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Abstract: In this work we present an alternative method to construct diverse non-relativistic Chern-Simons supergravity theories in three spacetime dimensions. To this end, we apply the Lie algebra expansion method based on semigroups to a supersymmetric extension of the Nappi-Witten algebra. Two different families of non-relativistic superalgebras are obtained, corresponding to generalizations of the extended Bargmann superalgebra and extended Newton-Hooke superalgebra, respectively. The expansion method considered here allows to obtain known and new non-relativistic supergravity models in a systematic way. In particular, it immediately provides an invariant tensor for the expanded superalgebra, which is essential to construct the corresponding Chern-Simons supergravity action. We show that the extended Bargmann supergravity and its Maxwellian generalization appear as particular subcases of a generalized extended Bargmann supergravity theory. In addition, we demonstrate that the generalized extended Bargmann and generalized extended Newton-Hooke supergravity families are related through a contraction process.

Keywords: Chern-Simons Theories, Supergravity Models, Gauge Symmetry, Classical Theories of Gravity

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Contents

1 Introduction

The formulation of a non-relativistic (NR) three-dimensional supergravity theory has recently been approached in [1] and subsequently developed in [2, 3]. These last two years, the construction of NR supergravity actions has received a growing interest [4–8] considering different procedures.[1] Such supergravity models correspond to supersymmetric extensions of diverse NR gravity theories. Unlike their bosonic counterparts, the construction of NR supergravity theories remains as a challenging task mainly motivated by the diverse applications of these models in the context of holography and relativistic field theory.

The first proposed NR supergravity theory corresponds to a supersymmetric extension of the Newton-Cartan gravity which was obtained by gauging a Bargmann superalgebra [1]. At the bosonic level, the Newton-Cartan formalism allows to formulate in a geometric way a Newtonian gravity model that resembles General Relativity [10, 11]. Newton-Cartan gravity theories have been largely studied and extended with diverse purposes [12–27]. Remarkably, Newton-Cartan geometry has been useful to approach strongly coupled condensed matter systems [28–38] and NR effective field theories [39–43]. Nevertheless, an action principle for a NR supergravity theory requires to consider an approach different from the Newton-Cartan supergravity one. To this end, a Chern-Simons (CS) formalism

[1] See [9] for a classification of $\mathcal{N}=1$ and $\mathcal{N}=2$ supersymmetric extensions of the Bargmann and extended Newton-Hooke algebras in $(3+1)$ dimensions.

was considered in [3] to construct a three-dimensional NR supergravity action invariant under an extended Bargmann superalgebra. Such superalgebra can be seen as the supersymmetric extension of the extended Bargmann algebra [44–48]. The extended Bargmann superalgebra admits an invariant bilinear form which ensures the proper construction of a well-defined CS action. Furthermore, the extended Bargmann gravity differs from the Newton-Cartan gravity at the matter coupling level, allowing all components of the Ricci tensor to be non-vanishing. On the other hand, the CS formalism has the advantage of offering a gauge-invariant action, this being an interesting three-dimensional toy model [49–51].

To go beyond Poincaré supergravity is a natural step to explore more general supergravity theories. Analogously, at the NR level, it is possible to extend the extended Bargmann supergravity to approach other features. The inclusion of a cosmological constant in a NR supergravity theory was presented in [6] which, in the flat limit, reproduces the extended Bargmann supergravity. The NR action is based on the extended Newton-Hooke superalgebra which appears as a Lie algebra expansion [52] of the $\mathcal{N} = 2$ AdS superalgebra. More recently, a Maxwellian generalization of the extended Bargmann supergravity and its extension to an enlarged extended Bargmann supergravity were studied in [7] and [8], respectively. They can be seen as supersymmetric extensions of the Maxwellian extended Bargmann (MEB) [19] and enlarged extended Bargmann (EEB) gravity theories [22]. While the MEB supergravity action has been obtained by hand, the EEB supergravity theory has been found from the relativistic $\mathcal{N} = 2$ AdS-Lorentz superalgebra through the semigroup expansion method [53].

The Lie algebra expansion procedure has been introduced in [54] and subsequently developed by expanding Maurer-Cartan forms [52, 55]. An expansion method based on semigroups (S-expansion) has been then introduced in [53] and subsequently studied in [56–61]. Within the S-expansion procedure the expanded (super)algebra is obtained by combining the structure constants of a Lie (super)algebra with the multiplication law of a semigroup S. In addition, the S-expansion method provides us with the non-vanishing components of the invariant tensor for the expanded (super)algebra, which are crucial to construct CS actions. The S-expansion mechanism not only has been useful at the NR level[2] [22, 27, 65–68] but also to obtain novel relativistic symmetries [69–73], superalgebras [74–79], and asymptotic symmetries [80–82], among others.

In this work, we present an alternative procedure to construct various NR supergravity theories by considering the S-expansion method. We extend the results obtained in [22, 65] in which diverse NR symmetries are obtained by expanding the Nappi-Witten algebra. The Nappi-Witten symmetry was introduced in [83, 84] and can be seen as a central extension of the homogeneous part of the Galilei algebra. Here, we apply the S-expansion procedure to the Nappi-Witten superalgebra introduced in [8] to obtain known and new NR superalgebras. We get two families of NR superalgebras by considering two different semigroup families. In particular, we first show that the extended Bargmann superalgebra and its generalizations can be obtained as an S-expansion of the super Nappi-Witten algebra. Then,

[2] Application at the NR level of the Lie algebra expansion method considering the Maurer-Cartan equations can be found in [5, 26, 62–64].

using the same method but different semigroups, we get the extended Newton-Hooke superalgebra and its generalizations. The construction of NR CS supergravity actions for each NR superalgebra is also presented. Interestingly, we prove that the extended Bargmann supergravity along with its Maxwellian version correspond to particular subcases of a generalized extended Bargmann supergravity theory. Furthermore, they can alternatively be obtained as an Inönü-Wigner (IW) contraction [85] of the generalized extended Newton-Hooke supergravity theory presented here. Our construction offers a systematic way to obtain different NR supergravity models which are supersymmetric extensions of distinct NR gravity theories.

The organization of the paper is as follows: in section 2, we discuss the supersymmetric extension of the Nappi-Witten algebra. In section 3, we obtain the extended Bargmann supergravity theory, its Maxwellian version, and further generalizations by applying the S-expansion method to the super Nappi-Witten algebra and its invariant tensor. In section 4, we recover an extended Newton-Hooke supergravity along with generalizations of the latter by considering different semigroups. Section 5 is devoted to some concluding remarks.

2 Nappi-Witten superalgebra and Chern-Simons action

A supersymmetric extension of the Nappi-Witten algebra was recently introduced in [8]. In addition to the Nappi-Witten generators $\{J, G_a, S\}$, it contains two additional bosonic generators $\{T_1, T_2\}$ and three Majorana fermionic generators given by Q^+_α, Q^-_α, and R_α (with $\alpha, \beta, \ldots = 1, 2$). The extra bosonic content assures not only the Jacobi identities of the superalgebra but also the non-degeneracy of the invariant tensor. The super Nappi-Witten generators satisfy the following (anti-)commutation relations:[3]

$$
\begin{aligned}
[J, G_a] &= \epsilon_{ab} G_b \,, & [G_a, G_b] &= -\epsilon_{ab} S \,, \\
[J, Q^\pm_\alpha] &= -\frac{1}{2} (\gamma_0)_\alpha{}^\beta Q^\pm_\beta \,, & [J, R_\alpha] &= -\frac{1}{2} (\gamma_0)_\alpha{}^\beta R_\beta \,, \\
\left[G_a, Q^+_\alpha\right] &= -\frac{1}{2} (\gamma_a)_\alpha{}^\beta Q^-_\beta \,, & [G_a, Q^-_\alpha] &= -\frac{1}{2} (\gamma_a)_\alpha{}^\beta R_\beta \,, \\
\left[S, Q^+_\alpha\right] &= -\frac{1}{2} (\gamma_0)_\alpha{}^\beta R_\beta \,, & [T_1, Q^\pm_\alpha] &= \pm\frac{1}{2} (\gamma_0)_{\alpha\beta} Q^\pm_\beta \,, \\
\left[T_2, Q^+_\alpha\right] &= \frac{1}{2} (\gamma_0)_{\alpha\beta} R_\beta \,, & [T_1, R_\alpha] &= \frac{1}{2} (\gamma_0)_{\alpha\beta} R_\beta \,, \\
\left\{Q^+_\alpha, Q^-_\beta\right\} &= -(\gamma^a C)_{\alpha\beta} G_a \,, \\
\left\{Q^+_\alpha, Q^+_\beta\right\} &= -\left(\gamma^0 C\right)_{\alpha\beta} J - \left(\gamma^0 C\right)_{\alpha\beta} T_1 \,, \\
\left\{Q^-_\alpha, Q^-_\beta\right\} &= -\left(\gamma^0 C\right)_{\alpha\beta} S + \left(\gamma^0 C\right)_{\alpha\beta} T_2 \,, \\
\left\{Q^+_\alpha, R_\beta\right\} &= -\left(\gamma^0 C\right)_{\alpha\beta} S - \left(\gamma^0 C\right)_{\alpha\beta} T_2 \,,
\end{aligned}
\tag{2.1}
$$

where $a, b, \ldots = 1, 2$, $\epsilon_{ab} = \epsilon_{0ab}$, $\epsilon^{ab} = \epsilon^{0ab}$, γ^0 and γ^a are the Dirac gamma matrices in three dimensions, and C is the charge conjugation matrix. Let us note that the Nappi-Witten superalgebra (2.1) can be decomposed in a bosonic subspace $V_0 = \{J, G_a, S, T_1, T_2\}$

[3] In this work, we are using a mostly plus metric.

and a fermionic one $V_1 = \{Q^+_\alpha, Q^-_\alpha, R_\alpha\}$ such that they satisfy

$$[V_0, V_0] \subset V_0\,, \quad [V_0, V_1] \subset V_1\,, \quad [V_1, V_1] \subset V_0\,. \tag{2.2}$$

In addition, a non-degenerate invariant supertrace on the Nappi-Witten superalgebra is given in terms of the following non-vanishing components:

$$\begin{aligned} \langle G_a G_b \rangle &= \delta_{ab}\,, \\ \langle JS \rangle &= -1\,, \\ \langle T_1 T_2 \rangle &= 1\,, \\ \left\langle Q^-_\alpha Q^-_\beta \right\rangle = \left\langle Q^+_\alpha R_\beta \right\rangle &= 2C_{\alpha\beta}\,. \end{aligned} \tag{2.3}$$

Then, the three-dimensional CS action based on the Nappi-Witten superalgebra can be constructed introducing the gauge connection one-form

$$A = \omega J + \omega^a G_a + sS + t_1 T_1 + t_2 T_2 + \bar{\psi}^+ Q^+ + \bar{\psi}^- Q^- + \bar{\rho} R \tag{2.4}$$

and the previously defined invariant supertrace in the general expression of a CS action in three spacetime dimensions,

$$I_{\text{CS}} = \int \left\langle A \wedge dA + \frac{1}{3} A \wedge [A, A] \right\rangle\,. \tag{2.5}$$

The supersymmetric CS action invariant under the Nappi-Witten superalgebra (2.1) is given by[4]

$$I_{\text{NW}} = \int \omega_a R^a(\omega^b) - 2sR(\omega) + 2t_1 dt_2 + 2\bar{\psi}^- \nabla\psi^- + 2\bar{\psi}^+ \nabla\rho + 2\bar{\rho}\nabla\psi^+\,, \tag{2.6}$$

where

$$\begin{aligned} R(\omega) &= d\omega\,, \\ R^a\left(\omega^b\right) &= d\omega^a + \epsilon^{ac}\omega\omega \quad {}_c\,, \end{aligned} \tag{2.7}$$

while the covariant derivatives of the spinor 1-form fields appearing in (2.6) are

$$\begin{aligned} \nabla\psi^+ &= d\psi^+ + \frac{1}{2}\omega\gamma_0\psi^+ - \frac{1}{2}t_1\gamma_0\psi^+\,, \\ \nabla\psi^- &= d\psi^- + \frac{1}{2}\omega\gamma_0\psi^- + \frac{1}{2}\omega^a\gamma_a\psi^+ + \frac{1}{2}t_1\gamma_0\psi^-\,, \\ \nabla\rho &= d\rho + \frac{1}{2}\omega\gamma_0\rho + \frac{1}{2}\omega^a\gamma_a\psi^- + \frac{1}{2}s\gamma_0\psi^+ - \frac{1}{2}t_2\gamma_0\psi^+ - \frac{1}{2}t_1\gamma_0\rho\,. \end{aligned} \tag{2.8}$$

Let us note that having a non-degenerate invariant trace correspond to the physical requirement that the CS supersymmetric action (2.6) involves a kinematical term for each

[4] For the sake of simplicity, here and in the sequel we will omit writing the wedge product between forms and the spinor index α as well.

gauge field and the field equations reduce to the vanishing of all the curvatures of the model. In the present case, the curvatures for each gauge field are given by

$$
\begin{aligned}
F(\omega) &= R(\omega) + \frac{1}{2}\bar{\psi}^{+}\gamma^{0}\psi^{+}\,, \\
F^{a}\left(\omega^{b}\right) &= R^{a}\left(\omega^{b}\right) + \bar{\psi}^{+}\gamma^{a}\psi^{-}\,, \\
F(s) &= ds + \frac{1}{2}\bar{\psi}^{-}\gamma^{0}\psi^{-} + \bar{\psi}^{+}\gamma^{0}\rho\,, \\
F(t_1) &= dt_1 + \frac{1}{2}\bar{\psi}^{+}\gamma^{0}\psi^{+}\,, \\
F(t_2) &= dt_2 - \frac{1}{2}\bar{\psi}^{-}\gamma^{0}\psi^{-} + \bar{\psi}^{+}\gamma^{0}\rho\,,
\end{aligned}
\tag{2.9}
$$

along with (2.8). Then, the field equations coming from the variation of the CS action (2.6) correspond, as it is expected, to the vanishing of all curvatures (2.8) and (2.9).

In the following sections, we will apply the S-expansion method [53] to the super Nappi-Witten algebra in order to obtain different NR superalgebras. For our purpose, we will consider two different types of semigroup, that are $S_E^{(2N)} = \{\lambda_0, \lambda_1, \ldots, \lambda_{2N}, \lambda_{2N+1}\}$ and $S_{\mathcal{M}}^{(2N)} = \{\lambda_0, \lambda_1, \lambda_2, \ldots, \lambda_{2N-1}, \lambda_{2N}\}$. In both cases, for different values of N, known and new NR superalgebras will appear, corresponding to distinct supersymmetric extensions of known NR algebras.

One of the advantages of working with the S-expansion is that it provides us with the non-degenerate invariant tensors of the S-expanded (super)algebras, which are given in terms of the invariant tensors for the original (super)algebra. This turns out to be essential in the construction of NR CS (super)gravity actions.

3 Generalized extended Bargmann supergravity theory and semigroup expansion method

In this section, we shall derive different NR superalgebras whose bosonic sectors include as subalgebra the extended Bargmann algebra and its generalizations, depending on the case. The aforementioned NR superalgebras appear as the result of applying the S-expansion procedure to the Nappi-Witten superalgebra introduced in the previous section, considering $S_E^{(2N)} = \{\lambda_0, \lambda_1, \ldots, \lambda_{2N}, \lambda_{2N+1}\}$ as the relevant semigroup. We will show that all known and new NR superalgebras belong to a family of NR superalgebras, which we call as generalized extended Bargmann (GEB$^{(N)}$) superalgebras. As we will see, they correspond to supersymmetric extensions of the NR counterparts of the $\mathfrak{B}_{N+2}$ algebras enlarged with $(N+1)\,U_1$ generators [27]. At the relativistic level, the $\mathfrak{B}_{N+2}$ algebras were first introduced in [69, 86] for obtaining standard General Relativity from Chern-Simons gravity. Furthermore, as we have mentioned before, an important advantage of the S-expansion is that it allows to derive the non-vanishing components of an invariant tensor of the expanded (super)algebra. Then, we will exploit this powerful feature to derive the NR invariant supertraces and the CS actions invariant under the aforesaid expanded NR superalgebras.

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3.1 Extended Bargmann supergravity

The so-called extended Bargmann algebra [3] can be obtained as the NR limit of the Poincaré algebra (corresponding to the $\mathfrak{B}_3$ algebra) enlarged with two U_1 generators. An alternative method has been proposed in [65] in which the extended Bargmann algebra can be derived as an S-expansion of the Nappi-Witten algebra. Here we show that a supersymmetric extension of the extended Bargmann algebra can be obtained by performing an S-expansion of the Nappi-Witten superalgebra (2.1). To this end, we consider $S_E^{(2)} = \{\lambda_0, \lambda_1, \lambda_2, \lambda_3\}$ as the relevant semigroup whose elements satisfy the following multiplication table:

$$\begin{array}{c|cccc} \lambda_3 & \lambda_3 & \lambda_3 & \lambda_3 & \lambda_3 \\ \lambda_2 & \lambda_2 & \lambda_3 & \lambda_3 & \lambda_3 \\ \lambda_1 & \lambda_1 & \lambda_2 & \lambda_3 & \lambda_3 \\ \lambda_0 & \lambda_0 & \lambda_1 & \lambda_2 & \lambda_3 \\ \hline & \lambda_0 & \lambda_1 & \lambda_2 & \lambda_3 \end{array} \tag{3.1}$$

Indeed, one can consider a resonant decomposition,

$$\begin{aligned} S_0 &= \{\lambda_0, \lambda_2, \lambda_3\} \,, \\ S_1 &= \{\lambda_1, \lambda_3\} \,, \end{aligned} \tag{3.2}$$

with $\lambda_3 = 0_S$ being the zero element of the semigroup. Let us note that the decomposition (3.2) is said to be resonant since it satisfies the same structure as the super Nappi-Witten subspaces,

$$S_0 \cdot S_0 \subset S_0 \,, \qquad S_0 \cdot S_1 \subset S_1 \,, \qquad S_1 \cdot S_1 \subset S_0 \,. \tag{3.3}$$

Then, following the definitions of [53], after considering a resonant $S_E^{(2)}$-expansion followed by a 0_S-reduction to the Nappi-Witten superalgebra (2.1), we find an expanded superalgebra spanned by the set of generators[5] $\left\{\tilde{J}, \tilde{G}_a, \tilde{S}, \tilde{H}, \tilde{P}_a, \tilde{M}, \tilde{Y}_1, \tilde{Y}_2, \tilde{U}_1, \tilde{U}_2, \tilde{Q}_\alpha^+, \tilde{Q}_\alpha^-, \tilde{R}_\alpha\right\}$ which are related to the super Nappi-Witten ones through the semigroup elements as follows:

$$\begin{aligned} &\tilde{J} = \lambda_0 J \,, && \tilde{G}_a = \lambda_0 G_a \,, && \tilde{S} = \lambda_0 S \,, && \tilde{Y}_1 = \lambda_0 T_1 \,, && \tilde{U}_1 = \lambda_2 T_1 \,, \\ &\tilde{H} = \lambda_2 J \,, && \tilde{P}_a = \lambda_2 G_a \,, && \tilde{M} = \lambda_2 S \,, && \tilde{Y}_2 = \lambda_0 T_2 \,, && \tilde{U}_2 = \lambda_2 T_2 \,, \\ &\tilde{Q}_\alpha^+ = \lambda_1 Q_\alpha^+ \,, && \tilde{Q}_\alpha^- = \lambda_1 Q_\alpha^- \,, && \tilde{R}_\alpha = \lambda_1 R_\alpha \,. \end{aligned} \tag{3.4}$$

Such expanded generators satisfy the following non-vanishing (anti-)commutation relations:

$$\begin{aligned} &\left[\tilde{J}, \tilde{G}_a\right] = \epsilon_{ab} \tilde{G}_b \,, && \left[\tilde{G}_a, \tilde{G}_b\right] = -\epsilon_{ab} \tilde{S} \,, && \left[\tilde{J}, \tilde{P}_a\right] = \epsilon_{ab} \tilde{P}_b \\ &\left[\tilde{H}, \tilde{G}_a\right] = \epsilon_{ab} \tilde{P}_b \,, && \left[\tilde{G}_a, \tilde{P}_b\right] = -\epsilon_{ab} \tilde{M} \,, && \left[\tilde{J}, \tilde{Q}_\alpha^\pm\right] = -\frac{1}{2} \left(\gamma_0\right)_\alpha{}^\beta \tilde{Q}_\beta^\pm \,, \\ &\left[\tilde{J}, \tilde{R}_\alpha\right] = -\frac{1}{2} \left(\gamma_0\right)_\alpha{}^\beta \tilde{R}_\beta \,, && \left[\tilde{G}_a, \tilde{Q}_\alpha^+\right] = -\frac{1}{2} \left(\gamma_a\right)_\alpha{}^\beta \tilde{Q}_\beta^- \,, && \left[\tilde{G}_a, \tilde{Q}_\alpha^-\right] = -\frac{1}{2} \left(\gamma_a\right)_\alpha{}^\beta \tilde{R}_\beta \,, \\ &\left[\tilde{S}, \tilde{Q}_\alpha^+\right] = -\frac{1}{2} \left(\gamma_0\right)_\alpha{}^\beta \tilde{R}_\beta \,, && \left[\tilde{Y}_1, Q_\alpha^\pm\right] = \pm\frac{1}{2} \left(\gamma_0\right)_{\alpha\beta} Q_\beta^\pm \,, && \left[\tilde{Y}_1, R_\alpha\right] = \frac{1}{2} \left(\gamma_0\right)_{\alpha\beta} R_\beta \,, \end{aligned}$$

[5] Here and in the sequel, we denote the generators of the expanded algebra with a tilde symbol.

$$
\begin{aligned}
\left[\tilde{Y}_2, \tilde{Q}^+_\alpha\right] &= \frac{1}{2}\left(\gamma_0\right)_{\alpha\beta} R_\beta \\
\left\{\tilde{Q}^+_\alpha, \tilde{Q}^-_\beta\right\} &= -\left(\gamma^a C\right)_{\alpha\beta} \tilde{P}_a \,, \\
\left\{\tilde{Q}^+_\alpha, \tilde{Q}^+_\beta\right\} &= -\left(\gamma^0 C\right)_{\alpha\beta} \tilde{H} - \left(\gamma^0 C\right)_{\alpha\beta} \tilde{U}_1 \,, \\
\left\{\tilde{Q}^-_\alpha, \tilde{Q}^-_\beta\right\} &= -\left(\gamma^0 C\right)_{\alpha\beta} \tilde{M} + \left(\gamma^0 C\right)_{\alpha\beta} \tilde{U}_2 \,, \\
\left\{\tilde{Q}^+_\alpha, \tilde{R}_\beta\right\} &= -\left(\gamma^0 C\right)_{\alpha\beta} \tilde{M} - \left(\gamma^0 C\right)_{\alpha\beta} \tilde{U}_2 \,.
\end{aligned} \tag{3.5}
$$

This superalgebra corresponds to an extension of the extended Bargmann superalgebra introduced in [3].[6] In fact, by considering $\tilde{Y}_1 = \tilde{Y}_2 = \tilde{U}_1 = \tilde{U}_2 = 0$, one can exactly reproduce the extended Bargmann superalgebra of [3]. Let us note that the additional bosonic generators $\tilde{Y}_1$, $\tilde{Y}_2$, $\tilde{U}_1$, and $\tilde{U}_2$ appear as a consequence of the expansion of the Nappi-Witten generators T_1 and T_2. They allow to get a non-degenerate invariant tensor within our framework. In this regards, observe that the central charge $\tilde{U}_1$ may be trivially reabsorbed through a redefinition of $\tilde{H}$. Nevertheless, we retain $\tilde{U}_1$ in the sequel, since it directly follows from the expansion we have performed.

The invariant tensor for the extended Bargmann superalgebra can be obtained by applying the S-expansion method to the invariant tensor of the Nappi-Witten superalgebra, given in (2.3). In this way, one finds that the non-vanishing components of a non-degenerate invariant tensor for the extended Bargmann superalgebra are given by

$$
\begin{aligned}
\left\langle \tilde{G}_a \tilde{G}_b \right\rangle &= \alpha_0 \delta_{ab} \,, & \left\langle \tilde{J}\tilde{S} \right\rangle &= -\alpha_0 \,, \\
\left\langle \tilde{G}_a \tilde{P}_b \right\rangle &= \alpha_1 \delta_{ab} \,, & \left\langle \tilde{Y}_1\tilde{Y}_2 \right\rangle &= \alpha_0 \,, \\
\left\langle \tilde{J}\tilde{M} \right\rangle = \left\langle \tilde{H}\tilde{S} \right\rangle &= -\alpha_1 \,, & \left\langle \tilde{Y}_1\tilde{U}_2 \right\rangle = \left\langle \tilde{U}_1\tilde{Y}_2 \right\rangle &= \alpha_1 \,, \\
\left\langle \tilde{Q}^-_\alpha \tilde{Q}^-_\beta \right\rangle = \left\langle \tilde{Q}^+_\alpha \tilde{R}_\beta \right\rangle &= 2\alpha_1 C_{\alpha\beta} \,,
\end{aligned} \tag{3.6}
$$

where the α's are arbitrary constants and appear as a consequence of the S-expansion procedure. Then, the three-dimensional CS action invariant under the superalgebra (3.5) can be directly constructed by introducing the gauge connection one-form

$$
\begin{aligned}
A = {} & \omega\tilde{J} + h\tilde{H} + \omega^a\tilde{G}_a + e^a\tilde{P}_a + s\tilde{S} + m\tilde{M} + y_1\tilde{Y}_1 + y_2\tilde{Y}_2 + u_1\tilde{U}_1 + u_2\tilde{U}_2 \\
& + \bar{\psi}^+\tilde{Q}^+ + \bar{\psi}^-\tilde{Q}^- + \bar{\rho}\tilde{R}
\end{aligned} \tag{3.7}
$$

and the invariant tensor (3.6) in the general expression for a CS action (2.5). The NR CS supergravity action reads

$$
I_{\text{EB}} = \alpha_0 I_0 + \alpha_1 I_1 \,, \tag{3.8}
$$

where

$$
\begin{aligned}
I_0 &= \int \omega_a R^a(\omega^b) - 2sR\left(\omega\right) + 2y_1 dy_2 \,, \\
I_1 &= \int 2e_a R^a(\omega^b) - 2mR(\omega) - 2\tau R(s) + 2y_1 du_2 + 2u_1 dy_2 \\
&\quad + 2\bar{\psi}^+\nabla\rho + 2\bar{\rho}\nabla\psi^+ + 2\bar{\psi}^-\nabla\psi^- \;,
\end{aligned} \tag{3.9}
$$

[6] Note that we have some differences in signs with respect to the commutation relations of [3], due to different conventions, but this can be solved by just setting $\epsilon_{ab} \to -\epsilon_{ab}$; also, let us recall that $\gamma_0 = -\gamma^0$.

and

$$R(s) = ds + \frac{1}{2}\epsilon^{ac}\omega_a\omega_c\,, \tag{3.10}$$

while $R(\omega)$ and $R^a(\omega^b)$ are given by (2.7). Besides, the covariant derivatives of the spinor 1-form fields read

$$\begin{aligned}
\nabla\psi^+ &= d\psi^+ + \frac{1}{2}\omega\gamma_0\psi^+ - \frac{1}{2}y_1\gamma_0\psi^+\,,\\
\nabla\psi^- &= d\psi^- + \frac{1}{2}\omega\gamma_0\psi^- + \frac{1}{2}\omega^a\gamma_a\psi^+ + \frac{1}{2}y_1\gamma_0\psi^-\,,\\
\nabla\rho &= d\rho + \frac{1}{2}\omega\gamma_0\rho + \frac{1}{2}\omega^a\gamma_a\psi^- + \frac{1}{2}s\gamma_0\psi^+ - \frac{1}{2}y_2\gamma_0\psi^+ - \frac{1}{2}y_1\gamma_0\rho\,.
\end{aligned} \tag{3.11}$$

The CS action (3.8) describes an extension of the so-called extended Bargmann supergravity theory [3]. Indeed, the CS action I_1 corresponds to the extended Bargmann supergravity action introduced in [3], endowed with some additional terms involving the extra bosonic 1-form fields y_1, y_2, u_1, and u_2. Furthermore, the term along α_0 corresponds to a NR exotic Lagrangian. The extra bosonic field content is related to the additional bosonic generators which allow to define the non-degenerate invariant tensor (3.6). The non-degeneracy of the invariant tensor implies that the equations of motion are given by the vanishing of the curvature two-forms of the model, which are given by (3.11) and

$$\begin{aligned}
F\left(\omega\right) &= R(\omega)\,, & F^a(\omega^b) &= R^a(\omega^b)\,,\\
F\left(s\right) &= R(s)\,, & F\left(\tau\right) &= d\left(\tau\right) + \frac{1}{2}\bar{\psi}^+\gamma^0\psi^+\,,\\
F^a(e^b) &= de^a + \epsilon^{ac}\omega e_c + \epsilon^{ac}\tau\omega_c + \bar{\psi}^+\gamma^a\psi^-\,, &&\\
F\left(m\right) &= dm + \epsilon^{ac}\omega_a e_c + \frac{1}{2}\bar{\psi}^-\gamma^0\psi^- + \bar{\psi}^+\gamma^0\rho\,, &&\\
F\left(y_1\right) &= dy_1\,, & F\left(y_2\right) &= dy_2\,,\\
F\left(u_1\right) &= du_1 + \frac{1}{2}\bar{\psi}^+\gamma^0\psi^+\,, &&\\
F\left(u_2\right) &= du_2 - \frac{1}{2}\bar{\psi}^-\gamma^0\psi^- + \bar{\psi}^+\gamma^0\rho\,.
\end{aligned} \tag{3.12}$$

Let us note that the present NR theory can be seen as the most general three-dimensional extended Bargmann supergravity theory containing both exotic and standard sectors. Nonetheless, the formulation of a NR supergravity theory is not unique and can be generalized beyond the extended Bargmann one.

3.2 Maxwellian extended Bargmann supergravity

A Maxwellian version of the extended Bargmann algebra was recently presented in [19]. It was denoted as MEB algebra and it was obtained as a NR limit of the Maxwell algebra (also called as $\mathfrak{B}_4$ algebra) enlarged with three U_1 generators. At the relativistic level, the Maxwell symmetry appears in the description of a particle in a Minkowski spacetime in the presence of an electromagnetic field background [87–89]. Subsequently, a supersymmetric extension of the MEB algebra was introduced in [7]. In order to construct a well-defined

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supergravity theory in this context it was necessary to construct by hand the aforementioned supersymmetric extension. Now, we are going to show that the MEB superalgebra and the corresponding NR supergravity theory can be derived by means of the S-expansion method. Indeed, let us consider the $S_E^{(4)}$-expansion of the Nappi-Witten superalgebra (2.1). The elements of the $S_E^{(4)}$ semigroup satisfy

$$\begin{array}{c|cccccc}
\lambda_5 & \lambda_5 & \lambda_5 & \lambda_5 & \lambda_5 & \lambda_5 & \lambda_5 \\
\lambda_4 & \lambda_4 & \lambda_5 & \lambda_5 & \lambda_5 & \lambda_5 & \lambda_5 \\
\lambda_3 & \lambda_3 & \lambda_4 & \lambda_5 & \lambda_5 & \lambda_5 & \lambda_5 \\
\lambda_2 & \lambda_2 & \lambda_3 & \lambda_4 & \lambda_5 & \lambda_5 & \lambda_5 \\
\lambda_1 & \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 & \lambda_5 & \lambda_5 \\
\lambda_0 & \lambda_0 & \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 & \lambda_5 \\
\hline
 & \lambda_0 & \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 & \lambda_5
\end{array} \tag{3.13}$$

where $\lambda_5 = 0_S$ is the zero element of the semigroup. Let $S_E^{(4)} = S_0 \cup S_1$ be a resonant decomposition with

$$\begin{aligned}
S_0 &= \{\lambda_0, \lambda_2, \lambda_4, \lambda_5\} \,, \\
S_1 &= \{\lambda_1, \lambda_3, \lambda_5\} \,.
\end{aligned} \tag{3.14}$$

After considering a resonant $S_E^{(4)}$-expansion followed by a 0_S-reduction of the Nappi-Witten superalgebra, we find an expanded superalgebra spanned by the generators

$$\left\{\tilde{J}, \tilde{G}_a, \tilde{S}, \tilde{H}, \tilde{P}_a, \tilde{M}, \tilde{Z}, \tilde{Z}_a, \tilde{T}, \tilde{Y}_1, \tilde{Y}_2, \tilde{U}_1, \tilde{U}_2, \tilde{B}_1, \tilde{B}_2, \tilde{Q}^+_\alpha, \tilde{Q}^-_\alpha, \tilde{R}_\alpha, \tilde{\Sigma}^+_\alpha, \tilde{\Sigma}^-_\alpha, \tilde{W}_\alpha\right\} ,$$

which are related to the super Nappi-Witten ones through

$$\begin{array}{lllll}
\tilde{J} = \lambda_0 J \,, & \tilde{G}_a = \lambda_0 G_a \,, & \tilde{S} = \lambda_0 S \,, & \tilde{Y}_1 = \lambda_0 T_1 \,, & \tilde{Y}_2 = \lambda_0 T_2 \,, \\
\tilde{H} = \lambda_2 J \,, & \tilde{P}_a = \lambda_2 G_a \,, & \tilde{M} = \lambda_2 S \,, & \tilde{U}_1 = \lambda_2 T_1 \,, & \tilde{U}_2 = \lambda_2 T_2 \,, \\
\tilde{Z} = \lambda_4 J \,, & \tilde{Z}_a = \lambda_4 G_a \,, & \tilde{T} = \lambda_4 S \,, & \tilde{B}_1 = \lambda_4 T_1 \,, & \tilde{B}_2 = \lambda_4 T_2 \,, \\
\tilde{Q}^+_\alpha = \lambda_1 Q^+_\alpha \,, & \tilde{Q}^-_\alpha = \lambda_1 Q^-_\alpha \,, & \tilde{\Sigma}^+_\alpha = \lambda_3 Q^+_\alpha \,, & \tilde{\Sigma}^-_\alpha = \lambda_3 Q^-_\alpha \,, & \tilde{R}_\alpha = \lambda_1 R_\alpha \,, \\
\tilde{W}_\alpha = \lambda_3 R_\alpha \,. & & & &
\end{array} \tag{3.15}$$

Using the multiplication law (3.13) and the original (anti-)commutation relations of the Nappi-Witten superalgebra (2.1), one can show that the expanded generators satisfy an expanded NR superalgebra whose non-trivial (anti-)commutation relations are given by (3.5) along with

$$\begin{aligned}
&\left[\tilde{P}_a, \tilde{P}_b\right] = -\epsilon_{ab}\tilde{T} \,, && \left[\tilde{G}_a, \tilde{Z}_b\right] = -\epsilon_{ab}\tilde{T} \,, && \left[\tilde{H}, \tilde{P}_a\right] = \epsilon_{ab}\tilde{Z}_b \,, \\
&\left[\tilde{J}, \tilde{Z}_a\right] = \epsilon_{ab}\tilde{Z}_b \,, && \left[\tilde{Z}, \tilde{G}_a\right] = \epsilon_{ab}\tilde{Z}_b \,, && \left[\tilde{U}_2, \tilde{Q}^+_\alpha\right] = \frac{1}{2}(\gamma_0)_{\alpha\beta}\tilde{W}_\beta \,, \\
&\left[\tilde{J}, \tilde{\Sigma}^\pm_\alpha\right] = -\frac{1}{2}(\gamma_0)_\alpha^{\ \beta}\tilde{\Sigma}^\pm_\beta \,, && \left[\tilde{S}, \tilde{\Sigma}^+_\alpha\right] = -\frac{1}{2}(\gamma_0)_\alpha^{\ \beta}\tilde{W}_\beta \,, && \left[\tilde{Y}_1, \tilde{W}_\alpha\right] = \frac{1}{2}(\gamma_0)_{\alpha\beta}\tilde{W}_\beta \,, \\
&\left[\tilde{H}, \tilde{Q}^\pm_\alpha\right] = -\frac{1}{2}(\gamma_0)_\alpha^{\ \beta}\tilde{\Sigma}^\pm_\beta \,, && \left[\tilde{P}_a, \tilde{Q}^+_\alpha\right] = -\frac{1}{2}(\gamma_a)_\alpha^{\ \beta}\tilde{\Sigma}^-_\beta \,, && \left[\tilde{Y}_1, \tilde{\Sigma}^+_\alpha\right] = \frac{1}{2}(\gamma_0)_{\alpha\beta}\tilde{\Sigma}^+_\beta \,,
\end{aligned}$$

$$
\begin{aligned}
\left[\tilde{G}_a,\tilde{\Sigma}^+_\alpha\right] &= -\frac{1}{2}\left(\gamma_a\right)_\alpha^{\ \beta}\tilde{\Sigma}^-_\beta\,, & \left[\tilde{G}_a,\tilde{\Sigma}^-_\alpha\right] &= -\frac{1}{2}\left(\gamma_a\right)_\alpha^{\ \beta}\tilde{W}_\beta\,, & \left[\tilde{Y}_1,\tilde{\Sigma}^-_\alpha\right] &= -\frac{1}{2}\left(\gamma_0\right)_{\alpha\beta}\tilde{\Sigma}^-_\beta\,,\\
\left[P_a,\tilde{Q}^-_\alpha\right] &= -\frac{1}{2}\left(\gamma_a\right)_\alpha^{\ \beta}\tilde{W}_\beta\,, & \left[\tilde{J},\tilde{R}_\alpha\right] &= -\frac{1}{2}\left(\gamma_0\right)_\alpha^{\ \beta}\tilde{R}_\beta\,, & \left[\tilde{U}_1,\tilde{Q}^+_\alpha\right] &= \frac{1}{2}\left(\gamma_0\right)_{\alpha\beta}\tilde{\Sigma}^+_\beta\,,\\
\left[\tilde{J},\tilde{W}_\alpha\right] &= -\frac{1}{2}\left(\gamma_0\right)_\alpha^{\ \beta}\tilde{W}_\beta\,, & \left[\tilde{H},\tilde{R}_\alpha\right] &= -\frac{1}{2}\left(\gamma_0\right)_\alpha^{\ \beta}\tilde{W}_\beta\,, & \left[\tilde{U}_1,\tilde{Q}^-_\alpha\right] &= -\frac{1}{2}\left(\gamma_0\right)_{\alpha\beta}\tilde{\Sigma}^-_\beta\,,\\
\left[\tilde{M},\tilde{Q}^+_\alpha\right] &= -\frac{1}{2}\left(\gamma_0\right)_\alpha^{\ \beta}\tilde{W}_\beta\,, & \left[\tilde{Y}_2,\tilde{\Sigma}^+_\alpha\right] &= \frac{1}{2}\left(\gamma_0\right)_{\alpha\beta}\tilde{W}_\beta\,, & \left[\tilde{U}_1,\tilde{R}_\alpha\right] &= \frac{1}{2}\left(\gamma_0\right)_{\alpha\beta}\tilde{W}_\beta\,,
\end{aligned}
$$

$$
\begin{aligned}
\left\{\tilde{Q}^-_\alpha,\tilde{\Sigma}^-_\beta\right\} &= -\left(\gamma^0C\right)_{\alpha\beta}\tilde{T}+\left(\gamma^0C\right)_{\alpha\beta}\tilde{B}_2\,,\\
\left\{\tilde{Q}^\pm_\alpha,\tilde{\Sigma}^\mp_\beta\right\} &= -\left(\gamma^aC\right)_{\alpha\beta}\tilde{Z}_a\,,\\
\left\{\tilde{Q}^+_\alpha,\tilde{\Sigma}^+_\beta\right\} &= -\left(\gamma^0C\right)_{\alpha\beta}\tilde{Z}-\left(\gamma^0C\right)_{\alpha\beta}\tilde{B}_1\,,\\
\left\{\tilde{Q}^+_\alpha,\tilde{W}_\beta\right\} &= \left\{\tilde{\Sigma}^+_\alpha,\tilde{R}_\beta\right\} = -\left(\gamma^0C\right)_{\alpha\beta}\tilde{T}-\left(\gamma^0C\right)_{\alpha\beta}\tilde{B}_2\,.
\end{aligned}
\tag{3.16}
$$

The superalgebra just obtained corresponds to the Maxwellian extended Bargmann superalgebra first presented in [7]. The MEB superalgebra is characterized by the presence of three additional fermionic generators, besides $\tilde{Q}^\pm_\alpha$ and $\tilde{R}_\alpha$, namely $\tilde{\Sigma}^+_\alpha,\tilde{\Sigma}^-_\alpha$, and $\tilde{W}_\alpha$. Furthermore, unlike the extended Bargmann superalgebra, $\tilde{U}_1$ and $\tilde{U}_2$ are no longer central charges but act non-trivially on the fermionic generators $\tilde{Q}^\pm_\alpha$ and $\tilde{R}_\alpha$. The non-vanishing components of an invariant supertrace for the MEB superalgebra are obtained from the Nappi-Witten ones through the S-expansion method. These components are given by (3.6) together with

$$
\begin{aligned}
\left\langle\tilde{P}_a\tilde{P}_b\right\rangle = \left\langle\tilde{G}_a\tilde{Z}_b\right\rangle &= \alpha_2\delta_{ab}\,,\\
\left\langle\tilde{J}\tilde{T}\right\rangle = \left\langle\tilde{H}\tilde{M}\right\rangle &= \left\langle\tilde{S}\tilde{Z}\right\rangle = -\alpha_2\,,\\
\left\langle\tilde{Y}_1\tilde{B}_2\right\rangle = \left\langle\tilde{U}_1\tilde{U}_2\right\rangle &= \left\langle\tilde{B}_1\tilde{Y}_2\right\rangle = \alpha_2\,,\\
\left\langle\tilde{Q}^-_\alpha\tilde{\Sigma}^-_\beta\right\rangle = \left\langle\tilde{\Sigma}^+_\alpha\tilde{R}_\beta\right\rangle &= \left\langle\tilde{Q}^+_\alpha\tilde{W}_\beta\right\rangle = 2\alpha_2C_{\alpha\beta}\,,
\end{aligned}
\tag{3.17}
$$

where α_2 is an arbitrary constant. The NR CS supergravity action invariant under the MEB superalgebra, which was presented in [7], is obtained by considering the following gauge connection one-form:

$$
\begin{aligned}
A =\,& \omega\tilde{J}+\omega^a\tilde{G}_a+\tau\tilde{H}+e^a\tilde{P}_a+k\tilde{Z}+k^a\tilde{Z}_a+m\tilde{M}+s\tilde{S}+t\tilde{T}\\
&+y_1\tilde{Y}_1+y_2\tilde{Y}_2+b_1\tilde{B}_1+b_2\tilde{B}_2+u_1\tilde{U}_1+u_2\tilde{U}_2\\
&+\psi^+\tilde{Q}^++\psi^-\tilde{Q}^-+\xi^+\tilde{\Sigma}^++\xi^-\tilde{\Sigma}^-+\rho\tilde{R}+\chi\tilde{W}
\end{aligned}
\tag{3.18}
$$

and the non-vanishing components of the invariant tensor (3.6) and (3.17) in the CS expression (2.5). It reads, up to boundary terms, as follows:

$$
I_{\text{MEB}} = \alpha_0 I_0+\alpha_1 I_1+\alpha_2 I_2\,,
\tag{3.19}
$$

where I_0 and I_1 are given in (3.9), while the term along α_2 reads

$$
\begin{aligned}
I_2 = \int\ & e_aR^a\left(e^b\right)+k_aR^a\left(\omega^b\right)+\omega_aR^a\left(k^b\right)-2sR\left(k\right)\\
&-\ 2mR\left(\tau\right)-2tR\left(\omega\right)+2y_1db_2+2u_1du_2+2y_2db_1\\
&+2\bar{\psi}^-\nabla\xi^-+2\bar{\xi}^-\nabla\psi^-+2\bar{\psi}^+\nabla\chi+2\bar{\chi}\nabla\psi^++2\bar{\xi}^+\nabla\rho+2\bar{\rho}\nabla\xi^+\ .
\end{aligned}
\tag{3.20}
$$

Here, $R(\omega)$ and $R^a(\omega^b)$ were defined in (2.7), while

$$
\begin{aligned}
R\left(\tau\right) &= d\tau\,,\\
R^{a}(e^{b}) &= de^{a}+\epsilon^{ac}\omega e_{c}+\epsilon^{ac}\tau\omega_{c}\,,\\
R\left(k\right) &= dk\,,\\
R^{a}(k^{b}) &= dk^{a}+\epsilon^{ac}\omega k_{c}+\epsilon^{ac}\tau e_{c}+\epsilon^{ac}k\omega_{c}\,.
\end{aligned}
\tag{3.21}
$$

On the other hand, the covariant derivatives of the spinor 1-form fields appearing in I_2 are given by

$$
\begin{aligned}
\nabla\xi^{+} &= d\xi^{+}+\frac{1}{2}\omega\gamma_{0}\xi^{+}+\frac{1}{2}\tau\gamma_{0}\psi^{+}-\frac{1}{2}y_{1}\gamma_{0}\xi^{+}-\frac{1}{2}u_{1}\gamma_{0}\psi^{+}\,,\\
\nabla\xi^{-} &= d\xi^{-}+\frac{1}{2}\omega\gamma_{0}\xi^{-}+\frac{1}{2}\tau\gamma_{0}\psi^{-}+\frac{1}{2}e^{a}\gamma_{a}\psi^{+}+\frac{1}{2}\omega^{a}\gamma_{a}\xi^{+}+\frac{1}{2}y_{1}\gamma_{0}\xi^{-}+\frac{1}{2}u_{1}\gamma_{0}\psi^{-}\,,\\
\nabla\chi &= d\chi+\frac{1}{2}\omega\gamma_{0}\chi+\frac{1}{2}\omega^{a}\gamma_{a}\xi^{-}+\frac{1}{2}e^{a}\gamma_{a}\psi^{-}+\frac{1}{2}\tau\gamma_{0}\rho+\frac{1}{2}s\gamma_{0}\xi^{+}+\frac{1}{2}m\gamma_{0}\psi^{+}\\
&\quad-\frac{1}{2}y_{2}\gamma_{0}\xi^{+}-\frac{1}{2}y_{1}\gamma_{0}\chi-\frac{1}{2}u_{2}\gamma_{0}\psi^{+}-\frac{1}{2}u_{1}\gamma_{0}\rho\,,
\end{aligned}
\tag{3.22}
$$

along with (3.11). The CS action (3.19) describes the Maxwellian extended Bargmann supergravity theory first presented in [7]. As we can see, the S-expansion of the super Nappi-Witten algebra (2.1) with the $S_E^{(4)}$ semigroup adds a new sector to the action with respect to the case of the extended Bargmann supergravity theory previously studied. This new sector along the arbitrary constant α_2 corresponds to the CS action for a new NR Maxwell superalgebra, whose bosonic part is the MEB gravity presented in [19], supplemented with some bosonic 1-form fields. Let us note that the extended Bargmann supergravity action (3.9) appears as a particular subcase along α_0 and α_1. The equations of motion of the theory are given by the vanishing of the curvature two-forms, which, in the case under analysis, are given by (3.11), (3.12), (3.22), and

$$
\begin{aligned}
F\left(k\right) &= R\left(k\right)+\bar{\psi}^{+}\gamma^{0}\xi^{+}\,,\\
F^{a}(k^{b}) &= R^{a}(k^{b})+\bar{\psi}^{+}\gamma^{a}\xi^{-}+\bar{\psi}^{-}\gamma^{a}\xi^{+}\,,\\
F\left(t\right) &= dt+\epsilon^{ac}\omega_{a}k_{c}+\frac{1}{2}\epsilon^{ac}e_{a}e_{c}+\bar{\psi}^{-}\gamma^{0}\xi^{-}+\bar{\psi}^{+}\gamma^{0}\chi+\bar{\xi}^{+}\gamma^{0}\rho\,,\\
F\left(b_{1}\right) &= db_{1}+\bar{\psi}^{+}\gamma^{0}\xi^{+}\,,\\
F\left(b_{2}\right) &= db_{2}-\bar{\psi}^{-}\gamma^{0}\xi^{-}+\bar{\psi}^{+}\gamma^{0}\chi+\bar{\xi}^{+}\gamma^{0}\rho\,.
\end{aligned}
\tag{3.23}
$$

This is indeed expected for a well-defined and consistent CS (super)gravity model.

3.3 Generalized Maxwellian extended Bargmann supergravity

A generalization of the Maxwellian extended Bargmann algebra was introduced very recently in [27]. The aforesaid algebra was obtained by considering a NR limit of a generalized Maxwell algebra (also denoted as $\mathfrak{B}_5$) defined in three spacetime dimensions. Here, we will show that a supersymmetric extension of the generalized Maxwellian extended Bargmann (GMEB) algebra can be obtained considering an $S_E^{(6)}$-expansion of the Nappi-Witten superalgebra (2.1). Furthermore, the S-expansion method will allow us to construct the supergravity theory invariant under the GMEB superalgebra, as it provides with the non-degenerate invariant supertrace.

The elements of the $S_E^{(6)}$ semigroup satisfy the multiplication law

$$\lambda_\alpha \lambda_\beta = \begin{cases} \lambda_{\alpha+\beta} & \text{if} \quad \alpha+\beta \leq 6\,, \\ \lambda_7 & \text{if} \quad \alpha+\beta > 6\,, \end{cases} \tag{3.24}$$

with $\lambda_7 = 0_S$ being the zero element of the semigroup. Let $S_E^{(6)} = S_0 \cup S_1$ be a resonant decomposition with

$$\begin{aligned} S_0 &= \{\lambda_0, \lambda_2, \lambda_4, \lambda_6, \lambda_7\}\ , \\ S_1 &= \{\lambda_1, \lambda_3, \lambda_5, \lambda_7\}\ . \end{aligned} \tag{3.25}$$

After applying a resonant $S_E^{(6)}$-expansion and a 0_S-reduction to the Nappi-Witten superalgebra, we find an expanded superalgebra spanned by the bosonic generators

$$\left\{\tilde{J}, \tilde{G}_a, \tilde{S}, \tilde{H}, \tilde{P}_a, \tilde{M}, \tilde{Z}, \tilde{Z}_a, \tilde{N}, \tilde{N}_a, \tilde{T}, \tilde{V}, \tilde{Y}_1, \tilde{Y}_2, \tilde{U}_1, \tilde{U}_2, \tilde{B}_1, \tilde{B}_2, \tilde{C}_1, \tilde{C}_2\right\}$$

along with the fermionic charges

$$\left\{\tilde{Q}^+_\alpha, \tilde{Q}^-_\alpha, \tilde{R}_\alpha, \tilde{\Sigma}^+_\alpha, \tilde{\Sigma}^-_\alpha, \tilde{W}_\alpha, \tilde{\Xi}^+_\alpha, \tilde{\Xi}^-_\alpha, \tilde{K}_\alpha\right\}\ ,$$

which are related to the super Nappi-Witten ones through

$$\begin{array}{lllll} \tilde{J} = \lambda_0 J\,, & \tilde{G}_a = \lambda_0 G_a\,, & \tilde{S} = \lambda_0 S\,, & \tilde{Y}_1 = \lambda_0 T_1\,, & \tilde{Y}_2 = \lambda_0 T_2\,, \\ \tilde{H} = \lambda_2 J\,, & \tilde{P}_a = \lambda_2 G_a\,, & \tilde{M} = \lambda_2 S\,, & \tilde{U}_1 = \lambda_2 T_1\,, & \tilde{U}_2 = \lambda_2 T_2\,, \\ \tilde{Z} = \lambda_4 J\,, & \tilde{Z}_a = \lambda_4 G_a\,, & \tilde{T} = \lambda_4 S\,, & \tilde{B}_1 = \lambda_4 T_1\,, & \tilde{B}_2 = \lambda_4 T_2\,, \\ \tilde{N} = \lambda_6 J\,, & \tilde{N}_a = \lambda_6 G_a\,, & \tilde{V} = \lambda_6 S\,, & \tilde{C}_1 = \lambda_6 T_1\,, & \tilde{C}_2 = \lambda_6 T_2\,, \\ \tilde{Q}^+_\alpha = \lambda_1 Q^+_\alpha\,, & \tilde{Q}^-_\alpha = \lambda_1 Q^-_\alpha\,, & \tilde{\Sigma}^+_\alpha = \lambda_3 Q^+_\alpha\,, & \tilde{\Sigma}^-_\alpha = \lambda_3 Q^-_\alpha\,, & \tilde{R}_\alpha = \lambda_1 R_\alpha\,, \\ \tilde{W}_\alpha = \lambda_3 R_\alpha\,, & \tilde{\Xi}^+_\alpha = \lambda_5 Q^+_\alpha\,, & \tilde{\Xi}^-_\alpha = \lambda_5 Q^-_\alpha\,, & \tilde{K}_\alpha = \lambda_5 R_\alpha\,. & \end{array} \tag{3.26}$$

One can show that, using the multiplication law (3.24) and the original (anti-)commutation relations of the Nappi-Witten superalgebra (2.1), the expanded generators satisfy the (anti-)commutation relations (3.5), (3.16) along with

$$\begin{array}{lll} \left[\tilde{J}, \tilde{N}_a\right] = \epsilon_{ab}\tilde{N}_b\,, & \left[\tilde{P}_a, \tilde{Z}_b\right] = -\epsilon_{ab}\tilde{V}\,, & \left[\tilde{H}, \tilde{Z}_a\right] = \epsilon_{ab}\tilde{N}_b\,, \\ \left[\tilde{Z}, \tilde{P}_a\right] = \epsilon_{ab}\tilde{N}_b\,, & \left[\tilde{G}_a, \tilde{N}_b\right] = -\epsilon_{ab}\tilde{V}\,, & \left[\tilde{N}, \tilde{G}_a\right] = \epsilon_{ab}\tilde{N}_b\,, \\ \left[\tilde{J}, \tilde{\Xi}^\pm_\alpha\right] = -\frac{1}{2}\left(\gamma_0\right)_\alpha^{\ \beta}\tilde{\Xi}^\pm_\beta\,, & \left[\tilde{H}, \tilde{\Sigma}^\pm_\alpha\right] = -\frac{1}{2}\left(\gamma_0\right)_\alpha^{\ \beta}\tilde{\Xi}^\pm_\beta\,, & \left[\tilde{Z}, \tilde{Q}^\pm_\alpha\right] = \frac{1}{2}\left(\gamma_0\right)_\alpha^{\ \beta}\tilde{\Xi}^\pm_\beta\,, \\ \left[\tilde{J}, \tilde{K}_\alpha\right] = -\frac{1}{2}\left(\gamma_0\right)_\alpha^{\ \beta}\tilde{K}_\beta\,, & \left[\tilde{H}, \tilde{W}_\alpha\right] = -\frac{1}{2}\left(\gamma_0\right)_\alpha^{\ \beta}\tilde{K}_\beta\,, & \left[\tilde{Z}, \tilde{R}_\alpha\right] = \frac{1}{2}\left(\gamma_0\right)_\alpha^{\ \beta}\tilde{K}_\beta\,, \end{array}$$

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$$
\begin{aligned}
&\left[\tilde{G}_a,\tilde{\Xi}^+_\alpha\right]=-\frac{1}{2}\left(\gamma_a\right)_\alpha^{\ \beta}\tilde{\Xi}^-_\beta\,, &&\left[\tilde{P}_a,\tilde{\Sigma}^+_\alpha\right]=-\frac{1}{2}\left(\gamma_a\right)_\alpha^{\ \beta}\tilde{\Xi}^-_\beta\,, &&\left[\tilde{Z}_a,\tilde{Q}^+_\alpha\right]=\frac{1}{2}\left(\gamma_a\right)_\alpha^{\ \beta}\tilde{\Xi}^-_\beta\,,\\
&\left[\tilde{G}_a,\tilde{\Xi}^-_\alpha\right]=-\frac{1}{2}\left(\gamma_a\right)_\alpha^{\ \beta}\tilde{K}_\beta\,, &&\left[\tilde{P}_a,\tilde{\Sigma}^-_\alpha\right]=-\frac{1}{2}\left(\gamma_a\right)_\alpha^{\ \beta}\tilde{K}_\beta\,, &&\left[\tilde{Z}_a,\tilde{Q}^-_\alpha\right]=\frac{1}{2}\left(\gamma_a\right)_\alpha^{\ \beta}\tilde{K}_\beta\,,\\
&\left[\tilde{S},\tilde{\Xi}^+_\alpha\right]=-\frac{1}{2}\left(\gamma_0\right)_\alpha^{\ \beta}\tilde{K}_\beta\,, &&\left[\tilde{M},\tilde{\Sigma}^+_\alpha\right]=-\frac{1}{2}\left(\gamma_0\right)_\alpha^{\ \beta}\tilde{K}_\beta\,, &&\left[\tilde{T},\tilde{Q}^+_\alpha\right]=\frac{1}{2}\left(\gamma_0\right)_\alpha^{\ \beta}\tilde{K}_\beta\,,\\
&\left[\tilde{Y}_1,\tilde{\Xi}^\pm_\alpha\right]=-\frac{1}{2}\left(\gamma_0\right)_{\alpha\beta}\tilde{\Xi}^\pm_\beta\,, &&\left[\tilde{U}_1,\tilde{\Sigma}^\pm_\alpha\right]=-\frac{1}{2}\left(\gamma_0\right)_{\alpha\beta}\tilde{\Xi}^\pm_\beta\,, &&\left[\tilde{B}_1,\tilde{Q}^\pm_\alpha\right]=\frac{1}{2}\left(\gamma_0\right)_{\alpha\beta}\tilde{\Xi}^\pm_\beta\,,\\
&\left[\tilde{Y}_2,\tilde{\Xi}^+_\alpha\right]=-\frac{1}{2}\left(\gamma_0\right)_{\alpha\beta}\tilde{K}_\beta\,, &&\left[\tilde{U}_2,\tilde{\Sigma}^+_\alpha\right]=-\frac{1}{2}\left(\gamma_0\right)_{\alpha\beta}\tilde{K}_\beta\,, &&\left[\tilde{B}_2,\tilde{Q}^+_\alpha\right]=\frac{1}{2}\left(\gamma_0\right)_{\alpha\beta}\tilde{K}_\beta\,,\\
&\left[\tilde{Y}_1,\tilde{K}_\alpha\right]=-\frac{1}{2}\left(\gamma_0\right)_{\alpha\beta}\tilde{K}_\beta\,, &&\left[\tilde{U}_1,\tilde{W}_\alpha\right]=-\frac{1}{2}\left(\gamma_0\right)_{\alpha\beta}\tilde{K}_\beta\,, &&\left[\tilde{B}_1,\tilde{R}_\alpha\right]=\frac{1}{2}\left(\gamma_0\right)_{\alpha\beta}\tilde{K}_\beta\,,
\end{aligned}
$$
$$
\begin{aligned}
&\left\{\tilde{Q}^-_\alpha,\tilde{\Xi}^-_\beta\right\}=\left\{\tilde{\Sigma}^-_\alpha,\tilde{\Sigma}^-_\beta\right\}=-\left(\gamma^0C\right)_{\alpha\beta}\tilde{V}+\left(\gamma^0C\right)_{\alpha\beta}\tilde{C}_2\,,\\
&\left\{\tilde{Q}^\pm_\alpha,\tilde{\Xi}^\mp_\beta\right\}=\left\{\tilde{\Sigma}^+_\alpha,\tilde{\Sigma}^-_\beta\right\}=-\left(\gamma^aC\right)_{\alpha\beta}\tilde{N}_a\,,\\
&\left\{\tilde{Q}^+_\alpha,\tilde{\Xi}^+_\beta\right\}=\left\{\tilde{\Sigma}^+_\alpha,\tilde{\Sigma}^+_\beta\right\}=-\left(\gamma^0C\right)_{\alpha\beta}\tilde{N}-\left(\gamma^0C\right)_{\alpha\beta}\tilde{C}_1\,,\\
&\left\{\tilde{Q}^+_\alpha,\tilde{K}_\beta\right\}=\left\{\tilde{\Sigma}^+_\alpha,\tilde{W}_\beta\right\}=\left\{\tilde{\Xi}^+_\alpha,\tilde{R}_\beta\right\}=-\left(\gamma^0C\right)_{\alpha\beta}\tilde{V}-\left(\gamma^0C\right)_{\alpha\beta}\tilde{C}_2\,.
\end{aligned}
\tag{3.27}
$$

This superalgebra corresponds to a supersymmetric extension of the GMEB algebra introduced in [27]. Note that this is a new NR superalgebra, unlike the previous ones which had already been presented in the literature in previous works. The GMEB superalgebra is characterized by the presence of three extra fermionic generators, which are $\tilde{\Xi}^\pm_\alpha$ and $\tilde{K}_\alpha$, with respect to the MEB superalgebra. $\tilde{C}_1$ and $\tilde{C}_2$ play the role of central charges, while $\tilde{B}_1$ and $\tilde{B}_2$ act non-trivially on the fermionic generators $\tilde{Q}^\pm_\alpha$ and $\tilde{R}_\alpha$. The non-vanishing components of an invariant supertrace are given by (3.6), (3.17), and

$$
\begin{aligned}
\left\langle\tilde{G}_a\tilde{N}_b\right\rangle=\left\langle\tilde{P}_a\tilde{Z}_b\right\rangle&=\alpha_3\delta_{ab}\,,\\
\left\langle\tilde{J}\tilde{V}\right\rangle=\left\langle\tilde{H}\tilde{T}\right\rangle&=\left\langle\tilde{M}\tilde{Z}\right\rangle=\left\langle\tilde{S}\tilde{N}\right\rangle=-\alpha_3\,,\\
\left\langle\tilde{Y}_1\tilde{C}_2\right\rangle=\left\langle\tilde{U}_1\tilde{B}_2\right\rangle&=\left\langle\tilde{B}_1\tilde{U}_2\right\rangle=\left\langle\tilde{C}_1\tilde{Y}_2\right\rangle=\alpha_3\,,\\
\left\langle\tilde{Q}^-_\alpha\tilde{\Xi}^-_\beta\right\rangle=\left\langle\tilde{\Sigma}^+_\alpha\tilde{W}_\beta\right\rangle&=\left\langle\tilde{\Xi}^+_\alpha\tilde{R}_\beta\right\rangle=\left\langle\tilde{Q}^+_\alpha\tilde{K}_\beta\right\rangle=\left\langle\tilde{\Sigma}^-_\alpha\tilde{\Sigma}^-_\beta\right\rangle=2\alpha_3C_{\alpha\beta}\,.
\end{aligned}
\tag{3.28}
$$

The NR one-form gauge connection for the GMEB superalgebra reads

$$
\begin{aligned}
A=\;&\tau\tilde{H}+e^a\tilde{P}_a+\omega\tilde{J}+\omega^a\tilde{G}_a+k\tilde{Z}+k^a\tilde{Z}_a+f\tilde{N}+f^a\tilde{N}_a+m\tilde{M}+s\tilde{S}\\
&+t\tilde{T}+v\tilde{V}+y_1\tilde{Y}_1+y_2\tilde{Y}_2+b_1\tilde{B}_1+b_2\tilde{B}_2+u_1\tilde{U}_1+u_2\tilde{U}_2+c_1\tilde{C}_1+c_2\tilde{C}_2\\
&+\psi^+\tilde{Q}^++\psi^-\tilde{Q}^-+\xi^+\tilde{\Sigma}^++\xi^-\tilde{\Sigma}^-+\zeta^+\tilde{\Xi}^++\zeta^-\tilde{\Xi}^-+\rho\tilde{R}+\chi\tilde{W}+\kappa\tilde{K}\,.
\end{aligned}
\tag{3.29}
$$

The corresponding NR CS action can be obtained by inserting the above gauge connection and the invariant supertrace given by (3.6), (3.17), and (3.28) into the general expression for the CS action in three spacetime dimensions (2.5). The aforesaid action reads, up to boundary terms, as follows:

$$
I_{\text{GMEB}}=\alpha_0I_0+\alpha_1I_1+\alpha_2I_2+\alpha_3I_3\,,
\tag{3.30}
$$

where I_0 and I_1 are given in (3.9), I_2 is given in (3.20), while the term along α_3 reads

$$
\begin{aligned}
I_3 = \int & \omega_a R^a\left(f^b\right) + f_a R^a\left(\omega^b\right) + e_a R^a\left(k^b\right) + k_a R^a\left(e^b\right) - 2sR\left(f\right) \\
& - 2vR\left(\omega\right) - 2mR\left(k\right) - 2tR\left(\tau\right) + 2y_1 dc_2 + 2u_1 db_2 + 2y_2 dc_1 + 2b_1 du_2 \\
& + 2\bar{\psi}^- \nabla\zeta^- + 2\bar{\zeta}^- \nabla\psi^- + 2\bar{\xi}^- \nabla\xi^- + 2\bar{\psi}^+ \nabla\kappa + 2\bar{\zeta}^+ \nabla\rho + 2\bar{\xi}^+ \nabla\chi \\
& + 2\bar{\rho}\nabla\zeta^+ + 2\bar{\chi}\nabla\xi^+ + 2\bar{\kappa}\nabla\psi^+ \,.
\end{aligned} \tag{3.31}
$$

In the above action, the expressions for $R(\omega)$ and $R^a(\omega^b)$ were defined in (2.7), while those for $R(\tau)$, $R(k)$, $R^a(e^b)$, and $R^a(k^b)$ are given in (3.21). Furthermore,

$$
\begin{aligned}
R\left(f\right) &= df \,, \\
R^a\left(f^b\right) &= df^a + \epsilon^{ac}\omega f_c + \epsilon^{ac}\tau k_c + \epsilon^{ac} k e_c + \epsilon^{ac} f \omega_c \,,
\end{aligned} \tag{3.32}
$$

and the covariant derivatives of the spinor 1-form fields appearing in I_3 are given by

$$
\begin{aligned}
\nabla\zeta^+ &= d\zeta^+ + \frac{1}{2}\omega\gamma_0\zeta^+ + \frac{1}{2}\tau\gamma_0\xi^+ + \frac{1}{2}k\gamma_0\psi^+ - \frac{1}{2}y_1\gamma_0\zeta^+ - \frac{1}{2}u_1\gamma_0\xi^+ - \frac{1}{2}b_1\gamma_0\psi^+ \,, \\
\nabla\zeta^- &= d\zeta^- + \frac{1}{2}\omega\gamma_0\zeta^- + \frac{1}{2}\tau\gamma_0\xi^- + \frac{1}{2}k\gamma_0\psi^- + \frac{1}{2}e^a\gamma_a\xi^+ + \frac{1}{2}\omega^a\gamma_a\zeta^+ + \frac{1}{2}k^a\gamma_a\psi^+ \\
&\quad + \frac{1}{2}y_1\gamma_0\zeta^- + \frac{1}{2}u_1\gamma_0\xi^- + \frac{1}{2}b_1\gamma_0\psi^- \,, \\
\nabla\kappa &= d\kappa + \frac{1}{2}\omega\gamma_0\kappa + \frac{1}{2}\tau\gamma_0\chi + \frac{1}{2}k\gamma_0\rho + \frac{1}{2}\omega^a\gamma_a\zeta^- + \frac{1}{2}e^a\gamma_a\xi^- + \frac{1}{2}k^a\gamma_a\psi^- + \frac{1}{2}s\gamma_0\zeta^+ \\
&\quad + \frac{1}{2}m\gamma_0\xi^+ + \frac{1}{2}t\gamma_0\psi^+ - \frac{1}{2}y_2\gamma_0\zeta^+ - \frac{1}{2}u_2\gamma_0\xi^+ - \frac{1}{2}b_2\gamma_0\psi^+ - \frac{1}{2}y_1\gamma_0\kappa - \frac{1}{2}u_1\gamma_0\chi \\
&\quad - \frac{1}{2}b_1\gamma_0\rho \,,
\end{aligned} \tag{3.33}
$$

along with (3.11) and (3.22). From (3.30), we see that the CS action contains four independent sectors. The new gauge fields f_a, f, v, ζ^+, ζ^-, and κ appear explicitly in the last term proportional to α_3, which corresponds to the CS action for the new NR generalized Maxwell superalgebra. Note that the GMEB superalgebra allows to include a cosmological constant term along α_3 different from the one appearing in the case of the extended NR supergravity presented in [8]. One can see that the field equations imply the vanishing of the curvature two forms given by (3.11), (3.12), (3.22), (3.23), (3.33), and

$$
\begin{aligned}
F\left(f\right) &= R\left(f\right) + \bar{\psi}^+\gamma^0\zeta^+ + \frac{1}{2}\bar{\xi}^+\gamma^0\xi^+ \,, \\
F^a(f^b) &= R^a(f^b) + \bar{\psi}^+\gamma^a\zeta^- + \bar{\psi}^-\gamma^a\zeta^+ + \bar{\xi}^+\gamma^a\xi^- \,, \\
F\left(v\right) &= dv + \epsilon^{ac}\omega_a k_c + \epsilon^{ac} e_a k_c + \bar{\psi}^-\gamma^0\zeta^- + \frac{1}{2}\bar{\xi}^-\gamma^0\xi^- + \bar{\psi}^+\gamma^0\kappa + \bar{\xi}^+\gamma^0\chi + \bar{\zeta}^+\gamma^0\rho \,, \\
F\left(c_1\right) &= dc_1 + \bar{\psi}^+\gamma^0\zeta^+ + \frac{1}{2}\bar{\xi}^+\gamma^0\xi^+ \,, \\
F\left(c_2\right) &= dc_2 - \bar{\psi}^-\gamma^0\zeta^- - \frac{1}{2}\bar{\xi}^-\gamma^0\xi^- + \bar{\psi}^+\gamma^0\kappa + +\bar{\xi}^+\gamma^0\chi + \bar{\zeta}^+\gamma^0\rho \,,
\end{aligned} \tag{3.34}
$$

which are the GMEB supercurvatures. A natural subsequent step would be to continue expanding the Nappi-Witten superalgebra with bigger semigroups. As we will show in the next section, all NR superalgebras coming from the semigroup expansion of (2.1), when the semigroup under consideration is of the S_E-type, can be written in a very compact and general way in terms of the original (anti-)commutation relations (2.1) and the elements of the relevant semigroup.

3.4 Generalized extended Bargmann supergravity

As it was shown in [27], the extended Bargmann, the MEB, and the GMEB algebras can be seen as particular cases of the so-called generalized extended Bargmann (GEB$^{(N)}$) algebra, which corresponds to the NR version of the $\mathfrak{B}_{N+2}$ algebra enlarged with U_1 generators. In this section, we present a supersymmetric extension of the GEB$^{(N)}$ algebra which can be obtained by performing an S-expansion of the super Nappi-Witten algebra (2.1). Similarly to the purely bosonic case, the new NR superalgebra contains the extended Bargmann, the MEB, and the GMEB superalgebras previously introduced as particular cases.

Let $S_E^{(2N)} = \{\lambda_0, \lambda_1, \lambda_2, \ldots, \lambda_{2N-1}, \lambda_{2N}, \lambda_{2N+1}\}$ be the relevant semigroup whose elements satisfy

$$\lambda_\mu \lambda_\nu = \begin{cases} \lambda_{\mu+\nu} & \text{if} \quad \mu+\nu \leq 2N \,, \\ \lambda_{2N+1} & \text{if} \quad \mu+\nu > 2N \,, \end{cases} \tag{3.35}$$

where $\lambda_{2N+1} = 0_S$ is the zero element of the semigroup such that $0_S \lambda_\mu = 0_S$. Let $S_E^{(2N)} = S_0 \cup S_1$ be a semigroup decomposition with

$$\begin{aligned} S_0 &= \{\lambda_{2i}, \ i = 0, \ldots, N\} \cup \lambda_{2N+1} \,, \\ S_1 &= \{\lambda_{2m-1}, \ m = 1, \ldots, N\} \cup \lambda_{2N+1} \,. \end{aligned} \tag{3.36}$$

Such decomposition is said to be resonant since it behaves as the subspace decomposition of the Nappi-Witten superalgebra (2.2),

$$S_0 \cdot S_0 \subset S_0 \,, \qquad S_0 \cdot S_1 \subset S_1 \,, \qquad S_1 \cdot S_1 \subset S_0 \,. \tag{3.37}$$

Then, after performing a resonant $S_E^{(2N)}$-expansion to the super Nappi-Witten algebra (2.1) and considering a 0_S-reduction, we find a new NR superalgebra. The expanded NR generators are related to the super Nappi-Witten ones through the semigroup elements as follows:

$$\begin{aligned} \tilde{J}^{(i)} &= \lambda_{2i} J \,, & \tilde{Q}_\alpha^{+(m)} &= \lambda_{2m-1} Q_\alpha^+ \,, \\ \tilde{G}_a^{(i)} &= \lambda_{2i} G_a \,, & \tilde{Q}_\alpha^{-(m)} &= \lambda_{2m-1} Q_\alpha^- \,, \\ \tilde{S}^{(i)} &= \lambda_{2i} S \,, & \tilde{R}_\alpha^{(m)} &= \lambda_{2m-1} R_\alpha \,, \\ \tilde{T}_1^{(i)} &= \lambda_{2i} T_1 & \tilde{T}_2^{(i)} &= \lambda_{2i} T_2 \,. \end{aligned} \tag{3.38}$$

One can prove that the expanded generators satisfy the following (anti-)commutation relations:

$$\begin{aligned} \left[\tilde{J}^{(i)}, \tilde{G}_a^{(j)}\right] &= \epsilon_{ab} \tilde{G}_b^{(i+j)} \,, & \left[\tilde{G}_a^{(i)}, \tilde{G}_b^{(j)}\right] &= -\epsilon_{ab} \tilde{S}^{(i+j)} \,, \\ \left[\tilde{J}^{(i)}, \tilde{Q}_\alpha^{\pm(m)}\right] &= -\frac{1}{2} \left(\gamma_0\right)_\alpha^{\ \beta} \tilde{Q}_\beta^{\pm(i+m)} \,, & \left[\tilde{J}^{(i)}, \tilde{R}_\alpha^{(m)}\right] &= -\frac{1}{2} \left(\gamma_0\right)_\alpha^{\ \beta} \tilde{R}_\beta^{(i+m)} \,, \end{aligned}$$

$$
\begin{aligned}
\left[\tilde{G}_a^{(i)},\tilde{Q}_\alpha^{+(m)}\right] &= -\frac{1}{2}\left(\gamma_a\right)_\alpha^{\ \beta}\tilde{Q}_\beta^{-(i+m)}\,, & \left[\tilde{G}_a^{(i)},\tilde{Q}_\alpha^{-(m)}\right] &= -\frac{1}{2}\left(\gamma_a\right)_\alpha^{\ \beta}\tilde{R}_\beta^{(i+m)}\,,\\
\left[\tilde{S}^{(i)},\tilde{Q}_\alpha^{+(m)}\right] &= -\frac{1}{2}\left(\gamma_0\right)_\alpha^{\ \beta}\tilde{R}_\beta^{(i+m)}\,, & \left[\tilde{T}_1^{(i)},\tilde{Q}_\alpha^{\pm(m)}\right] &= \pm\frac{1}{2}\left(\gamma_0\right)_{\alpha\beta}\tilde{Q}_\beta^{\pm(i+m)}\,,\\
\left[\tilde{T}_2^{(i)},\tilde{Q}_\alpha^{+(m)}\right] &= \frac{1}{2}\left(\gamma_0\right)_{\alpha\beta}\tilde{R}_\beta^{(i+m)}\,, & \left[\tilde{T}_1^{(i)},\tilde{R}_\alpha^{(m)}\right] &= \frac{1}{2}\left(\gamma_0\right)_{\alpha\beta}\tilde{R}_\beta^{(i+m)}\,,\\
\left\{\tilde{Q}_\alpha^{+(m)},\tilde{Q}_\beta^{-(n)}\right\} &= -\left(\gamma^a C\right)_{\alpha\beta}\tilde{G}_a^{(m+n-1)}\,,\\
\left\{\tilde{Q}_\alpha^{+(m)},\tilde{Q}_\beta^{+(n)}\right\} &= -\left(\gamma^0 C\right)_{\alpha\beta}\tilde{J}^{(m+n-1)}-\left(\gamma^0 C\right)_{\alpha\beta}\tilde{T}_1^{(m+n-1)}\,,\\
\left\{\tilde{Q}_\alpha^{-(m)},\tilde{Q}_\beta^{-(n)}\right\} &= -\left(\gamma^0 C\right)_{\alpha\beta}\tilde{S}^{(m+n-1)}+\left(\gamma^0 C\right)_{\alpha\beta}\tilde{T}_2^{(m+n-1)}\,,\\
\left\{\tilde{Q}_\alpha^{+(m)},\tilde{R}_\beta^{(n)}\right\} &= -\left(\gamma^0 C\right)_{\alpha\beta}\tilde{S}^{(m+n-1)}-\left(\gamma^0 C\right)_{\alpha\beta}\tilde{T}_2^{(m+n-1)}\,.
\end{aligned}
\tag{3.39}
$$

This is obtained by using the multiplication law of the $S_E^{(2N)}$ semigroup (3.35) and the super Nappi-Witten (anti-)commutation relations (2.1).

The expanded superalgebra above generalizes the extended Bargmann superalgebra [3] and corresponds to the supersymmetric extension of the GEB$^{(N)}$ algebra introduced in [27]. The NR superalgebra (3.39) contains $2N+2$ bosonic generators in addition to the GEB$^{(N)}$ bosonic generators and $3N$ fermionic charges. The extra bosonic generators $\tilde{T}_1^{(i)}$ and $\tilde{T}_2^{(i)}$ act non-trivially on the fermionic charges $\tilde{Q}_\alpha^{(m)}$ if $i+m\leq N$. On the other hand, both $\tilde{T}_1^{(N)}$ and $\tilde{T}_2^{(N)}$ are central charges. One can see that the case $N=1$ reproduces the extended Bargmann superalgebra, while the $N=2$ and $N=3$ ones correspond to the MEB and GMEB superalgebras, respectively. At the bosonic level, as it was shown in [27], the so-called $\mathfrak{B}_{N+2}$ algebra [69, 86] is the relativistic counterpart of the GEB$^{(N)}$ algebra. Then, one could deduce that the present generalization is the respective NR version of the supersymmetric extension of the $\mathfrak{B}_{N+2}$ algebra [90].

The gauge connection one-form A for the GEB$^{(N)}$ superalgebra reads

$$
\begin{aligned}
A =&\,\omega^{(i)}\tilde{J}^{(i)}+\omega^{a(i)}\tilde{G}_a^{(i)}+s^{(i)}\tilde{S}^{(i)}+t_1^{(i)}\tilde{T}_1^{(i)}+t_2^{(i)}T_2^{(i)}\\
&+\bar{\psi}^{+(m)}\tilde{Q}^{+(m)}+\bar{\psi}^{-(m)}\tilde{Q}^{-(m)}+\bar{\rho}^{(m)}\tilde{R}^{(m)}\,.
\end{aligned}
\tag{3.40}
$$

The presence of the additional bosonic generators $\{\tilde{T}_1^{(i)},\tilde{T}_2^{(i)}\}$ is required to ensure a non-degenerate invariant supertrace allowing the construction of a proper NR CS supergravity action. In particular, the GEB$^{(N)}$ superalgebra admits the following non-degenerate invariant tensor:

$$
\begin{aligned}
\left\langle\tilde{G}_a^{(i)}\tilde{G}_b^{(j)}\right\rangle &= \alpha_{i+j}\delta_{ab}\,,\\
\left\langle\tilde{J}^{(i)}\tilde{S}^{(j)}\right\rangle &= -\alpha_{i+j}\,,\\
\left\langle\tilde{T}_1^{(i)}\tilde{T}_2^{(j)}\right\rangle &= \alpha_{i+j}\,,\\
\left\langle\tilde{Q}_\alpha^{-(m)}\tilde{Q}_\beta^{-(n)}\right\rangle &= 2\alpha_{m+n-1}C_{\alpha\beta}\;=\;\left\langle\tilde{Q}_\alpha^{+(m)}\tilde{R}_\beta^{(n)}\right\rangle\,,
\end{aligned}
\tag{3.41}
$$

where $i,j=0,1,\ldots,N$; $m,n=1,2,\ldots,N$, and $i+j<N+1$, $i+m<N+1$, and $m+n<N+1$. Here, the non-vanishing components of an invariant tensor for the GEB$^{(N)}$

superalgebra are obtained from the super Nappi-Witten ones (2.3) using the definitions of the S-expansion method [53]. Then, a GEB$^{(N)}$ CS supergravity action can be constructed by inserting the gauge connection one-form (3.40) and the non-vanishing components of the invariant tensor (3.41) in the general expression for the CS action (2.5), that yields

$$I_{\mathrm{GEB}^{(N)}}=\alpha_i I_i\ =\alpha_0 I_0+\alpha_1 I_1+\ldots+\alpha_N I_N\,, \tag{3.42}$$

with

$$\begin{aligned}I_i=\int &\omega_a^{(j)}d\omega^{a(k)}\delta^i_{j+k}+\epsilon^{ac}\omega_a^{(j)}\omega^{(k)}\omega_c^{(l)}\delta^i_{j+k+l}-2s^{(j)}d\omega^{(k)}\delta^i_{j+k}+2t_1^{(j)}dt_2^{(k)}\delta^i_{j+k}\\&+2\bar{\psi}^{-(m)}\nabla\psi^{-(n)}\delta^i_{m+n-1}+2\bar{\psi}^{+(m)}\nabla\rho^{(n)}\delta^i_{m+n-1}+2\bar{\rho}^{(m)}\nabla\psi^{+(n)}\delta^i_{m+n-1}\,,\end{aligned} \tag{3.43}$$

where the covariant derivatives of the spinor 1-forms for the GEB$^{(N)}$ superalgebra read

$$\begin{aligned}\nabla\psi^{+(m)}&=d\psi^{+(m)}+\frac{1}{2}\omega^{(i)}\gamma_0\psi^{+(n)}\delta^m_{i+n}-\frac{1}{2}t_1^{(i)}\gamma_0\psi^{+(n)}\delta^m_{i+n}\,,\\\nabla\psi^{-(m)}&=d\psi^{-(m)}+\frac{1}{2}\omega^{(i)}\gamma_0\psi^{-(n)}\delta^m_{i+n}+\frac{1}{2}\omega^{a(i)}\gamma_a\psi^{+(n)}\delta^m_{i+n}+\frac{1}{2}t_1^{(i)}\gamma_0\psi^{-(n)}\delta^m_{i+n}\,,\\\nabla\rho^{(m)}&=d\rho^{(m)}+\frac{1}{2}\omega^{(i)}\gamma_0\rho^{(n)}\delta^m_{i+n}+\frac{1}{2}\omega^{a(i)}\gamma_a\psi^{-(n)}\delta^m_{i+n}+\frac{1}{2}s^{(i)}\gamma_0\psi^{+(n)}\delta^m_{i+n}\\&\quad-\frac{1}{2}t_2^{(i)}\gamma_0\psi^{+(n)}\delta^m_{i+n}-\frac{1}{2}t_1^{(i)}\gamma_0\rho^{(n)}\delta^m_{i+n}\,.\end{aligned} \tag{3.44}$$

The new NR CS action is invariant under the GEB$^{(N)}$ superalgebra (3.39) and contains $N+1$ independent sectors proportional to the α_i's. Interestingly, one can see that the GEB$^{(i)}$ CS action, for $i<N$, appears as a particular subcase. Indeed, the I_0 and I_1 CS terms describe the most general extended Bargmann supergravity action, where I_0 corresponds to a NR exotic term. The explicit extended Bargmann CS action (3.8) is recovered by identifying the gauge field one-forms as

$$\begin{aligned}&\omega^{(0)}=\omega\,, && \omega_a^{(0)}=\omega_a\,, && s^{(0)}=s\,, && t_1^{(0)}=y_1\,, && t_2^{(0)}=y_2\,,\\&\omega^{(1)}=\tau\,, && \omega_a^{(1)}=e_a\,, && s^{(1)}=m\,, && t_1^{(1)}=u_1\,, && t_2^{(1)}=u_2\,,\\&\psi^{+(1)}=\psi^+\,, && \psi^{-(1)}=\psi^-\,, && \rho^{(1)}=\rho\,.\end{aligned} \tag{3.45}$$

On the other hand, the I_2 term along with I_1 and I_0 describe the MEB CS supergravity action [7]. In particular, the I_2 CS action coincides with the one obtained in (3.20) by identifying the gauge field one-forms as in (3.45) together with

$$\begin{aligned}&\omega^{(2)}=k\,, && \omega_a^{(2)}=k_a\,, && s^{(2)}=t\,, && t_1^{(2)}=b_1\,, && t_2^{(2)}=b_2\,,\\&\psi^{+(2)}=\xi^+\,, && \psi^{-(2)}=\xi^-\,, && \rho^{(2)}=\chi\,.\end{aligned} \tag{3.46}$$

In addition, one can see that the GMEB CS supergravity theory is described by I_0, I_1, I_2, and I_3 by considering the redefinitions of the gauge fields as in (3.45), (3.46), and

$$\begin{aligned}&\omega^{(3)}=f\,, && \omega_a^{(3)}=f_a\,, && s^{(3)}=u\,, && t_1^{(3)}=c_1\,, && t_2^{(3)}=c_2\,,\\&\psi^{+(3)}=\zeta^+\,, && \psi^{-(3)}=\zeta^-\,, && \rho^{(3)}=\kappa\,.\end{aligned} \tag{3.47}$$

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Novel generalizations of the extended Bargmann supergravity theory are obtained for $N > 3$ and correspond to supersymmetric extensions of the GEB$^{(N)}$ gravity theory presented in [27]. Let us note that the equations of motion for a specific GEB$^{(i)}$ superalgebra are given by the vanishing of the curvature two-forms associated with the respective superalgebra, which are given by (3.44) along with

$$\begin{aligned}
F\left(\omega^{(i)}\right) &= d\omega^{i} + \frac{1}{2}\bar{\psi}^{+(m)}\gamma^{0}\psi^{+(n)}\delta^{i}_{m+n-1}\,, \\
F^{a}\left(\omega^{b(i)}\right) &= d\omega^{a(i)} + \epsilon^{ac}\omega^{(j)}\omega_{c}^{(k)}\delta^{i}_{j+k} + \bar{\psi}^{+(m)}\gamma^{a}\psi^{-(n)}\delta^{i}_{m+n-1}\,, \\
F\left(s^{(i)}\right) &= ds^{(i)} + \frac{1}{2}\bar{\psi}^{-(m)}\gamma^{0}\psi^{-(n)}\delta^{i}_{m+n-1} + \bar{\psi}^{+(m)}\gamma^{0}\rho^{(n)}\delta^{i}_{m+n-1}\,, \\
F\left(t_{1}^{(i)}\right) &= dt_{1}^{(i)} + \frac{1}{2}\bar{\psi}^{+(m)}\gamma^{0}\psi^{+(n)}\delta^{i}_{m+n-1}\,, \\
F\left(t_{2}^{(i)}\right) &= dt_{2}^{(i)} - \frac{1}{2}\bar{\psi}^{-(m)}\gamma^{0}\psi^{-(n)}\delta^{i}_{m+n-1} + \bar{\psi}^{+(m)}\gamma^{0}\rho^{(n)}\delta^{i}_{m+n-1}\,.
\end{aligned} \tag{3.48}$$

The physical implications of the additional bosonic and fermionic content remains as an interesting open issue that shall be considered in a future work.

4 Generalized extended Newton-Hooke supergravity theory and semi-group expansion method

In this section, we apply the S-expansion method to the Nappi-Witten superalgebra (2.1) to find a different family of NR superalgebras. In particular, we shall see that the extended Newton-Hooke superalgebra [6] and its generalizations can be obtained considering $S_L^{(1)}$ and $S_{\mathcal{M}}^{(N)}$ as the relevant semigroups, respectively. The semigroup choice is not arbitrary and comes from the expansion relation presented at the level of the asymptotic symmetry. Indeed, as it was shown in [82], the conformal superalgebra can be obtained by expanding the super Virasoro one using $S_L^{(1)}$ as the relevant semigroup, while generalizations of the superconformal symmetry are found using $S_{\mathcal{M}}$. It is interesting to notice that the same semigroup used to relate diverse infinite-dimensional superalgebras can be considered at the NR level. Furthermore, we will show that the generalized extended Newton-Hooke superalgebras, which we have denoted as GNH$^{(N)}$, are related to the GEB$^{(N)}$ ones (3.39) through an IW contraction. The construction of a NR CS supergravity action based on the GNH$^{(N)}$ superalgebra is also presented.

4.1 Extended Newton-Hooke supergravity

From the super Nappi-Witten algebra (2.1), one can obtain a supersymmetric extension of the Newton-Hooke algebra (precisely, the extended Newton-Hooke superalgebra obtained in [8]) through the S-expansion method considering $S_L^{(1)} = \{\lambda_0, \lambda_1, \lambda_2\}$ as the relevant semigroup, whose elements satisfy the following multiplication law:

$$\begin{array}{c|ccc}
\lambda_2 & \lambda_2 & \lambda_2 & \lambda_2 \\
\lambda_1 & \lambda_2 & \lambda_1 & \lambda_2 \\
\lambda_0 & \lambda_0 & \lambda_2 & \lambda_2 \\
\hline
 & \lambda_0 & \lambda_1 & \lambda_2
\end{array} \tag{4.1}$$

with $\lambda_2 = 0_S$ being the zero element of the semigroup. Let $S_L^{(1)} = S_0 \cup S_1$ be a decomposition of the semigroup

$$
\begin{aligned}
S_0 &= \{\lambda_0, \lambda_1, \lambda_2\} \,, \\
S_1 &= \{\lambda_1, \lambda_2\} \,,
\end{aligned}
\tag{4.2}
$$

which is said to be resonant [53] since it satisfies the same structure as the subspaces of the super Nappi-Witten algebra (2.2),

$$
\begin{aligned}
S_0 \cdot S_0 &\subset S_0 \,, \\
S_0 \cdot S_1 &\subset S_1 \,, \\
S_1 \cdot S_1 &\subset S_0 \,.
\end{aligned}
\tag{4.3}
$$

An expanded superalgebra spanned by $\{L, L_a, N, X_1, X_2, \tilde{L}, \tilde{L}_a, \tilde{N}, \tilde{X}_1, \tilde{X}_2, \mathcal{Q}_\alpha^+, \mathcal{Q}_\alpha^-, \mathcal{R}_\alpha\}$ is obtained after considering a resonant $S_L^{(1)}$-expansion of the super Nappi-Witten algebra and performing a 0_S-reduction. The expanded generators are related to the super Nappi-Witten ones through the semigroup elements as

$$
\begin{aligned}
L &= \lambda_1 J \,, & L_a &= \lambda_1 G_a \,, & N &= \lambda_1 S \,, \\
\tilde{L} &= \lambda_0 J \,, & \tilde{L}_a &= \lambda_0 G_a \,, & \tilde{N} &= \lambda_0 S \,, \\
\mathcal{Q}_\alpha^+ &= \lambda_1 Q_\alpha^+ \,, & \mathcal{Q}_\alpha^- &= \lambda_1 Q_\alpha^- \,, & \mathcal{R}_\alpha &= \lambda_1 R_\alpha \,, \\
X_1 &= \lambda_1 T_1 \,, & X_2 &= \lambda_1 T_2 \,, \\
\tilde{X}_1 &= \lambda_0 T_1 \,, & \tilde{X}_2 &= \lambda_0 T_2 \,.
\end{aligned}
\tag{4.4}
$$

Then, one can see that the expanded NR generators satisfy two copies of the Nappi-Witten algebra, one of which is augmented by supersymmetry,

$$
\begin{aligned}
[L, L_a] &= \epsilon_{ab} L_b \,, & [L_a, L_b] &= -\epsilon_{ab} N \,, \\
\left[\tilde{L}, \tilde{L}_a\right] &= \epsilon_{ab} \tilde{L}_b \,, & \left[\tilde{L}_a, \tilde{L}_b\right] &= -\epsilon_{ab} \tilde{N} \,, \\
[L, \mathcal{Q}_\alpha^\pm] &= -\frac{1}{2} (\gamma_0)_\alpha{}^\beta \, \mathcal{Q}_\beta^\pm \,, & [L, \mathcal{R}_\alpha] &= -\frac{1}{2} (\gamma_0)_\alpha{}^\beta \, \mathcal{R}_\beta \,, \\
\left[L_a, \mathcal{Q}_\alpha^+\right] &= -\frac{1}{2} (\gamma_a)_\alpha{}^\beta \, \mathcal{Q}_\beta^- \,, & [L_a, \mathcal{Q}_\alpha^-] &= -\frac{1}{2} (\gamma_a)_\alpha{}^\beta \, \mathcal{R}_\beta \,, \\
\left[N, \mathcal{Q}_\alpha^+\right] &= -\frac{1}{2} (\gamma_0)_\alpha{}^\beta \, \mathcal{R}_\beta \,, & [X_1, \mathcal{Q}_\alpha^\pm] &= \pm\frac{1}{2} (\gamma_0)_{\alpha\beta} \, \mathcal{Q}_\beta^\pm \,, \\
\left[X_2, \mathcal{Q}_\alpha^+\right] &= \frac{1}{2} (\gamma_0)_{\alpha\beta} \, \mathcal{R}_\beta \,, & [X_1, \mathcal{R}_\alpha] &= \frac{1}{2} (\gamma_0)_{\alpha\beta} \, \mathcal{R}_\beta \,, \\
\left\{\mathcal{Q}_\alpha^+, \mathcal{Q}_\beta^-\right\} &= -\left(\gamma^a C\right)_{\alpha\beta} L_a \,, \\
\left\{\mathcal{Q}_\alpha^+, \mathcal{Q}_\beta^+\right\} &= -\left(\gamma^0 C\right)_{\alpha\beta} L - \left(\gamma^0 C\right)_{\alpha\beta} X_1 \,, \\
\left\{\mathcal{Q}_\alpha^-, \mathcal{Q}_\beta^-\right\} &= -\left(\gamma^0 C\right)_{\alpha\beta} N + \left(\gamma^0 C\right)_{\alpha\beta} X_2 , \\
\left\{\mathcal{Q}_\alpha^+, \mathcal{R}_\beta\right\} &= -\left(\gamma^0 C\right)_{\alpha\beta} N - \left(\gamma^0 C\right)_{\alpha\beta} X_2 \,,
\end{aligned}
\tag{4.5}
$$

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where we have used the multiplication law of the semigroup $S_L^{(1)}$ (4.1) and the (anti-)commutation relations of the Nappi-Witten superalgebra (2.1). On the other hand, let us note that the purely bosonic copy of the Nappi-Witen algebra is endowed with two $\mathfrak{u}(1)$ generators, namely $\tilde{X}_1$ and $\tilde{X}_2$. Considering the definitions of [53], one can see that the superalgebra (4.5) admits the following non-vanishing components of the invariant tensor:

$$
\begin{aligned}
\langle L_a L_b \rangle &= \mu \delta_{ab} \,, & \left\langle \tilde{L}_a \tilde{L}_b \right\rangle &= \nu \delta_{ab} \,, \\
\langle L N \rangle &= -\mu \,, & \left\langle \tilde{L} \tilde{N} \right\rangle &= -\nu \,, \\
\langle X_1 X_2 \rangle &= \mu \,, & \left\langle \tilde{X}_1 \tilde{X}_2 \right\rangle &= \nu \,, \\
\left\langle \mathcal{Q}_\alpha^- \mathcal{Q}_\beta^- \right\rangle &= 2\mu C_{\alpha\beta} \,, & \left\langle \mathcal{Q}_\alpha^+ \mathcal{R}_\beta \right\rangle &= 2\mu c_{\alpha\beta} \,,
\end{aligned}
\tag{4.6}
$$

where μ and ν are arbitrary independent constants related to the ones of the super Nappi-Witten and Nappi-Witten algebras, respectively. Interestingly, the superalgebra (4.5) can be related to a central extension of the extended Bargmann superalgebra (3.5) after a suitable redefinition of the NR generators and after having performed a vanishing cosmological constant limit. Indeed, let us consider the following redefinition of the generators:

$$
\begin{aligned}
\tilde{G}_a &= L_a - \tilde{L}_a \,, & \tilde{P}_a &= \frac{1}{\ell}\left(L_a + \tilde{L}_a\right) \,, & \tilde{Q}_\alpha^+ &= \sqrt{\frac{2}{\ell}}\mathcal{Q}_\alpha^+ \,, \\
\tilde{J} &= L + \tilde{L} \,, & \tilde{H} &= \frac{1}{\ell}\left(L - \tilde{L}\right) \,, & \tilde{Q}_\alpha^- &= \sqrt{\frac{2}{\ell}}\mathcal{Q}_\alpha^- \,, \\
\tilde{S} &= N + \tilde{N} \,, & \tilde{M} &= \frac{1}{\ell}\left(N - \tilde{N}\right) \,, & \tilde{R}_\alpha &= \sqrt{\frac{2}{\ell}}\mathcal{R}_\alpha \,, \\
\tilde{Y}_1 &= X_1 - \tilde{X}_1 \,, & \tilde{U}_1 &= \frac{1}{\ell}\left(X_1 + \tilde{X}_1\right) \,, \\
\tilde{Y}_2 &= X_2 - \tilde{X}_2 \,, & \tilde{U}_2 &= \frac{1}{\ell}\left(X_2 + \tilde{X}_2\right) \,,
\end{aligned}
\tag{4.7}
$$

where ℓ is a length parameter related to the cosmological constant through $\Lambda \propto \pm\frac{1}{\ell^2}$. Thus, the expanded superalgebra (4.5) can be rewritten as

$$
\begin{aligned}
\left[\tilde{J}, \tilde{G}_a\right] &= \epsilon_{ab}\tilde{G}_b \,, & \left[\tilde{G}_a, \tilde{G}_b\right] &= -\epsilon_{ab}\tilde{S} \,, & \left[\tilde{J}, \tilde{P}_a\right] &= \epsilon_{ab}\tilde{P}_b \,, \\
\left[\tilde{H}, \tilde{G}_a\right] &= \epsilon_{ab}\tilde{P}_b \,, & \left[\tilde{G}_a, \tilde{P}_b\right] &= -\epsilon_{ab}\tilde{M} \,, & \left[\tilde{H}, \tilde{P}_a\right] &= \frac{1}{\ell^2}\epsilon_{ab}\tilde{G}_b \,, \\
\left[\tilde{P}_a, \tilde{P}_b\right] &= -\frac{1}{\ell^2}\epsilon_{ab}\tilde{S} \,, & \left[\tilde{J}, \tilde{Q}_\alpha^\pm\right] &= -\frac{1}{2}\left(\gamma_0\right)_\alpha{}^\beta \tilde{Q}_\beta^\pm \,, & \left[\tilde{H}, \tilde{Q}_\alpha^\pm\right] &= -\frac{1}{2\ell}\left(\gamma_0\right)_\alpha{}^\beta \tilde{Q}_\beta^\pm \,, \\
\left[\tilde{J}, \tilde{R}_\alpha\right] &= -\frac{1}{2}\left(\gamma_0\right)_\alpha{}^\beta \tilde{R}_\beta \,, & \left[\tilde{H}, \tilde{R}_\alpha\right] &= -\frac{1}{2\ell}\left(\gamma_0\right)_\alpha{}^\beta \tilde{R}_\beta \,, & \left[\tilde{G}_a, \tilde{Q}_\alpha^+\right] &= -\frac{1}{2}\left(\gamma_a\right)_\alpha{}^\beta \tilde{Q}_\beta^- \,, \\
\left[\tilde{P}_a, \tilde{Q}_\alpha^+\right] &= -\frac{1}{2\ell}\left(\gamma_a\right)_\alpha{}^\beta \tilde{Q}_\beta^- \,, & \left[\tilde{G}_a, \tilde{Q}_\alpha^-\right] &= -\frac{1}{2}\left(\gamma_a\right)_\alpha{}^\beta \tilde{R}_\beta \,, & \left[\tilde{P}_a, \tilde{Q}_\alpha^-\right] &= -\frac{1}{2\ell}\left(\gamma_a\right)_\alpha{}^\beta \tilde{R}_\beta \,, \\
\left[\tilde{S}, \tilde{Q}_\alpha^+\right] &= -\frac{1}{2}\left(\gamma_0\right)_\alpha{}^\beta \tilde{R}_\beta \,, & \left[\tilde{M}, \tilde{Q}_\alpha^+\right] &= -\frac{1}{2\ell}\left(\gamma_0\right)_\alpha{}^\beta \tilde{R}_\beta \,, & \left[\tilde{Y}_1, \tilde{Q}_\alpha^\pm\right] &= \pm\frac{1}{2}\left(\gamma_0\right)_{\alpha\beta} \tilde{Q}_\beta^\pm \,, \\
\left[\tilde{U}_1, \tilde{Q}_\alpha^\pm\right] &= \pm\frac{1}{2\ell}\left(\gamma_0\right)_{\alpha\beta} \tilde{Q}_\beta^\pm \,, & \left[\tilde{Y}_2, \tilde{Q}_\alpha^+\right] &= \frac{1}{2}\left(\gamma_0\right)_{\alpha\beta} \tilde{R}_\beta \,, & \left[\tilde{U}_2, \tilde{Q}_\alpha^+\right] &= \frac{1}{2\ell}\left(\gamma_0\right)_{\alpha\beta} \tilde{R}_\beta \,, \\
\left[\tilde{Y}_1, \tilde{R}_\alpha\right] &= \frac{1}{2}\left(\gamma_0\right)_{\alpha\beta} \tilde{R}_\beta \,, & \left[\tilde{U}_1, \tilde{R}_\alpha\right] &= \frac{1}{2\ell}\left(\gamma_0\right)_{\alpha\beta} \tilde{R}_\beta \,,
\end{aligned}
$$

$$
\begin{aligned}
\left\{\tilde{Q}^+_\alpha,\tilde{Q}^-_\beta\right\} &= -\frac{1}{\ell}\left(\gamma^a C\right)_{\alpha\beta}\tilde{G}_a - \left(\gamma^a C\right)_{\alpha\beta}\tilde{P}_a\,,\\
\left\{\tilde{Q}^+_\alpha,\tilde{Q}^+_\beta\right\} &= -\frac{1}{\ell}\left(\gamma^0 C\right)_{\alpha\beta}\tilde{J} - \left(\gamma^0 C\right)_{\alpha\beta}\tilde{H} - \frac{1}{\ell}\left(\gamma^0 C\right)_{\alpha\beta}\tilde{Y}_1 - \left(\gamma^0 C\right)_{\alpha\beta}\tilde{U}_1\,,\\
\left\{\tilde{Q}^-_\alpha,\tilde{Q}^-_\beta\right\} &= -\frac{1}{\ell}\left(\gamma^0 C\right)_{\alpha\beta}\tilde{S} - \left(\gamma^0 C\right)_{\alpha\beta}\tilde{M} + \frac{1}{\ell}\left(\gamma^0 C\right)_{\alpha\beta}\tilde{Y}_2 + \left(\gamma^0 C\right)_{\alpha\beta}\tilde{U}_2\,,\\
\left\{\tilde{Q}^+_\alpha,\tilde{R}_\beta\right\} &= -\frac{1}{\ell}\left(\gamma^0 C\right)_{\alpha\beta}\tilde{S} - \left(\gamma^0 C\right)_{\alpha\beta}\tilde{M} - \frac{1}{\ell}\left(\gamma^0 C\right)_{\alpha\beta}\tilde{Y}_2 - \left(\gamma^0 C\right)_{\alpha\beta}\tilde{U}_2\,.
\end{aligned}
\tag{4.8}
$$

This superalgebra corresponds to a supersymmetric extension of the extended Newton-Hooke algebra [91–98] and coincides with the one presented in [8]. One can notice that it contains two additional bosonic generators, namely $\tilde{Y}_1$ and $\tilde{Y}_2$, with respect to the extended Newton-Hooke superalgebra introduced in [6]. Such generators, along with $\tilde{U}_1$ and $\tilde{U}_2$, act non-trivially on the fermionic generators $\tilde{Q}^\pm_\alpha$ and $\tilde{R}_\alpha$. Remarkably, in the flat limit $\ell\to\infty$ we recover the centrally extended version of the extended Bargmann superalgebra (3.5) with central charges $\tilde{U}_1$ and $\tilde{U}_2$.

The presence of the additional bosonic generators ensures to have a non-degenerate invariant tensor allowing us to construct the most general CS action invariant under the extended Newton-Hooke superalgebra. In particular, the invariant tensor can be obtained from the invariant tensor of the two copies of the Nappi-Witten algebra, one of which is supersymmetric. Indeed, considering the redefinition of the generators (4.7) in the invariant tensor (4.6), we find that the extended Newton-Hooke superalgebra admits the following non-vanishing components of the invariant tensor:

$$
\begin{aligned}
&\left\langle \tilde{G}_a\tilde{G}_b\right\rangle = \alpha_0\delta_{\alpha\beta}\,, && \left\langle \tilde{J}\tilde{S}\right\rangle = -\alpha_0\,, && \left\langle \tilde{Y}_1\tilde{Y}_2\right\rangle = \alpha_0\,,\\
&\left\langle \tilde{G}_a\tilde{P}_b\right\rangle = \alpha_1\delta_{\alpha\beta}\,, && \left\langle \tilde{J}\tilde{M}\right\rangle = -\alpha_1\,, && \left\langle \tilde{Y}_1\tilde{U}_2\right\rangle = \alpha_1\,,\\
&\left\langle \tilde{P}_a\tilde{P}_b\right\rangle = \frac{\alpha_0}{\ell^2}\delta_{\alpha\beta}\,, && \left\langle \tilde{H}\tilde{S}\right\rangle = -\alpha_1\,, && \left\langle \tilde{U}_1\tilde{Y}_2\right\rangle = \alpha_1\,,\\
&\left\langle \tilde{H}\tilde{M}\right\rangle = -\frac{\alpha_0}{\ell^2}\,, && \left\langle \tilde{U}_1\tilde{U}_2\right\rangle = \frac{\alpha_0}{\ell^2}\,,
\end{aligned}
\tag{4.9}
$$

along with

$$
\begin{aligned}
\left\langle \tilde{Q}^-_\alpha\tilde{Q}^-_\beta\right\rangle &= 2\left(\frac{\alpha_0}{\ell}+\alpha_1\right)C_{\alpha\beta}\,,\\
\left\langle \tilde{Q}^+_\alpha\tilde{R}_\beta\right\rangle &= 2\left(\frac{\alpha_0}{\ell}+\alpha_1\right)C_{\alpha\beta}\,,
\end{aligned}
\tag{4.10}
$$

where the Newton-Hooke parameters are related to the Nappi-Witten ones through

$$
\alpha_0 = \mu+\nu\,,\qquad \alpha_1 = \frac{1}{\ell}\left(\mu-\nu\right)\,. \tag{4.11}
$$

One can see that the invariant tensor (3.6) for the extended Bargmann superalgebra are recovered in the vanishing cosmological constant limit $\ell\to\infty$.

Let us now consider the gauge connection one-form A for the extended Newton-Hooke superalgebra (4.8), that is

$$
\begin{aligned}
A &= \omega\tilde{J} + \tau\tilde{H} + \omega^a\tilde{G}_a + e^a\tilde{P}_a + s\tilde{S} + m\tilde{M} + y_1\tilde{Y}_1 + y_2\tilde{Y}_2 + u_1\tilde{U}_1 + u_2\tilde{U}_2\\
&\quad + \bar{\psi}^+\tilde{Q}^+ + \bar{\psi}^-\tilde{Q}^- + \bar{\rho}\tilde{R}\,.
\end{aligned}
\tag{4.12}
$$

Then, the three-dimensional extended Newton-Hooke CS supergravity theory is obtained considering the gauge connection one-form (4.12) and the non-vanishing components of the invariant tensor (4.9)–(4.10) and plugging them into the general expression for a CS action (2.5). By doing so, one gets

$$I_{\text{NH}} = \alpha_0 I_0 + \alpha_1 I_1 \,, \tag{4.13}$$

where

$$\begin{aligned} I_0 = \int & \omega_a R^a(\omega^b) - 2sR(\omega) + \frac{1}{\ell^2} e_a R^a\left(e^b\right) - \frac{1}{\ell^2} 2mR(\tau) + 2y_1 dy_2 + \frac{2}{\ell^2} u_1 du_2 \\ & - \frac{2}{\ell}\bar{\psi}^- \nabla\psi^- - \frac{2}{\ell}\bar{\psi}^+ \nabla\rho - \frac{2}{\ell}\bar{\rho}\nabla\psi^+ \,, \end{aligned} \tag{4.14}$$

$$\begin{aligned} I_1 = \int & 2e_a R^a(\omega^b) - 2mR(\omega) - 2\tau R(s) + \frac{1}{\ell^2}\epsilon_{ab}\tau e^a e^b + 2y_1 du_2 + 2u_1 dy_2 \\ & - 2\bar{\psi}^- \nabla\psi^- - 2\bar{\psi}^+ \nabla\rho - 2\bar{\rho}\nabla\psi^+ \,. \end{aligned} \tag{4.15}$$

Here, the curvature two-form $R(s)$ is given by

$$R(s) = ds + \frac{1}{2}\epsilon^{ac}\omega_a\omega_c \,, \tag{4.16}$$

while $R(\omega)$ and $R^a(\omega^b)$ are given by (2.7). Besides, the covariant derivatives of the spinor 1-form fields read

$$\begin{aligned} \nabla\psi^+ &= d\psi^+ + \frac{1}{2}\omega\gamma_0\psi^+ + \frac{1}{2\ell}\tau\gamma_0\psi^+ - \frac{1}{2}y_1\gamma_0\psi^+ - \frac{1}{2\ell}u_1\gamma_0\psi^+ \,, \\ \nabla\psi^- &= d\psi^- + \frac{1}{2}\omega\gamma_0\psi^- + \frac{1}{2}\omega^a\gamma_a\psi^+ + \frac{1}{2\ell}\tau\gamma_0\psi^- + \frac{1}{2\ell}e^a\gamma_a\psi^+ + \frac{1}{2}y_1\gamma_0\psi^- + \frac{1}{2\ell}u_1\gamma_0\psi^- \,, \\ \nabla\rho &= d\rho + \frac{1}{2}\omega\gamma_0\rho + \frac{1}{2}\omega^a\gamma_a\psi^- + \frac{1}{2}s\gamma_0\psi^+ + \frac{1}{2\ell}\tau\gamma_0\rho + \frac{1}{2\ell}e^a\gamma_a\psi^- + \frac{1}{2\ell}m\gamma_0\psi^+ \\ & \quad - \frac{1}{2}y_2\gamma_0\psi^+ - \frac{1}{2}y_1\gamma_0\rho - \frac{1}{2\ell}u_2\gamma_0\psi^+ - \frac{1}{2\ell}u_1\gamma_0\rho \,. \end{aligned} \tag{4.17}$$

The three-dimensional CS extended Newton-Hooke supergravity action (4.13) can be written in terms of two independent CS actions. The CS term I_0 corresponds to an exotic CS NR supergravity action [6] and can be seen as the NR version of the exotic $\mathcal{N}=2$ CS AdS supergravity action [49, 99, 100],

$$I^{\text{AdS}}_{\text{exotic}} = \int \omega_A d\omega^A + \frac{1}{3}\epsilon_{ABC}\omega^A\omega^B\omega^C + \frac{1}{\ell^2}e_A T^A + \mathsf{t}d\mathsf{t} + \frac{1}{\ell^2}\mathsf{u}d\mathsf{u} - \frac{2}{\ell}\bar{\psi}^i\nabla\psi^i \,, \tag{4.18}$$

where $A=0,1,2$, $i=1,2$, and $\{\mathsf{t},\mathsf{u}\}$ are $\mathfrak{so}(2)$ internal symmetry gauge fields. On the other hand, the CS term I_1 is the supersymmetric extension of the Newton-Hooke gravity action and resembles the extended Newton-Hooke supergravity action introduced in [6] except for the presence of the additional bosonic gauge fields y_1, y_2, u_1, and u_2. At the bosonic level, the relativistic exotic term is related to the Pontryagin density, while the term along α_1 corresponds to an Euler CS form [101]. The combination of both families allows to write the most general CS action not only for the AdS (or Poincaré) algebra but also for the Maxwell-like symmetries [102]. Interestingly, the exotic Pontryagin CS term can be extended at the supersymmetric and NR level, corresponding to the I_0 CS action. As we shall see in the next section, the NR exotic terms can also be introduced into a generalized extended Newton-Hooke supergravity theory.

Let us note that the non-degeneracy of the invariant tensor (4.9)–(4.10) is related to the requirement that the CS supergravity action involves a kinematical term for each gauge field. Then, the equations of motion of the extended Newton-Hooke supergravity theory are given by the vanishing of the corresponding curvature two-forms, which are given by (4.17) along with

$$\begin{aligned}
F\left(\omega\right) &= R(\omega) + \frac{1}{2\ell}\bar{\psi}^{+}\gamma^{0}\psi^{+}\,, \\
F\left(\tau\right) &= d\left(\tau\right) + \frac{1}{2}\bar{\psi}^{+}\gamma^{0}\psi^{+}\,, \\
F^{a}(\omega^{b}) &= R^{a}(\omega^{b}) + \frac{1}{\ell^{2}}\epsilon^{ac}\tau e_{c} + \frac{1}{2}\bar{\psi}^{+}\gamma^{0}\psi^{-}\,, \\
F\left(s\right) &= R(s) + \frac{1}{2\ell}\epsilon^{ac}e_{a}e_{c} + \frac{1}{2\ell}\bar{\psi}^{-}\gamma^{0}\psi^{-} + \frac{1}{\ell}\bar{\psi}^{+}\gamma^{0}\rho\,, \\
F\left(y_{1}\right) &= dy_{1} + \frac{1}{2\ell}\bar{\psi}^{+}\gamma^{0}\psi^{+}\,, \\
F\left(y_{2}\right) &= dy_{2} - \frac{1}{2\ell}\bar{\psi}^{-}\gamma^{0}\psi^{-} + \frac{1}{\ell}\bar{\psi}^{+}\gamma^{0}\rho\,, \\
F^{a}(e^{b}) &= de^{a} + \epsilon^{ac}\omega e_{c} + \epsilon^{ac}\tau\omega_{c} + \bar{\psi}^{+}\gamma^{a}\psi^{-}\,, \\
F\left(m\right) &= dm + \epsilon^{ac}\omega_{a}e_{c} + \frac{1}{2}\bar{\psi}^{-}\gamma^{0}\psi^{-} + \bar{\psi}^{+}\gamma^{0}\rho\,, \\
F\left(u_{1}\right) &= du_{1} + \frac{1}{2}\bar{\psi}^{+}\gamma^{0}\psi^{+}\,, \\
F\left(u_{2}\right) &= du_{2} - \frac{1}{2}\bar{\psi}^{-}\gamma^{0}\psi^{-} + \bar{\psi}^{+}\gamma^{0}\rho\,.
\end{aligned} \tag{4.19}$$

Here, one can prove that the vanishing cosmological constant limit $\ell \to \infty$ reproduces the equations of motion of the extended Bargmann supergravity theory. Thus,the extended Newton-Hooke supergravity theory reproduces, in the flat limit, the extended Bargmann supergravity theory (3.8). In particular, I_0 reduces to the exotic extended Bargmann gravity action, while the I_1 term leads us to the usual extended Bargmann supergravity [3] endowed with extra bosonic gauge fields.

4.2 Generalized extended Newton-Hooke supergravity

A generalization of the extended Newton-Hooke superalgebra, denoted as $\mathrm{GNH}^{(N)}$, can be obtained by performing an S-expansion of the super Nappi-Witten algebra (2.1). To this end, let us first consider $S_{\mathcal{M}}^{(2N)} = \{\lambda_0, \lambda_1, \lambda_2, \dots, \lambda_{2N-1}, \lambda_{2N}\}$ as the relevant semigroup whose elements satisfy the following multiplication law:

$$\lambda_{\alpha}\lambda_{\beta} = \begin{cases} \lambda_{\alpha+\beta} & \text{if} \quad \alpha+\beta \leq 2N\,, \\ \lambda_{\alpha+\beta-2N} & \text{if} \quad \alpha+\beta > 2N\,, \end{cases} \tag{4.20}$$

with $N \geq 2$. Let us note that, unlike the semigroups $S_{E}^{(2N)}$ and $S_{L}^{(1)}$, there is no zero element in the present semigroup. Let us now consider a semigroup decomposition $S_{\mathcal{M}}^{(2N)} =$

$S_0 \cup S_1$ where

$$
\begin{aligned}
S_0 &= \{\lambda_{2i},\ i=0,\dots,N\}\,,\\
S_1 &= \{\lambda_{2m-1},\ m=1,\dots,N\}\,.
\end{aligned}
\tag{4.21}
$$

Such decomposition is said to be resonant since it satisfies the same subspace decomposition of the Nappi-Witten superalgebra (2.2),

$$
\begin{aligned}
S_0\cdot S_0 &\subset S_0\,,\\
S_0\cdot S_1 &\subset S_1\,,\\
S_1\cdot S_1 &\subset S_0\,.
\end{aligned}
\tag{4.22}
$$

One can see that a novel family of NR superalgebras is obtained after performing a resonant $S_{\mathcal{M}}^{(2N)}$-expansion to the super Nappi-Witten algebra (2.1). In particular, the expanded NR generators are related to the super Nappi-Witten ones through the semigroup elements as follows:

$$
\begin{aligned}
\tilde{J}^{(i)} &= \lambda_{2i}J\,, & \tilde{Q}_\alpha^{+(m)} &= \lambda_{2m-1}Q_\alpha^{+}\,,\\
\tilde{G}_a^{(i)} &= \lambda_{2i}G_a\,, & \tilde{Q}_\alpha^{-(m)} &= \lambda_{2m-1}Q_\alpha^{-}\,,\\
\tilde{S}^{(i)} &= \lambda_{2i}S\,, & \tilde{R}_\alpha^{(m)} &= \lambda_{2m-1}R_\alpha\,,\\
\tilde{T}_1^{(i)} &= \lambda_{2i}T_1 & \tilde{T}_2^{(i)} &= \lambda_{2i}T_2\,.
\end{aligned}
\tag{4.23}
$$

Then, considering the multiplication law of the $S_{\mathcal{M}}^{(2N)}$ semigroup (4.20) and the original commutation relations of the super Nappi-Witten algebra (2.1), one can then show that the expanded NR superalgebra satisfy the following (anti-)commutation relations:

$$
\begin{aligned}
\left[\tilde{J}^{(i)},\tilde{G}_a^{(j)}\right] &= \epsilon_{ab}\tilde{G}_b^{(i*j)}\,, & \left[\tilde{G}_a^{(i)},\tilde{G}_b^{(j)}\right] &= -\epsilon_{ab}\tilde{S}^{(i*j)}\,,\\
\left[\tilde{J}^{(i)},\tilde{Q}_\alpha^{\pm(m)}\right] &= -\frac{1}{2}\left(\gamma_0\right)_\alpha^{\ \beta}\tilde{Q}_\beta^{\pm(i*m)}\,, & \left[\tilde{J}^{(i)},\tilde{R}_\alpha^{(m)}\right] &= -\frac{1}{2}\left(\gamma_0\right)_\alpha^{\ \beta}\tilde{R}_\beta^{(i*m)}\,,\\
\left[\tilde{G}_a^{(i)},\tilde{Q}_\alpha^{+(m)}\right] &= -\frac{1}{2}\left(\gamma_a\right)_\alpha^{\ \beta}\tilde{Q}_\beta^{-(i*m)}\,, & \left[\tilde{G}_a^{(i)},\tilde{Q}_\alpha^{-(m)}\right] &= -\frac{1}{2}\left(\gamma_a\right)_\alpha^{\ \beta}\tilde{R}_\beta^{(i*m)}\,,\\
\left[\tilde{S}^{(i)},\tilde{Q}_\alpha^{+(m)}\right] &= -\frac{1}{2}\left(\gamma_0\right)_\alpha^{\ \beta}\tilde{R}_\beta^{(i*m)}\,, & \left[\tilde{T}_1^{(i)},\tilde{Q}_\alpha^{\pm(m)}\right] &= \pm\frac{1}{2}\left(\gamma_0\right)_{\alpha\beta}\tilde{Q}_\beta^{\pm(i*m)}\,,\\
\left[\tilde{T}_2^{(i)},\tilde{Q}_\alpha^{+(m)}\right] &= \frac{1}{2}\left(\gamma_0\right)_{\alpha\beta}\tilde{R}_\beta^{(i*m)}\,, & \left[\tilde{T}_1^{(i)},\tilde{R}_\alpha^{(m)}\right] &= \frac{1}{2}\left(\gamma_0\right)_{\alpha\beta}\tilde{R}_\beta^{(i*m)}\,,\\
\left\{\tilde{Q}_\alpha^{+(m)},\tilde{Q}_\beta^{-(n)}\right\} &= -\left(\gamma^a C\right)_{\alpha\beta}\tilde{G}_a^{(m*[n-1])}\,,\\
\left\{\tilde{Q}_\alpha^{+(m)},\tilde{Q}_\beta^{+(n)}\right\} &= -\left(\gamma^0 C\right)_{\alpha\beta}\tilde{J}^{(m*[n-1])}-\left(\gamma^0 C\right)_{\alpha\beta}\tilde{T}_1^{(m*[n-1])}\,,\\
\left\{\tilde{Q}_\alpha^{-(m)},\tilde{Q}_\beta^{-(n)}\right\} &= -\left(\gamma^0 C\right)_{\alpha\beta}\tilde{S}^{(m*[n-1])}+\left(\gamma^0 C\right)_{\alpha\beta}\tilde{T}_2^{(m*[n-1])}\,,\\
\left\{\tilde{Q}_\alpha^{+(m)},\tilde{R}_\beta^{(n)}\right\} &= -\left(\gamma^0 C\right)_{\alpha\beta}\tilde{S}^{(m*[n-1])}-\left(\gamma^0 C\right)_{\alpha\beta}\tilde{T}_2^{(m*[n-1])}\,,
\end{aligned}
\tag{4.24}
$$

where we have defined

$$
i*j=\begin{cases} i+j & \text{if}\ \ i+j\leq N\,,\\ i+j-N & \text{if}\ \ i+j>N\,.\end{cases}
\tag{4.25}
$$

The NR superalgebra (4.24) corresponds to a generalized extended Newton-Hooke superalgebra which we have denoted as GNH$^{(N)}$ superalgebra. The GNH$^{(N)}$ superalgebra is characterized by $3N$ fermionic generators and can be seen as the supersymmetric extension of the generalized Newton-Hooke algebra presented in [65]. Furthermore, as a consequence of the S-expansion procedure, the GNH$^{(N)}$ superalgebra contains $2N$ additional bosonic generators with respect to the generalized Newton-Hooke ones. As we shall see, their presence ensures the non-degeneracy of the invariant tensor. The present superalgebra can be seen as the NR counterpart of the supersymmetric extension of the $\mathfrak{C}_{N+2}$ algebra [72, 103]. However, the construction of a NR limit reproducing the GNH$^{(N)}$ superalgebra shall not be explored here.

It is important to clarify that $N \geq 2$, while the usual extended Newton-Hooke superalgebra (4.8) does not appear as a particular subcase of GNH$^{(N)}$ since it has been obtained with a different semigroup, namely $S_L^{(1)}$. Nevertheless, since at the bosonic level the extended Newton-Hooke algebra belongs to the bosonic generalized Newton-Hooke family, we shall consider the extended Newton-Hooke superalgebra as the GNH$^{(1)}$ superalgebra. The first case obtained with the present procedure corresponds to the GNH$^{(2)}$ superalgebra. In particular, the $N = 2$ case reproduces the enlarged extended Bargmann (EEB) superalgebra recently introduced in [8]. Remarkably, similarly to the extended Newton-Hooke superalgebra, the EEB superalgebra can be written as three copies of the Nappi-Witten algebra, two of which are augmented by supersymmetry [8]. One could argue that the GNH$^{(N)}$ superalgebra could be written as $N + 1$ copies of Nappi-Witten algebras, N of which should be augmented by supersymmetry. Nevertheless, this does not hold true anymore for $N > 2$.

Let us note that the $2N$ extra bosonic generators $\tilde{T}_1^{(i)}$ and $\tilde{T}_2^{(i)}$ allow to have a non-degenerate invariant tensor which is essential to the proper construction of a NR CS supergravity action. Indeed, following the definitions of [53], it is possible to show that the GNH$^{(N)}$ superalgebra admits the following non-vanishing components of the invariant tensor:

$$
\begin{aligned}
\left\langle \tilde{G}_a^{(i)} \tilde{G}_b^{(j)} \right\rangle &= \alpha_{i*j}\delta_{ab} \,, \\
\left\langle \tilde{J}^{(i)} \tilde{S}^{(j)} \right\rangle &= -\alpha_{i*j} \,, \\
\left\langle \tilde{T}_1^{(i)} \tilde{T}_2^{(j)} \right\rangle &= \alpha_{i*j} \,, \\
\left\langle \tilde{Q}_\alpha^{-(m)} \tilde{Q}_\beta^{-(n)} \right\rangle &= 2\alpha_{m*(n-1)} C_{\alpha\beta} \;=\; \left\langle \tilde{Q}_\alpha^{+(m)} \tilde{R}_\beta^{(n)} \right\rangle \,,
\end{aligned}
\tag{4.26}
$$

where the NR parameters appear as a consequence of the S-expansion procedure [53]. On the other hand, the GNH$^{(N)}$ gauge connection one-form A reads

$$
\begin{aligned}
A =& \,\omega^{(i)} \tilde{J}^{(i)} + \omega^{a(i)} \tilde{G}_a^{(i)} + s^{(i)} \tilde{S}^{(i)} + t_1^{(i)} \tilde{T}_1^{(i)} + t_2^{(i)} T_2^{(i)} \\
&+ \bar{\psi}^{+(m)} \tilde{Q}^{+(m)} + \bar{\psi}^{-(m)} \tilde{Q}^{-(m)} + \bar{\rho}^{(m)} \tilde{R}^{(m)} \,.
\end{aligned}
\tag{4.27}
$$

Then, the three-dimensional CS action based on the super GNH$^{(N)}$ algebra is obtained by combining the gauge connection one-form (4.27) and the non-vanishing components of the invariant tensor (4.26) into the general expression for a CS action (2.5), which thus yields

$$I_{\mathrm{GNH}^{(N)}} = \alpha_i I_i \; = \alpha_0 I_0 + \alpha_1 I_1 + \ldots + \alpha_N I_N \,, \tag{4.28}$$

where

$$\begin{aligned} I_i = \int & \omega_a^{(j)} d\omega^{a(k)} \delta^i_{j*k} + \epsilon^{ac} \omega_a^{(j)} \omega^{(k)} \omega_c^{(l)} \delta^i_{j*k*l} - 2 s^{(j)} d\omega^{(k)} \delta^i_{j*k} + 2 t_1^{(j)} dt_2^{(k)} \delta^i_{j*k} \\ & + 2\bar{\psi}^{-(m)} \nabla \psi^{-(n)} \delta^i_{m*(n-1)} + 2\bar{\psi}^{+(m)} \nabla \rho^{(n)} \delta^i_{m*(n-1)} + 2\bar{\rho}^{(m)} \nabla \psi^{+(n)} \delta^i_{m*(n-1)} \,. \end{aligned} \tag{4.29}$$

In particular, the covariant derivatives of the spinor 1-forms for the $\mathrm{GNH}^{(N)}$ superalgebra read

$$\begin{aligned} \nabla \psi^{+(m)} &= d\psi^{+(m)} + \frac{1}{2} \omega^{(i)} \gamma_0 \psi^{+(n)} \delta^m_{i*n} - \frac{1}{2} t_1^{(i)} \gamma_0 \psi^{+(n)} \delta^m_{i*n} \,, \\ \nabla \psi^{-(m)} &= d\psi^{-(m)} + \frac{1}{2} \omega^{(i)} \gamma_0 \psi^{-(n)} \delta^m_{i*n} + \frac{1}{2} \omega^{a(i)} \gamma_a \psi^{+(n)} \delta^m_{i*n} + \frac{1}{2} t_1^{(i)} \gamma_0 \psi^{-(n)} \delta^m_{i*n} \,, \\ \nabla \rho^{(m)} &= d\rho^{(m)} + \frac{1}{2} \omega^{(i)} \gamma_0 \rho^{(n)} \delta^m_{i*n} + \frac{1}{2} \omega^{a(i)} \gamma_a \psi^{-(n)} \delta^m_{i*n} + \frac{1}{2} s^{(i)} \gamma_0 \psi^{+(n)} \delta^m_{i*n} \\ & \quad - \frac{1}{2} t_2^{(i)} \gamma_0 \psi^{+(n)} \delta^m_{i*n} - \frac{1}{2} t_1^{(i)} \gamma_0 \rho^{(n)} \delta^m_{i*n} \,. \end{aligned} \tag{4.30}$$

The CS supergravity action (4.29) is the most general CS action based on the $\mathrm{GNH}^{(N)}$ superalgebra. In particular, as in the extended Newton-Hooke supergravity, the CS expression (4.28) can be split into two families: the CS term proportional to α_{2k} corresponds to NR exotic terms whose bosonic counterparts are related to the Pontryagin density; on the other hand, the contributions proportional to α_{2k+1} belong to the Euler CS family. In the particular case $N = 2$, the CS supergravity action reproduces the EEB supergravity theory presented in [8] by identifying the gauge field one-forms as

$$\begin{aligned} &\omega^{(0)} = \omega \,, && \omega_a^{(0)} = \omega_a \,, && s^{(0)} = s \,, && t_1^{(0)} = y_1 \,, && t_2^{(0)} = y_2 \,, \\ &\omega^{(1)} = \tau \,, && \omega_a^{(1)} = e_a \,, && s^{(1)} = m \,, && t_1^{(1)} = u_1 \,, && t_2^{(1)} = u_2 \,, \\ &\omega^{(2)} = k \,, && \omega_a^{(2)} = k_a \,, && s^{(2)} = t \,, && t_1^{(2)} = b_1 \,, && t_2^{(2)} = b_2 \,, \\ &\psi^{+(1)} = \psi^+ \,, && \psi^{-(1)} = \psi^- \,, && \rho^{(1)} = \rho \,, && && \\ &\psi^{+(2)} = \xi^+ \,, && \psi^{-(2)} = \xi^- \,, && \rho^{(2)} = \chi \,. && && \end{aligned} \tag{4.31}$$

The EEB supergravity action contains three independent sectors proportional to α_0, α_1, and α_2. In such NR model, I_0 describes the NR exotic gravity coupled to the additional bosonic gauge fields y_1 and y_2. On the other hand, I_1 and I_2 describe a NR extended supergravity model in the presence of a cosmological constant and of extra bosonic content given by k, k_a, and t (for further details about the EEB supergravity theory see [8]). The $N = 3$ case would now reproduce four independent sectors as in the GMEB supergravity theory previously introduced.

Let us note that, due to the non-degeneracy of the invariant tensor, the equations of motion of the $\mathrm{GNH}^{(N)}$ supergravity theory are given by the vanishing of the associated curvature two-forms, which read

$$F\left(\omega^{(i)}\right) = d\omega^i + \frac{1}{2} \bar{\psi}^{+(m)} \gamma^0 \psi^{+(n)} \delta^i_{m*(n-1)} \,,$$

$$
\begin{aligned}
F^{a}\left(\omega^{b(i)}\right) &= d\omega^{a(i)} + \epsilon^{ac}\omega^{(j)}\omega_{c}^{(k)}\delta_{j*k}^{i} + \bar{\psi}^{+(m)}\gamma^{a}\psi^{-(n)}\delta_{m*(n-1)}^{i} \,, \\
F\left(s^{(i)}\right) &= ds^{(i)} + \frac{1}{2}\bar{\psi}^{-(m)}\gamma^{0}\psi^{-(n)}\delta_{m*(n-1)}^{i} + \bar{\psi}^{+(m)}\gamma^{0}\rho^{(n)}\delta_{m*(n-1)}^{i} \,, \\
F\left(t_{1}^{(i)}\right) &= dt_{1}^{(i)} + \frac{1}{2}\bar{\psi}^{+(m)}\gamma^{0}\psi^{+(n)}\delta_{m*(n-1)}^{i} \,, \\
F\left(t_{2}^{(i)}\right) &= dt_{2}^{(i)} - \frac{1}{2}\bar{\psi}^{-(m)}\gamma^{0}\psi^{-(n)}\delta_{m*(n-1)}^{i} + \bar{\psi}^{+(m)}\gamma^{0}\rho^{(n)}\delta_{m*(n-1)}^{i} \,,
\end{aligned}
\tag{4.32}
$$

along with (4.30).

It is interesting to notice that the $\text{GNH}^{(N)}$ superalgebras are related to the $\text{GEB}^{(N)}$ ones through an IW contraction procedure. The following diagram summarizes the expansion and contraction relations:

$$
\begin{tikzcd}
 & \boxed{\text{super extended Newton-Hooke}} \arrow[dd, "\ell\to\infty"] & & & \boxed{\text{super GNH}^{(N)}} \arrow[dd, "\text{IW}"] \\
\boxed{\text{super Nappi-Witten}} \arrow[ur, "S_L^{(1)}"] \arrow[dr, "S_E^{(2)}"'] & & \xrightarrow{\text{Generalization}} & \boxed{\text{super Nappi-Witten}} \arrow[ur, "S_{\mathcal{M}}^{(2N)}"] \arrow[dr, "S_E^{(2N)}"'] & \\
 & \boxed{\text{super extended Bargmann}} & & & \boxed{\text{super GEB}^{(N)}}
\end{tikzcd}
$$

In particular, the $\text{GEB}^{(N)}$ superalgebras appear as an IW contraction of the generalized extended Newton-Hooke ones by rescaling the super $\text{GNH}^{(N)}$ generators as follows:

$$
\begin{aligned}
\tilde{J}^{(i)} &\to \sigma^{2i}\tilde{J}^{(i)} \,, & \tilde{Q}_{\alpha}^{+(m)} &\to \sigma^{2m-1}\tilde{Q}_{\alpha}^{+(m)} \,, \\
\tilde{G}_{a}^{(i)} &\to \sigma^{2i}\tilde{G}_{a}^{(i)} \,, & \tilde{Q}_{\alpha}^{-(m)} &\to \sigma^{2m-1}\tilde{Q}_{\alpha}^{-(m)} \,, \\
\tilde{S}^{(i)} &\to \sigma^{2i}\tilde{S}^{(i)} \,, & \tilde{R}_{\alpha}^{(m)} &\to \sigma^{2m-1}\tilde{R}_{\alpha}^{(m)} \,,
\end{aligned}
\tag{4.33}
$$

and considering the limit $\sigma\to\infty$. Note that the vanishing cosmological constant limit $\ell\to\infty$ performed in the $N=1$ case to recover the extended Bargmann superalgebra can also be seen as an IW contraction. At the CS action level, the proper IW contraction requires rescaling, in addition, the NR parameters in the invariant tensor (4.26) as $\alpha_i\to\sigma^{i}\alpha_i$.

As an ending remark, let us note that the S_E and $S_{\mathcal{M}}$ semigroups used to obtain the respective generalizations of the extended Bargmann and extended Newton-Hooke supergravity theories are the same used to find generalizations of the super Poincaré and super AdS algebra. Furthermore, as it was shown in [27, 65], the same semigroups were considered at the bosonic level, allowing to define the $\text{GEB}^{(N)}$ and $\text{GNH}^{(N)}$ algebras.

5 Concluding remarks

In this paper, we have introduced an alternative procedure to obtain diverse NR CS supergravity theories in three spacetime dimensions. Known and new NR superalgebras have been obtained considering the expansion method based on semigroups, the so-called S-expansion, to a Nappi-Witten superalgebra introduced in [8]. Interestingly, the S-expansion allows to immediately obtain the non-vanishing components of the invariant tensor of an expanded superalgebra in term of the original ones. Such advantage has allowed us to construct, in a systematic way, the respective NR CS supergravity actions for each NR superalgebra presented.

We have shown that our resulting theories can be split into two NR supergravity families. Indeed, the extended Bargmann supergravity along with its Maxwellian version can be seen as particular subcases of a generalized extended Bargmann supergravity theory which we have denoted as GEB$^{(N)}$ supergravity. In particular, for $N = 1$ we recover the extended Bargmann supergravity theory. On the other hand, the extended Newton-Hooke supergravity belongs to a generalized extended Newton-Hooke theory which we have called as GNH$^{(N)}$. Such generalizations correspond to supersymmetric extensions of the GEB$^{(N)}$ and GNH$^{(N)}$ bosonic algebras recently introduced in [27]. Remarkably, both families are related through an IW contraction process, similarly as their bosonic counterparts.

It would be worth considering further studies on the Maxwellian version and generalizations of the extended Bargmann supergravity theory. One could analyze, for instance, the Schrödinger extension [2, 104] of the NR generalized superalgebras presented here in a similar way as it was done in the case of the extended Schrödinger supergravity [6]. In particular, the map between Newton-Cartan geometry and Horava-Lifshitz gravity [97] could suggest a superconformal non-projectable Horava-Lifshitz gravity. The Schrödinger version of our results would allow us to approach an off-shell formulation of the respective NR supergravity actions which could serve to construct NR effective field theories on curved backgrounds by means of localization [105, 106]. On the other hand, it would be intriguing to explore the possibility to apply the S-expansion method in the context of the Schrödinger superalgebra families, in order to establish a systematic way to obtain generalized Schrödinger supergravity actions, in a very similar way to the construction presented here.

A future development could also consist in generalizing our results to the extended Newtonian family. One could expect to obtain novel Newtonian supergravity theories, different from the extended Newtonian one presented in [4], being supersymmetric extensions of the exotic and Maxwellian extended Newtonian gravity recently introduced in [107] and [68], respectively. Newtonian gravity models are worth studying as they offer an action principle for Newtonian gravity through the CS formalism different from the one introduced in [108]. It would be interesting to study the matter coupling of the new Newtonian supergravity theories as well.

The procedure considered here could be extended to the ultra-relativistic (UR) regime. In particular, the construction of UR supergravity models remains poorly explored [109, 110]. The S-expansion method could be applied to a Carrollian version of the Nappi-Witten symmetry to obtain new UR superalgebras (work in progress). One could expect to find two UR superalgebra families being the respective UR versions of the relativistic $\mathfrak{B}_{N+2}$ and $\mathfrak{C}_{N+2}$ superalgebras. At the bosonic level, the Carrollian symmetries emerge in the framework of flat holography and fluid/gravity correspondence [111–114], whose applications motivate us to explore supersymmetric extensions of the Carrollian symmetries in the context of supergravity.

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References

[1] R. Andringa, E.A. Bergshoeff, J. Rosseel and E. Sezgin, *3D Newton-Cartan supergravity*, *Class. Quant. Grav.* **30** (2013) 205005 [arXiv:1305.6737] [INSPIRE].

[2] E. Bergshoeff, J. Rosseel and T. Zojer, *Newton-Cartan supergravity with torsion and Schrödinger supergravity*, *JHEP* **11** (2015) 180 [arXiv:1509.04527] [INSPIRE].

[3] E.A. Bergshoeff and J. Rosseel, *Three-dimensional extended Bargmann supergravity*, *Phys. Rev. Lett.* **116** (2016) 251601 [arXiv:1604.08042] [INSPIRE].

[4] N. Ozdemir, M. Ozkan, O. Tunca and U. Zorba, *Three-dimensional extended Newtonian (super)gravity*, *JHEP* **05** (2019) 130 [arXiv:1903.09377] [INSPIRE].

[5] J.A. de Azcárraga, D. Gútiez and J.M. Izquierdo, *Extended $D = 3$ Bargmann supergravity from a Lie algebra expansion*, *Nucl. Phys. B* **946** (2019) 114706 [arXiv:1904.12786] [INSPIRE].

[6] N. Ozdemir, M. Ozkan and U. Zorba, *Three-dimensional extended Lifshitz, Schrödinger and Newton-Hooke supergravity*, *JHEP* **11** (2019) 052 [arXiv:1909.10745] [INSPIRE].

[7] P. Concha, L. Ravera and E. Rodríguez, *Three-dimensional Maxwellian extended Bargmann supergravity*, *JHEP* **04** (2020) 051 [arXiv:1912.09477] [INSPIRE].

[8] P. Concha, L. Ravera and E. Rodríguez, *Three-dimensional non-relativistic extended supergravity with cosmological constant*, *Eur. Phys. J. C* **80** (2020) 1105 [arXiv:2008.08655] [INSPIRE].

[9] R. Grassie, *Generalised Bargmann superalgebras*, arXiv:2010.01894 [INSPIRE].

[10] E. Cartan, *Sur les variétés à connexion affine et la théorie de la relativité généralisée (première partie)*, *Ann. Ecole Norm. Sup.* **40** (1923) 325.

[11] E. Cartan, *Sur les variétés à connexion affine et la théorie de la relativité généralisée (première partie) (suite)*, *Ann. Ecole Norm. Sup.* **41** (1924) 1.

[12] C. Duval and H.P. Kunzle, *Minimal gravitational coupling in the Newtonian theory and the covariant Schrödinger equation*, *Gen. Rel. Grav.* **16** (1984) 333 [INSPIRE].

[13] C. Duval, G. Burdet, H.P. Kunzle and M. Perrin, *Bargmann structures and Newton-Cartan theory*, *Phys. Rev. D* **31** (1985) 1841 [INSPIRE].

[14] C. Duval and P.A. Horvathy, *Non-relativistic conformal symmetries and Newton-Cartan structures*, J. Phys. A **42** (2009) 465206 [arXiv:0904.0531] [INSPIRE].

[15] R. Andringa, E. Bergshoeff, S. Panda and M. de Roo, *Newtonian gravity and the Bargmann algebra*, Class. Quant. Grav. **28** (2011) 105011 [arXiv:1011.1145] [INSPIRE].

[16] R. Banerjee, A. Mitra and P. Mukherjee, *Localization of the Galilean symmetry and dynamical realization of Newton-Cartan geometry*, Class. Quant. Grav. **32** (2015) 045010 [arXiv:1407.3617] [INSPIRE].

[17] R. Banerjee and P. Mukherjee, *Torsional Newton-Cartan geometry from Galilean gauge theory*, Class. Quant. Grav. **33** (2016) 225013 [arXiv:1604.06893] [INSPIRE].

[18] E. Bergshoeff, A. Chatzistavrakidis, L. Romano and J. Rosseel, *Newton-Cartan gravity and torsion*, JHEP **10** (2017) 194 [arXiv:1708.05414] [INSPIRE].

[19] L. Avilés, E. Frodden, J. Gomis, D. Hidalgo and J. Zanelli, *Non-relativistic Maxwell Chern-Simons gravity*, JHEP **05** (2018) 047 [arXiv:1802.08453] [INSPIRE].

[20] L. Avilés, J. Gomis and D. Hidalgo, *Stringy (Galilei) Newton-Hooke Chern-Simons Gravities*, JHEP **09** (2019) 015 [arXiv:1905.13091] [INSPIRE].

[21] D. Chernyavsky and D. Sorokin, *Three-dimensional (higher-spin) gravities with extended Schrödinger and l-conformal Galilean symmetries*, JHEP **07** (2019) 156 [arXiv:1905.13154] [INSPIRE].

[22] P. Concha and E. Rodríguez, *Non-relativistic gravity theory based on an enlargement of the extended Bargmann algebra*, JHEP **07** (2019) 085 [arXiv:1906.00086] [INSPIRE].

[23] T. Harmark, J. Hartong, L. Menculini, N.A. Obers and G. Oling, *Relating non-relativistic string theories*, JHEP **11** (2019) 071 [arXiv:1907.01663] [INSPIRE].

[24] D. Hansen, J. Hartong and N.A. Obers, *Non-relativistic gravity and its coupling to matter*, JHEP **06** (2020) 145 [arXiv:2001.10277] [INSPIRE].

[25] M. Ergen, E. Hamamci and D. Van den Bleeken, *Oddity in nonrelativistic, strong gravity*, Eur. Phys. J. C **80** (2020) 563 [*Erratum ibid.* **80** (2020) 657] [arXiv:2002.02688] [INSPIRE].

[26] O. Kasikci, N. Ozdemir, M. Ozkan and U. Zorba, *Three-dimensional higher-order Schrödinger algebras and Lie algebra expansions*, JHEP **04** (2020) 067 [arXiv:2002.03558] [INSPIRE].

[27] P. Concha, M. Ipinza and E. Rodríguez, *Generalized Maxwellian exotic Bargmann gravity theory in three spacetime dimensions*, Phys. Lett. B **807** (2020) 135593 [arXiv:2004.01203] [INSPIRE].

[28] D.T. Son, *Toward an AdS/cold atoms correspondence: a geometric realization of the Schrödinger symmetry*, Phys. Rev. D **78** (2008) 046003 [arXiv:0804.3972] [INSPIRE].

[29] K. Balasubramanian and J. McGreevy, *Gravity duals for non-relativistic CFTs*, Phys. Rev. Lett. **101** (2008) 061601 [arXiv:0804.4053] [INSPIRE].

[30] S. Kachru, X. Liu and M. Mulligan, *Gravity duals of Lifshitz-like fixed points*, Phys. Rev. D **78** (2008) 106005 [arXiv:0808.1725] [INSPIRE].

[31] A. Bagchi and R. Gopakumar, *Galilean conformal algebras and AdS/CFT*, JHEP **07** (2009) 037 [arXiv:0902.1385] [INSPIRE].

JHEP02(2021)094

[32] A. Bagchi, R. Gopakumar, I. Mandal and A. Miwa, *GCA in 2d*, *JHEP* **08** (2010) 004 [arXiv:0912.1090] [INSPIRE].

[33] M.H. Christensen, J. Hartong, N.A. Obers and B. Rollier, *Torsional Newton-Cartan geometry and Lifshitz holography*, *Phys. Rev. D* **89** (2014) 061901 [arXiv:1311.4794] [INSPIRE].

[34] M.H. Christensen, J. Hartong, N.A. Obers and B. Rollier, *Boundary stress-energy tensor and Newton-Cartan geometry in Lifshitz holography*, *JHEP* **01** (2014) 057 [arXiv:1311.6471] [INSPIRE].

[35] J. Hartong, E. Kiritsis and N.A. Obers, *Lifshitz space-times for Schrödinger holography*, *Phys. Lett. B* **746** (2015) 318 [arXiv:1409.1519] [INSPIRE].

[36] J. Hartong, E. Kiritsis and N.A. Obers, *Schrödinger invariance from Lifshitz isometries in holography and field theory*, *Phys. Rev. D* **92** (2015) 066003 [arXiv:1409.1522] [INSPIRE].

[37] J. Hartong, E. Kiritsis and N.A. Obers, *Field theory on Newton-Cartan backgrounds and symmetries of the Lifshitz vacuum*, *JHEP* **08** (2015) 006 [arXiv:1502.00228] [INSPIRE].

[38] M. Taylor, *Lifshitz holography*, *Class. Quant. Grav.* **33** (2016) 033001 [arXiv:1512.03554] [INSPIRE].

[39] C. Hoyos and D.T. Son, *Hall viscosity and electromagnetic response*, *Phys. Rev. Lett.* **108** (2012) 066805 [arXiv:1109.2651] [INSPIRE].

[40] D.T. Son, *Newton-Cartan geometry and the quantum Hall effect*, arXiv:1306.0638 [INSPIRE].

[41] A.G. Abanov and A. Gromov, *Electromagnetic and gravitational responses of two-dimensional noninteracting electrons in a background magnetic field*, *Phys. Rev. B* **90** (2014) 014435 [arXiv:1401.3703] [INSPIRE].

[42] M. Geracie, K. Prabhu and M.M. Roberts, *Curved non-relativistic spacetimes, Newtonian gravitation and massive matter*, *J. Math. Phys.* **56** (2015) 103505 [arXiv:1503.02682] [INSPIRE].

[43] A. Gromov, K. Jensen and A.G. Abanov, *Boundary effective action for quantum Hall states*, *Phys. Rev. Lett.* **116** (2016) 126802 [arXiv:1506.07171] [INSPIRE].

[44] D.R. Grigore, *The projective unitary irreducible representations of the Galilei group in* $(1+2)$*-dimensions*, *J. Math. Phys.* **37** (1996) 460 [hep-th/9312048] [INSPIRE].

[45] S.K. Bose, *The Galilean group in* $(2+1)$ *space-times and its central extension*, *Commun. Math. Phys.* **169** (1995) 385 [INSPIRE].

[46] C. Duval and P.A. Horvathy, *The 'Peierls substitution' and the exotic Galilei group*, *Phys. Lett. B* **479** (2000) 284 [hep-th/0002233] [INSPIRE].

[47] R. Jackiw and V.P. Nair, *Anyon spin and the exotic central extension of the planar Galilei group*, *Phys. Lett. B* **480** (2000) 237 [hep-th/0003130] [INSPIRE].

[48] G. Papageorgiou and B.J. Schroers, *A Chern-Simons approach to Galilean quantum gravity in* $2+1$ *dimensions*, *JHEP* **11** (2009) 009 [arXiv:0907.2880] [INSPIRE].

[49] A. Achucarro and P.K. Townsend, *A Chern-Simons action for three-dimensional Anti-de Sitter supergravity theories*, *Phys. Lett. B* **180** (1986) 89 [INSPIRE].

[50] E. Witten, $(2+1)$*-dimensional gravity as an exactly soluble system*, *Nucl. Phys. B* **311** (1988) 46 [INSPIRE].

JHEP02(2021)094

[51] J. Zanelli, *Lecture notes on Chern-Simons (super-)gravities. Second edition (February 2008)*, hep-th/0502193 [INSPIRE].

[52] J.A. de Azcarraga, J.M. Izquierdo, M. Picón and O. Varela, *Generating Lie and gauge free differential (super)algebras by expanding Maurer-Cartan forms and Chern-Simons supergravity*, Nucl. Phys. B **662** (2003) 185 [hep-th/0212347] [INSPIRE].

[53] F. Izaurieta, E. Rodriguez and P. Salgado, *Expanding Lie (super)algebras through Abelian semigroups*, J. Math. Phys. **47** (2006) 123512 [hep-th/0606215] [INSPIRE].

[54] M. Hatsuda and M. Sakaguchi, *Wess-Zumino term for the AdS superstring and generalized Inonu-Wigner contraction*, Prog. Theor. Phys. **109** (2003) 853 [hep-th/0106114] [INSPIRE].

[55] J.A. de Azcarraga, J.M. Izquierdo, M. Picón and O. Varela, *Expansions of algebras and superalgebras and some applications*, Int. J. Theor. Phys. **46** (2007) 2738 [hep-th/0703017] [INSPIRE].

[56] R. Caroca, I. Kondrashuk, N. Merino and F. Nadal, *Bianchi spaces and their three-dimensional isometries as S-expansions of two-dimensional isometries*, J. Phys. A **46** (2013) 225201 [arXiv:1104.3541] [INSPIRE].

[57] L. Andrianopoli, N. Merino, F. Nadal and M. Trigiante, *General properties of the expansion methods of Lie algebras*, J. Phys. A **46** (2013) 365204 [arXiv:1308.4832] [INSPIRE].

[58] M. Artebani, R. Caroca, M.C. Ipinza, D.M. Peñafiel and P. Salgado, *Geometrical aspects of the Lie algebra S-expansion procedure*, J. Math. Phys. **57** (2016) 023516 [arXiv:1602.04525] [INSPIRE].

[59] M.C. Ipinza, F. Lingua, D.M. Peñafiel and L. Ravera, *An analytic method for S-expansion involving resonance and reduction*, Fortsch. Phys. **64** (2016) 854 [arXiv:1609.05042] [INSPIRE].

[60] C. Inostroza, I. Kondrashuk, N. Merino and F. Nadal, *A Java library to perform S-expansions of Lie algebras*, arXiv:1703.04036 [INSPIRE].

[61] C. Inostroza, I. Kondrashuk, N. Merino and F. Nadal, *On the algorithm to find S-related Lie algebras*, J. Phys. Conf. Ser. **1085** (2018) 052011 [arXiv:1802.05765] [INSPIRE].

[62] E. Bergshoeff, J.M. Izquierdo, T. Ortín and L. Romano, *Lie algebra expansions and actions for non-relativistic gravity*, JHEP **08** (2019) 048 [arXiv:1904.08304] [INSPIRE].

[63] L. Romano, *Non-relativistic four dimensional p-brane supersymmetric theories and Lie algebra expansion*, arXiv:1906.08220 [INSPIRE].

[64] A. Fontanella and L. Romano, *Lie algebra expansion and integrability in superstring σ-models*, JHEP **07** (2020) 083 [arXiv:2005.01736] [INSPIRE].

[65] D.M. Peñafiel and P. Salgado-ReboLledó, *Non-relativistic symmetries in three space-time dimensions and the Nappi-Witten algebra*, Phys. Lett. B **798** (2019) 135005 [arXiv:1906.02161] [INSPIRE].

[66] J. Gomis, A. Kleinschmidt, J. Palmkvist and P. Salgado-ReboLledó, *Newton-Hooke/Carrollian expansions of (A)dS and Chern-Simons gravity*, JHEP **02** (2020) 009 [arXiv:1912.07564] [INSPIRE].

[67] E. Bergshoeff, J. Gomis and P. Salgado-ReboLledó, *Non-relativistic limits and three-dimensional coadjoint Poincaré gravity*, Proc. Roy. Soc. Lond. A **476** (2020) 20200106 [arXiv:2001.11790] [INSPIRE].

[68] P. Concha, L. Ravera, E. Rodríguez and G. Rubio, *Three-dimensional Maxwellian extended Newtonian gravity and flat limit*, JHEP **10** (2020) 181 [arXiv:2006.13128] [INSPIRE].

[69] F. Izaurieta, E. Rodriguez, P. Minning, P. Salgado and A. Perez, *Standard general relativity from Chern-Simons gravity*, Phys. Lett. B **678** (2009) 213 [arXiv:0905.2187] [INSPIRE].

[70] J. Diaz, O. Fierro, F. Izaurieta, N. Merino, E. Rodriguez, P. Salgado et al., *A generalized action for* $(2+1)$*-dimensional Chern-Simons gravity*, J. Phys. A **45** (2012) 255207 [arXiv:1311.2215] [INSPIRE].

[71] P.K. Concha, D.M. Peñafiel, E.K. Rodríguez and P. Salgado, *Even-dimensional general relativity from Born-Infeld gravity*, Phys. Lett. B **725** (2013) 419 [arXiv:1309.0062] [INSPIRE].

[72] P. Salgado and S. Salgado, $\mathfrak{so}(D-1,1)\otimes\mathfrak{so}(D-1,2)$ *algebras and gravity*, Phys. Lett. B **728** (2014) 5 [INSPIRE].

[73] R. Caroca, P. Concha, O. Fierro, E. Rodríguez and P. Salgado-ReboLledó, *Generalized Chern-Simons higher-spin gravity theories in three dimensions*, Nucl. Phys. B **934** (2018) 240 [arXiv:1712.09975] [INSPIRE].

[74] F. Izaurieta, E. Rodriguez and P. Salgado, *Eleven-dimensional gauge theory for the M algebra as an Abelian semigroup expansion of* $osp(32|1)$, Eur. Phys. J. C **54** (2008) 675 [hep-th/0606225] [INSPIRE].

[75] O. Fierro, F. Izaurieta, P. Salgado and O. Valdivia, *Minimal AdS-Lorentz supergravity in three-dimensions*, Phys. Lett. B **788** (2019) 198 [arXiv:1401.3697] [INSPIRE].

[76] P.K. Concha and E.K. Rodríguez, $N=1$ *supergravity and Maxwell superalgebras*, JHEP **09** (2014) 090 [arXiv:1407.4635] [INSPIRE].

[77] P.K. Concha, O. Fierro and E.K. Rodríguez, *Inönü-Wigner contraction and* $D=2+1$ *supergravity*, Eur. Phys. J. C **77** (2017) 48 [arXiv:1611.05018] [INSPIRE].

[78] A. Banaudi and L. Ravera, *Generalized AdS-Lorentz deformed supergravity on a manifold with boundary*, Eur. Phys. J. Plus **133** (2018) 514 [arXiv:1803.08738] [INSPIRE].

[79] P. Concha, D.M. Peñafiel and E. Rodríguez, *On the Maxwell supergravity and flat limit in* $2+1$ *dimensions*, Phys. Lett. B **785** (2018) 247 [arXiv:1807.00194] [INSPIRE].

[80] R. Caroca, P. Concha, E. Rodríguez and P. Salgado-ReboLledó, *Generalizing the* bms_3 *and 2D-conformal algebras by expanding the Virasoro algebra*, Eur. Phys. J. C **78** (2018) 262 [arXiv:1707.07209] [INSPIRE].

[81] R. Caroca, P. Concha, O. Fierro and E. Rodríguez, *Three-dimensional Poincaré supergravity and* N*-extended supersymmetric* BMS_3 *algebra*, Phys. Lett. B **792** (2019) 93 [arXiv:1812.05065] [INSPIRE].

[82] R. Caroca, P. Concha, O. Fierro and E. Rodríguez, *On the supersymmetric extension of asymptotic symmetries in three spacetime dimensions*, Eur. Phys. J. C **80** (2020) 29 [arXiv:1908.09150] [INSPIRE].

[83] C.R. Nappi and E. Witten, *A WZW model based on a nonsemisimple group*, Phys. Rev. Lett. **71** (1993) 3751 [hep-th/9310112] [INSPIRE].

[84] J.M. Figueroa-O'Farrill and S. Stanciu, *More D-branes in the Nappi-Witten background*, JHEP **01** (2000) 024 [hep-th/9909164] [INSPIRE].

JHEP02(2021)094

[85] E. Inonu and E.P. Wigner, *On the contraction of groups and their represenations*, Proc. Nat. Acad. Sci. **39** (1953) 510 [INSPIRE].

[86] J.D. Edelstein, M. Hassaine, R. Troncoso and J. Zanelli, *Lie-algebra expansions, Chern-Simons theories and the Einstein-Hilbert Lagrangian*, Phys. Lett. B **640** (2006) 278 [hep-th/0605174] [INSPIRE].

[87] R. Schrader, *The Maxwell group and the quantum theory of particles in classical homogeneous electromagnetic fields*, Fortsch. Phys. **20** (1972) 701 [INSPIRE].

[88] H. Bacry, P. Combe and J.L. Richard, *Group-theoretical analysis of elementary particles in an external electromagnetic field. 1. The relativistic particle in a constant and uniform field*, Nuovo Cim. A **67** (1970) 267 [INSPIRE].

[89] J. Gomis and A. Kleinschmidt, *On free Lie algebras and particles in electro-magnetic fields*, JHEP **07** (2017) 085 [arXiv:1705.05854] [INSPIRE].

[90] P.K. Concha and E.K. Rodríguez, *Maxwell superalgebras and Abelian semigroup expansion*, Nucl. Phys. B **886** (2014) 1128 [arXiv:1405.1334] [INSPIRE].

[91] R. Aldrovandi, A.L. Barbosa, L.C.B. Crispino and J.G. Pereira, *Non-relativistic spacetimes with cosmological constant*, Class. Quant. Grav. **16** (1999) 495 [gr-qc/9801100] [INSPIRE].

[92] G.W. Gibbons and C.E. Patricot, *Newton-Hooke space-times, Hpp waves and the cosmological constant*, Class. Quant. Grav. **20** (2003) 5225 [hep-th/0308200] [INSPIRE].

[93] J. Brugues, J. Gomis and K. Kamimura, *Newton-Hooke algebras, non-relativistic branes and generalized pp-wave metrics*, Phys. Rev. D **73** (2006) 085011 [hep-th/0603023] [INSPIRE].

[94] P.D. Alvarez, J. Gomis, K. Kamimura and M.S. Plyushchay, $(2+1)D$ *exotic Newton-Hooke symmetry, duality and projective phase*, Annals Phys. **322** (2007) 1556 [hep-th/0702014] [INSPIRE].

[95] G. Papageorgiou and B.J. Schroers, *Galilean quantum gravity with cosmological constant and the extended q-Heisenberg algebra*, JHEP **11** (2010) 020 [arXiv:1008.0279] [INSPIRE].

[96] C. Duval and P. Horvathy, *Conformal Galilei groups, Veronese curves, and Newton-Hooke spacetimes*, J. Phys. A **44** (2011) 335203 [arXiv:1104.1502] [INSPIRE].

[97] J. Hartong, Y. Lei and N.A. Obers, *Nonrelativistic Chern-Simons theories and three-dimensional Hořava-Lifshitz gravity*, Phys. Rev. D **94** (2016) 065027 [arXiv:1604.08054] [INSPIRE].

[98] C. Duval, G. Gibbons and P. Horvathy, *Conformal and projective symmetries in Newtonian cosmology*, J. Geom. Phys. **112** (2017) 197 [arXiv:1605.00231] [INSPIRE].

[99] P.S. Howe, J.M. Izquierdo, G. Papadopoulos and P.K. Townsend, *New supergravities with central charges and Killing spinors in* $(2+1)$*-dimensions*, Nucl. Phys. B **467** (1996) 183 [hep-th/9505032] [INSPIRE].

[100] A. Giacomini, R. Troncoso and S. Willison, *Three-dimensional supergravity reloaded*, Class. Quant. Grav. **24** (2007) 2845 [hep-th/0610077] [INSPIRE].

[101] R. Troncoso and J. Zanelli, *Higher dimensional gravity, propagating torsion and AdS gauge invariance*, Class. Quant. Grav. **17** (2000) 4451 [hep-th/9907109] [INSPIRE].

[102] P.K. Concha, D.M. Peñafiel, E.K. Rodríguez and P. Salgado, *Generalized Poincaré algebras and Lovelock-Cartan gravity theory*, Phys. Lett. B **742** (2015) 310 [arXiv:1405.7078] [INSPIRE].

JHEP02(2021)094

[103] P.K. Concha, R. Durka, N. Merino and E.K. Rodríguez, *New family of Maxwell like algebras*, *Phys. Lett. B* **759** (2016) 507 [arXiv:1601.06443] [INSPIRE].

[104] H.R. Afshar, E.A. Bergshoeff, A. Mehra, P. Parekh and B. Rollier, *A Schrödinger approach to Newton-Cartan and Hořava-Lifshitz gravities*, *JHEP* **04** (2016) 145 [arXiv:1512.06277] [INSPIRE].

[105] V. Pestun, *Localization of gauge theory on a four-sphere and supersymmetric Wilson loops*, *Commun. Math. Phys.* **313** (2012) 71 [arXiv:0712.2824] [INSPIRE].

[106] G. Festuccia and N. Seiberg, *Rigid supersymmetric theories in curved superspace*, *JHEP* **06** (2011) 114 [arXiv:1105.0689] [INSPIRE].

[107] P. Concha, L. Ravera and E. Rodríguez, *Three-dimensional exotic Newtonian gravity with cosmological constant*, *Phys. Lett. B* **804** (2020) 135392 [arXiv:1912.02836] [INSPIRE].

[108] D. Hansen, J. Hartong and N.A. Obers, *Action principle for Newtonian gravity*, *Phys. Rev. Lett.* **122** (2019) 061106 [arXiv:1807.04765] [INSPIRE].

[109] L. Ravera, *AdS Carroll Chern-Simons supergravity in* $2+1$ *dimensions and its flat limit*, *Phys. Lett. B* **795** (2019) 331 [arXiv:1905.00766] [INSPIRE].

[110] F. Ali and L. Ravera, $\mathcal{N}$*-extended Chern-Simons Carrollian supergravities in* $2+1$ *spacetime dimensions*, *JHEP* **02** (2020) 128 [arXiv:1912.04172] [INSPIRE].

[111] L. Ciambelli, C. Marteau, A.C. Petkou, P.M. Petropoulos and K. Siampos, *Covariant Galilean versus Carrollian hydrodynamics from relativistic fluids*, *Class. Quant. Grav.* **35** (2018) 165001 [arXiv:1802.05286] [INSPIRE].

[112] L. Ciambelli, C. Marteau, A.C. Petkou, P.M. Petropoulos and K. Siampos, *Flat holography and Carrollian fluids*, *JHEP* **07** (2018) 165 [arXiv:1802.06809] [INSPIRE].

[113] L. Ciambelli and C. Marteau, *Carrollian conservation laws and Ricci-flat gravity*, *Class. Quant. Grav.* **36** (2019) 085004 [arXiv:1810.11037] [INSPIRE].

[114] A. Campoleoni, L. Ciambelli, C. Marteau, P.M. Petropoulos and K. Siampos, *Two-dimensional fluids and their holographic duals*, *Nucl. Phys. B* **946** (2019) 114692 [arXiv:1812.04019] [INSPIRE].