebras by considering two different semigroup families. In particular, we first show that the extended Bargmann superalgebra and its generalizations can be obtained as an $S$-expansion of the super Nappi-Witten algebra. Then,

[^1]using the same method but different semigroups, we get the extended Newton-Hooke superalgebra and its generalizations. The construction of NR CS supergravity actions for each NR superalgebra is also presented. Interestingly, we prove that the extended Bargmann supergravity along with its Maxwellian version correspond to particular subcases of a generalized extended Bargmann supergravity theory. Furthermore, they can alternatively be obtained as an Inönü-Wigner (IW) contraction [85] of the generalized extended NewtonHooke supergravity theory presented here. Our construction offers a systematic way to obtain different NR supergravity models which are supersymmetric extensions of distinct NR gravity theories.

The organization of the paper is as follows: in section 2, we discuss the supersymmetric extension of the Nappi-Witten algebra. In section 3, we obtain the extended Bargmann supergravity theory, its Maxwellian version, and further generalizations by applying the $S$ expansion method to the super Nappi-Witten algebra and its invariant tensor. In section 4, we recover an extended Newton-Hooke supergravity along with generalizations of the latter by considering different semigroups. Section 5 is devoted to some concluding remarks.

## 2 Nappi-Witten superalgebra and Chern-Simons action

A supersymmetric extension of the Nappi-Witten algebra was recently introduced in [8]. In addition to the Nappi-Witten generators $\left\{J, G_{a}, S\right\}$, it contains two additional bosonic generators $\left\{T_{1}, T_{2}\right\}$ and three Majorana fermionic generators given by $Q_{\alpha}^{+}, Q_{\alpha}^{-}$, and $R_{\alpha}$ (with $\alpha, \beta, \ldots=1,2$ ). The extra bosonic content assures not only the Jacobi identities of the superalgebra but also the non-degeneracy of the invariant tensor. The super NappiWitten generators satisfy the following (anti-)commutation relations: ${ }^{3}$

$$
\begin{aligned}
{\left[J, G_{a}\right] } & =\epsilon_{a b} G_{b}, \\
{\left[J, Q_{\alpha}^{ \pm}\right] } & =-\frac{1}{2}\left(\gamma_{0}\right)_{\alpha}{ }^{\beta} Q_{\beta}^{ \pm}, \\
{\left[G_{a}, Q_{\alpha}^{+}\right] } & =-\frac{1}{2}\left(\gamma_{a}\right)_{\alpha}{ }^{\beta} Q_{\beta}^{-}, \\
{\left[S, Q_{\alpha}^{+}\right] } & =-\frac{1}{2}\left(\gamma_{0}\right)_{\alpha}^{\beta} R_{\beta}, \\
{\left[T_{2}, Q_{\alpha}^{+}\right] } & =\frac{1}{2}\left(\gamma_{0}\right)_{\alpha \beta} R_{\beta}, \\
\left\{Q_{\alpha}^{+}, Q_{\beta}^{-}\right\} & =-\left(\gamma^{a} C\right)_{\alpha \beta} G_{a}, \\
\left\{Q_{\alpha}^{+}, Q_{\beta}^{+}\right\} & =-\left(\gamma^{0} C\right)_{\alpha \beta} J-\left(\gamma^{0} C\right)_{\alpha \beta} T_{1}, \\
\left\{Q_{\alpha}^{-}, Q_{\beta}^{-}\right\} & =-\left(\gamma^{0} C\right)_{\alpha \beta} S+\left(\gamma^{0} C\right)_{\alpha \beta} T_{2}, \\
\left\{Q_{\alpha}^{+}, R_{\beta}\right\} & =-\left(\gamma^{0} C\right)_{\alpha \beta} S-\left(\gamma^{0} C\right)_{\alpha \beta} T_{2},
\end{aligned}
$$

$$
\begin{aligned}
{\left[G_{a}, G_{b}\right] } & =-\epsilon_{a b} S, \\
{\left[J, R_{\alpha}\right] } & =-\frac{1}{2}\left(\gamma_{0}\right)_{\alpha}^{\beta} R_{\beta}, \\
{\left[G_{a}, Q_{\alpha}^{-}\right] } & =-\frac{1}{2}\left(\gamma_{a}\right)_{\alpha}^{\beta} R_{\beta}, \\
{\left[T_{1}, Q_{\alpha}^{ \pm}\right] } & = \pm \frac{1}{2}\left(\gamma_{0}\right)_{\alpha \beta} Q_{\beta}^{ \pm}, \\
{\left[T_{1}, R_{\alpha}\right] } & =\frac{1}{2}\left(\gamma_{0}\right)_{\alpha \beta} R_{\beta},
\end{aligned}
$$

where $a, b, \ldots=1,2, \epsilon_{a b}=\epsilon_{0 a b}, \epsilon^{a b}=\epsilon^{0 a b}, \gamma^{0}$ and $\gamma^{a}$ are the Dirac gamma matrices in three dimensions, and $C$ is the charge conjugation matrix. Let us note that the NappiWitten superalgebra (2.1) can be decomposed in a bosonic subspace $V_{0}=\left\{J, G_{a}, S, T_{1}, T_{2}\right\}$

[^2]and a fermionic one $V_{1}=\left\{Q_{\alpha}^{+}, Q_{\alpha}^{-}, R_{\alpha}\right\}$ such that they satisfy
\[

$$
\begin{equation*}
\left[V_{0}, V_{0}\right] \subset V_{0}, \quad\left[V_{0}, V_{1}\right] \subset V_{1}, \quad\left[V_{1}, V_{1}\right] \subset V_{0} . \tag{2.2}
\end{equation*}
$$

\]

In addition, a non-degenerate invariant supertrace on the Nappi-Witten superalgebra is given in terms of the following non-vanishing components:

$$
\begin{align*}
\left\langle G_{a} G_{b}\right\rangle & =\delta_{a b}, \\
\langle J S\rangle & =-1, \\
\left\langle T_{1} T_{2}\right\rangle & =1, \\
\left\langle Q_{\alpha}^{-} Q_{\beta}^{-}\right\rangle & =\left\langle Q_{\alpha}^{+} R_{\beta}\right\rangle=2 C_{\alpha \beta} . \tag{2.3}
\end{align*}
$$

Then, the three-dimensional CS action based on the Nappi-Witten superalgebra can be constructed introducing the gauge connection one-form

$$
\begin{equation*}
A=\omega J+\omega^{a} G_{a}+s S+t_{1} T_{1}+t_{2} T_{2}+\bar{\psi}^{+} Q^{+}+\bar{\psi}^{-} Q^{-}+\bar{\rho} R \tag{2.4}
\end{equation*}
$$

and the previously defined invariant supertrace in the general expression of a CS action in three spacetime dimensions,

$$
\begin{equation*}
I_{\mathrm{CS}}=\int\left\langle A \wedge d A+\frac{1}{3} A \wedge[A, A]\right\rangle . \tag{2.5}
\end{equation*}
$$

The supersymmetric CS action invariant under the Nappi-Witten superalgebra (2.1) is given by ${ }^{4}$

$$
\begin{equation*}
I_{\mathrm{NW}}=\int \omega_{a} R^{a}\left(\omega^{b}\right)-2 s R(\omega)+2 t_{1} d t_{2}+2 \bar{\psi}^{-} \nabla \psi^{-}+2 \bar{\psi}^{+} \nabla \rho+2 \bar{\rho} \nabla \psi^{+} \tag{2.6}
\end{equation*}
$$

where

$$
\begin{align*}
R(\omega) & =d \omega \\
R^{a}\left(\omega^{b}\right) & =d \omega^{a}+\epsilon^{a c} \omega \omega \quad, \tag{2.7}
\end{align*}
$$

while the covariant derivatives of the spinor 1-form fields appearing in (2.6) are

$$
\begin{align*}
\nabla \psi^{+} & =d \psi^{+}+\frac{1}{2} \omega \gamma_{0} \psi^{+}-\frac{1}{2} t_{1} \gamma_{0} \psi^{+}, \\
\nabla \psi^{-} & =d \psi^{-}+\frac{1}{2} \omega \gamma_{0} \psi^{-}+\frac{1}{2} \omega^{a} \gamma_{a} \psi^{+}+\frac{1}{2} t_{1} \gamma_{0} \psi^{-}, \\
\nabla \rho & =d \rho+\frac{1}{2} \omega \gamma_{0} \rho+\frac{1}{2} \omega^{a} \gamma_{a} \psi^{-}+\frac{1}{2} s \gamma_{0} \psi^{+}-\frac{1}{2} t_{2} \gamma_{0} \psi^{+}-\frac{1}{2} t_{1} \gamma_{0} \rho . \tag{2.8}
\end{align*}
$$

Let us note that having a non-degenerate invariant trace correspond to the physical requirement that the CS supersymmetric action (2.6) involves a kinematical term for each

[^3]gauge field and the field equations reduce to the vanishing of all the curvatures of the model. In the present case, the curvatures for each gauge field are given by
\[

$$
\begin{align*}
F(\omega) & =R(\omega)+\frac{1}{2} \bar{\psi}^{+} \gamma^{0} \psi^{+}, \\
F^{a}\left(\omega^{b}\right) & =R^{a}\left(\omega^{b}\right)+\bar{\psi}^{+} \gamma^{a} \psi^{-}, \\
F(s) & =d s+\frac{1}{2} \bar{\psi}^{-} \gamma^{0} \psi^{-}+\bar{\psi}^{+} \gamma^{0} \rho, \\
F\left(t_{1}\right) & =d t_{1}+\frac{1}{2} \bar{\psi}^{+} \gamma^{0} \psi^{+}, \\
F\left(t_{2}\right) & =d t_{2}-\frac{1}{2} \bar{\psi}^{-} \gamma^{0} \psi^{-}+\bar{\psi}^{+} \gamma^{0} \rho, \tag{2.9}
\end{align*}
$$
\]

along with (2.8). Then, the field equations coming from the variation of the CS action (2.6) correspond, as it is expected, to the vanishing of all curvatures (2.8) and (2.9).

In the following sections, we will apply the $S$-expansion method [53] to the super NappiWitten algebra in order to obtain different NR superalgebras. For our purpose, we will consider two different types of semigroup, that are $S_{E}^{(2 N)}=\left\{\lambda_{0}, \lambda_{1}, \ldots, \lambda_{2 N}, \lambda_{2 N+1}\right\}$ and $S_{\mathcal{M}}^{(2 N)}=\left\{\lambda_{0}, \lambda_{1}, \lambda_{2}, \ldots, \lambda_{2 N-1}, \lambda_{2 N}\right\}$. In both cases, for different values of $N$, known and new NR superalgebras will appear, corresponding to distinct supersymmetric extensions of known NR algebras.

One of the advantages of working with the $S$-expansion is that it provides us with the non-degenerate invariant tensors of the $S$-expanded (super)algebras, which are given in terms of the invariant tensors for the original (super)algebra. This turns out to be essential in the construction of NR CS (super)gravity actions.

## 3 Generalized extended Bargmann supergravity theory and semigroup expansion method

In this section, we shall derive different NR superalgebras whose bosonic sectors include as subalgebra the extended Bargmann algebra and its generalizations, depending on the case. The aforementioned NR superalgebras appear as the result of applying the $S$-expansion procedure to the Nappi-Witten superalgebra introduced in the previous section, considering $S_{E}^{(2 N)}=\left\{\lambda_{0}, \lambda_{1}, \ldots, \lambda_{2 N}, \lambda_{2 N+1}\right\}$ as the relevant semigroup. We will show that all known and new NR superalgebras belong to a family of NR superalgebras, which we call as generalized extended Bargmann (GEB ${ }^{(N)}$ ) superalgebras. As we will see, they correspond to supersymmetric extensions of the NR counterparts of the $\mathfrak{B}_{N+2}$ algebras enlarged with $(N+1) U_{1}$ generators [27]. At the relativistic level, the $\mathfrak{B}_{N+2}$ algebras were first introduced in $[69,86]$ for obtaining standard General Relativity from Chern-Simons gravity. Furthermore, as we have mentioned before, an important advantage of the $S$-expansion is that it allows to derive the non-vanishing components of an invariant tensor of the expanded (super)algebra. Then, we will exploit this powerful feature to derive the NR invariant supertraces and the CS actions invariant under the aforesaid expanded NR superalgebras.

### 3.1 Extended Bargmann supergravity

The so-called extended Bargmann algebra [3] can be obtained as the NR limit of the Poincaré algebra (corresponding to the $\mathfrak{B}_{3}$ algebra) enlarged with two $U_{1}$ generators. An alternative method has been proposed in [65] in which the extended Bargmann algebra can be derived as an $S$-expansion of the Nappi-Witten algebra. Here we show that a supersymmetric extension of the extended Bargmann algebra can be obtained by performing an $S$-expansion of the Nappi-Witten superalgebra (2.1). To this end, we consider $S_{E}^{(2)}=\left\{\lambda_{0}, \lambda_{1}, \lambda_{2}, \lambda_{3}\right\}$ as the relevant semigroup whose elements satisfy the following multiplication table:

| $\lambda_{3}$ | $\lambda_{3}$ | $\lambda_{3}$ | $\lambda_{3}$ | $\lambda_{3}$ |
| :---: | :---: | :---: | :---: | :---: |
| $\lambda_{2}$ | $\lambda_{2}$ | $\lambda_{3}$ | $\lambda_{3}$ | $\lambda_{3}$ |
| $\lambda_{1}$ | $\lambda_{1}$ | $\lambda_{2}$ | $\lambda_{3}$ | $\lambda_{3}$ |
| $\lambda_{0}$ | $\lambda_{0}$ | $\lambda_{1}$ | $\lambda_{2}$ | $\lambda_{3}$ |
|  | $\lambda_{0}$ | $\lambda_{1}$ | $\lambda_{2}$ | $\lambda_{3}$ |

Indeed, one can consider a resonant decomposition,

$$
\begin{align*}
& S_{0}=\left\{\lambda_{0}, \lambda_{2}, \lambda_{3}\right\} \\
& S_{1}=\left\{\lambda_{1}, \lambda_{3}\right\} \tag{3.2}
\end{align*}
$$

with $\lambda_{3}=0_{S}$ being the zero element of the semigroup. Let us note that the decomposition (3.2) is said to be resonant since it satisfies the same structure as the super Nappi-Witten subspaces,

$$
\begin{equation*}
S_{0} \cdot S_{0} \subset S_{0}, \quad S_{0} \cdot S_{1} \subset S_{1}, \quad S_{1} \cdot S_{1} \subset S_{0} \tag{3.3}
\end{equation*}
$$

Then, following the definitions of [53], after considering a resonant $S_{E}^{(2)}$-expansion followed by a $0_{S}$-reduction to the Nappi-Witten superalgebra (2.1), we find an expanded superalgebra spanned by the set of generators ${ }^{5}\left\{\tilde{J}, \tilde{G}_{a}, \tilde{S}, \tilde{H}, \tilde{P}_{a}, \tilde{M}, \tilde{Y}_{1}, \tilde{Y}_{2}, \tilde{U}_{1}, \tilde{U}_{2}, \tilde{Q}_{\alpha}^{+}, \tilde{Q}_{\alpha}^{-}, \tilde{R}_{\alpha}\right\}$ which are related to the super Nappi-Witten ones through the semigroup elements as follows:

$$
\begin{array}{rlrlrl}
\tilde{J} & =\lambda_{0} J, & & \tilde{G}_{a}=\lambda_{0} G_{a}, & & \tilde{S}=\lambda_{0} S, \\
\tilde{H} & =\lambda_{2} J, & \tilde{P}_{a}=\lambda_{2} G_{a}, & & \tilde{Y}_{1}=\lambda_{0} T_{1}, & \\
\tilde{Q}_{\alpha}^{+} & =\lambda_{1} Q_{\alpha}^{+}, & & \tilde{Q}_{\alpha}^{-}=\lambda_{2} S, & \tilde{Y}_{2} Q_{\alpha}=\lambda_{0} T_{2}, &  \tag{3.4}\\
\tilde{U}_{2}=\lambda_{2} T_{2} \\
& & \tilde{R}_{\alpha}=\lambda_{1} R_{\alpha} & &
\end{array}
$$

Such expanded generators satisfy the following non-vanishing (anti-)commutation relations:

$$
\begin{aligned}
& {\left[\tilde{J}, \tilde{G}_{a}\right]=\epsilon_{a b} \tilde{G}_{b}, \quad\left[\tilde{G}_{a}, \tilde{G}_{b}\right]=-\epsilon_{a b} \tilde{S}, \quad\left[\tilde{J}, \tilde{P}_{a}\right]=\epsilon_{a b} \tilde{P}_{b}} \\
& {\left[\tilde{H}, \tilde{G}_{a}\right]=\epsilon_{a b} \tilde{P}_{b}, \quad\left[\tilde{G}_{a}, \tilde{P}_{b}\right]=-\epsilon_{a b} \tilde{M}, \quad\left[\tilde{J}, \tilde{Q}_{\alpha}^{ \pm}\right]=-\frac{1}{2}\left(\gamma_{0}\right)_{\alpha}{ }^{\beta} \tilde{Q}_{\beta}^{ \pm},} \\
& {\left[\tilde{J}, \tilde{R}_{\alpha}\right]=-\frac{1}{2}\left(\gamma_{0}\right)_{\alpha}{ }^{\beta} \tilde{R}_{\beta}, \quad\left[\tilde{G}_{a}, \tilde{Q}_{\alpha}^{+}\right]=-\frac{1}{2}\left(\gamma_{a}\right)_{\alpha}{ }^{\beta} \tilde{Q}_{\beta}^{-}, \quad\left[\tilde{G}_{a}, \tilde{Q}_{\alpha}^{-}\right]=-\frac{1}{2}\left(\gamma_{a}\right)_{\alpha}{ }^{\beta} \tilde{R}_{\beta},} \\
& {\left[\tilde{S}, \tilde{Q}_{\alpha}^{+}\right]=-\frac{1}{2}\left(\gamma_{0}\right)_{\alpha}{ }^{\beta} \tilde{R}_{\beta}, \quad\left[\tilde{Y}_{1}, Q_{\alpha}^{ \pm}\right]= \pm \frac{1}{2}\left(\gamma_{0}\right)_{\alpha \beta} Q_{\beta}^{ \pm}, \quad\left[\tilde{Y}_{1}, R_{\alpha}\right]=\frac{1}{2}\left(\gamma_{0}\right)_{\alpha \beta} R_{\beta},}
\end{aligned}
$$

[^4]\[

$$
\begin{align*}
{\left[\tilde{Y}_{2}, \tilde{Q}_{\alpha}^{+}\right] } & =\frac{1}{2}\left(\gamma_{0}\right)_{\alpha \beta} R_{\beta} \\
\left\{\tilde{Q}_{\alpha}^{+}, \tilde{Q}_{\beta}^{-}\right\} & =-\left(\gamma^{a} C\right)_{\alpha \beta} \tilde{P}_{a}, \\
\left\{\tilde{Q}_{\alpha}^{+}, \tilde{Q}_{\beta}^{+}\right\} & =-\left(\gamma^{0} C\right)_{\alpha \beta} \tilde{H}-\left(\gamma^{0} C\right)_{\alpha \beta} \tilde{U}_{1}, \\
\left\{\tilde{Q}_{\alpha}^{-}, \tilde{Q}_{\beta}^{-}\right\} & =-\left(\gamma^{0} C\right)_{\alpha \beta} \tilde{M}+\left(\gamma^{0} C\right)_{\alpha \beta} \tilde{U}_{2}, \\
\left\{\tilde{Q}_{\alpha}^{+}, \tilde{R}_{\beta}\right\} & =-\left(\gamma^{0} C\right)_{\alpha \beta} \tilde{M}-\left(\gamma^{0} C\right)_{\alpha \beta} \tilde{U}_{2} . \tag{3.5}
\end{align*}
$$
\]

This superalgebra corresponds to an extension of the extended Bargmann superalgebra introduced in $[3] .{ }^{6}$ In fact, by considering $\tilde{Y}_{1}=\tilde{Y}_{2}=\tilde{U}_{1}=\tilde{U}_{2}=0$, one can exactly reproduce the extended Bargmann superalgebra of [3]. Let us note that the additional bosonic generators $\tilde{Y}_{1}, \tilde{Y}_{2}, \tilde{U}_{1}$, and $\tilde{U}_{2}$ appear as a consequence of the expansion of the Nappi-Witten generators $T_{1}$ and $T_{2}$. They allow to get a non-degenerate invariant tensor within our framework. In this regards, observe that the central charge $\tilde{U}_{1}$ may be trivially reabsorbed through a redefinition of $\tilde{H}$. Nevertheless, we retain $\tilde{U}_{1}$ in the sequel, since it directly follows from the expansion we have performed.

The invariant tensor for the extended Bargmann superalgebra can be obtained by applying the $S$-expansion method to the invariant tensor of the Nappi-Witten superalgebra, given in (2.3). In this way, one finds that the non-vanishing components of a non-degenerate invariant tensor for the extended Bargmann superalgebra are given by

$$
\begin{align*}
& \left\langle\tilde{G}_{a} \tilde{G}_{b}\right\rangle=\alpha_{0} \delta_{a b}, \quad\langle\tilde{J} \tilde{S}\rangle=-\alpha_{0}, \\
& \left\langle\tilde{G}_{a} \tilde{P}_{b}\right\rangle=\alpha_{1} \delta_{a b}, \quad\left\langle\tilde{Y}_{1} \tilde{Y}_{2}\right\rangle=\alpha_{0}, \\
& \langle\tilde{J} \tilde{M}\rangle=\langle\tilde{H} \tilde{S}\rangle=-\alpha_{1}, \quad\left\langle\tilde{Y}_{1} \tilde{U}_{2}\right\rangle=\left\langle\tilde{U}_{1} \tilde{Y}_{2}\right\rangle=\alpha_{1}, \\
& \left\langle\tilde{Q}_{\alpha}^{-} \tilde{Q}_{\beta}^{-}\right\rangle=\left\langle\tilde{Q}_{\alpha}^{+} \tilde{R}_{\beta}\right\rangle=2 \alpha_{1} C_{\alpha \beta}, \tag{3.6}
\end{align*}
$$

where the $\alpha$ 's are arbitrary constants and appear as a consequence of the $S$-expansion procedure. Then, the three-dimensional CS action invariant under the superalgebra (3.5) can be directly constructed by introducing the gauge connection one-form

$$
\begin{align*}
A= & \omega \tilde{J}+h \tilde{H}+\omega^{a} \tilde{G}_{a}+e^{a} \tilde{P}_{a}+s \tilde{S}+m \tilde{M}+y_{1} \tilde{Y}_{1}+y_{2} \tilde{Y}_{2}+u_{1} \tilde{U}_{1}+u_{2} \tilde{U}_{2} \\
& +\bar{\psi}^{+} \tilde{Q}^{+}+\bar{\psi}^{-} \tilde{Q}^{-}+\tilde{\rho} \tilde{R} \tag{3.7}
\end{align*}
$$

and the invariant tensor (3.6) in the general expression for a CS action (2.5). The NR CS supergravity action reads

$$
\begin{equation*}
I_{\mathrm{EB}}=\alpha_{0} I_{0}+\alpha_{1} I_{1}, \tag{3.8}
\end{equation*}
$$

where

$$
\begin{align*}
I_{0}= & \int \omega_{a} R^{a}\left(\omega^{b}\right)-2 s R(\omega)+2 y_{1} d y_{2}, \\
I_{1}= & \int 2 e_{a} R^{a}\left(\omega^{b}\right)-2 m R(\omega)-2 \tau R(s)+2 y_{1} d u_{2}+2 u_{1} d y_{2} \\
& +2 \bar{\psi}^{+} \nabla \rho+2 \bar{\rho} \nabla \psi^{+}+2 \bar{\psi}^{-} \nabla \psi^{-}, \tag{3.9}
\end{align*}
$$

[^5]and
\[

$$
\begin{equation*}
R(s)=d s+\frac{1}{2} \epsilon^{a c} \omega_{a} \omega_{c} \tag{3.10}
\end{equation*}
$$

\]

while $R(\omega)$ and $R^{a}\left(\omega^{b}\right)$ are given by (2.7). Besides, the covariant derivatives of the spinor 1-form fields read

$$
\begin{align*}
\nabla \psi^{+} & =d \psi^{+}+\frac{1}{2} \omega \gamma_{0} \psi^{+}-\frac{1}{2} y_{1} \gamma_{0} \psi^{+} \\
\nabla \psi^{-} & =d \psi^{-}+\frac{1}{2} \omega \gamma_{0} \psi^{-}+\frac{1}{2} \omega^{a} \gamma_{a} \psi^{+}+\frac{1}{2} y_{1} \gamma_{0} \psi^{-} \\
\nabla \rho & =d \rho+\frac{1}{2} \omega \gamma_{0} \rho+\frac{1}{2} \omega^{a} \gamma_{a} \psi^{-}+\frac{1}{2} s \gamma_{0} \psi^{+}-\frac{1}{2} y_{2} \gamma_{0} \psi^{+}-\frac{1}{2} y_{1} \gamma_{0} \rho \tag{3.11}
\end{align*}
$$

The CS action (3.8) describes an extension of the so-called extended Bargmann supergravity theory [3]. Indeed, the CS action $I_{1}$ corresponds to the extended Bargmann supergravity action introduced in [3], endowed with some additional terms involving the extra bosonic 1 -form fields $y_{1}, y_{2}, u_{1}$, and $u_{2}$. Furthermore, the term along $\alpha_{0}$ corresponds to a NR exotic Lagrangian. The extra bosonic field content is related to the additional bosonic generators which allow to define the non-degenerate invariant tensor (3.6). The non-degeneracy of the invariant tensor implies that the equations of motion are given by the vanishing of the curvature two-forms of the model, which are given by (3.11) and

$$
\begin{array}{rlrl}
F(\omega) & =R(\omega), & F\left(\omega^{b}\right) & =R^{a}\left(\omega^{b}\right), \\
F(s) & =R(s), & F(\tau)=d(\tau)+\frac{1}{2} \bar{\psi}^{+} \gamma^{0} \psi^{+}, \\
F^{a}\left(e^{b}\right) & =d e^{a}+\epsilon^{a c} \omega e_{c}+\epsilon^{a c} \tau \omega_{c}+\bar{\psi}^{+} \gamma^{a} \psi^{-}, & & \\
F(m) & =d m+\epsilon^{a c} \omega_{a} e_{c}+\frac{1}{2} \bar{\psi}^{-} \gamma^{0} \psi^{-}+\bar{\psi}^{+} \gamma^{0} \rho, & \\
F\left(y_{1}\right) & =d y_{1}, & F\left(y_{2}\right)=d y_{2}, \\
F\left(u_{1}\right) & =d u_{1}+\frac{1}{2} \bar{\psi}^{+} \gamma^{0} \psi^{+}, \\
F\left(u_{2}\right) & =d u_{2}-\frac{1}{2} \bar{\psi}^{-} \gamma^{0} \psi^{-}+\bar{\psi}^{+} \gamma^{0} \rho . &
\end{array}
$$

Let us note that the present NR theory can be seen as the most general three-dimensional extended Bargmann supergravity theory containing both exotic and standard sectors. Nonetheless, the formulation of a NR supergravity theory is not unique and can be generalized beyond the extended Bargmann one.

### 3.2 Maxwellian extended Bargmann supergravity

A Maxwellian version of the extended Bargmann algebra was recently presented in [19]. It was denoted as MEB algebra and it was obtained as a NR limit of the Maxwell algebra (also called as $\mathfrak{B}_{4}$ algebra) enlarged with three $U_{1}$ generators. At the relativistic level, the Maxwell symmetry appears in the description of a particle in a Minkowski spacetime in the presence of an electromagnetic field background [87-89]. Subsequently, a supersymmetric extension of the MEB algebra was introduced in [7]. In order to construct a well-defined
supergravity theory in this context it was necessary to construct by hand the aforementioned supersymmetric extension. Now, we are going to show that the MEB superalgebra and the corresponding NR supergravity theory can be derived by means of the $S$-expansion method. Indeed, let us consider the $S_{E}^{(4)}$-expansion of the Nappi-Witten superalgebra (2.1). The elements of the $S_{E}^{(4)}$ semigroup satisfy

| $\lambda_{5}$ | $\lambda_{5}$ | $\lambda_{5}$ | $\lambda_{5}$ | $\lambda_{5}$ | $\lambda_{5}$ | $\lambda_{5}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\lambda_{4}$ | $\lambda_{4}$ | $\lambda_{5}$ | $\lambda_{5}$ | $\lambda_{5}$ | $\lambda_{5}$ | $\lambda_{5}$ |
| $\lambda_{3}$ | $\lambda_{3}$ | $\lambda_{4}$ | $\lambda_{5}$ | $\lambda_{5}$ | $\lambda_{5}$ | $\lambda_{5}$ |
| $\lambda_{2}$ | $\lambda_{2}$ | $\lambda_{3}$ | $\lambda_{4}$ | $\lambda_{5}$ | $\lambda_{5}$ | $\lambda_{5}$ |
| $\lambda_{1}$ | $\lambda_{1}$ | $\lambda_{2}$ | $\lambda_{3}$ | $\lambda_{4}$ | $\lambda_{5}$ | $\lambda_{5}$ |
| $\lambda_{0}$ | $\lambda_{0}$ | $\lambda_{1}$ | $\lambda_{2}$ | $\lambda_{3}$ | $\lambda_{4}$ | $\lambda_{5}$ |
|  | $\lambda_{0}$ | $\lambda_{1}$ | $\lambda_{2}$ | $\lambda_{3}$ | $\lambda_{4}$ | $\lambda_{5}$ |

where $\lambda_{5}=0_{S}$ is the zero element of the semigroup. Let $S_{E}^{(4)}=S_{0} \cup S_{1}$ be a resonant decomposition with

$$
\begin{align*}
& S_{0}=\left\{\lambda_{0}, \lambda_{2}, \lambda_{4}, \lambda_{5}\right\}, \\
& S_{1}=\left\{\lambda_{1}, \lambda_{3}, \lambda_{5}\right\} . \tag{3.14}
\end{align*}
$$

After considering a resonant $S_{E}^{(4)}$-expansion followed by a $0_{S}$-reduction of the Nappi-Witten superalgebra, we find an expanded superalgebra spanned by the generators

$$
\left\{\tilde{J}, \tilde{G}_{a}, \tilde{S}, \tilde{H}, \tilde{P}_{a}, \tilde{M}, \tilde{Z}, \tilde{Z}_{a}, \tilde{T}, \tilde{Y}_{1}, \tilde{Y}_{2}, \tilde{U}_{1}, \tilde{U}_{2}, \tilde{B}_{1}, \tilde{B}_{2}, \tilde{Q}_{\alpha}^{+}, \tilde{Q}_{\alpha}^{-}, \tilde{R}_{\alpha}, \tilde{\Sigma}_{\alpha}^{+}, \tilde{\Sigma}_{\alpha}^{-}, \tilde{W}_{\alpha}\right\}
$$

which are related to the super Nappi-Witten ones through

$$
\begin{array}{rlrlrlrl}
\tilde{J} & =\lambda_{0} J, & \tilde{G}_{a} & =\lambda_{0} G_{a}, & \tilde{S}=\lambda_{0} S, & & \tilde{Y}_{1}=\lambda_{0} T_{1}, & \\
\tilde{H}=\lambda_{2} J, & \tilde{P}_{a}=\lambda_{0} T_{2}, \\
\tilde{Z} G_{a}, & \tilde{M}=\lambda_{2} S, & & \tilde{U}_{1}=\lambda_{2} T_{1}, & & \tilde{U}_{2}=\lambda_{2} T_{2}, \\
\tilde{Z}=\lambda_{4} J, & \tilde{Z}_{a}=\lambda_{4} G_{a}, & \tilde{T}=\lambda_{4} S, & & \tilde{B}_{1}=\lambda_{4} T_{1}, & & \tilde{B}_{2}=\lambda_{4} T_{2}, \\
\tilde{Q}_{\alpha}^{+}=\lambda_{1} Q_{\alpha}^{+}, & \tilde{Q}_{\alpha}^{-}=\lambda_{1} Q_{\alpha}^{-}, & \tilde{\Sigma}_{\alpha}^{+}=\lambda_{3} Q_{\alpha}^{+}, & & \tilde{\Sigma}_{\alpha}^{-}=\lambda_{3} Q_{\alpha}^{-}, & & \tilde{R}_{\alpha}=\lambda_{1} R_{\alpha},  \tag{3.15}\\
\tilde{W}_{\alpha}=\lambda_{3} R_{\alpha} . & & & & &
\end{array}
$$

Using the multiplication law (3.13) and the original (anti-)commutation relations of the Nappi-Witten superalgebra (2.1), one can show that the expanded generators satisfy an expanded NR superalgebra whose non-trivial (anti-)commutation relations are given by (3.5) along with

$$
\begin{array}{rlrl}
{\left[\tilde{P}_{a}, \tilde{P}_{b}\right]} & =-\epsilon_{a b} \tilde{T}, & {\left[\tilde{G}_{a}, \tilde{Z}_{b}\right]} & =-\epsilon_{a b} \tilde{T}, \\
{\left[\tilde{J}, \tilde{Z}_{a}\right]} & =\epsilon_{a b} \tilde{Z}_{b}, & {\left[\tilde{H}, \tilde{P}_{a}\right]} & =\epsilon_{a b} \tilde{Z}_{b}, \\
{\left[\tilde{J}, \tilde{\Sigma}_{a}^{ \pm}\right]} & =-\frac{1}{2}\left(\epsilon_{a b}\right)_{\alpha}^{\beta} \tilde{\Sigma}_{b}^{ \pm}, & {\left[\tilde{U}_{2}, \tilde{Q}_{\alpha}^{+}\right]} & =\frac{1}{2}\left(\gamma_{0}\right)_{\alpha \beta} \tilde{W}_{\beta}, \\
{\left[\tilde{S}, \tilde{Q}_{\alpha}^{+}\right]=-\frac{1}{2}\left(\gamma_{0}\right)_{\alpha}^{\beta} \tilde{W}_{\beta},} & {\left[\tilde{Y}_{1}, \tilde{W}_{\alpha}\right]=\frac{1}{2}\left(\gamma_{0}\right)_{\alpha \beta} \tilde{W}_{\beta},} \\
{\left[\frac{1}{2}\left(\gamma_{0}\right)_{\alpha}^{\beta} \tilde{\Sigma}_{\beta}^{ \pm},\right.} & {\left[\tilde{P}_{a}, \tilde{Q}_{\alpha}^{+}\right]=-\frac{1}{2}\left(\gamma_{a}\right)_{\alpha}^{\beta} \tilde{\Sigma}_{\beta}^{-},} & {\left[\tilde{Y}_{1}, \tilde{\Sigma}_{\alpha}^{+}\right]=\frac{1}{2}\left(\gamma_{0}\right)_{\alpha \beta} \tilde{\Sigma}_{\beta}^{+},}
\end{array}
$$

$$
\begin{align*}
{\left[\tilde{G}_{a}, \tilde{\Sigma}_{\alpha}^{+}\right] } & =-\frac{1}{2}\left(\gamma_{a}\right)_{\alpha}^{\beta} \tilde{\Sigma}_{\beta}^{-},
\end{align*} \quad\left[\tilde{G}_{a}, \tilde{\Sigma}_{\alpha}^{-}\right]=-\frac{1}{2}\left(\gamma_{a}\right)_{\alpha}^{\beta} \tilde{W}_{\beta}, \quad\left[\begin{array}{cc}
\left.\tilde{Y}_{1}, \tilde{\Sigma}_{\alpha}^{-}\right]=-\frac{1}{2}\left(\gamma_{0}\right)_{\alpha \beta} \tilde{\Sigma}_{\beta}^{-}, \\
{\left[P_{a}, \tilde{Q}_{\alpha}^{-}\right]} & =-\frac{1}{2}\left(\gamma_{a}\right)_{\alpha}^{\beta} \tilde{W}_{\beta}, \\
{\left[\tilde{J}, \tilde{R}_{\alpha}\right]=-\frac{1}{2}\left(\gamma_{0}\right)_{\alpha}^{\beta} \tilde{R}_{\beta},} & {\left[\tilde{U}_{1}, \tilde{Q}_{\alpha}^{+}\right]=\frac{1}{2}\left(\gamma_{0}\right)_{\alpha \beta} \tilde{\Sigma}_{\beta}^{+},} \\
{\left[\tilde{W}_{\alpha}\right]} & =-\frac{1}{2}\left(\gamma_{0}\right)_{\alpha}^{\beta} \tilde{W}_{\beta}, \\
{\left[\tilde{M}, \tilde{Q}^{2}, \tilde{R}_{\alpha}\right]=-\frac{1}{2}\left(\gamma_{0}\right)_{\alpha}^{\beta} \tilde{W}_{\beta},} & {\left[\tilde{U}_{1}, \tilde{Q}_{\alpha}^{-}\right]=-\frac{1}{2}\left(\gamma_{0}\right)_{\alpha \beta} \tilde{\Sigma}_{\beta}^{-},} \\
\left\{\tilde{Q}_{\alpha}^{-}, \tilde{\Sigma}_{\beta}^{-}\right\} & =-\left(\gamma^{0} C\right)_{\alpha \beta} \tilde{T}\left(\gamma_{0}\right)_{\alpha}^{\beta} \tilde{W}_{\beta}, \quad\left[\tilde{Y}_{2}, \tilde{\Sigma}_{\alpha}^{+}\right]=\frac{1}{2}\left(\gamma_{0} C\right)_{\alpha \beta} \tilde{W}_{\beta}, \\
\tilde{B}_{2}, & {\left[\tilde{U}_{1}, \tilde{R}_{\alpha}\right]=\frac{1}{2}\left(\gamma_{0}\right)_{\alpha \beta} \tilde{W}_{\beta},} \\
\left\{\tilde{Q}_{\alpha}^{ \pm}, \tilde{\Sigma}_{\beta}^{\mp}\right\} & =-\left(\gamma^{a} C\right)_{\alpha \beta} \tilde{Z}_{a},  \tag{3.16}\\
\left\{\tilde{Q}_{\alpha}^{+}, \tilde{\Sigma}_{\beta}^{+}\right\} & =-\left(\gamma^{0} C\right)_{\alpha \beta} \tilde{Z}-\left(\gamma^{0} C\right)_{\alpha \beta} \tilde{B}_{1}, \\
\left\{\tilde{Q}_{\alpha}^{+}, \tilde{W}_{\beta}\right\} & =\left\{\tilde{\Sigma}_{\alpha}^{+}, \tilde{R}_{\beta}\right\}=-\left(\gamma^{0} C\right)_{\alpha \beta} \tilde{T}-\left(\gamma^{0} C\right)_{\alpha \beta} \tilde{B}_{2} .
\end{array}\right.
$$

The superalgebra just obtained corresponds to the Maxwellian extended Bargmann superalgebra first presented in [7]. The MEB superalgebra is characterized by the presence of three additional fermionic generators, besides $\tilde{Q}_{\alpha}^{ \pm}$and $\tilde{R}_{\alpha}$, namely $\tilde{\Sigma}_{\alpha}^{+}, \tilde{\Sigma}_{\alpha}^{-}$, and $\tilde{W}_{\alpha}$. Furthermore, unlike the extended Bargmann superalgebra, $\tilde{U}_{1}$ and $\tilde{U}_{2}$ are no longer central charges but act non-trivially on the fermionic generators $\tilde{Q}_{\alpha}^{ \pm}$and $\tilde{R}_{\alpha}$. The non-vanishing components of an invariant supertrace for the MEB superalgebra are obtained from the Nappi-Witten ones through the $S$-expansion method. These components are given by (3.6) together with

$$
\begin{align*}
\left\langle\tilde{P}_{a} \tilde{P}_{b}\right\rangle & =\left\langle\tilde{G}_{a} \tilde{Z}_{b}\right\rangle=\alpha_{2} \delta_{a b}, \\
\langle\tilde{J} \tilde{T}\rangle & =\langle\tilde{H} \tilde{M}\rangle=\langle\tilde{S} \tilde{Z}\rangle=-\alpha_{2}, \\
\left\langle\tilde{Y}_{1} \tilde{B}_{2}\right\rangle & =\left\langle\tilde{U}_{1} \tilde{U}_{2}\right\rangle=\left\langle\tilde{B}_{1} \tilde{Y}_{2}\right\rangle=\alpha_{2}, \\
\left\langle\tilde{Q}_{\alpha}^{-} \tilde{\Sigma}_{\beta}^{-}\right\rangle & =\left\langle\tilde{\Sigma}_{\alpha}^{+} \tilde{R}_{\beta}\right\rangle=\left\langle\tilde{Q}_{\alpha}^{+} \tilde{W}_{\beta}\right\rangle=2 \alpha_{2} C_{\alpha \beta}, \tag{3.17}
\end{align*}
$$

where $\alpha_{2}$ is an arbitrary constant. The NR CS supergravity action invariant under the MEB superalgebra, which was presented in [7], is obtained by considering the following gauge connection one-form:

$$
\begin{align*}
A= & \omega \tilde{J}+\omega^{a} \tilde{G}_{a}+\tau \tilde{H}+e^{a} \tilde{P}_{a}+k \tilde{Z}+k^{a} \tilde{Z}_{a}+m \tilde{M}+s \tilde{S}+t \tilde{T} \\
& +y_{1} \tilde{Y}_{1}+y_{2} \tilde{Y}_{2}+b_{1} \tilde{B}_{1}+b_{2} \tilde{B}_{2}+u_{1} \tilde{U}_{1}+u_{2} \tilde{U}_{2} \\
& +\psi^{+} \tilde{Q}^{+}+\psi^{-} \tilde{Q}^{-}+\xi^{+} \tilde{\Sigma}^{+}+\xi^{-} \tilde{\Sigma}^{-}+\rho \tilde{R}+\chi \tilde{W} \tag{3.18}
\end{align*}
$$

and the non-vanishing components of the invariant tensor (3.6) and (3.17) in the CS expression (2.5). It reads, up to boundary terms, as follows:

$$
\begin{equation*}
I_{\mathrm{MEB}}=\alpha_{0} I_{0}+\alpha_{1} I_{1}+\alpha_{2} I_{2}, \tag{3.19}
\end{equation*}
$$

where $I_{0}$ and $I_{1}$ are given in (3.9), while the term along $\alpha_{2}$ reads

$$
\begin{align*}
I_{2}= & \int e_{a} R^{a}\left(e^{b}\right)+k_{a} R^{a}\left(\omega^{b}\right)+\omega_{a} R^{a}\left(k^{b}\right)-2 s R(k) \\
& -2 m R(\tau)-2 t R(\omega)+2 y_{1} d b_{2}+2 u_{1} d u_{2}+2 y_{2} d b_{1} \\
& +2 \bar{\psi}^{-} \nabla \xi^{-}+2 \bar{\xi}^{-} \nabla \psi^{-}+2 \bar{\psi}^{+} \nabla \chi+2 \bar{\chi} \nabla \psi^{+}+2 \bar{\xi}^{+} \nabla \rho+2 \bar{\rho} \nabla \xi^{+} . \tag{3.20}
\end{align*}
$$

Here, $R(\omega)$ and $R^{a}\left(\omega^{b}\right)$ were defined in (2.7), while

$$
\begin{align*}
R(\tau) & =d \tau \\
R^{a}\left(e^{b}\right) & =d e^{a}+\epsilon^{a c} \omega e_{c}+\epsilon^{a c} \tau \omega_{c} \\
R(k) & =d k \\
R^{a}\left(k^{b}\right) & =d k^{a}+\epsilon^{a c} \omega k_{c}+\epsilon^{a c} \tau e_{c}+\epsilon^{a c} k \omega_{c} . \tag{3.21}
\end{align*}
$$

On the other hand, the covariant derivatives of the spinor 1-form fields appearing in $I_{2}$ are given by

$$
\begin{align*}
\nabla \xi^{+}= & d \xi^{+}+\frac{1}{2} \omega \gamma_{0} \xi^{+}+\frac{1}{2} \tau \gamma_{0} \psi^{+}-\frac{1}{2} y_{1} \gamma_{0} \xi^{+}-\frac{1}{2} u_{1} \gamma_{0} \psi^{+} \\
\nabla \xi^{-}= & d \xi^{-}+\frac{1}{2} \omega \gamma_{0} \xi^{-}+\frac{1}{2} \tau \gamma_{0} \psi^{-}+\frac{1}{2} e^{a} \gamma_{a} \psi^{+}+\frac{1}{2} \omega^{a} \gamma_{a} \xi^{+}+\frac{1}{2} y_{1} \gamma_{0} \xi^{-}+\frac{1}{2} u_{1} \gamma_{0} \psi^{-} \\
\nabla \chi= & d \chi+\frac{1}{2} \omega \gamma_{0} \chi+\frac{1}{2} \omega^{a} \gamma_{a} \xi^{-}+\frac{1}{2} e^{a} \gamma_{a} \psi^{-}+\frac{1}{2} \tau \gamma_{0} \rho+\frac{1}{2} s \gamma_{0} \xi^{+}+\frac{1}{2} m \gamma_{0} \psi^{+} \\
& -\frac{1}{2} y_{2} \gamma_{0} \xi^{+}-\frac{1}{2} y_{1} \gamma_{0} \chi-\frac{1}{2} u_{2} \gamma_{0} \psi^{+}-\frac{1}{2} u_{1} \gamma_{0} \rho \tag{3.22}
\end{align*}
$$

along with (3.11). The CS action (3.19) describes the Maxwellian extended Bargmann supergravity theory first presented in [7]. As we can see, the $S$-expansion of the super Nappi-Witten algebra (2.1) with the $S_{E}^{(4)}$ semigroup adds a new sector to the action with respect to the case of the extended Bargmann supergravity theory previously studied. This new sector along the arbitrary constant $\alpha_{2}$ corresponds to the CS action for a new NR Maxwell superalgebra, whose bosonic part is the MEB gravity presented in [19], supplemented with some bosonic 1-form fields. Let us note that the extended Bargmann supergravity action (3.9) appears as a particular subcase along $\alpha_{0}$ and $\alpha_{1}$. The equations of motion of the theory are given by the vanishing of the curvature two-forms, which, in the case under analysis, are given by (3.11), (3.12), (3.22), and

$$
\begin{align*}
F(k) & =R(k)+\bar{\psi}^{+} \gamma^{0} \xi^{+} \\
F^{a}\left(k^{b}\right) & =R^{a}\left(k^{b}\right)+\bar{\psi}^{+} \gamma^{a} \xi^{-}+\bar{\psi}^{-} \gamma^{a} \xi^{+} \\
F(t) & =d t+\epsilon^{a c} \omega_{a} k_{c}+\frac{1}{2} \epsilon^{a c} e_{a} e_{c}+\bar{\psi}^{-} \gamma^{0} \xi^{-}+\bar{\psi}^{+} \gamma^{0} \chi+\bar{\xi}^{+} \gamma^{0} \rho \\
F\left(b_{1}\right) & =d b_{1}+\bar{\psi}^{+} \gamma^{0} \xi^{+} \\
F\left(b_{2}\right) & =d b_{2}-\bar{\psi}^{-} \gamma^{0} \xi^{-}+\bar{\psi}^{+} \gamma^{0} \chi+\bar{\xi}^{+} \gamma^{0} \rho \tag{3.23}
\end{align*}
$$

This is indeed expected for a well-defined and consistent CS (super)gravity model.

### 3.3 Generalized Maxwellian extended Bargmann supergravity

A generalization of the Maxwellian extended Bargmann algebra was introduced very recently in [27]. The aforesaid algebra was obtained by considering a NR limit of a generalized Maxwell algebra (also denoted as $\mathfrak{B}_{5}$ ) defined in three spacetime dimensions. Here, we will
show that a supersymmetric extension of the generalized Maxwellian extended Bargmann (GMEB) algebra can be obtained considering an $S_{E}^{(6)}$-expansion of the Nappi-Witten superalgebra (2.1). Furthermore, the $S$-expansion method will allow us to construct the supergravity theory invariant under the GMEB superalgebra, as it provides with the nondegenerate invariant supertrace.

The elements of the $S_{E}^{(6)}$ semigroup satisfy the multiplication law

$$
\lambda_{\alpha} \lambda_{\beta}= \begin{cases}\lambda_{\alpha+\beta} & \text { if } \alpha+\beta \leq 6,  \tag{3.24}\\ \lambda_{7} & \text { if } \alpha+\beta>6,\end{cases}
$$

with $\lambda_{7}=0_{S}$ being the zero element of the semigroup. Let $S_{E}^{(6)}=S_{0} \cup S_{1}$ be a resonant decomposition with

$$
\begin{align*}
& S_{0}=\left\{\lambda_{0}, \lambda_{2}, \lambda_{4}, \lambda_{6}, \lambda_{7}\right\}, \\
& S_{1}=\left\{\lambda_{1}, \lambda_{3}, \lambda_{5}, \lambda_{7}\right\} . \tag{3.25}
\end{align*}
$$

After applying a resonant $S_{E}^{(6)}$-expansion and a $0_{S}$-reduction to the Nappi-Witten superalgebra, we find an expanded superalgebra spanned by the bosonic generators

$$
\left\{\tilde{J}, \tilde{G}_{a}, \tilde{S}, \tilde{H}, \tilde{P}_{a}, \tilde{M}, \tilde{Z}, \tilde{Z}_{a}, \tilde{N}, \tilde{N} a, \tilde{T}, \tilde{V}, \tilde{Y}_{1}, \tilde{Y}_{2}, \tilde{U}_{1}, \tilde{U}_{2}, \tilde{B}_{1}, \tilde{B}_{2}, \tilde{C}_{1}, \tilde{C}_{2}\right\}
$$

along with the fermionic charges

$$
\left\{\tilde{Q}_{\alpha}^{+}, \tilde{Q}_{\alpha}^{-}, \tilde{R}_{\alpha}, \tilde{\Sigma}_{\alpha}^{+}, \tilde{\Sigma}_{\alpha}^{-}, \tilde{W}_{\alpha}, \tilde{\Xi}_{\alpha}^{+}, \tilde{\Xi}_{\alpha}^{-}, \tilde{K}_{\alpha}\right\}
$$

which are related to the super Nappi-Witten ones through

$$
\begin{array}{rlrlrlr}
\tilde{J}=\lambda_{0} J, & \tilde{G}_{a}=\lambda_{0} G_{a}, & \tilde{S}=\lambda_{0} S, & & \tilde{Y}_{1}=\lambda_{0} T_{1}, & & \tilde{Y}_{2}=\lambda_{0} T_{2}, \\
\tilde{H}=\lambda_{2} J, & & \tilde{P}_{a}=\lambda_{2} G_{a}, & \tilde{M}=\lambda_{2} S, & & \tilde{U}_{1}=\lambda_{2} T_{1}, & \\
\tilde{Z}=\tilde{U}_{2}=\lambda_{2} T_{2}, \\
\tilde{Z}=\lambda_{4} J, & \tilde{Z}_{a}=\lambda_{4} G_{a}, & \tilde{T}=\lambda_{4} S, & & \tilde{B}_{1}=\lambda_{4} T_{1}, & & \tilde{B}_{2}=\lambda_{4} T_{2}, \\
\tilde{N}=\lambda_{6} J, & & \tilde{N}_{a}=\lambda_{6} G_{a}, & \tilde{V}=\lambda_{6} S, & & \tilde{C}_{1}=\lambda_{6} T_{1}, & \\
\tilde{C}_{2}=\lambda_{6} T_{2},  \tag{3.26}\\
\tilde{Q}_{\alpha}^{+}=\lambda_{1} Q_{\alpha}^{+}, & \tilde{Q}_{\alpha}^{-}=\lambda_{1} Q_{\alpha}^{-}, & \tilde{\Sigma}_{\alpha}^{+}=\lambda_{3} Q_{\alpha}^{+}, & \tilde{\Sigma}_{\alpha}^{-}=\lambda_{3} Q_{\alpha}^{-}, & & \tilde{R}_{\alpha}=\lambda_{1} R_{\alpha}, \\
\tilde{W}_{\alpha}=\lambda_{3} R_{\alpha}, & & \tilde{\Xi}_{\alpha}^{+}=\lambda_{5} Q_{\alpha}^{+}, & \tilde{\Xi}_{\alpha}^{-}=\lambda_{5} Q_{\alpha}^{-}, & \tilde{K}_{\alpha}=\lambda_{5} R_{\alpha} . & &
\end{array}
$$

One can show that, using the multiplication law (3.24) and the original (anti-)commutation relations of the Nappi-Witten superalgebra (2.1), the expanded generators satisfy the (anti)commutation relations (3.5), (3.16) along with

$$
\begin{array}{lll}
{\left[\tilde{J}, \tilde{N}_{a}\right]=\epsilon_{a b} \tilde{N}_{b},} & {\left[\tilde{P}_{a}, \tilde{Z}_{b}\right]=-\epsilon_{a b} \tilde{V},} & {\left[\tilde{H}, \tilde{Z}_{a}\right]=\epsilon_{a b} \tilde{N}_{b},} \\
{\left[\tilde{Z}, \tilde{P}_{a}\right]=\epsilon_{a b} \tilde{N}_{b},} & {\left[\tilde{G}_{a}, \tilde{N}_{b}\right]=-\epsilon_{a b} \tilde{V},} & {\left[\tilde{N}, \tilde{G}_{a}\right]=\epsilon_{a b} \tilde{N}_{b},} \\
{\left[\tilde{J}, \tilde{\Xi}_{\alpha}^{ \pm}\right]=-\frac{1}{2}\left(\gamma_{0}\right)_{\alpha}^{\beta} \tilde{\Xi}_{\beta}^{ \pm},} & {\left[\tilde{H}, \tilde{\Sigma}_{\alpha}^{ \pm}\right]=-\frac{1}{2}\left(\gamma_{0}\right)_{\alpha}^{\beta} \tilde{\Xi}_{\beta}^{ \pm},} & {\left[\tilde{Z}, \tilde{Q}_{\alpha}^{ \pm}\right]=\frac{1}{2}\left(\gamma_{0}\right)_{\alpha}^{\beta} \tilde{\Xi}_{\beta}^{ \pm},} \\
{\left[\tilde{J}, \tilde{K}_{\alpha}\right]=-\frac{1}{2}\left(\gamma_{0}\right)_{\alpha}^{\beta} \tilde{K}_{\beta},} & {\left[\tilde{H}, \tilde{W}_{\alpha}\right]=-\frac{1}{2}\left(\gamma_{0}\right)_{\alpha}^{\beta} \tilde{K}_{\beta},} & {\left[\tilde{Z}, \tilde{R}_{\alpha}\right]=\frac{1}{2}\left(\gamma_{0}\right)_{\alpha}^{\beta} \tilde{K}_{\beta},}
\end{array}
$$

$$
\begin{align*}
& {\left[\tilde{G}_{a}, \tilde{\Xi}_{\alpha}^{+}\right]=-\frac{1}{2}\left(\gamma_{a}\right)_{\alpha}^{\beta} \tilde{\Xi}_{\beta}^{-}, \quad\left[\tilde{P}_{a}, \tilde{\Sigma}_{\alpha}^{+}\right]=-\frac{1}{2}\left(\gamma_{a}\right)_{\alpha}^{\beta} \tilde{\Xi}_{\beta}^{-}, \quad\left[\tilde{Z}_{a}, \tilde{Q}_{\alpha}^{+}\right]=\frac{1}{2}\left(\gamma_{a}\right)_{\alpha}^{\beta} \tilde{\Xi}_{\beta}^{-},} \\
& {\left[\tilde{G}_{a}, \tilde{\Xi}_{\alpha}^{-}\right]=-\frac{1}{2}\left(\gamma_{a}\right)_{\alpha}^{\beta} \tilde{K}_{\beta}, \quad\left[\tilde{P}_{a}, \tilde{\Sigma}_{\alpha}^{-}\right]=-\frac{1}{2}\left(\gamma_{a}\right)_{\alpha}^{\beta} \tilde{K}_{\beta}, \quad\left[\tilde{Z}_{a}, \tilde{Q}_{\alpha}^{-}\right]=\frac{1}{2}\left(\gamma_{a}\right)_{\alpha}^{\beta} \tilde{K}_{\beta},} \\
& {\left[\tilde{S}, \tilde{\Xi}_{\alpha}^{+}\right]=-\frac{1}{2}\left(\gamma_{0}\right)_{\alpha}^{\beta} \tilde{K}_{\beta}, \quad\left[\tilde{M}, \tilde{\Sigma}_{\alpha}^{+}\right]=-\frac{1}{2}\left(\gamma_{0}\right)_{\alpha}^{\beta} \tilde{K}_{\beta}, \quad\left[\tilde{T}, \tilde{Q}_{\alpha}^{+}\right]=\frac{1}{2}\left(\gamma_{0}\right)_{\alpha}^{\beta} \tilde{K}_{\beta},} \\
& {\left[\tilde{Y}_{1}, \tilde{\Xi}_{\alpha}^{ \pm}\right]=-\frac{1}{2}\left(\gamma_{0}\right)_{\alpha \beta} \tilde{\Xi}_{\beta}^{ \pm}, \quad\left[\tilde{U}_{1}, \tilde{\Sigma}_{\alpha}^{ \pm}\right]=-\frac{1}{2}\left(\gamma_{0}\right)_{\alpha \beta} \tilde{\Xi}_{\beta}^{ \pm}, \quad\left[\tilde{B}_{1}, \tilde{Q}_{\alpha}^{ \pm}\right]=\frac{1}{2}\left(\gamma_{0}\right)_{\alpha \beta} \tilde{\Xi}_{\beta}^{ \pm},} \\
& {\left[\tilde{Y}_{2}, \tilde{\Xi}_{\alpha}^{+}\right]=-\frac{1}{2}\left(\gamma_{0}\right)_{\alpha \beta} \tilde{K}_{\beta}, \quad\left[\tilde{U}_{2}, \tilde{\Sigma}_{\alpha}^{+}\right]=-\frac{1}{2}\left(\gamma_{0}\right)_{\alpha \beta} \tilde{K}_{\beta}, \quad\left[\tilde{B}_{2}, \tilde{Q}_{\alpha}^{+}\right]=\frac{1}{2}\left(\gamma_{0}\right)_{\alpha \beta} \tilde{K}_{\beta},} \\
& {\left[\tilde{Y}_{1}, \tilde{K}_{\alpha}\right]=-\frac{1}{2}\left(\gamma_{0}\right)_{\alpha \beta} \tilde{K}_{\beta}, \quad\left[\tilde{U}_{1}, \tilde{W}_{\alpha}\right]=-\frac{1}{2}\left(\gamma_{0}\right)_{\alpha \beta} \tilde{K}_{\beta}, \quad\left[\tilde{B}_{1}, \tilde{R}_{\alpha}\right]=\frac{1}{2}\left(\gamma_{0}\right)_{\alpha \beta} \tilde{K}_{\beta},} \\
& \left\{\tilde{Q}_{\alpha}^{-}, \tilde{\Xi}_{\beta}^{-}\right\}=\left\{\tilde{\Sigma}_{\alpha}^{-}, \tilde{\Sigma}_{\beta}^{-}\right\}=-\left(\gamma^{0} C\right)_{\alpha \beta} \tilde{V}+\left(\gamma^{0} C\right)_{\alpha \beta} \tilde{C}_{2}, \\
& \left\{\tilde{Q}_{\alpha}^{ \pm}, \tilde{\Xi}_{\beta}^{\mp}\right\}=\left\{\tilde{\Sigma}_{\alpha}^{+}, \tilde{\Sigma}_{\beta}^{-}\right\}=-\left(\gamma^{a} C\right)_{\alpha \beta} \tilde{N}_{a}, \\
& \left\{\tilde{Q}_{\alpha}^{+}, \tilde{\Xi}_{\beta}^{+}\right\}=\left\{\tilde{\Sigma}_{\alpha}^{+}, \tilde{\Sigma}_{\beta}^{+}\right\}=-\left(\gamma^{0} C\right)_{\alpha \beta} \tilde{N}-\left(\gamma^{0} C\right)_{\alpha \beta} \tilde{C}_{1}, \\
& \left\{\tilde{Q}_{\alpha}^{+}, \tilde{K}_{\beta}\right\}=\left\{\tilde{\Sigma}_{\alpha}^{+}, \tilde{W}_{\beta}\right\}=\left\{\tilde{\Xi}_{\alpha}^{+}, \tilde{R}_{\beta}\right\}=-\left(\gamma^{0} C\right)_{\alpha \beta} \tilde{V}-\left(\gamma^{0} C\right)_{\alpha \beta} \tilde{C}_{2} . \tag{3.27}
\end{align*}
$$

This superalgebra corresponds to a supersymmetric extension of the GMEB algebra introduced in [27]. Note that this is a new NR superalgebra, unlike the previous ones which had already been presented in the literature in previous works. The GMEB superalgebra is characterized by the presence of three extra fermionic generators, which are $\tilde{\Xi}_{\alpha}^{ \pm}$and $\tilde{K}_{\alpha}$, with respect to the MEB superalgebra. $\tilde{C}_{1}$ and $\tilde{C}_{2}$ play the role of central charges, while $\tilde{B}_{1}$ and $\tilde{B}_{2}$ act non-trivially on the fermionic generators $\tilde{Q}_{\alpha}^{ \pm}$and $\tilde{R}_{\alpha}$. The non-vanishing components of an invariant supertrace are given by (3.6), (3.17), and

$$
\begin{align*}
\left\langle\tilde{G}_{a} \tilde{N}_{b}\right\rangle & =\left\langle\tilde{P}_{a} \tilde{Z}_{b}\right\rangle=\alpha_{3} \delta_{a b}, \\
\langle\tilde{J} \tilde{V}\rangle & =\langle\tilde{H} \tilde{T}\rangle=\langle\tilde{M} \tilde{Z}\rangle=\langle\tilde{S} \tilde{N}\rangle=-\alpha_{3}, \\
\left\langle\tilde{Y}_{1} \tilde{C}_{2}\right\rangle & =\left\langle\tilde{U}_{1} \tilde{B}_{2}\right\rangle=\left\langle\tilde{B}_{1} \tilde{U}_{2}\right\rangle=\left\langle\tilde{C}_{1} \tilde{Y}_{2}\right\rangle=\alpha_{3}, \\
\left\langle\tilde{Q}_{\alpha}^{-} \tilde{\Xi}_{\beta}^{-}\right\rangle & =\left\langle\tilde{\Sigma}_{\alpha}^{+} \tilde{W}_{\beta}\right\rangle=\left\langle\tilde{\Xi}_{\alpha}^{+} \tilde{R}_{\beta}\right\rangle=\left\langle\tilde{Q}_{\alpha}^{+} \tilde{K}_{\beta}\right\rangle=\left\langle\tilde{\Sigma}_{\alpha}^{-} \tilde{\Sigma}_{\beta}^{-}\right\rangle=2 \alpha_{3} C_{\alpha \beta} . \tag{3.28}
\end{align*}
$$

The NR one-form gauge connection for the GMEB superalgebra reads

$$
\begin{align*}
A= & \tau \tilde{H}+e^{a} \tilde{P}_{a}+\omega \tilde{J}+\omega^{a} \tilde{G}_{a}+k \tilde{Z}+k^{a} \tilde{Z}_{a}+f \tilde{N}+f^{a} \tilde{N}_{a}+m \tilde{M}+s \tilde{S} \\
& +t \tilde{T}+v \tilde{V}+y_{1} \tilde{Y}_{1}+y_{2} \tilde{Y}_{2}+b_{1} \tilde{B}_{1}+b_{2} \tilde{B}_{2}+u_{1} \tilde{U}_{1}+u_{2} \tilde{U}_{2}+c_{1} \tilde{C}_{1}+c_{2} \tilde{C}_{2} \\
& +\psi^{+} \tilde{Q}^{+}+\psi^{-} \tilde{Q}^{-}+\xi^{+} \tilde{\Sigma}^{+}+\xi^{-} \tilde{\Sigma}^{-}+\zeta^{+} \tilde{\Xi}^{+}+\zeta^{-} \tilde{\Xi}^{-}+\rho \tilde{R}+\chi \tilde{W}+\kappa \tilde{K} . \tag{3.29}
\end{align*}
$$

The corresponding NR CS action can be obtained by inserting the above gauge connection and the invariant supertrace given by (3.6), (3.17), and (3.28) into the general expression for the CS action in three spacetime dimensions (2.5). The aforesaid action reads, up to boundary terms, as follows:

$$
\begin{equation*}
I_{\mathrm{GMEB}}=\alpha_{0} I_{0}+\alpha_{1} I_{1}+\alpha_{2} I_{2}+\alpha_{3} I_{3}, \tag{3.30}
\end{equation*}
$$

where $I_{0}$ and $I_{1}$ are given in (3.9), $I_{2}$ is given in (3.20), while the term along $\alpha_{3}$ reads

$$
\begin{align*}
I_{3}= & \int \omega_{a} R^{a}\left(f^{b}\right)+f_{a} R^{a}\left(\omega^{b}\right)+e_{a} R^{a}\left(k^{b}\right)+k_{a} R^{a}\left(e^{b}\right)-2 s R(f) \\
& -2 v R(\omega)-2 m R(k)-2 t R(\tau)+2 y_{1} d c_{2}+2 u_{1} d b_{2}+2 y_{2} d c_{1}+2 b_{1} d u_{2} \\
& +2 \bar{\psi}^{-} \nabla \zeta^{-}+2 \bar{\zeta}^{-} \nabla \psi^{-}+2 \bar{\xi}^{-} \nabla \xi^{-}+2 \bar{\psi}^{+} \nabla \kappa+2 \bar{\zeta}^{+} \nabla \rho+2 \bar{\xi}^{+} \nabla \chi \\
& +2 \bar{\rho} \nabla \zeta^{+}+2 \bar{\chi} \nabla \xi^{+}+2 \bar{\kappa} \nabla \psi^{+} \tag{3.31}
\end{align*}
$$

In the above action, the expressions for $R(\omega)$ and $R^{a}\left(\omega^{b}\right)$ were defined in (2.7), while those for $R(\tau), R(k), R^{a}\left(e^{b}\right)$, and $R^{a}\left(k^{b}\right)$ are given in (3.21). Furthermore,

$$
\begin{align*}
R(f) & =d f \\
R^{a}\left(f^{b}\right) & =d f^{a}+\epsilon^{a c} \omega f_{c}+\epsilon^{a c} \tau k_{c}+\epsilon^{a c} k e_{c}+\epsilon^{a c} f \omega_{c} \tag{3.32}
\end{align*}
$$

and the covariant derivatives of the spinor 1-form fields appearing in $I_{3}$ are given by

$$
\begin{align*}
\nabla \zeta^{+}= & d \zeta^{+}+\frac{1}{2} \omega \gamma_{0} \zeta^{+}+\frac{1}{2} \tau \gamma_{0} \xi^{+}+\frac{1}{2} k \gamma_{0} \psi^{+}-\frac{1}{2} y_{1} \gamma_{0} \zeta^{+}-\frac{1}{2} u_{1} \gamma_{0} \xi^{+}-\frac{1}{2} b_{1} \gamma_{0} \psi^{+} \\
\nabla \zeta^{-}= & d \zeta^{-}+\frac{1}{2} \omega \gamma_{0} \zeta^{-}+\frac{1}{2} \tau \gamma_{0} \xi^{-}+\frac{1}{2} k \gamma_{0} \psi^{-}+\frac{1}{2} e^{a} \gamma_{a} \xi^{+}+\frac{1}{2} \omega^{a} \gamma_{a} \zeta^{+}+\frac{1}{2} k^{a} \gamma_{a} \psi^{+} \\
& +\frac{1}{2} y_{1} \gamma_{0} \zeta^{-}+\frac{1}{2} u_{1} \gamma_{0} \xi^{-}+\frac{1}{2} b_{1} \gamma_{0} \psi^{-}, \\
\nabla \kappa= & d \kappa+\frac{1}{2} \omega \gamma_{0} \kappa+\frac{1}{2} \tau \gamma_{0} \chi+\frac{1}{2} k \gamma_{0} \rho+\frac{1}{2} \omega^{a} \gamma_{a} \zeta^{-}+\frac{1}{2} e^{a} \gamma_{a} \xi^{-}+\frac{1}{2} k^{a} \gamma_{a} \psi^{-}+\frac{1}{2} s \gamma_{0} \zeta^{+} \\
& +\frac{1}{2} m \gamma_{0} \xi^{+}+\frac{1}{2} t \gamma_{0} \psi^{+}-\frac{1}{2} y_{2} \gamma_{0} \zeta^{+}-\frac{1}{2} u_{2} \gamma_{0} \xi^{+}-\frac{1}{2} b_{2} \gamma_{0} \psi^{+}-\frac{1}{2} y_{1} \gamma_{0} \kappa-\frac{1}{2} u_{1} \gamma_{0} \chi \\
& -\frac{1}{2} b_{1} \gamma_{0} \rho \tag{3.33}
\end{align*}
$$

along with (3.11) and (3.22). From (3.30), we see that the CS action contains four independent sectors. The new gauge fields $f_{a}, f, v, \zeta^{+}, \zeta^{-}$, and $\kappa$ appear explicitly in the last term proportional to $\alpha_{3}$, which corresponds to the CS action for the new NR generalized Maxwell superalgebra. Note that the GMEB superalgebra allows to include a cosmological constant term along $\alpha_{3}$ different from the one appearing in the case of the extended NR supergravity presented in [8]. One can see that the field equations imply the vanishing of the curvature two forms given by $(3.11),(3.12),(3.22),(3.23),(3.33)$, and

$$
\begin{align*}
F(f) & =R(f)+\bar{\psi}^{+} \gamma^{0} \zeta^{+}+\frac{1}{2} \bar{\xi}^{+} \gamma^{0} \xi^{+} \\
F^{a}\left(f^{b}\right) & =R^{a}\left(f^{b}\right)+\bar{\psi}^{+} \gamma^{a} \zeta^{-}+\bar{\psi}^{-} \gamma^{a} \zeta^{+}+\bar{\xi}^{+} \gamma^{a} \xi^{-} \\
F(v) & =d v+\epsilon^{a c} \omega_{a} k_{c}+\epsilon^{a c} e e_{a}+\bar{\psi}^{-} \gamma^{0} \zeta^{-}+\frac{1}{2} \bar{\xi}^{-} \gamma^{0} \xi^{-}+\bar{\psi}^{+} \gamma^{0} \kappa+\bar{\xi}^{+} \gamma^{0} \chi+\bar{\zeta}^{+} \gamma^{0} \rho \\
F\left(c_{1}\right) & =d c_{1}+\bar{\psi}^{+} \gamma^{0} \zeta^{+}+\frac{1}{2} \bar{\xi}^{+} \gamma^{0} \xi^{+} \\
F\left(c_{2}\right) & =d c_{2}-\bar{\psi}-\gamma^{0} \zeta^{-}-\frac{1}{2} \bar{\xi}^{-} \gamma^{0} \xi^{-}+\bar{\psi}^{+} \gamma^{0} \kappa++\bar{\xi}^{+} \gamma^{0} \chi+\bar{\zeta}^{+} \gamma^{0} \rho \tag{3.34}
\end{align*}
$$

which are the GMEB supercurvatures. A natural subsequent step would be to continue expanding the Nappi-Witten superalgebra with bigger semigroups. As we will show in the
next section, all NR superalgebras coming from the semigroup expansion of (2.1), when the semigroup under consideration is of the $S_{E}$-type, can be written in a very compact and general way in terms of the original (anti-)commutation relations (2.1) and the elements of the relevant semigroup.

### 3.4 Generalized extended Bargmann supergravity

As it was shown in [27], the extended Bargmann, the MEB, and the GMEB algebras can be seen as particular cases of the so-called generalized extended Bargmann ( $\mathrm{GEB}^{(N)}$ ) algebra, which corresponds to the NR version of the $\mathfrak{B}_{N+2}$ algebra enlarged with $U_{1}$ generators. In this section, we present a supersymmetric extension of the GEB ${ }^{(N)}$ algebra which can be obtained by performing an $S$-expansion of the super Nappi-Witten algebra (2.1). Similarly to the purely bosonic case, the new NR superalgebra contains the extended Bargmann, the MEB, and the GMEB superalgebras previously introduced as particular cases.

Let $S_{E}^{(2 N)}=\left\{\lambda_{0}, \lambda_{1}, \lambda_{2}, \ldots, \lambda_{2 N-1}, \lambda_{2 N}, \lambda_{2 N+1}\right\}$ be the relevant semigroup whose elements satisfy

$$
\lambda_{\mu} \lambda_{\nu}= \begin{cases}\lambda_{\mu+\nu} & \text { if } \quad \mu+\nu \leq 2 N,  \tag{3.35}\\ \lambda_{2 N+1} & \text { if } \mu+\nu>2 N,\end{cases}
$$

where $\lambda_{2 N+1}=0_{S}$ is the zero element of the semigroup such that $0_{S} \lambda_{\mu}=0_{S}$. Let $S_{E}^{(2 N)}=$ $S_{0} \cup S_{1}$ be a semigroup decomposition with

$$
\begin{align*}
& S_{0}=\left\{\lambda_{2 i}, i=0, \ldots, N\right\} \cup \lambda_{2 N+1}, \\
& S_{1}=\left\{\lambda_{2 m-1}, m=1, \ldots, N\right\} \cup \lambda_{2 N+1} . \tag{3.36}
\end{align*}
$$

Such decomposition is said to be resonant since it behaves as the subspace decomposition of the Nappi-Witten superalgebra (2.2),

$$
\begin{equation*}
S_{0} \cdot S_{0} \subset S_{0}, \quad S_{0} \cdot S_{1} \subset S_{1}, \quad S_{1} \cdot S_{1} \subset S_{0} \tag{3.37}
\end{equation*}
$$

Then, after performing a resonant $S_{E}^{(2 N)}$-expansion to the super Nappi-Witten algebra (2.1) and considering a $0_{S}$-reduction, we find a new NR superalgebra. The expanded NR generators are related to the super Nappi-Witten ones through the semigroup elements as follows:

$$
\begin{array}{ll}
\tilde{J}^{(i)}=\lambda_{2 i} J, & \tilde{Q}_{\alpha}^{+(m)}=\lambda_{2 m-1} Q_{\alpha}^{+}, \\
\tilde{G}_{a}^{(i)}=\lambda_{2 i} G_{a}, & \tilde{Q}_{\alpha}^{-(m)}=\lambda_{2 m-1} Q_{\alpha}^{-}, \\
\tilde{S}^{(i)}=\lambda_{2 i} S, & \tilde{R}_{\alpha}^{(m)}=\lambda_{2 m-1} R_{\alpha}, \\
\tilde{T}_{1}^{(i)}=\lambda_{2 i} T_{1} & \tilde{T}_{2}^{(i)}=\lambda_{2 i} T_{2} . \tag{3.38}
\end{array}
$$

One can prove that the expanded generators satisfy the following (anti-)commutation relations:

$$
\begin{aligned}
& {\left[\tilde{J}^{(i)}, \tilde{G}_{a}^{(j)}\right]=\epsilon_{a b} \tilde{G}_{b}^{(i+j)},} \\
& {\left[\tilde{G}_{a}^{(i)}, \tilde{G}_{b}^{(j)}\right]=-\epsilon_{a b} \tilde{S}^{(i+j)},} \\
& {\left[\tilde{J}^{(i)}, \tilde{Q}_{\alpha}^{ \pm(m)}\right]=-\frac{1}{2}\left(\gamma_{0}\right)_{\alpha}{ }^{\beta} \tilde{Q}_{\beta}^{ \pm(i+m)},} \\
& {\left[\tilde{J}^{(i)}, \tilde{R}_{\alpha}^{(m)}\right]=-\frac{1}{2}\left(\gamma_{0}\right)_{\alpha}^{\beta} \tilde{R}_{\beta}^{(i+m)},}
\end{aligned}
$$

$$
\begin{array}{rlrl}
{\left[\tilde{G}_{a}^{(i)}, \tilde{Q}_{\alpha}^{+(m)}\right]} & =-\frac{1}{2}\left(\gamma_{a}\right)_{\alpha}{ }^{\beta} \tilde{Q}_{\beta}^{-(i+m)}, & & {\left[\tilde{G}_{a}^{(i)}, \tilde{Q}_{\alpha}^{-(m)}\right]=-\frac{1}{2}\left(\gamma_{a}\right)_{\alpha}{ }^{\beta} \tilde{R}_{\beta}^{(i+m)},} \\
{\left[\tilde{S}^{(i)}, \tilde{Q}_{\alpha}^{+(m)}\right]} & =-\frac{1}{2}\left(\gamma_{0}\right)_{\alpha}{ }_{\alpha} \tilde{R}_{\beta}^{(i+m)}, & {\left[\tilde{T}_{1}^{(i)}, \tilde{Q}_{\alpha}^{ \pm(m)}\right]= \pm \frac{1}{2}\left(\gamma_{0}\right)_{\alpha \beta} \tilde{Q}_{\beta}^{ \pm(i+m)},} \\
{\left[\tilde{T}_{2}^{(i)}, \tilde{Q}_{\alpha}^{+(m)}\right]} & =\frac{1}{2}\left(\gamma_{0}\right)_{\alpha \beta} \tilde{R}_{\beta}^{(i+m)}, & {\left[\tilde{T}_{1}^{(i)}, \tilde{R}_{\alpha}^{(m)}\right]=\frac{1}{2}\left(\gamma_{0}\right)_{\alpha \beta} \tilde{R}_{\beta}^{(i+m)},} \\
\left\{\tilde{Q}_{\alpha}^{+(m)}, \tilde{Q}_{\beta}^{-(n)}\right\} & =-\left(\gamma^{a} C\right)_{\alpha \beta} \tilde{G}_{a}^{(m+n-1)}, & \\
\left\{\tilde{Q}_{\alpha}^{+(m)}, \tilde{Q}_{\beta}^{+(n)}\right\} & =-\left(\gamma^{0} C\right)_{\alpha \beta} \tilde{J}^{(m+n-1)}-\left(\gamma^{0} C\right)_{\alpha \beta} \tilde{T}_{1}^{(m+n-1)}, \\
\left\{\tilde{Q}_{\alpha}^{-(m)}, \tilde{Q}_{\beta}^{-(n)}\right\} & =-\left(\gamma^{0} C\right)_{\alpha \beta} \tilde{S}^{(m+n-1)}+\left(\gamma^{0} C\right)_{\alpha \beta} \tilde{T}_{2}^{(m+n-1)}, \\
\left\{\tilde{Q}_{\alpha}^{+(m)}, \tilde{R}_{\beta}^{(n)}\right\} & =-\left(\gamma^{0} C\right)_{\alpha \beta} \tilde{S}^{(m+n-1)}-\left(\gamma^{0} C\right)_{\alpha \beta} \tilde{T}_{2}^{(m+n-1)} . \tag{3.39}
\end{array}
$$

This is obtained by using the multiplication law of the $S_{E}^{(2 N)}$ semigroup (3.35) and the super Nappi-Witten (anti-)commutation relations (2.1).

The expanded superalgebra above generalizes the extended Bargmann superalgebra [3] and corresponds to the supersymmetric extension of the $\mathrm{GEB}^{(N)}$ algebra introduced in [27]. The NR superalgebra (3.39) contains $2 N+2$ bosonic generators in addition to the $\operatorname{GEB}^{(N)}$ bosonic generators and $3 N$ fermionic charges. The extra bosonic generators $\tilde{T}_{1}^{(i)}$ and $\tilde{T}_{2}^{(i)}$ act non-trivially on the fermionic charges $\tilde{Q}_{\alpha}^{(m)}$ if $i+m \leq N$. On the other hand, both $\tilde{T}_{1}^{(N)}$ and $\tilde{T}_{2}^{(N)}$ are central charges. One can see that the case $N=1$ reproduces the extended Bargmann superalgebra, while the $N=2$ and $N=3$ ones correspond to the MEB and GMEB superalgebras, respectively. At the bosonic level, as it was shown in [27], the so-called $\mathfrak{B}_{N+2}$ algebra $[69,86]$ is the relativistic counterpart of the $\mathrm{GEB}^{(N)}$ algebra. Then, one could deduce that the present generalization is the respective NR version of the supersymmetric extension of the $\mathfrak{B}_{N+2}$ algebra [90].

The gauge connection one-form $A$ for the $\mathrm{GEB}^{(N)}$ superalgebra reads

$$
\begin{align*}
A= & \omega^{(i)} \tilde{J}^{(i)}+\omega^{a(i)} \tilde{G}_{a}^{(i)}+s^{(i)} \tilde{S}^{(i)}+t_{1}^{(i)} \tilde{T}_{1}^{(i)}+t_{2}^{(i)} T_{2}^{(i)} \\
& +\bar{\psi}^{+(m)} \tilde{Q}^{+(m)}+\bar{\psi}^{-(m)} \tilde{Q}^{-(m)}+\bar{\rho}^{(m)} \tilde{R}^{(m)} . \tag{3.40}
\end{align*}
$$

The presence of the additional bosonic generators $\left\{\tilde{T}_{1}^{(i)}, \tilde{T}_{2}^{(i)}\right\}$ is required to ensure a nondegenerate invariant supertrace allowing the construction of a proper NR CS supergravity action. In particular, the $\mathrm{GEB}^{(N)}$ superalgebra admits the following non-degenerate invariant tensor:

$$
\begin{align*}
\left\langle\tilde{G}_{a}^{(i)} \tilde{G}_{b}^{(j)}\right\rangle & =\alpha_{i+j} \delta_{a b}, \\
\left\langle\tilde{J}^{(i)} \tilde{S}^{(j)}\right\rangle & =-\alpha_{i+j}, \\
\left\langle\tilde{T}_{1}^{(i)} \tilde{T}_{2}^{(j)}\right\rangle & =\alpha_{i+j}, \\
\left\langle\tilde{Q}_{\alpha}^{-(m)} \tilde{Q}_{\beta}^{-(n)}\right\rangle & =2 \alpha_{m+n-1} C_{\alpha \beta}=\left\langle\tilde{Q}_{\alpha}^{+(m)} \tilde{R}_{\beta}^{(n)}\right\rangle, \tag{3.41}
\end{align*}
$$

where $i, j=0,1, \ldots, N ; m, n=1,2, \ldots, N$, and $i+j<N+1, i+m<N+1$, and $m+n<N+1$. Here, the non-vanishing components of an invariant tensor for the GEB ${ }^{(N)}$
superalgebra are obtained from the super Nappi-Witten ones (2.3) using the definitions of the $S$-expansion method [53]. Then, a GEB ${ }^{(N)}$ CS supergravity action can be constructed by inserting the gauge connection one-form (3.40) and the non-vanishing components of the invariant tensor (3.41) in the general expression for the CS action (2.5), that yields

$$
\begin{equation*}
I_{\mathrm{GEB}^{(N)}}=\alpha_{i} I_{i}=\alpha_{0} I_{0}+\alpha_{1} I_{1}+\ldots+\alpha_{N} I_{N}, \tag{3.42}
\end{equation*}
$$

with

$$
\begin{align*}
I_{i}= & \int \omega_{a}^{(j)} d \omega^{a(k)} \delta_{j+k}^{i}+\epsilon^{a c} \omega_{a}^{(j)} \omega^{(k)} \omega_{c}^{(l)} \delta_{j+k+l}^{i}-2 s^{(j)} d \omega^{(k)} \delta_{j+k}^{i}+2 t_{1}^{(j)} d t_{2}^{(k)} \delta_{j+k}^{i} \\
& +2 \bar{\psi}^{-(m)} \nabla \psi^{-(n)} \delta_{m+n-1}^{i}+2 \bar{\psi}^{+(m)} \nabla \rho^{(n)} \delta_{m+n-1}^{i}+2 \bar{\rho}^{(m)} \nabla \psi^{+(n)} \delta_{m+n-1}^{i} \tag{3.43}
\end{align*}
$$

where the covariant derivatives of the spinor 1-forms for the GEB ${ }^{(N)}$ superalgebra read

$$
\begin{align*}
\nabla \psi^{+(m)}= & d \psi^{+(m)}+\frac{1}{2} \omega^{(i)} \gamma_{0} \psi^{+(n)} \delta_{i+n}^{m}-\frac{1}{2} t_{1}^{(i)} \gamma_{0} \psi^{+(n)} \delta_{i+n}^{m}, \\
\nabla \psi^{-(m)}= & d \psi^{-(m)}+\frac{1}{2} \omega^{(i)} \gamma_{0} \psi^{-(n)} \delta_{i+n}^{m}+\frac{1}{2} \omega^{a(i)} \gamma_{a} \psi^{+(n)} \delta_{i+n}^{m}+\frac{1}{2} t_{1}^{(i)} \gamma_{0} \psi^{-(n)} \delta_{i+n}^{m}, \\
\nabla \rho^{(m)}= & d \rho^{(m)}+\frac{1}{2} \omega^{(i)} \gamma_{0} \rho^{(n)} \delta_{i+n}^{m}+\frac{1}{2} \omega^{a(i)} \gamma_{a} \psi^{-(n)} \delta_{i+n}^{m}+\frac{1}{2} s^{(i)} \gamma_{0} \psi^{+(n)} \delta_{i+n}^{m} \\
& -\frac{1}{2} t_{2}^{(i)} \gamma_{0} \psi^{+(n)} \delta_{i+n}^{m}-\frac{1}{2} t_{1}^{(i)} \gamma_{0} \rho^{(n)} \delta_{i+n}^{m} . \tag{3.44}
\end{align*}
$$

The new NR CS action is invariant under the GEB ${ }^{(N)}$ superalgebra (3.39) and contains $N+1$ independent sectors proportional to the $\alpha_{i}$ 's. Interestingly, one can see that the GEB ${ }^{(i)}$ CS action, for $i<N$, appears as a particular subcase. Indeed, the $I_{0}$ and $I_{1}$ CS terms describe the most general extended Bargmann supergravity action, where $I_{0}$ corresponds to a NR exotic term. The explicit extended Bargmann CS action (3.8) is recovered by identifying the gauge field one-forms as

$$
\begin{align*}
& \omega^{(0)}=\omega, \quad \omega_{a}^{(0)}=\omega_{a}, \quad s^{(0)}=s, \quad t_{1}^{(0)}=y_{1}, \quad t_{2}^{(0)}=y_{2}, \\
& \omega^{(1)}=\tau, \quad \omega_{a}^{(1)}=e_{a}, \quad s^{(1)}=m, \quad t_{1}^{(1)}=u_{1}, \quad t_{2}^{(1)}=u_{2}, \\
& \psi^{+(1)}=\psi^{+}, \quad \psi^{-(1)}=\psi^{-}, \quad \rho^{(1)}=\rho . \tag{3.45}
\end{align*}
$$

On the other hand, the $I_{2}$ term along with $I_{1}$ and $I_{0}$ describe the MEB CS supergravity action [7]. In particular, the $I_{2} \mathrm{CS}$ action coincides with the one obtained in (3.20) by identifying the gauge field one-forms as in (3.45) together with

$$
\begin{array}{rlrlr}
\omega^{(2)} & =k, & \omega_{a}^{(2)} & =k_{a}, & s^{(2)}=t,
\end{array} t_{1}^{(2)}=b_{1}, \quad t_{2}^{(2)}=b_{2},
$$

In addition, one can see that the GMEB CS supergravity theory is described by $I_{0}, I_{1}, I_{2}$, and $I_{3}$ by considering the redefinitions of the gauge fields as in (3.45), (3.46), and

$$
\begin{align*}
\omega^{(3)} & =f, & \omega_{a}^{(3)} & =f_{a}, & s^{(3)} & =u, \\
\psi^{+(3)} & =\zeta^{+}, & \psi^{-(3)} & =\zeta^{-}, & \rho_{1}^{(3)} & =\kappa . \tag{3.47}
\end{align*}
$$

Novel generalizations of the extended Bargmann supergravity theory are obtained for $N>$ 3 and correspond to supersymmetric extensions of the $\mathrm{GEB}^{(N)}$ gravity theory presented in [27]. Let us note that the equations of motion for a specific $\mathrm{GEB}^{(i)}$ superalgebra are given by the vanishing of the curvature two-forms associated with the respective superalgebra, which are given by (3.44) along with

$$
\begin{align*}
F\left(\omega^{(i)}\right) & =d \omega^{i}+\frac{1}{2} \bar{\psi}^{+(m)} \gamma^{0} \psi^{+(n)} \delta_{m+n-1}^{i} \\
F^{a}\left(\omega^{b(i)}\right) & =d \omega^{a(i)}+\epsilon^{a c} \omega^{(j)} \omega_{c}^{(k)} \delta_{j+k}^{i}+\bar{\psi}^{+(m)} \gamma^{a} \psi^{-(n)} \delta_{m+n-1}^{i} \\
F\left(s^{(i)}\right) & =d s^{(i)}+\frac{1}{2} \bar{\psi}^{-(m)} \gamma^{0} \psi^{-(n)} \delta_{m+n-1}^{i}+\bar{\psi}^{+(m)} \gamma^{0} \rho^{(n)} \delta_{m+n-1}^{i} \\
F\left(t_{1}^{(i)}\right) & =d t_{1}^{(i)}+\frac{1}{2} \bar{\psi}^{+(m)} \gamma^{0} \psi^{+(n)} \delta_{m+n-1}^{i} \\
F\left(t_{2}^{(i)}\right) & =d t_{2}^{(i)}-\frac{1}{2} \bar{\psi}^{-(m)} \gamma^{0} \psi^{-(n)} \delta_{m+n-1}^{i}+\bar{\psi}^{+(m)} \gamma^{0} \rho^{(n)} \delta_{m+n-1}^{i} \tag{3.48}
\end{align*}
$$

The physical implications of the additional bosonic and fermionic content remains as an interesting open issue that shall be considered in a future work.

## 4 Generalized extended Newton-Hooke supergravity theory and semigroup expansion method

In this section, we apply the $S$-expansion method to the Nappi-Witten superalgebra (2.1) to find a different family of NR superalgebras. In particular, we shall see that the extended Newton-Hooke superalgebra [6] and its generalizations can be obtained considering $S_{L}^{(1)}$ and $S_{\mathcal{M}}^{(N)}$ as the relevant semigroups, respectively. The semigroup choice is not arbitrary and comes from the expansion relation presented at the level of the asymptotic symmetry. Indeed, as it was shown in [82], the conformal superalgebra can be obtained by expanding the super Virasoro one using $S_{L}^{(1)}$ as the relevant semigroup, while generalizations of the superconformal symmetry are found using $S_{\mathcal{M}}$. It is interesting to notice that the same semigroup used to relate diverse infinite-dimensional superalgebras can be considered at the NR level. Furthermore, we will show that the generalized extended Newton-Hooke superalgebras, which we have denoted as $\mathrm{GNH}^{(N)}$, are related to the $\mathrm{GEB}^{(N)}$ ones (3.39) through an IW contraction. The construction of a NR CS supergravity action based on the $\mathrm{GNH}^{(N)}$ superalgebra is also presented.

### 4.1 Extended Newton-Hooke supergravity

From the super Nappi-Witten algebra (2.1), one can obtain a supersymmetric extension of the Newton-Hooke algebra (precisely, the extended Newton-Hooke superalgebra obtained in [8]) through the $S$-expansion method considering $S_{L}^{(1)}=\left\{\lambda_{0}, \lambda_{1}, \lambda_{2}\right\}$ as the relevant semigroup, whose elements satisfy the following multiplication law:

$$
\begin{array}{c|ccc}
\lambda_{2} & \lambda_{2} & \lambda_{2} & \lambda_{2}  \tag{4.1}\\
\lambda_{1} & \lambda_{2} & \lambda_{1} & \lambda_{2} \\
\lambda_{0} & \lambda_{0} & \lambda_{2} & \lambda_{2} \\
\hline & \lambda_{0} & \lambda_{1} & \lambda_{2}
\end{array}
$$

with $\lambda_{2}=0_{S}$ being the zero element of the semigroup. Let $S_{L}^{(1)}=S_{0} \cup S_{1}$ be a decomposition of the semigroup

$$
\begin{align*}
& S_{0}=\left\{\lambda_{0}, \lambda_{1}, \lambda_{2}\right\}, \\
& S_{1}=\left\{\lambda_{1}, \lambda_{2}\right\}, \tag{4.2}
\end{align*}
$$

which is said to be resonant [53] since it satisfies the same structure as the subspaces of the super Nappi-Witten algebra (2.2),

$$
\begin{align*}
& S_{0} \cdot S_{0} \subset S_{0}, \\
& S_{0} \cdot S_{1} \subset S_{1}, \\
& S_{1} \cdot S_{1} \subset S_{0} \tag{4.3}
\end{align*}
$$

An expanded superalgebra spanned by $\left\{L, L_{a}, N, X_{1}, X_{2}, \tilde{L}, \tilde{L}_{a}, \tilde{N}, \tilde{X}_{1}, \tilde{X}_{2}, \mathcal{Q}_{\alpha}^{+}, \mathcal{Q}_{\alpha}^{-}, \mathcal{R}_{\alpha}\right\}$ is obtained after considering a resonant $S_{L}^{(1)}$-expansion of the super Nappi-Witten algebra and performing a $0_{S}$-reduction. The expanded generators are related to the super NappiWitten ones through the semigroup elements as

Then, one can see that the expanded NR generators satisfy two copies of the Nappi-Witten algebra, one of which is augmented by supersymmetry,

$$
\begin{align*}
{\left[L, L_{a}\right] } & =\epsilon_{a b} L_{b}, \\
{\left[\tilde{L}, \tilde{L}_{a}\right] } & =\epsilon_{a b} \tilde{L}_{b}, \\
{\left[L, \mathcal{Q}_{\alpha}^{ \pm}\right] } & =-\frac{1}{2}\left(\gamma_{0}\right)_{\alpha}^{\beta} \mathcal{Q}_{\beta}^{ \pm}, \\
{\left[L_{a}, \mathcal{Q}_{\alpha}^{+}\right] } & =-\frac{1}{2}\left(\gamma_{a}\right)_{\alpha}^{\beta} \mathcal{Q}_{\beta}^{-}, \\
{\left[N, \mathcal{Q}_{\alpha}^{+}\right] } & =-\frac{1}{2}\left(\gamma_{0}\right)_{\alpha}^{\beta} \mathcal{R}_{\beta}, \\
{\left[X_{2}, \mathcal{Q}_{\alpha}^{+}\right] } & =\frac{1}{2}\left(\gamma_{0}\right)_{\alpha \beta} \mathcal{R}_{\beta}, \\
\left\{\mathcal{Q}_{\alpha}^{+}, \mathcal{Q}_{\beta}^{-}\right\} & =-\left(\gamma^{a} C\right)_{\alpha \beta} L_{a}, \\
\left\{\mathcal{Q}_{\alpha}^{+}, \mathcal{Q}_{\beta}^{+}\right\} & =-\left(\gamma^{0} C\right)_{\alpha \beta} L-\left(\gamma^{0} C\right)_{\alpha \beta} X_{1}, \\
\left\{\mathcal{Q}_{\alpha}^{-}, \mathcal{Q}_{\beta}^{-}\right\} & =-\left(\gamma^{0} C\right)_{\alpha \beta} N+\left(\gamma^{0} C\right)_{\alpha \beta} X_{2}, \\
\left\{\mathcal{Q}_{\alpha}^{+}, \mathcal{R}_{\beta}\right\} & =-\left(\gamma^{0} C\right)_{\alpha \beta} N-\left(\gamma^{0} C\right)_{\alpha \beta} X_{2},
\end{align*}
$$

$$
\left[L_{a}, L_{b}\right]=-\epsilon_{a b} N
$$

$$
\begin{align*}
& L=\lambda_{1} J, \quad L_{a}=\lambda_{1} G_{a}, \quad N=\lambda_{1} S, \\
& \tilde{L}=\lambda_{0} J \text {, } \\
& \tilde{L}_{a}=\lambda_{0} G_{a}, \\
& \tilde{N}=\lambda_{0} S, \\
& \mathcal{Q}_{\alpha}^{+}=\lambda_{1} Q_{\alpha}^{+}, \\
& \mathcal{Q}_{\alpha}^{-}=\lambda_{1} Q_{\alpha}^{-} \text {, } \\
& \mathcal{R}_{\alpha}=\lambda_{1} R_{\alpha}, \\
& X_{1}=\lambda_{1} T_{1} \text {, } \\
& X_{2}=\lambda_{1} T_{2} \text {, } \\
& \tilde{X}_{1}=\lambda_{0} T_{1} \text {, } \\
& \tilde{X}_{2}=\lambda_{0} T_{2} \text {. } \tag{4.4}
\end{align*}
$$

where we have used the multiplication law of the semigroup $S_{L}^{(1)}$ (4.1) and the (anti)commutation relations of the Nappi-Witten superalgebra (2.1). On the other hand, let us note that the purely bosonic copy of the Nappi-Witen algebra is endowed with two $\mathfrak{u}(1)$ generators, namely $\tilde{X}_{1}$ and $\tilde{X}_{2}$. Considering the definitions of [53], one can see that the superalgebra (4.5) admits the following non-vanishing components of the invariant tensor:

$$
\begin{align*}
\left\langle L_{a} L_{b}\right\rangle & =\mu \delta_{a b}, & \left\langle\tilde{L}_{a} \tilde{L}_{b}\right\rangle & =\nu \delta_{a b} \\
\langle L N\rangle & =-\mu, & \langle\tilde{L} \tilde{N}\rangle & =-\nu \\
\left\langle X_{1} X_{2}\right\rangle & =\mu, & \left\langle\tilde{X}_{1} \tilde{X}_{2}\right\rangle & =\nu \\
\left\langle\mathcal{Q}_{\alpha}^{-} \mathcal{Q}_{\beta}^{-}\right\rangle & =2 \mu C_{\alpha \beta}, & \left\langle\mathcal{Q}_{\alpha}^{+} \mathcal{R}_{\beta}\right\rangle & =2 \mu c_{\alpha \beta}, \tag{4.6}
\end{align*}
$$

where $\mu$ and $\nu$ are arbitrary independent constants related to the ones of the super NappiWitten and Nappi-Witten algebras, respectively. Interestingly, the superalgebra (4.5) can be related to a central extension of the extended Bargmann superalgebra (3.5) after a suitable redefinition of the NR generators and after having performed a vanishing cosmological constant limit. Indeed, let us consider the following redefinition of the generators:

$$
\begin{array}{rlr}
\tilde{G}_{a}=L_{a}-\tilde{L}_{a}, & \tilde{P}_{a}=\frac{1}{\ell}\left(L_{a}+\tilde{L}_{a}\right), & \tilde{Q}_{\alpha}^{+}=\sqrt{\frac{2}{\ell}} \mathcal{Q}_{\alpha}^{+}, \\
\tilde{J}=L+\tilde{L}, & \tilde{H}=\frac{1}{\ell}(L-\tilde{L}), & \tilde{Q}_{\alpha}^{-}=\sqrt{\frac{2}{\ell}} \mathcal{Q}_{\alpha}^{-}, \\
\tilde{S}=N+\tilde{N}, & \tilde{M}=\frac{1}{\ell}(N-\tilde{N}), & \tilde{R}_{\alpha}=\sqrt{\frac{2}{\ell}} \mathcal{R}_{\alpha}, \\
\tilde{Y}_{1}=X_{1}-\tilde{X}_{1}, & \tilde{U}_{1}=\frac{1}{\ell}\left(X_{1}+\tilde{X}_{1}\right), & \\
\tilde{Y}_{2}=X_{2}-\tilde{X}_{2}, & \tilde{U}_{2}=\frac{1}{\ell}\left(X_{2}+\tilde{X}_{2}\right), &
\end{array}
$$

where $\ell$ is a length parameter related to the cosmological constant through $\Lambda \propto \pm \frac{1}{\ell^{2}}$. Thus, the expanded superalgebra (4.5) can be rewritten as

$$
\begin{array}{rlrl}
{\left[\tilde{J}, \tilde{G}_{a}\right]=\epsilon_{a b} \tilde{G}_{b},} & & {\left[\tilde{G}_{a}, \tilde{G}_{b}\right]=-\epsilon_{a b} \tilde{S},} & {\left[\tilde{J}, \tilde{P}_{a}\right]=\epsilon_{a b} \tilde{P}_{b},} \\
{\left[\tilde{H}^{\prime}, \tilde{G}_{a}\right]=\epsilon_{a b} \tilde{P}_{b},} & & {\left[\tilde{G}_{a}, \tilde{P}_{b}\right]=-\epsilon_{a b} \tilde{M},} & {\left[\tilde{H}, \tilde{P}_{a}\right]=\frac{1}{\ell^{2}} \epsilon_{a b} \tilde{G}_{b},} \\
{\left[\tilde{P}_{a}, \tilde{P}_{b}\right]=-\frac{1}{\ell^{2}} \epsilon_{a b} \tilde{S},} & & {\left[\tilde{J}, \tilde{Q}_{\alpha}^{ \pm}\right]=-\frac{1}{2}\left(\gamma_{0}\right)_{\alpha}{ }^{\beta} \tilde{Q}_{\beta}^{ \pm},} & {\left[\tilde{H}, \tilde{Q}_{\alpha}^{ \pm}\right]=-\frac{1}{2 \ell}\left(\gamma_{0}\right)_{\alpha}{ }^{\beta} \tilde{Q}_{\beta}^{ \pm},} \\
{\left[\tilde{J}, \tilde{R}_{\alpha}\right]=-\frac{1}{2}\left(\gamma_{0}\right)_{\alpha}{ }^{\beta} \tilde{R}_{\beta},} & & {\left[\tilde{H}, \tilde{R}_{\alpha}\right]=-\frac{1}{2 \ell}\left(\gamma_{0}\right)_{\alpha}{ }^{\beta} \tilde{R}_{\beta},} & {\left[\tilde{G}_{a}, \tilde{Q}_{\alpha}^{+}\right]=-\frac{1}{2}\left(\gamma_{a}\right)_{\alpha}{ }^{\beta} \tilde{Q}_{\beta}^{-},} \\
{\left[\tilde{P}_{a}, \tilde{Q}_{\alpha}^{+}\right]=-\frac{1}{2 \ell}\left(\gamma_{a}\right)_{\alpha}{ }^{\beta} \tilde{Q}_{\beta}^{-},} & & {\left[\tilde{G}_{a}, \tilde{Q}_{\alpha}^{-}\right]=-\frac{1}{2}\left(\gamma_{a}\right)_{\alpha}{ }^{\beta} \tilde{R}_{\beta},} & {\left[\tilde{P}_{a}, \tilde{Q}_{\alpha}^{-}\right]=-\frac{1}{2 \ell}\left(\gamma_{a}\right)_{\alpha}{ }^{\beta} \tilde{R}_{\beta},} \\
{\left[\tilde{S}, \tilde{Q}_{\alpha}^{+}\right]=-\frac{1}{2}\left(\gamma_{0}\right)_{\alpha}{ }^{\beta} \tilde{R}_{\beta},} & & {\left[\tilde{M}, \tilde{Q}_{\alpha}^{+}\right]=-\frac{1}{2 \ell}\left(\gamma_{0}\right)_{\alpha}{ }^{\beta} \tilde{R}_{\beta},} & {\left[\tilde{Y}_{1}, \tilde{Q}_{\alpha}^{ \pm}\right]= \pm \frac{1}{2}\left(\gamma_{0}\right)_{\alpha \beta} \tilde{Q}_{\beta}^{ \pm}} \\
{\left[\tilde{U}_{1}, \tilde{Q}_{\alpha}^{ \pm}\right]= \pm \frac{1}{2 \ell}\left(\gamma_{0}\right)_{\alpha \beta} \tilde{Q}_{\beta}^{ \pm,}} & {\left[\tilde{Y}_{2}, \tilde{Q}_{\alpha}^{+}\right]=\frac{1}{2}\left(\gamma_{0}\right)_{\alpha \beta} \tilde{R}_{\beta},} & {\left[\tilde{U}_{2}, \tilde{Q}_{\alpha}^{+}\right]=\frac{1}{2 \ell}\left(\gamma_{0}\right)_{\alpha \beta} \tilde{R}_{\beta},} \\
{\left[\tilde{Y}_{1}, \tilde{R}_{\alpha}\right]=\frac{1}{2}\left(\gamma_{0}\right)_{\alpha \beta} \tilde{R}_{\beta},} & & {\left[\tilde{U}_{1}, \tilde{R}_{\alpha}\right]=\frac{1}{2 \ell}\left(\gamma_{0}\right)_{\alpha \beta} \tilde{R}_{\beta},} &
\end{array}
$$

$$
\begin{align*}
& \left\{\tilde{Q}_{\alpha}^{+}, \tilde{Q}_{\beta}^{-}\right\}=-\frac{1}{\ell}\left(\gamma^{a} C\right)_{\alpha \beta} \tilde{G}_{a}-\left(\gamma^{a} C\right)_{\alpha \beta} \tilde{P}_{a}, \\
& \left\{\tilde{Q}_{\alpha}^{+}, \tilde{Q}_{\beta}^{+}\right\}=-\frac{1}{\ell}\left(\gamma^{0} C\right)_{\alpha \beta} \tilde{J}-\left(\gamma^{0} C\right)_{\alpha \beta} \tilde{H}-\frac{1}{\ell}\left(\gamma^{0} C\right)_{\alpha \beta} \tilde{Y}_{1}-\left(\gamma^{0} C\right)_{\alpha \beta} \tilde{U}_{1}, \\
& \left\{\tilde{Q}_{\alpha}^{-}, \tilde{Q}_{\beta}^{-}\right\}=-\frac{1}{\ell}\left(\gamma^{0} C\right)_{\alpha \beta} \tilde{S}-\left(\gamma^{0} C\right)_{\alpha \beta} \tilde{M}+\frac{1}{\ell}\left(\gamma^{0} C\right)_{\alpha \beta} \tilde{Y}_{2}+\left(\gamma^{0} C\right)_{\alpha \beta} \tilde{U}_{2}, \\
& \left\{\tilde{Q}_{\alpha}^{+}, \tilde{R}_{\beta}\right\}=-\frac{1}{\ell}\left(\gamma^{0} C\right)_{\alpha \beta} \tilde{S}-\left(\gamma^{0} C\right)_{\alpha \beta} \tilde{M}-\frac{1}{\ell}\left(\gamma^{0} C\right)_{\alpha \beta} \tilde{Y}_{2}-\left(\gamma^{0} C\right)_{\alpha \beta} \tilde{U}_{2} . \tag{4.8}
\end{align*}
$$

This superalgebra corresponds to a supersymmetric extension of the extended NewtonHooke algebra [91-98] and coincides with the one presented in [8]. One can notice that it contains two additional bosonic generators, namely $\tilde{Y}_{1}$ and $\tilde{Y}_{2}$, with respect to the extended Newton-Hooke superalgebra introduced in [6]. Such generators, along with $\tilde{U}_{1}$ and $\tilde{U}_{2}$, act non-trivially on the fermionic generators $\tilde{Q}_{\alpha}^{ \pm}$and $\tilde{R}_{\alpha}$. Remarkably, in the flat limit $\ell \rightarrow \infty$ we recover the centrally extended version of the extended Bargmann superalgebra (3.5) with central charges $\tilde{U}_{1}$ and $\tilde{U}_{2}$.

The presence of the additional bosonic generators ensures to have a non-degenerate invariant tensor allowing us to construct the most general CS action invariant under the extended Newton-Hooke superalgebra. In particular, the invariant tensor can be obtained from the invariant tensor of the two copies of the Nappi-Witten algebra, one of which is supersymmetric. Indeed, considering the redefinition of the generators (4.7) in the invariant tensor (4.6), we find that the extended Newton-Hooke superalgebra admits the following non-vanishing components of the invariant tensor:

$$
\begin{align*}
& \left\langle\tilde{G}_{a} \tilde{G}_{b}\right\rangle=\alpha_{0} \delta_{\alpha \beta}, \\
& \langle\tilde{J} \tilde{S}\rangle=-\alpha_{0}, \\
& \left\langle\tilde{Y}_{1} \tilde{Y}_{2}\right\rangle=\alpha_{0}, \\
& \left\langle\tilde{G}_{a} \tilde{P}_{b}\right\rangle=\alpha_{1} \delta_{\alpha \beta}, \\
& \langle\tilde{J} \tilde{M}\rangle=-\alpha_{1} \\
& \left\langle\tilde{Y}_{1} \tilde{U}_{2}\right\rangle=\alpha_{1}, \\
& \left\langle\tilde{P}_{a} \tilde{P}_{b}\right\rangle=\frac{\alpha_{0}}{\ell^{2}} \delta_{\alpha \beta}, \\
& \langle\tilde{H} \tilde{S}\rangle=-\alpha_{1}, \quad\left\langle\tilde{U}_{1} \tilde{Y}_{2}\right\rangle=\alpha_{1}, \\
& \langle\tilde{H} \tilde{M}\rangle=-\frac{\alpha_{0}}{\ell^{2}},  \tag{4.9}\\
& \left\langle\tilde{U}_{1} \tilde{U}_{2}\right\rangle=\frac{\alpha_{0}}{\ell^{2}},
\end{align*}
$$

along with

$$
\begin{align*}
\left\langle\tilde{Q}_{\alpha}^{-} \tilde{Q}_{\beta}^{-}\right\rangle & =2\left(\frac{\alpha_{0}}{\ell}+\alpha_{1}\right) C_{\alpha \beta}, \\
\left\langle\tilde{Q}_{\alpha}^{+} \tilde{R}_{\beta}\right\rangle & =2\left(\frac{\alpha_{0}}{\ell}+\alpha_{1}\right) C_{\alpha \beta}, \tag{4.10}
\end{align*}
$$

where the Newton-Hooke parameters are related to the Nappi-Witten ones through

$$
\begin{equation*}
\alpha_{0}=\mu+\nu, \quad \alpha_{1}=\frac{1}{\ell}(\mu-\nu) . \tag{4.11}
\end{equation*}
$$

One can see that the invariant tensor (3.6) for the extended Bargmann superalgebra are recovered in the vanishing cosmological constant limit $\ell \rightarrow \infty$.

Let us now consider the gauge connection one-form $A$ for the extended Newton-Hooke superalgebra (4.8), that is

$$
\begin{align*}
A= & \omega \tilde{J}+\tau \tilde{H}+\omega^{a} \tilde{G}_{a}+e^{a} \tilde{P}_{a}+s \tilde{S}+m \tilde{M}+y_{1} \tilde{Y}_{1}+y_{2} \tilde{Y}_{2}+u_{1} \tilde{U}_{1}+u_{2} \tilde{U}_{2} \\
& +\bar{\psi}^{+} \tilde{Q}^{+}+\bar{\psi}^{-} \tilde{Q}^{-}+\tilde{\rho} \tilde{R} . \tag{4.12}
\end{align*}
$$

Then, the three-dimensional extended Newton-Hooke CS supergravity theory is obtained considering the gauge connection one-form (4.12) and the non-vanishing components of the invariant tensor (4.9)-(4.10) and plugging them into the general expression for a CS action (2.5). By doing so, one gets

$$
\begin{equation*}
I_{\mathrm{NH}}=\alpha_{0} I_{0}+\alpha_{1} I_{1} \tag{4.13}
\end{equation*}
$$

where

$$
\begin{align*}
I_{0}= & \int \omega_{a} R^{a}\left(\omega^{b}\right)-2 s R(\omega)+\frac{1}{\ell^{2}} e_{a} R^{a}\left(e^{b}\right)-\frac{1}{\ell^{2}} 2 m R(\tau)+2 y_{1} d y_{2}+\frac{2}{\ell^{2}} u_{1} d u_{2} \\
& -\frac{2}{\ell} \bar{\psi}^{-} \nabla \psi^{-}-\frac{2}{\ell} \bar{\psi}^{+} \nabla \rho-\frac{2}{\ell} \bar{\rho} \nabla \psi^{+}  \tag{4.14}\\
I_{1}= & \int 2 e_{a} R^{a}\left(\omega^{b}\right)-2 m R(\omega)-2 \tau R(s)+\frac{1}{\ell^{2}} \epsilon_{a b} \tau e^{a} e^{b}+2 y_{1} d u_{2}+2 u_{1} d y_{2} \\
& -2 \bar{\psi}^{-} \nabla \psi^{-}-2 \bar{\psi}^{+} \nabla \rho-2 \bar{\rho} \nabla \psi^{+} . \tag{4.15}
\end{align*}
$$

Here, the curvature two-form $R(s)$ is given by

$$
\begin{equation*}
R(s)=d s+\frac{1}{2} \epsilon^{a c} \omega_{a} \omega_{c} \tag{4.16}
\end{equation*}
$$

while $R(\omega)$ and $R^{a}\left(\omega^{b}\right)$ are given by (2.7). Besides, the covariant derivatives of the spinor 1 -form fields read

$$
\begin{align*}
\nabla \psi^{+}= & d \psi^{+}+\frac{1}{2} \omega \gamma_{0} \psi^{+}+\frac{1}{2 \ell} \tau \gamma_{0} \psi^{+}-\frac{1}{2} y_{1} \gamma_{0} \psi^{+}-\frac{1}{2 \ell} u_{1} \gamma_{0} \psi^{+} \\
\nabla \psi^{-}= & d \psi^{-}+\frac{1}{2} \omega \gamma_{0} \psi^{-}+\frac{1}{2} \omega^{a} \gamma_{a} \psi^{+}+\frac{1}{2 \ell} \tau \gamma_{0} \psi^{-}+\frac{1}{2 \ell} e^{a} \gamma_{a} \psi^{+}+\frac{1}{2} y_{1} \gamma_{0} \psi^{-}+\frac{1}{2 \ell} u_{1} \gamma_{0} \psi^{-} \\
\nabla \rho= & d \rho+\frac{1}{2} \omega \gamma_{0} \rho+\frac{1}{2} \omega^{a} \gamma_{a} \psi^{-}+\frac{1}{2} s \gamma_{0} \psi^{+}+\frac{1}{2 \ell} \tau \gamma_{0} \rho+\frac{1}{2 \ell} e^{a} \gamma_{a} \psi^{-}+\frac{1}{2 \ell} m \gamma_{0} \psi^{+} \\
& -\frac{1}{2} y_{2} \gamma_{0} \psi^{+}-\frac{1}{2} y_{1} \gamma_{0} \rho-\frac{1}{2 \ell} u_{2} \gamma_{0} \psi^{+}-\frac{1}{2 \ell} u_{1} \gamma_{0} \rho \tag{4.17}
\end{align*}
$$

The three-dimensional CS extended Newton-Hooke supergravity action (4.13) can be written in terms of two independent CS actions. The CS term $I_{0}$ corresponds to an exotic CS NR supergravity action [6] and can be seen as the NR version of the exotic $\mathcal{N}=2 \operatorname{CS} \operatorname{AdS}$ supergravity action [49, 99, 100],

$$
\begin{equation*}
I_{\text {exotic }}^{\mathrm{AdS}}=\int \omega_{A} d \omega^{A}+\frac{1}{3} \epsilon_{A B C} \omega^{A} \omega^{B} \omega^{C}+\frac{1}{\ell^{2}} e_{A} T^{A}+\mathrm{t} d \mathrm{t}+\frac{1}{\ell^{2}} \mathbf{u} d \mathbf{u}-\frac{2}{\ell} \bar{\psi}^{i} \nabla \psi^{i}, \tag{4.18}
\end{equation*}
$$

where $A=0,1,2, i=1,2$, and $\{\mathrm{t}, \mathrm{u}\}$ are $\mathfrak{s o}(2)$ internal symmetry gauge fields. On the other hand, the CS term $I_{1}$ is the supersymmetric extension of the Newton-Hooke gravity action and resembles the extended Newton-Hooke supergravity action introduced in [6] except for the presence of the additional bosonic gauge fields $y_{1}, y_{2}, u_{1}$, and $u_{2}$. At the bosonic level, the relativistic exotic term is related to the Pontryagin density, while the term along $\alpha_{1}$ corresponds to an Euler CS form [101]. The combination of both families allows to write the most general CS action not only for the AdS (or Poincaré) algebra but also for the Maxwell-like symmetries [102]. Interestingly, the exotic Pontryagin CS term
can be extended at the supersymmetric and NR level, corresponding to the $I_{0} \mathrm{CS}$ action. As we shall see in the next section, the NR exotic terms can also be introduced into a generalized extended Newton-Hooke supergravity theory.

Let us note that the non-degeneracy of the invariant tensor (4.9)-(4.10) is related to the requirement that the CS supergravity action involves a kinematical term for each gauge field. Then, the equations of motion of the extended Newton-Hooke supergravity theory are given by the vanishing of the corresponding curvature two-forms, which are given by (4.17) along with

$$
\begin{align*}
F(\omega) & =R(\omega)+\frac{1}{2 \ell} \bar{\psi}^{+} \gamma^{0} \psi^{+}, \\
F(\tau) & =d(\tau)+\frac{1}{2} \bar{\psi}^{+} \gamma^{0} \psi^{+}, \\
F^{a}\left(\omega^{b}\right) & =R^{a}\left(\omega^{b}\right)+\frac{1}{\ell^{2}} \epsilon^{a c} \tau e_{c}+\frac{1}{2} \bar{\psi}^{+} \gamma^{0} \psi^{-}, \\
F(s) & =R(s)+\frac{1}{2 \ell} \epsilon^{a c} e_{a} e_{c}+\frac{1}{2 \ell} \bar{\psi}^{-} \gamma^{0} \psi^{-}+\frac{1}{\ell} \bar{\psi}^{+} \gamma^{0} \rho, \\
F\left(y_{1}\right) & =d y_{1}+\frac{1}{2 \ell} \bar{\psi}^{+} \gamma^{0} \psi^{+}, \\
F\left(y_{2}\right) & =d y_{2}-\frac{1}{2 \ell} \bar{\psi}^{-} \gamma^{0} \psi^{-}+\frac{1}{\ell} \bar{\psi}^{+} \gamma^{0} \rho, \\
F^{a}\left(e^{b}\right) & =d e^{a}+\epsilon^{a c} \omega e_{c}+\epsilon^{a c} \tau \omega_{c}+\bar{\psi}^{+} \gamma^{a} \psi^{-}, \\
F(m) & =d m+\epsilon^{a c} \omega_{a} e_{c}+\frac{1}{2} \bar{\psi}^{-} \gamma^{0} \psi^{-}+\bar{\psi}^{+} \gamma^{0} \rho, \\
F\left(u_{1}\right) & =d u_{1}+\frac{1}{2} \bar{\psi}^{+} \gamma^{0} \psi^{+}, \\
F\left(u_{2}\right) & =d u_{2}-\frac{1}{2} \bar{\psi}^{-} \gamma^{0} \psi^{-}+\bar{\psi}^{+} \gamma^{0} \rho . \tag{4.19}
\end{align*}
$$

Here, one can prove that the vanishing cosmological constant limit $\ell \rightarrow \infty$ reproduces the equations of motion of the extended Bargmann supergravity theory. Thus,the extended Newton-Hooke supergravity theory reproduces, in the flat limit, the extended Bargmann supergravity theory (3.8). In particular, $I_{0}$ reduces to the exotic extended Bargmann gravity action, while the $I_{1}$ term leads us to the usual extended Bargmann supergravity [3] endowed with extra bosonic gauge fields.

### 4.2 Generalized extended Newton-Hooke supergravity

A generalization of the extended Newton-Hooke superalgebra, denoted as GNH ${ }^{(N)}$, can be obtained by performing an $S$-expansion of the super Nappi-Witten algebra (2.1). To this end, let us first consider $S_{\mathcal{M}}^{(2 N)}=\left\{\lambda_{0}, \lambda_{1}, \lambda_{2}, \ldots, \lambda_{2 N-1}, \lambda_{2 N}\right\}$ as the relevant semigroup whose elements satisfy the following multiplication law:

$$
\lambda_{\alpha} \lambda_{\beta}= \begin{cases}\lambda_{\alpha+\beta} & \text { if } \alpha+\beta \leq 2 N  \tag{4.20}\\ \lambda_{\alpha+\beta-2 N} & \text { if } \quad \alpha+\beta>2 N\end{cases}
$$

with $N \geq 2$. Let us note that, unlike the semigroups $S_{E}^{(2 N)}$ and $S_{L}^{(1)}$, there is no zero element in the present semigroup. Let us now consider a semigroup decomposition $S_{\mathcal{M}}^{(2 N)}=$
$S_{0} \cup S_{1}$ where

$$
\begin{align*}
& S_{0}=\left\{\lambda_{2 i}, i=0, \ldots, N\right\}, \\
& S_{1}=\left\{\lambda_{2 m-1}, m=1, \ldots, N\right\} . \tag{4.21}
\end{align*}
$$

Such decomposition is said to be resonant since it satisfies the same subspace decomposition of the Nappi-Witten superalgebra (2.2),

$$
\begin{align*}
& S_{0} \cdot S_{0} \subset S_{0}, \\
& S_{0} \cdot S_{1} \subset S_{1}, \\
& S_{1} \cdot S_{1} \subset S_{0} . \tag{4.22}
\end{align*}
$$

One can see that a novel family of NR superalgebras is obtained after performing a resonant $S_{\mathcal{M}}^{(2 N)}$-expansion to the super Nappi-Witten algebra (2.1). In particular, the expanded NR generators are related to the super Nappi-Witten ones through the semigroup elements as follows:

$$
\begin{array}{ll}
\tilde{J}^{(i)}=\lambda_{2 i} J, & \tilde{Q}_{\alpha}^{+(m)}=\lambda_{2 m-1} Q_{\alpha}^{+}, \\
\tilde{G}_{a}^{(i)}=\lambda_{2 i} G_{a}, & \tilde{Q}_{\alpha}^{-(m)}=\lambda_{2 m-1} Q_{\alpha}^{-}, \\
\tilde{S}^{(i)}=\lambda_{2 i} S, & \tilde{R}_{\alpha}^{(m)}=\lambda_{2 m-1} R_{\alpha}, \\
\tilde{T}_{1}^{(i)}=\lambda_{2 i} T_{1} & \tilde{T}_{2}^{(i)}=\lambda_{2 i} T_{2} .
\end{array}
$$

Then, considering the multiplication law of the $S_{\mathcal{M}}^{(2 N)}$ semigroup (4.20) and the original commutation relations of the super Nappi-Witten algebra (2.1), one can then show that the expanded NR superalgebra satisfy the following (anti-)commutation relations:

$$
\begin{array}{rlrl}
{\left[\tilde{J}^{(i)}, \tilde{G}_{a}^{(j)}\right]} & =\epsilon_{a b} \tilde{G}_{b}^{(i * j)}, & {\left[\tilde{G}_{a}^{(i)}, \tilde{G}_{b}^{(j)}\right]} & =-\epsilon_{a b} \tilde{S}^{(i * j)}, \\
{\left[\tilde{J}^{(i)}, \tilde{Q}_{\alpha}^{ \pm(m)}\right]} & =-\frac{1}{2}\left(\gamma_{0}\right)_{\alpha}{ }^{\beta} \tilde{Q}_{\beta}^{ \pm(i * m)}, & {\left[\tilde{J}^{(i)}, \tilde{R}_{\alpha}^{(m)}\right]} & =-\frac{1}{2}\left(\gamma_{0}\right)_{\alpha}^{\beta} \tilde{R}_{\beta}^{(i * m)}, \\
{\left[\tilde{G}_{a}^{(i)}, \tilde{Q}_{\alpha}^{+(m)}\right]} & =-\frac{1}{2}\left(\gamma_{a}\right)_{\alpha}{ }^{\beta} \tilde{Q}_{\beta}^{-(i * m)}, & {\left[\tilde{G}_{a}^{(i)}, \tilde{Q}_{\alpha}^{-(m)}\right]=-\frac{1}{2}\left(\gamma_{a}\right)_{\alpha}^{\beta} \tilde{R}_{\beta}^{(i * m)},} \\
{\left[\tilde{S}^{(i)}, \tilde{Q}_{\alpha}^{+(m)}\right]} & =-\frac{1}{2}\left(\gamma_{0}\right)_{\alpha}^{\beta} \tilde{R}_{\beta}^{(i * m)}, & {\left[\tilde{T}_{1}^{(i)}, \tilde{Q}_{\alpha}^{ \pm(m)}\right]= \pm \frac{1}{2}\left(\gamma_{0}\right)_{\alpha \beta} \tilde{Q}_{\beta}^{ \pm(i * m)},} \\
{\left[\tilde{T}_{2}^{(i)}, \tilde{Q}_{\alpha}^{+(m)}\right]} & =\frac{1}{2}\left(\gamma_{0}\right)_{\alpha \beta} \tilde{R}_{\beta}^{(i * m)}, & {\left[\tilde{T}_{1}^{(i)}, \tilde{R}_{\alpha}^{(m)}\right]=\frac{1}{2}\left(\gamma_{0}\right)_{\alpha \beta} \tilde{R}_{\beta}^{(i * m)},} \\
\left\{\tilde{Q}_{\alpha}^{+(m)}, \tilde{Q}_{\beta}^{-(n)}\right\} & =-\left(\gamma^{a} C\right)_{\alpha \beta} \tilde{G}_{a}^{(m *[n-1])}, & \\
\left\{\tilde{Q}_{\alpha}^{+(m)}, \tilde{Q}_{\beta}^{+(n)}\right\} & =-\left(\gamma^{0} C\right)_{\alpha \beta} \tilde{J}^{(m *[n-1])}-\left(\gamma^{0} C\right)_{\alpha \beta} \tilde{T}_{1}^{(m *[n-1])}, \\
\left\{\tilde{Q}_{\alpha}^{-(m)}, \tilde{Q}_{\beta}^{-(n)}\right\} & =-\left(\gamma^{0} C\right)_{\alpha \beta} \tilde{S}^{(m *[n-1])}+\left(\gamma^{0} C\right)_{\alpha \beta} \tilde{T}_{2}^{(m *[n-1])}, \\
\left\{\tilde{Q}_{\alpha}^{+(m)}, \tilde{R}_{\beta}^{(n)}\right\} & =-\left(\gamma^{0} C\right)_{\alpha \beta} \tilde{S}^{(m *[n-1])}-\left(\gamma^{0} C\right)_{\alpha \beta} \tilde{T}_{2}^{(m *[n-1])},
\end{array}
$$

where we have defined

$$
i * j= \begin{cases}i+j & \text { if } \quad i+j \leq N,  \tag{4.25}\\ i+j-N & \text { if } \quad i+j>N\end{cases}
$$

The NR superalgebra (4.24) corresponds to a generalized extended Newton-Hooke superalgebra which we have denoted as $\mathrm{GNH}^{(N)}$ superalgebra. The $\mathrm{GNH}^{(N)}$ superalgebra is characterized by $3 N$ fermionic generators and can be seen as the supersymmetric extension of the generalized Newton-Hooke algebra presented in [65]. Furthermore, as a consequence of the $S$-expansion procedure, the $\mathrm{GNH}^{(N)}$ superalgebra contains $2 N$ additional bosonic generators with respect to the generalized Newton-Hooke ones. As we shall see, their presence ensures the non-degeneracy of the invariant tensor. The present superalgebra can be seen as the NR counterpart of the supersymmetric extension of the $\mathfrak{C}_{N+2}$ algebra [72, 103]. However, the construction of a NR limit reproducing the $\mathrm{GNH}^{(N)}$ superalgebra shall not be explored here.

It is important to clarify that $N \geq 2$, while the usual extended Newton-Hooke superalgebra (4.8) does not appear as a particular subcase of $\mathrm{GNH}^{(N)}$ since it has been obtained with a different semigroup, namely $S_{L}^{(1)}$. Nevertheless, since at the bosonic level the extended Newton-Hooke algebra belongs to the bosonic generalized Newton-Hooke family, we shall consider the extended Newton-Hooke superalgebra as the GNH ${ }^{(1)}$ superalgebra. The first case obtained with the present procedure corresponds to the GNH ${ }^{(2)}$ superalgebra. In particular, the $N=2$ case reproduces the enlarged extended Bargmann (EEB) superalgebra recently introduced in [8]. Remarkably, similarly to the extended Newton-Hooke superalgebra, the EEB superalgebra can be written as three copies of the Nappi-Witten algebra, two of which are augmented by supersymmetry [8]. One could argue that the $\mathrm{GNH}^{(N)}$ superalgebra could be written as $N+1$ copies of Nappi-Witten algebras, $N$ of which should be augmented by supersymmetry. Nevertheless, this does not hold true anymore for $N>2$.

Let us note that the $2 N$ extra bosonic generators $\tilde{T}_{1}^{(i)}$ and $\tilde{T}_{2}^{(i)}$ allow to have a non-degenerate invariant tensor which is essential to the proper construction of a NR CS supergravity action. Indeed, following the definitions of [53], it is possible to show that the $\mathrm{GNH}^{(N)}$ superalgebra admits the following non-vanishing components of the invariant tensor:

$$
\begin{align*}
\left\langle\tilde{G}_{a}^{(i)} \tilde{G}_{b}^{(j)}\right\rangle & =\alpha_{i * j} \delta_{a b}, \\
\left\langle\tilde{J}^{(i)} \tilde{S}^{(j)}\right\rangle & =-\alpha_{i * j}, \\
\left\langle\tilde{T}_{1}^{(i)} \tilde{T}_{2}^{(j)}\right\rangle & =\alpha_{i * j}, \\
\left\langle\tilde{Q}_{\alpha}^{-(m)} \tilde{Q}_{\beta}^{-(n)}\right\rangle & =2 \alpha_{m *(n-1)} C_{\alpha \beta}=\left\langle\tilde{Q}_{\alpha}^{+(m)} \tilde{R}_{\beta}^{(n)}\right\rangle, \tag{4.26}
\end{align*}
$$

where the NR parameters appear as a consequence of the $S$-expansion procedure [53]. On the other hand, the $\mathrm{GNH}^{(N)}$ gauge connection one-form $A$ reads

$$
\begin{align*}
& A=\omega^{(i)} \tilde{J}^{(i)}+\omega^{a(i)} \tilde{G}_{a}^{(i)}+s^{(i)} \tilde{S}^{(i)}+t_{1}^{(i)} \tilde{T}_{1}^{(i)}+t_{2}^{(i)} T_{2}^{(i)} \\
&+\bar{\psi}^{+(m)} \tilde{Q}^{+(m)}+\bar{\psi}^{-(m)} \tilde{Q}^{-(m)}+\bar{\rho}^{(m)} \tilde{R}^{(m)} . \tag{4.27}
\end{align*}
$$

Then, the three-dimensional CS action based on the super GNH ${ }^{(N)}$ algebra is obtained by combining the gauge connection one-form (4.27) and the non-vanishing components of the
invariant tensor (4.26) into the general expression for a CS action (2.5), which thus yields

$$
\begin{equation*}
I_{\mathrm{GNH}^{(N)}}=\alpha_{i} I_{i}=\alpha_{0} I_{0}+\alpha_{1} I_{1}+\ldots+\alpha_{N} I_{N}, \tag{4.28}
\end{equation*}
$$

where

$$
\begin{align*}
I_{i}= & \int \omega_{a}^{(j)} d \omega^{a(k)} \delta_{j * k}^{i}+\epsilon^{a c} \omega_{a}^{(j)} \omega^{(k)} \omega_{c}^{(l)} \delta_{j * k * l}^{i}-2 s^{(j)} d \omega^{(k)} \delta_{j * k}^{i}+2 t_{1}^{(j)} d t_{2}^{(k)} \delta_{j * k}^{i} \\
& +2 \bar{\psi}^{-(m)} \nabla \psi^{-(n)} \delta_{m *(n-1)}^{i}+2 \bar{\psi}^{+(m)} \nabla \rho^{(n)} \delta_{m *(n-1)}^{i}+2 \bar{\rho}^{(m)} \nabla \psi^{+(n)} \delta_{m *(n-1)}^{i} . \tag{4.29}
\end{align*}
$$

In particular, the covariant derivatives of the spinor 1-forms for the $\mathrm{GNH}^{(N)}$ superalgebra read

$$
\begin{align*}
\nabla \psi^{+(m)}= & d \psi^{+(m)}+\frac{1}{2} \omega^{(i)} \gamma_{0} \psi^{+(n)} \delta_{i * n}^{m}-\frac{1}{2} t_{1}^{(i)} \gamma_{0} \psi^{+(n)} \delta_{i * n}^{m}, \\
\nabla \psi^{-(m)}= & d \psi^{-(m)}+\frac{1}{2} \omega^{(i)} \gamma_{0} \psi^{-(n)} \delta_{i * n}^{m}+\frac{1}{2} \omega^{a(i)} \gamma_{a} \psi^{+(n)} \delta_{i * n}^{m}+\frac{1}{2} t_{1}^{(i)} \gamma_{0} \psi^{-(n)} \delta_{i * n}^{m}, \\
\nabla \rho^{(m)}= & d \rho^{(m)}+\frac{1}{2} \omega^{(i)} \gamma_{0} \rho^{(n)} \delta_{i * n}^{m}+\frac{1}{2} \omega^{a(i)} \gamma_{a} \psi^{-(n)} \delta_{i * n}^{m}+\frac{1}{2} s^{(i)} \gamma_{0} \psi^{+(n)} \delta_{i * n}^{m} \\
& -\frac{1}{2} t_{2}^{(i)} \gamma_{0} \psi^{+(n)} \delta_{i * n}^{m}-\frac{1}{2} t_{1}^{(i)} \gamma_{0} \rho^{(n)} \delta_{i * n}^{m} . \tag{4.30}
\end{align*}
$$

The CS supergravity action (4.29) is the most general CS action based on the GNH ${ }^{(N)}$ superalgebra. In particular, as in the extended Newton-Hooke supergravity, the CS expression (4.28) can be split into two families: the CS term proportional to $\alpha_{2 k}$ corresponds to NR exotic terms whose bosonic counterparts are related to the Pontryagin density; on the other hand, the contributions proportional to $\alpha_{2 k+1}$ belong to the Euler CS family. In the particular case $N=2$, the CS supergravity action reproduces the EEB supergravity theory presented in [8] by identifying the gauge field one-forms as

$$
\begin{align*}
& \omega^{(0)}=\omega, \quad \omega_{a}^{(0)}=\omega_{a}, \quad s^{(0)}=s, \quad t_{1}^{(0)}=y_{1}, \quad t_{2}^{(0)}=y_{2}, \\
& \omega^{(1)}=\tau, \quad \omega_{a}^{(1)}=e_{a}, \quad s^{(1)}=m, \quad t_{1}^{(1)}=u_{1}, \quad t_{2}^{(1)}=u_{2}, \\
& \omega^{(2)}=k, \quad \omega_{a}^{(2)}=k_{a}, \quad s^{(2)}=t, \quad t_{1}^{(2)}=b_{1}, \quad t_{2}^{(2)}=b_{2}, \\
& \psi^{+(1)}=\psi^{+}, \quad \psi^{-(1)}=\psi^{-}, \quad \rho^{(1)}=\rho, \\
& \psi^{+(2)}=\xi^{+}, \quad \psi^{-(2)}=\xi^{-}, \quad \rho^{(2)}=\chi . \tag{4.31}
\end{align*}
$$

The EEB supergravity action contains three independent sectors proportional to $\alpha_{0}, \alpha_{1}$, and $\alpha_{2}$. In such NR model, $I_{0}$ describes the NR exotic gravity coupled to the additional bosonic gauge fields $y_{1}$ and $y_{2}$. On the other hand, $I_{1}$ and $I_{2}$ describe a NR extended supergravity model in the presence of a cosmological constant and of extra bosonic content given by $k, k_{a}$, and $t$ (for further details about the EEB supergravity theory see [8]). The $N=3$ case would now reproduce four independent sectors as in the GMEB supergravity theory previously introduced.

Let us note that, due to the non-degeneracy of the invariant tensor, the equations of motion of the GNH ${ }^{(N)}$ supergravity theory are given by the vanishing of the associated curvature two-forms, which read

$$
F\left(\omega^{(i)}\right)=d \omega^{i}+\frac{1}{2} \bar{\psi}^{+(m)} \gamma^{0} \psi^{+(n)} \delta_{m *(n-1)}^{i},
$$

$$
\begin{align*}
F^{a}\left(\omega^{b(i)}\right) & =d \omega^{a(i)}+\epsilon^{a c} \omega^{(j)} \omega_{c}^{(k)} \delta_{j * k}^{i}+\bar{\psi}^{+(m)} \gamma^{a} \psi^{-(n)} \delta_{m *(n-1)}^{i} \\
F\left(s^{(i)}\right) & =d s^{(i)}+\frac{1}{2} \bar{\psi}^{-(m)} \gamma^{0} \psi^{-(n)} \delta_{m *(n-1)}^{i}+\bar{\psi}^{+(m)} \gamma^{0} \rho^{(n)} \delta_{m *(n-1)}^{i} \\
F\left(t_{1}^{(i)}\right) & =d t_{1}^{(i)}+\frac{1}{2} \bar{\psi}^{+(m)} \gamma^{0} \psi^{+(n)} \delta_{m *(n-1)}^{i} \\
F\left(t_{2}^{(i)}\right) & =d t_{2}^{(i)}-\frac{1}{2} \bar{\psi}^{-(m)} \gamma^{0} \psi^{-(n)} \delta_{m *(n-1)}^{i}+\bar{\psi}^{+(m)} \gamma^{0} \rho^{(n)} \delta_{m *(n-1)}^{i} \tag{4.32}
\end{align*}
$$

along with (4.30).
It is interesting to notice that the $\mathrm{GNH}^{(N)}$ superalgebras are related to the GEB ${ }^{(N)}$ ones through an IW contraction procedure. The following diagram summarizes the expansion and contraction relations:


In particular, the GEB ${ }^{(N)}$ superalgebras appear as an IW contraction of the generalized extended Newton-Hooke ones by rescaling the super GNH ${ }^{(N)}$ generators as follows:

$$
\begin{align*}
\tilde{J}^{(i)} & \rightarrow \sigma^{2 i} \tilde{J}^{(i)}, \\
\tilde{G}_{a}^{(i)} & \rightarrow \sigma^{2 i} \tilde{G}_{a}^{(i)} \\
\tilde{S}^{(i)} & \rightarrow \sigma^{2 i} \tilde{S}^{(i)}, \tag{4.33}
\end{align*}
$$

$$
\tilde{Q}_{\alpha}^{+(m)} \rightarrow \sigma^{2 m-1} \tilde{Q}_{\alpha}^{+(m)}
$$

$$
\tilde{Q}_{\alpha}^{-(m)} \rightarrow \sigma^{2 m-1} \tilde{Q}_{\alpha}^{-(m)}
$$

$$
\tilde{R}_{\alpha}^{(m)} \rightarrow \sigma^{2 m-1} \tilde{R}_{\alpha}^{(m)}
$$

and considering the limit $\sigma \rightarrow \infty$. Note that the vanishing cosmological constant limit $\ell \rightarrow$ $\infty$ performed in the $N=1$ case to recover the extended Bargmann superalgebra can also be seen as an IW contraction. At the CS action level, the proper IW contraction requires rescaling, in addition, the NR parameters in the invariant tensor (4.26) as $\alpha_{i} \rightarrow \sigma^{i} \alpha_{i}$.

As an ending remark, let us note that the $S_{E}$ and $S_{\mathcal{M}}$ semigroups used to obtain the respective generalizations of the extended Bargmann and extended Newton-Hooke supergravity theories are the same used to find generalizations of the super Poincaré and super AdS algebra. Furthermore, as it was shown in [27, 65], the same semigroups were considered at the bosonic level, allowing to define the $\mathrm{GEB}^{(N)}$ and $\mathrm{GNH}^{(N)}$ algebras.

## 5 Concluding remarks

In this paper, we have introduced an alternative procedure to obtain diverse NR CS supergravity theories in three spacetime dimensions. Known and new NR superalgebras have been obtained considering the expansion method based on semigroups, the so-called $S$ expansion, to a Nappi-Witten superalgebra introduced in [8]. Interestingly, the $S$-expansion
allows to immediately obtain the non-vanishing components of the invariant tensor of an expanded superalgebra in term of the original ones. Such advantage has allowed us to construct, in a systematic way, the respective NR CS supergravity actions for each NR superalgebra presented.

We have shown that our resulting theories can be split into two NR supergravity families. Indeed, the extended Bargmann supergravity along with its Maxwellian version can be seen as particular subcases of a generalized extended Bargmann supergravity theory which we have denoted as $\mathrm{GEB}^{(N)}$ supergravity. In particular, for $N=1$ we recover the extended Bargmann supergravity theory. On the other hand, the extended Newton-Hooke supergravity belongs to a generalized extended Newton-Hooke theory which we have called as $\mathrm{GNH}^{(N)}$. Such generalizations correspond to supersymmetric extensions of the GEB ${ }^{(N)}$ and $\mathrm{GNH}^{(N)}$ bosonic algebras recently introduced in [27]. Remarkably, both families are related through an IW contraction process, similarly as their bosonic counterparts.

It would be worth considering further studies on the Maxwellian version and generalizations of the extended Bargmann supergravity theory. One could analyze, for instance, the Schrödinger extension [2, 104] of the NR generalized superalgebras presented here in a similar way as it was done in the case of the extended Schrödinger supergravity [6]. In particular, the map between Newton-Cartan geometry and Horava-Lifshitz gravity [97] could suggest a superconformal non-projectable Horava-Lifshitz gravity. The Schrödinger version of our results would allow us to approach an off-shell formulation of the respective NR supergravity actions which could serve to construct NR effective field theories on curved backgrounds by means of localization [105, 106]. On the other hand, it would be intriguing to explore the possibility to apply the $S$-expansion method in the context of the Schrödinger superalgebra families, in order to establish a systematic way to obtain generalized Schrödinger supergravity actions, in a very similar way to the construction presented here.

A future development could also consist in generalizing our results to the extended Newtonian family. One could expect to obtain novel Newtonian supergravity theories, different from the extended Newtonian one presented in [4], being supersymmetric extensions of the exotic and Maxwellian extended Newtonian gravity recently introduced in [107] and [68], respectively. Newtonian gravity models are worth studying as they offer an action principle for Newtonian gravity through the CS formalism different from the one introduced in [108]. It would be interesting to study the matter coupling of the new Newtonian supergravity theories as well.

The procedure considered here could be extended to the ultra-relativistic (UR) regime. In particular, the construction of UR supergravity models remains poorly explored [109, 110]. The $S$-expansion method could be applied to a Carrollian version of the NappiWitten symmetry to obtain new UR superalgebras (work in progress). One could expect to find two UR superalgebra families being the respective UR versions of the relativistic $\mathfrak{B}_{N+2}$ and $\mathfrak{C}_{N+2}$ superalgebras. At the bosonic level, the Carrollian symmetries emerge in the framework of flat holography and fluid/gravity correspondence [111-114], whose applications motivate us to explore supersymmetric extensions of the Carrollian symmetries in the context of supergravity.

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## References

[1] R. Andringa, E.A. Bergshoeff, J. Rosseel and E. Sezgin, 3D Newton-Cartan supergravity, Class. Quant. Grav. 30 (2013) 205005 [arXiv:1305.6737] [inSPIRE].
[2] E. Bergshoeff, J. Rosseel and T. Zojer, Newton-Cartan supergravity with torsion and Schrödinger supergravity, JHEP 11 (2015) 180 [arXiv:1509.04527] [INSPIRE].
[3] E.A. Bergshoeff and J. Rosseel, Three-dimensional extended Bargmann supergravity, Phys. Rev. Lett. 116 (2016) 251601 [arXiv:1604.08042] [inSPIRE].
[4] N. Ozdemir, M. Ozkan, O. Tunca and U. Zorba, Three-dimensional extended Newtonian (super)gravity, JHEP 05 (2019) 130 [arXiv:1903.09377] [iNSPIRE].
[5] J.A. de Azcárraga, D. Gútiez and J.M. Izquierdo, Extended $D=3$ Bargmann supergravity from a Lie algebra expansion, Nucl. Phys. B 946 (2019) 114706 [arXiv:1904.12786] [INSPIRE].
[6] N. Ozdemir, M. Ozkan and U. Zorba, Three-dimensional extended Lifshitz, Schrödinger and Newton-Hooke supergravity, JHEP 11 (2019) 052 [arXiv:1909.10745] [INSPIRE].
[7] P. Concha, L. Ravera and E. Rodríguez, Three-dimensional Maxwellian extended Bargmann supergravity, JHEP 04 (2020) 051 [arXiv: 1912.09477] [INSPIRE].
[8] P. Concha, L. Ravera and E. Rodríguez, Three-dimensional non-relativistic extended supergravity with cosmological constant, Eur. Phys. J. C 80 (2020) 1105 [arXiv:2008.08655] [inSPIRE].
[9] R. Grassie, Generalised Bargmann superalgebras, arXiv:2010.01894 [inSPIRE].
[10] E. Cartan, Sur les variétés à connexion affine et la théorie de la relativité généralisée (première partie), Ann. Ecole Norm. Sup. 40 (1923) 325.
[11] E. Cartan, Sur les variétés à connexion affine et la théorie de la relativité généralisée (première partie) (suite), Ann. Ecole Norm. Sup. 41 (1924) 1.
[12] C. Duval and H.P. Kunzle, Minimal gravitational coupling in the Newtonian theory and the covariant Schrödinger equation, Gen. Rel. Grav. 16 (1984) 333 [inSPIRE].
[13] C. Duval, G. Burdet, H.P. Kunzle and M. Perrin, Bargmann structures and Newton-Cartan theory, Phys. Rev. D 31 (1985) 1841 [inSPIRE].
[14] C. Duval and P.A. Horvathy, Non-relativistic conformal symmetries and Newton-Cartan structures, J. Phys. A 42 (2009) 465206 [arXiv:0904.0531] [inSPIRE].
[15] R. Andringa, E. Bergshoeff, S. Panda and M. de Roo, Newtonian gravity and the Bargmann algebra, Class. Quant. Grav. 28 (2011) 105011 [arXiv:1011.1145] [inSPIRE].
[16] R. Banerjee, A. Mitra and P. Mukherjee, Localization of the Galilean symmetry and dynamical realization of Newton-Cartan geometry, Class. Quant. Grav. 32 (2015) 045010 [arXiv:1407.3617] [inSPIRE].
[17] R. Banerjee and P. Mukherjee, Torsional Newton-Cartan geometry from Galilean gauge theory, Class. Quant. Grav. 33 (2016) 225013 [arXiv:1604.06893] [INSPIRE].
[18] E. Bergshoeff, A. Chatzistavrakidis, L. Romano and J. Rosseel, Newton-Cartan gravity and torsion, JHEP 10 (2017) 194 [arXiv:1708.05414] [inSPIRE].
[19] L. Avilés, E. Frodden, J. Gomis, D. Hidalgo and J. Zanelli, Non-relativistic Maxwell Chern-Simons gravity, JHEP 05 (2018) 047 [arXiv:1802.08453] [InSPIRE].
[20] L. Avilés, J. Gomis and D. Hidalgo, Stringy (Galilei) Newton-Hooke Chern-Simons Gravities, JHEP 09 (2019) 015 [arXiv:1905.13091] [INSPIRE].
[21] D. Chernyavsky and D. Sorokin, Three-dimensional (higher-spin) gravities with extended Schrödinger and l-conformal Galilean symmetries, JHEP 07 (2019) 156 [arXiv:1905.13154] [inSPIRE].
[22] P. Concha and E. Rodríguez, Non-relativistic gravity theory based on an enlargement of the extended Bargmann algebra, JHEP 07 (2019) 085 [arXiv:1906.00086] [INSPIRE].
[23] T. Harmark, J. Hartong, L. Menculini, N.A. Obers and G. Oling, Relating non-relativistic string theories, JHEP 11 (2019) 071 [arXiv:1907.01663] [inSPIRE].
[24] D. Hansen, J. Hartong and N.A. Obers, Non-relativistic gravity and its coupling to matter, JHEP 06 (2020) 145 [arXiv:2001.10277] [InSPIRE].
[25] M. Ergen, E. Hamamci and D. Van den Bleeken, Oddity in nonrelativistic, strong gravity, Eur. Phys. J. C 80 (2020) 563 [Erratum ibid. 80 (2020) 657] [arXiv:2002.02688] [inSPIRE].
[26] O. Kasikci, N. Ozdemir, M. Ozkan and U. Zorba, Three-dimensional higher-order Schrödinger algebras and Lie algebra expansions, JHEP 04 (2020) 067 [arXiv:2002.03558] [inSPIRE].
[27] P. Concha, M. Ipinza and E. Rodríguez, Generalized Maxwellian exotic Bargmann gravity theory in three spacetime dimensions, Phys. Lett. B 807 (2020) 135593 [arXiv:2004.01203] [INSPIRE].
[28] D.T. Son, Toward an AdS/cold atoms correspondence: a geometric realization of the Schrödinger symmetry, Phys. Rev. D 78 (2008) 046003 [arXiv:0804.3972] [inSPIRE].
[29] K. Balasubramanian and J. McGreevy, Gravity duals for non-relativistic CFTs, Phys. Rev. Lett. 101 (2008) 061601 [arXiv:0804.4053] [inSPIRE].
[30] S. Kachru, X. Liu and M. Mulligan, Gravity duals of Lifshitz-like fixed points, Phys. Rev. D 78 (2008) 106005 [arXiv:0808.1725] [INSPIRE].
[31] A. Bagchi and R. Gopakumar, Galilean conformal algebras and AdS/CFT, JHEP 07 (2009) 037 [arXiv:0902.1385] [inSPIRE].
[32] A. Bagchi, R. Gopakumar, I. Mandal and A. Miwa, GCA in 2d, JHEP 08 (2010) 004 [arXiv:0912.1090] [INSPIRE].
[33] M.H. Christensen, J. Hartong, N.A. Obers and B. Rollier, Torsional Newton-Cartan geometry and Lifshitz holography, Phys. Rev. D 89 (2014) 061901 [arXiv:1311.4794] [inSPIRE].
[34] M.H. Christensen, J. Hartong, N.A. Obers and B. Rollier, Boundary stress-energy tensor and Newton-Cartan geometry in Lifshitz holography, JHEP 01 (2014) 057 [arXiv:1311.6471] [inSPIRE].
[35] J. Hartong, E. Kiritsis and N.A. Obers, Lifshitz space-times for Schrödinger holography, Phys. Lett. B 746 (2015) 318 [arXiv:1409.1519] [inSPIRE].
[36] J. Hartong, E. Kiritsis and N.A. Obers, Schrödinger invariance from Lifshitz isometries in holography and field theory, Phys. Rev. D 92 (2015) 066003 [arXiv:1409.1522] [INSPIRE].
[37] J. Hartong, E. Kiritsis and N.A. Obers, Field theory on Newton-Cartan backgrounds and symmetries of the Lifshitz vacuum, JHEP 08 (2015) 006 [arXiv:1502.00228] [inSPIRE].
[38] M. Taylor, Lifshitz holography, Class. Quant. Grav. 33 (2016) 033001 [arXiv:1512.03554] [inSPIRE].
[39] C. Hoyos and D.T. Son, Hall viscosity and electromagnetic response, Phys. Rev. Lett. 108 (2012) 066805 [arXiv:1109.2651] [INSPIRE].
[40] D.T. Son, Newton-Cartan geometry and the quantum Hall effect, arXiv:1306.0638 [inSPIRE].
[41] A.G. Abanov and A. Gromov, Electromagnetic and gravitational responses of two-dimensional noninteracting electrons in a background magnetic field, Phys. Rev. B 90 (2014) 014435 [arXiv:1401.3703] [INSPIRE].
[42] M. Geracie, K. Prabhu and M.M. Roberts, Curved non-relativistic spacetimes, Newtonian gravitation and massive matter, J. Math. Phys. 56 (2015) 103505 [arXiv:1503.02682] [INSPIRE].
[43] A. Gromov, K. Jensen and A.G. Abanov, Boundary effective action for quantum Hall states, Phys. Rev. Lett. 116 (2016) 126802 [arXiv:1506.07171] [InSPIRE].
[44] D.R. Grigore, The projective unitary irreducible representations of the Galilei group in (1+2)-dimensions, J. Math. Phys. 37 (1996) 460 [hep-th/9312048] [InSPIRE].
[45] S.K. Bose, The Galilean group in $(2+1)$ space-times and its central extension, Commun. Math. Phys. 169 (1995) 385 [INSPIRE].
[46] C. Duval and P.A. Horvathy, The 'Peierls substitution' and the exotic Galilei group, Phys. Lett. B 479 (2000) 284 [hep-th/0002233] [INSPIRE].
[47] R. Jackiw and V.P. Nair, Anyon spin and the exotic central extension of the planar Galilei group, Phys. Lett. B 480 (2000) 237 [hep-th/0003130] [inSPIRE].
[48] G. Papageorgiou and B.J. Schroers, A Chern-Simons approach to Galilean quantum gravity in $2+1$ dimensions, JHEP 11 (2009) 009 [arXiv:0907.2880] [inSPIRE].
[49] A. Achucarro and P.K. Townsend, A Chern-Simons action for three-dimensional Anti-de Sitter supergravity theories, Phys. Lett. B 180 (1986) 89 [INSPIRE].
[50] E. Witten, $(2+1)$-dimensional gravity as an exactly soluble system, Nucl. Phys. B 311 (1988) 46 [INSPIRE].
[51] J. Zanelli, Lecture notes on Chern-Simons (super-)gravities. Second edition (February 2008), hep-th/0502193 [INSPIRE].
[52] J.A. de Azcarraga, J.M. Izquierdo, M. Picón and O. Varela, Generating Lie and gauge free differential (super)algebras by expanding Maurer-Cartan forms and Chern-Simons supergravity, Nucl. Phys. B 662 (2003) 185 [hep-th/0212347] [inSPIRE].
[53] F. Izaurieta, E. Rodriguez and P. Salgado, Expanding Lie (super)algebras through Abelian semigroups, J. Math. Phys. 47 (2006) 123512 [hep-th/0606215] [inSPIRE].
[54] M. Hatsuda and M. Sakaguchi, Wess-Zumino term for the AdS superstring and generalized Inonu-Wigner contraction, Prog. Theor. Phys. 109 (2003) 853 [hep-th/0106114] [InSPIRE].
[55] J.A. de Azcarraga, J.M. Izquierdo, M. Picón and O. Varela, Expansions of algebras and superalgebras and some applications, Int. J. Theor. Phys. 46 (2007) 2738 [hep-th/0703017] [inSPIRE].
[56] R. Caroca, I. Kondrashuk, N. Merino and F. Nadal, Bianchi spaces and their three-dimensional isometries as $S$-expansions of two-dimensional isometries, J. Phys. A 46 (2013) 225201 [arXiv:1104.3541] [InSPIRE].
[57] L. Andrianopoli, N. Merino, F. Nadal and M. Trigiante, General properties of the expansion methods of Lie algebras, J. Phys. A 46 (2013) 365204 [arXiv:1308.4832] [InSPIRE].
[58] M. Artebani, R. Caroca, M.C. Ipinza, D.M. Peñafiel and P. Salgado, Geometrical aspects of the Lie algebra S-expansion procedure, J. Math. Phys. 57 (2016) 023516 [arXiv:1602.04525] [INSPIRE].
[59] M.C. Ipinza, F. Lingua, D.M. Peñafiel and L. Ravera, An analytic method for $S$-expansion involving resonance and reduction, Fortsch. Phys. 64 (2016) 854 [arXiv:1609.05042] [inSPIRE].
[60] C. Inostroza, I. Kondrashuk, N. Merino and F. Nadal, A Java library to perform $S$-expansions of Lie algebras, arXiv:1703. 04036 [InSPIRE].
[61] C. Inostroza, I. Kondrashuk, N. Merino and F. Nadal, On the algorithm to find S-related Lie algebras, J. Phys. Conf. Ser. 1085 (2018) 052011 [arXiv:1802.05765] [inSPIRE].
[62] E. Bergshoeff, J.M. Izquierdo, T. Ortín and L. Romano, Lie algebra expansions and actions for non-relativistic gravity, JHEP 08 (2019) 048 [arXiv:1904.08304] [InSPIRE].
[63] L. Romano, Non-relativistic four dimensional p-brane supersymmetric theories and Lie algebra expansion, arXiv:1906.08220 [INSPIRE].
[64] A. Fontanella and L. Romano, Lie algebra expansion and integrability in superstring $\sigma$-models, JHEP 07 (2020) 083 [arXiv:2005.01736] [inSPIRE].
[65] D.M. Peñafiel and P. Salgado-ReboLledó, Non-relativistic symmetries in three space-time dimensions and the Nappi-Witten algebra, Phys. Lett. B 798 (2019) 135005 [arXiv:1906.02161] [INSPIRE].
[66] J. Gomis, A. Kleinschmidt, J. Palmkvist and P. Salgado-ReboLledó, Newton-Hooke/Carrollian expansions of (A)dS and Chern-Simons gravity, JHEP 02 (2020) 009 [arXiv: 1912.07564] [inSPIRE].
[67] E. Bergshoeff, J. Gomis and P. Salgado-ReboLledó, Non-relativistic limits and three-dimensional coadjoint Poincaré gravity, Proc. Roy. Soc. Lond. A 476 (2020) 20200106 [arXiv:2001.11790] [INSPIRE].
[68] P. Concha, L. Ravera, E. Rodríguez and G. Rubio, Three-dimensional Maxwellian extended Newtonian gravity and flat limit, JHEP 10 (2020) 181 [arXiv:2006.13128] [INSPIRE].
[69] F. Izaurieta, E. Rodriguez, P. Minning, P. Salgado and A. Perez, Standard general relativity from Chern-Simons gravity, Phys. Lett. B 678 (2009) 213 [arXiv:0905.2187] [InSPIRE].
[70] J. Diaz, O. Fierro, F. Izaurieta, N. Merino, E. Rodriguez, P. Salgado et al., A generalized action for $(2+1)$-dimensional Chern-Simons gravity, J. Phys. A 45 (2012) 255207 [arXiv:1311.2215] [INSPIRE].
[71] P.K. Concha, D.M. Peñafiel, E.K. Rodríguez and P. Salgado, Even-dimensional general relativity from Born-Infeld gravity, Phys. Lett. B $\mathbf{7 2 5}$ (2013) 419 [arXiv:1309.0062] [inSPIRE].
[72] P. Salgado and S. Salgado, $\mathfrak{s o}(D-1,1) \otimes \mathfrak{s o}(D-1,2)$ algebras and gravity, Phys. Lett. $B$ 728 (2014) 5 [inSPIRE].
[73] R. Caroca, P. Concha, O. Fierro, E. Rodríguez and P. Salgado-ReboLledó, Generalized Chern-Simons higher-spin gravity theories in three dimensions, Nucl. Phys. B 934 (2018) 240 [arXiv:1712.09975] [inSPIRE].
[74] F. Izaurieta, E. Rodriguez and P. Salgado, Eleven-dimensional gauge theory for the $M$ algebra as an Abelian semigroup expansion of osp(32|1), Eur. Phys. J. C 54 (2008) 675 [hep-th/0606225] [inSPIRE].
[75] O. Fierro, F. Izaurieta, P. Salgado and O. Valdivia, Minimal AdS-Lorentz supergravity in three-dimensions, Phys. Lett. B 788 (2019) 198 [arXiv:1401.3697] [inSPIRE].
[76] P.K. Concha and E.K. Rodríguez, $N=1$ supergravity and Maxwell superalgebras, JHEP 09 (2014) 090 [arXiv:1407.4635] [inSPIRE].
[77] P.K. Concha, O. Fierro and E.K. Rodríguez, Inönü-Wigner contraction and $D=2+1$ supergravity, Eur. Phys. J. C $7 \mathbf{7 7}$ (2017) 48 [arXiv:1611.05018] [inSPIRE].
[78] A. Banaudi and L. Ravera, Generalized AdS-Lorentz deformed supergravity on a manifold with boundary, Eur. Phys. J. Plus 133 (2018) 514 [arXiv:1803.08738] [INSPIRE].
[79] P. Concha, D.M. Peñafiel and E. Rodríguez, On the Maxwell supergravity and flat limit in $2+1$ dimensions, Phys. Lett. B 785 (2018) 247 [arXiv:1807.00194] [INSPIRE].
[80] R. Caroca, P. Concha, E. Rodríguez and P. Salgado-ReboLledó, Generalizing the $b m s_{3}$ and 2D-conformal algebras by expanding the Virasoro algebra, Eur. Phys. J. C 78 (2018) 262 [arXiv:1707.07209] [inSPIRE].
[81] R. Caroca, P. Concha, O. Fierro and E. Rodríguez, Three-dimensional Poincaré supergravity and $N$-extended supersymmetric $B M S_{3}$ algebra, Phys. Lett. B 792 (2019) 93 [arXiv:1812.05065] [INSPIRE].
[82] R. Caroca, P. Concha, O. Fierro and E. Rodríguez, On the supersymmetric extension of asymptotic symmetries in three spacetime dimensions, Eur. Phys. J. C 80 (2020) 29 [arXiv:1908.09150] [INSPIRE].
[83] C.R. Nappi and E. Witten, A WZW model based on a nonsemisimple group, Phys. Rev. Lett. 71 (1993) 3751 [hep-th/9310112] [inSPIRE].
[84] J.M. Figueroa-O'Farrill and S. Stanciu, More D-branes in the Nappi-Witten background, JHEP 01 (2000) 024 [hep-th/9909164] [inSPIRE].
[85] E. Inonu and E.P. Wigner, On the contraction of groups and their represenations, Proc. Nat. Acad. Sci. 39 (1953) 510 [InSPIRE].
[86] J.D. Edelstein, M. Hassaine, R. Troncoso and J. Zanelli, Lie-algebra expansions, Chern-Simons theories and the Einstein-Hilbert Lagrangian, Phys. Lett. B 640 (2006) 278 [hep-th/0605174] [INSPIRE].
[87] R. Schrader, The Maxwell group and the quantum theory of particles in classical homogeneous electromagnetic fields, Fortsch. Phys. 20 (1972) 701 [InSPIRE].
[88] H. Bacry, P. Combe and J.L. Richard, Group-theoretical analysis of elementary particles in an external electromagnetic field. 1. The relativistic particle in a constant and uniform field, Nuovo Cim. A 67 (1970) 267 [InSPIRE].
[89] J. Gomis and A. Kleinschmidt, On free Lie algebras and particles in electro-magnetic fields, JHEP 07 (2017) 085 [arXiv:1705.05854] [inSPIRE].
[90] P.K. Concha and E.K. Rodríguez, Maxwell superalgebras and Abelian semigroup expansion, Nucl. Phys. B 886 (2014) 1128 [arXiv:1405.1334] [InSPIRE].
[91] R. Aldrovandi, A.L. Barbosa, L.C.B. Crispino and J.G. Pereira, Non-relativistic spacetimes with cosmological constant, Class. Quant. Grav. 16 (1999) 495 [gr-qc/9801100] [inSPIRE].
[92] G.W. Gibbons and C.E. Patricot, Newton-Hooke space-times, Hpp waves and the cosmological constant, Class. Quant. Grav. 20 (2003) 5225 [hep-th/0308200] [INSPIRE].
[93] J. Brugues, J. Gomis and K. Kamimura, Newton-Hooke algebras, non-relativistic branes and generalized pp-wave metrics, Phys. Rev. D 73 (2006) 085011 [hep-th/0603023] [INSPIRE].
[94] P.D. Alvarez, J. Gomis, K. Kamimura and M.S. Plyushchay, $(2+1) D$ exotic Newton-Hooke symmetry, duality and projective phase, Annals Phys. 322 (2007) 1556 [hep-th/0702014] [INSPIRE].
[95] G. Papageorgiou and B.J. Schroers, Galilean quantum gravity with cosmological constant and the extended $q$-Heisenberg algebra, JHEP 11 (2010) 020 [arXiv:1008.0279] [InSPIRE].
[96] C. Duval and P. Horvathy, Conformal Galilei groups, Veronese curves, and Newton-Hooke spacetimes, J. Phys. A 44 (2011) 335203 [arXiv:1104.1502] [InSPIRE].
[97] J. Hartong, Y. Lei and N.A. Obers, Nonrelativistic Chern-Simons theories and three-dimensional Hořava-Lifshitz gravity, Phys. Rev. D 94 (2016) 065027 [arXiv:1604.08054] [INSPIRE].
[98] C. Duval, G. Gibbons and P. Horvathy, Conformal and projective symmetries in Newtonian cosmology, J. Geom. Phys. 112 (2017) 197 [arXiv:1605.00231] [InSPIRE].
[99] P.S. Howe, J.M. Izquierdo, G. Papadopoulos and P.K. Townsend, New supergravities with central charges and Killing spinors in $(2+1)$-dimensions, Nucl. Phys. B 467 (1996) 183 [hep-th/9505032] [INSPIRE].
[100] A. Giacomini, R. Troncoso and S. Willison, Three-dimensional supergravity reloaded, Class. Quant. Grav. 24 (2007) 2845 [hep-th/0610077] [INSPIRE].
[101] R. Troncoso and J. Zanelli, Higher dimensional gravity, propagating torsion and AdS gauge invariance, Class. Quant. Grav. 17 (2000) 4451 [hep-th/9907109] [inSPIRE].
[102] P.K. Concha, D.M. Peñafiel, E.K. Rodríguez and P. Salgado, Generalized Poincaré algebras and Lovelock-Cartan gravity theory, Phys. Lett. B 742 (2015) 310 [arXiv:1405.7078] [inSPIRE].
[103] P.K. Concha, R. Durka, N. Merino and E.K. Rodríguez, New family of Maxwell like algebras, Phys. Lett. B 759 (2016) 507 [arXiv:1601.06443] [INSPIRE].
[104] H.R. Afshar, E.A. Bergshoeff, A. Mehra, P. Parekh and B. Rollier, A Schrödinger approach to Newton-Cartan and Hořava-Lifshitz gravities, JHEP 04 (2016) 145 [arXiv:1512.06277] [inSPIRE].
[105] V. Pestun, Localization of gauge theory on a four-sphere and supersymmetric Wilson loops, Commun. Math. Phys. 313 (2012) 71 [arXiv:0712.2824] [inSPIRE].
[106] G. Festuccia and N. Seiberg, Rigid supersymmetric theories in curved superspace, JHEP 06 (2011) 114 [arXiv:1105.0689] [inSPIRE].
[107] P. Concha, L. Ravera and E. Rodríguez, Three-dimensional exotic Newtonian gravity with cosmological constant, Phys. Lett. B 804 (2020) 135392 [arXiv:1912.02836] [inSPIRE].
[108] D. Hansen, J. Hartong and N.A. Obers, Action principle for Newtonian gravity, Phys. Rev. Lett. 122 (2019) 061106 [arXiv:1807.04765] [INSPIRE].
[109] L. Ravera, AdS Carroll Chern-Simons supergravity in $2+1$ dimensions and its flat limit, Phys. Lett. B 795 (2019) 331 [arXiv:1905.00766] [INSPIRE].
[110] F. Ali and L. Ravera, $\mathcal{N}$-extended Chern-Simons Carrollian supergravities in $2+1$ spacetime dimensions, JHEP 02 (2020) 128 [arXiv:1912.04172] [INSPIRE].
[111] L. Ciambelli, C. Marteau, A.C. Petkou, P.M. Petropoulos and K. Siampos, Covariant Galilean versus Carrollian hydrodynamics from relativistic fluids, Class. Quant. Grav. 35 (2018) 165001 [arXiv:1802.05286] [INSPIRE].
[112] L. Ciambelli, C. Marteau, A.C. Petkou, P.M. Petropoulos and K. Siampos, Flat holography and Carrollian fluids, JHEP 07 (2018) 165 [arXiv:1802.06809] [INSPIRE].
[113] L. Ciambelli and C. Marteau, Carrollian conservation laws and Ricci-flat gravity, Class. Quant. Grav. 36 (2019) 085004 [arXiv: 1810.11037] [INSPIRE].
[114] A. Campoleoni, L. Ciambelli, C. Marteau, P.M. Petropoulos and K. Siampos, Two-dimensional fluids and their holographic duals, Nucl. Phys. B 946 (2019) 114692 [arXiv:1812.04019] [INSPIRE].


[^0]:    ${ }^{1}$ See [9] for a classification of $\mathcal{N}=1$ and $\mathcal{N}=2$ supersymmetric extensions of the Bargmann and extended Newton-Hooke algebras in $(3+1)$ dimensions.

[^1]:    ${ }^{2}$ Application at the NR level of the Lie algebra expansion method considering the Maurer-Cartan equations can be found in $[5,26,62-64]$.

[^2]:    ${ }^{3}$ In this work, we are using a mostly plus metric.

[^3]:    ${ }^{4}$ For the sake of simplicity, here and in the sequel we will omit writing the wedge product between forms and the spinor index $\alpha$ as well.

[^4]:    ${ }^{5}$ Here and in the sequel, we denote the generators of the expanded algebra with a tilde symbol.

[^5]:    ${ }^{6}$ Note that we have some differences in signs with respect to the commutation relations of [3], due to different conventions, but this can be solved by just setting $\epsilon_{a b} \rightarrow-\epsilon_{a b}$; also, let us recall that $\gamma_{0}=-\gamma^{0}$.

