Who needs refined structural theories?

Who needs refined structural theories?

E. Carrera[1], I. Elishakoff[2], and M. Petrolo[1]

[1]MUL^2 Group, Department of Mechanical and Aerospace Engineering, Politecnico di Torino, Corso Duca degli Abruzzi 24, 10129 Torino, Italy
[2]Department of Ocean and Mechanical Engineering, Florida Atlantic University, Boca Raton, FL 33431-0991, USA

Submitted to Composite Structures

Corresponding author:
Marco Petrolo
MUL^2 Group, Department of Mechanical and Aerospace Engineering,
Politecnico di Torino,
Corso Duca degli Abruzzi 24,
10129 Torino, Italy,
tel: +39 011 090 6845,
fax: +39 011 090 6899,
e-mail: marco.petrolo@polito.it

Abstract

This paper discusses the question posed in the title and available options for the structural analysis of metallic and composite structures concerning the choice of 1D, 2D, and 3D theories. The focus is on the proper modeling of various types of mechanical behaviors and the associated solution's efficiency. The necessity and convenience of developing higher-order structural theories are discussed as compared to 3D models. Multiple problems are considered, including linear and nonlinear analyses and static and dynamic settings. Some possible guidelines on the proper selection of a model are outlined, and quantitative estimations on the accuracy are provided. It is demonstrated that the possibility of incorporating higher-order effects in 1D and 2D models continues to remain attractive in many structural engineering problems to alleviate the computational burdens of 3D analyses.

1 Introduction

This paper attempts to answer an often asked question posed in the title. It argues that the refined theories are not a mere intellectual exercise but stem from the very necessity of engineering practice. This section has two parts. First, a brief historical note concerning beam, plate, and shell models is given. Then, the primary methodologies to construct structural theories are presented together with this paper's aims and organization.

1.1 A brief history

The simplest dynamical refined theories were introduced by several investigators of the nineteenth century, namely by Jacques Antoine Charles Bresse in 1859 [1], followed by John William Strutt (Lord Rayleigh) in 1877 [2]. Static refined theories were suggested by Jean Victor Poncelet [3–5], William John Macquorn Rankine [6–9], Franz Grashof [10–12], Friedrich Engesser [13–22], August Föppl [23, 24], and, in the twentieth century, by Stephen Timoshenko [25–27] and Paul Ehrenfest. For Ehrenfest's contribution one has to consult references to the cooperation, made by Timoshenko [25, 27] and Grigolyuk [28]. The dynamic equations of beams incorporating the shear deformation and rotary inertia, as we know them today, were written by Timoshenko and Ehrenfest, and ought to be referred as Timoshenko-Ehrenfest equations (see Elishakoff [29, 30]). Alternatively, following Reddy et al. [31, 32], these could be referred as first-order shear deformation theory, or (FSDT).

As far as plates are concerned, refined theories started with Basset [33], Reissner [34–36], Hencky [37], Bolle [38], and Hildebrand et al. [39]. The dynamic equations of shear deformable plates with rotary inertia were first written by Uflyand [40] via the dynamic equilibrium method just as Timoshenko [25–27] with Ehrenfest did for beams. Later on, Mindlin [41] derived the same equations with the aid of Hamilton's principle. The plate equations incorporating the shear deformation and rotary inertia as we know them today were developed by Uflyand and Mindlin and hence ought to be called as Uflyend-Mindlin equations [29]. An equally valid name would be first order shear deformation theory [31, 32].

Reviews on shear deformable theories were written by Galinsh [42], Grigolyuk and Selezov[43], Reddy [44], Kapania and Raciti [45], Kant [46], Ghugal and Shimpi [47], Ambartsumian [48], Carrera [49], Reddy and Arciniega [50], and Annin and Volchkov [51]. Zig-zag theories were reviewed by Carrera [52], whereas buckling of moderately thick shells was reviewed by Simitses [53].

1.2 Refinement methodologies and aims of this paper

The two-dimensional (2D) and 1D structural theories constitute a powerful tool for analyzing and designing many engineering structures. The accuracy of structural theories depends on the problem in hand [54] as the reduction of the model from 3D to 2D or 1D introduces constraints concerning the mechanical behavior, e.g., neglect of shear deformations. A common modern solution strategy for engineering problems is the finite element method (FEM). The computational cost of a FEM model depends on the number of nodes and the nodal unknowns. The latter depends on the structural theory that is adopted. Finite elements in commercial codes rely on the classical theories of structures [35, 41, 55–58] with a maximum number of six nodal degrees of freedom (DOF), namely, three displacements and rotations. Classical theories consider unstretchable normals and may not be reliable to model the material's transverse mechanical characteristics and may fail at satisfying the boundary conditions on the external surfaces. The validity of such structural models is problem-dependent [48, 49]. To determine the applicability of classical models, one should consider several parameters [59, 60], e.g., the thickness ratio, gradients in the strain and stress fields, anisotropy, and/or inhomogeneity. The reliability of classical models increases when the structure is thin, there are no local effects, and the focus is on in-plane stress and transverse displacements. Conversely, caution is necessary whenever the focus is on edge-effects, local distortions, high frequency random oscillations, cracks, and contacts as transverse stresses and normal stretch become primarily important. Other examples of critical problems are those with multifield interactions such as thermal problems in which the material characteristics can change significantly and in an anisotropic manner.

Refined kinematics is necessary to overcome the limitations of classical models. A structural theory aims to reduce the 3D problem to 2D or 1D ones, preserve accuracy, and reduce computational overheads. In 2D, the approximation may be seen as a through-the-thickness integration, and, in 1D, it is an integral over the cross-section. The integration leads to unknowns dependent on the in-plane coordinates in the 2D case and the axial coordinate in the 1D case. Over the years, many approaches have emerged to refine structural theories, and a tentative classification may be as follows:

1. The method of hypotheses or axiomatic approach: the mechanical behavior is postulated and translated into mathematical constraints acting on the unknown variables, e.g., displacements, stresses, stress resultants. The classical models are the most known examples [35, 41, 58].

2. The method of expansions: the unknowns are defined via thickness or cross-section expansions. Usually, this method uses polynomials and leads to higher accuracy and computational costs as the number of expansion terms grows [61]. The choice of the expansion terms is critical, and the addition of new terms can be detrimental as soon as other equally important ones are not present [62, 63], and it is not always possible to prove the convergence to the exact solution. The expansion method is a generalization of the axiomatic one as the addition of terms to the expansion can remove the assumptions and widen the theory's applicability.

3. The asymptotic method [64, 65]: starting from the 3D equations, one or more characteristic parameters are identified, e.g., the thickness ratio, to build 2D or 1D expansions of the governing equations via that parameter and retain the terms up to a given order. This method provides a direct estimation of the accuracy of the solution compared to the 3D exact one, but it may be very cumbersome as soon as, for example, the expansion must consider many problem parameters simultaneously. The construction of approximate theories of beams, plates and shells via the asymptotic integration was considered by Goldenveiser et al. [66, 67], Ahalovyan [68, 69], Berdichevsky [70], Mekhtiev [71], and others.

The convenience in using a refined theory is not obvious given the availability of commercial software in which 3D FE can improve the accuracy of 2D and 1D in many cases. This paper aims to explore the capabilities of reduced models and enlist several structural problems to answer the following questions:

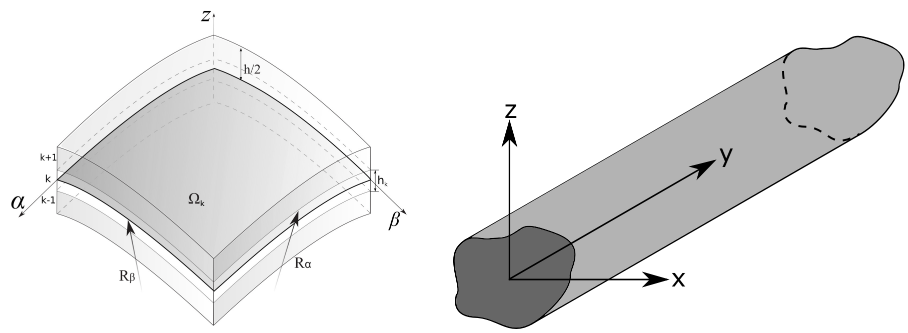
1. What are the problem features making classical 2D and 1D theories fail?

2. When are the refined 2D and 1D models more convenient than 3D?

3. When are 3D models unavoidable?

4. Are there any problems that 3D cannot handle but refined 2D or 1D do?

This paper starts with a brief overview of classes of refined theories in Section 2 and followed by examples of problems requiring either refined or 3D models, Section 3. Given that composite structures have features demanding peculiar refinement strategies, Sections 4 and 5 focus on theories and problems for composites. Section 6 attempts to provide guidelines to conduct a trade-off between accuracy and computational efficiency. Section 7 proposes a new approach to facilitate the evaluation of the accuracy and efficiency of refined models via machine learning. Some possible conclusions are then drawn in Section 8.



(a) 2D - Shell (b) 1D - Beam

Figure 1: Reference frames for 2D and 1D models

2 Brief overview on Higher-Order Structural Theories

This section aims to provide a brief overview of Higher-Order Structural Theories (HOST). For the sake of brevity, the focus is on 2D models, namely, plates and shells; in most cases, the assumptions used to develop such 2D models can be applied to 1D models as well. The introduction of shell theories will follow a standard notation in which the displacement field $\boldsymbol{u}$ has three components - u_α, u_β, and u_z - according to the reference system in Fig. 1 and using curvilinear coordinates. When referring to plate and beam models, the components are u_x, u_y, and u_z.

2.1 Classical models

As Zhavoronok [72] mentions, shell theories are constructed conventionally based on two approaches: a method of hypotheses and a ”regular” (or “hypotheses-free”) method. The first mentioned approach can be traced back to Kirchhoff and Love and the second one to Cauchy and Poisson. The hypothesis-based approach was efficiently used in most engineering methods of shell analysis, but to validate the shell theory, to analyze a complex stress state under local loadings, to model thick shells, to analyze high-frequency shell vibrations or transient dynamics, and for some other modern applications the “hypotheses-free” method is more useful. The “regular” methods of construction of the shell theories can be subdivided into two main groups: asymptotic methods using shell thickness as a small parameter and “formal” methods with no small parameters [42]. An approximate integration or a reduction technique of the 3D elasticity problem to the 2D problem is fundamental to all general shell theories.

The simplest theory is based on Kirchhoff-Love assumptions [55–58], and can be referred to as Love

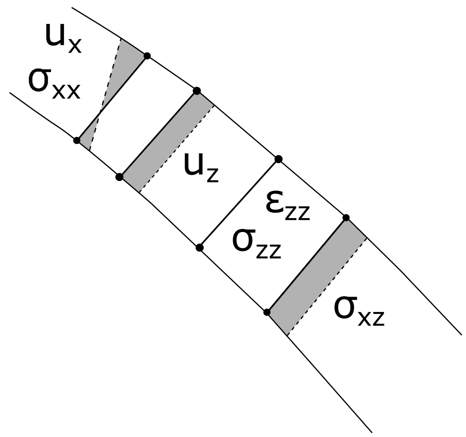


Figure 2: Through-the-thickness distributions via FSDT in isotropic structures; normal stretching and transverse stress are null

First Approximation Theory (LFAT) [73]. LFAT indicates models neglecting transverse shear stress and normal stretching. In laminated structures, the Kirchhoff-Love model coincides with the Classical Lamination Theory (CLT). The displacement field of CLT has 3 unknowns, namely, the displacements u_{α_1}, u_{β_1}, and u_{z_1},

$$\begin{aligned} u_\alpha &= u_{\alpha_1} - z\, u_{z_1,\alpha} \\ u_\beta &= u_{\beta_1} - z\, u_{z_1,\beta} \\ u_z &= u_{z_1} \end{aligned} \tag{1}$$

The derivatives of the transverse displacement have the physical meaning of rotations of the normals, i.e., $\phi_\alpha = -u_{z_1,\alpha}$ and $\phi_\beta = -u_{z_1,\beta}$.

The improvement of CLT by adding a constant shear distribution along the thickness leads to the First Order Shear Deformation Theory (FSDT) [35, 41],

$$\begin{aligned} u_\alpha &= u_{\alpha_1} + z\, u_{\alpha_2} \\ u_\beta &= u_{\beta_1} + z\, u_{\beta_2} \\ u_z &= u_{z_1} \end{aligned} \tag{2}$$

The FSDT has five unknowns and the additional two terms are rotations of the normals corrected by shear, i.e., $u_{\alpha_2} = \phi_\alpha = \epsilon_{\alpha z}-u_{z_1,\alpha}$. FSDT is commonly available in commercial codes due to its computational efficiency and reliability in many engineering applications. Following Kraus [73], the FSDT model falls in the Love Second Approximation Theory (LSAT) group. Figure 2 shows the typical distributions of displacement/strain/stress components provided by FSDT for a metallic structure. In-plane components can be linear, the transverse shear stress is constant, and the normal stretching and stress null. As shown in the next sections, there may be cases in which the latter hypotheses undermine the FSDT accuracy. The accuracy of FSDT for global responses is improved via shear correction factors [74]. Such factors are problem-dependent [75] as they depend on the distribution of shear stresses.

In this paper, CLT and FSDT are referred to as classical models or theories. In the case of 1D models, the same is valid for Euler-Bernoulli and Timoshenko-Ehrenfest models [27, 74, 76], and referred to as EBBT and TEBT, respectively. All other models going beyond the classical assumptions are referred to as Higher-Order Structural Theories (HOST).

2.2 Higher-order expansions

As Ambartsumian writes [48], the refined theories can be of special interest in shells and plates with localized singularities, such as inhomogeneous edges, lines of surface distortions, cracks, and contact zones. They can be extremely effective in contact problems. Here, along with transverse shears, the transverse normal stresses and transverse contractions are accounted for as well; this naturally satisfies the conditions of the contact surfaces. This is a case of special importance because the utilization of the classical theory for the contact problems of anisotropic shells and plates, routinely provided by many authors, is certainly incorrect. Moreover, the refined theories are very important for wave processes, oscillation problems, static and dynamic stability, especially when high-order oscillations and stability are concerned.

The enrichment of the displacement field is a common approach to remove assumptions from structural theory. According to Washizu [77], a 2D or 1D model based on an infinite expansion would guarantee the exact 3D solution. Truncated expansions are of practical interest, and the selection of the terms to add is not a trivial task.

Third-order expansions are among the most common, with Reddy [31] as one of the most important contributors. A third-order polynomial expansion, referred to as $N = 3$, has the following displacement field:

$$
\begin{aligned}
u_\alpha &= u_{\alpha_1} + z\, u_{\alpha_2} + z^2\, u_{\alpha_3} + z^3\, u_{\alpha_4} \\
u_\beta &= u_{\beta_1} + z\, u_{\beta_2} + z^2\, u_{\beta_3} + z^3\, u_{\beta_4} \\
u_z &= u_{z_1} + z\, u_{z_2} + z^2\, u_{z_3} + z^3\, u_{z_4}
\end{aligned}
\tag{3}
$$

In FE, such a model leads to twelve DOF per node. $N = 3$ improves FSDT by allowing parabolic distributions of transverse shear and thickness stretching. Many third-order models do not consider the thickness stretching, e.g.,

$$\begin{aligned} u_\alpha &= u_{\alpha_1} + z\, u_{\alpha_2} + z^2\, u_{\alpha_3} + z^3\, u_{\alpha_4} \\ u_\beta &= u_{\beta_1} + z\, u_{\beta_2} + z^2\, u_{\beta_3} + z^3\, u_{\beta_4} \\ u_z &= u_{z_1} \end{aligned} \tag{4}$$

Other examples of higher-order expansions enhance FSDT via parabolic transverse displacements, such as the following seven DOF model:

$$\begin{aligned} u_\alpha &= u_{\alpha_1} + z\, u_{\alpha_2} \\ u_\beta &= u_{\beta_1} + z\, u_{\beta_2} \\ u_z &= u_{z_1} + z\, u_{z_2} + z^2\, u_{z_3} \end{aligned} \tag{5}$$

The inclusion of the parabolic distribution is significant in many cases, such as a non-symmetric lamination.

Non-polynomial terms can be used to build expansions [78]. Such terms may be useful in the detection of more complex through-the-thickness distributions. Sinusoidal and exponential terms are typical choices, e.g.,

$$\begin{aligned} u_\alpha &= u_{\alpha_1} + z\, u_{\alpha_2} + sin(z\pi/h)\, u_{\alpha_3} + e^{z/h}\, u_{\alpha_4} \\ u_\beta &= u_{\beta_1} + z\, u_{\beta_2} + sin(z\pi/h)\, u_{\beta_3} + e^{z/h}\, u_{\beta_4} \\ u_z &= u_{z_1} + z\, u_{z_2} + sin(z\pi/h)\, u_{z_3} + e^{z/h}\, u_{z_4} \end{aligned} \tag{6}$$

Figure 3 shows typical distributions of various components. The use of HOST usually leads to improvements in the fulfillment of boundary conditions and full 3D fields. Higher-order theories of arbitrary order N were suggested by Zhavoronok [79], Egorova et al. [80], and Hadij et al. [81].

2.3 Carrera's unified formulation (CUF)

CUF emerged as a generalized approach to generate any expansions for structural theories [49, 82]. The CUF displacement field for a 2D model is

$$\mathbf{u}(\alpha, \beta, z) = F_\tau(z)\mathbf{u}_\tau(\alpha, \beta) \qquad \tau = 1, \ldots, M \tag{7}$$

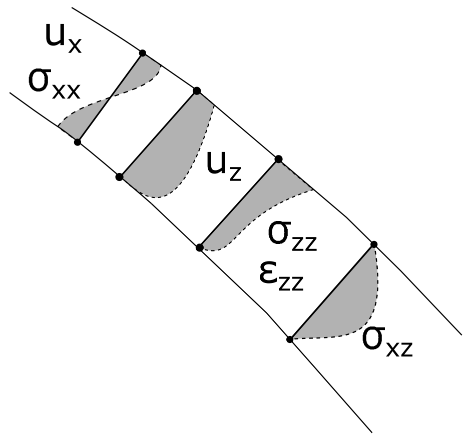


Figure 3: Through-the-thickness distributions via HOST in isotropic structures; the shape of the distributions depends on the expansion and its order

The Einstein notation acts on τ. F_τ are the thickness expansion functions. $\boldsymbol{u}_\tau$ is the vector of the generalized unknown displacements. M is the number of expansion terms. In the case of polynomial, Taylor-like expansions, a fourth-order model, referred to as $N = 4$, has the following displacement field:

$$
\begin{aligned}
u_\alpha &= u_{\alpha_1} + z\,u_{\alpha_2} + z^2\,u_{\alpha_3} + z^3\,u_{\alpha_4} + z^4\,u_{\alpha_5} \\
u_\beta &= u_{\beta_1} + z\,u_{\beta_2} + z^2\,u_{\beta_3} + z^3\,u_{\beta_4} + z^4\,u_{\beta_5} \\
u_z &= u_{z_1} + z\,u_{z_2} + z^2\,u_{z_3} + z^3\,u_{z_4} + z^4\,u_{z_5}
\end{aligned}
\tag{8}
$$

$N = 4$ has fifteen nodal DOF. The order and type of expansion is a free parameter.

2.4 Asymptotic approaches

Asymptotic methods define a characteristic parameter of the structure and problem [64, 65, 83]. A typical parameter is the ratio between the thickness and the in-plane dimension, i.e., ϵ = h/L. The displacement field, or any other unknown field, is introduced as an infinite expansion, such as

$$\boldsymbol{u} = \boldsymbol{u}_1 + z\,\boldsymbol{u}_2 + z^2\,\boldsymbol{u}_3 + ... \tag{9}$$

The displacement field is inserted in the governing equations in which same-order terms are isolated,

$$\boldsymbol{E} = \boldsymbol{E}_1 + \epsilon\,\boldsymbol{E}_2 + \epsilon^2\,\boldsymbol{E}_3 + ... \tag{10}$$

where $\boldsymbol{E}$ is the set of governing equations. The expansion is truncated to a given order, and the solution proceeds via the minimization of the strain energy. The main advantage of this approach is the possibility to control the accuracy of the solution against the 3D one.

3 Examples of problems in which the 3D stress field is necessary

This section presents a few cases in which the displacement or stress components usually neglected by classical theories are of primary importance for deriving meaningful results. Depending on the case, references are made to 1D or 2D structural theories, as the conclusions are usually valid for both cases.

3.1 Thermo-mechanical problems

By following the approach proposed in [84], let us consider a 3D temperature distribution, $T(x, y, z)$, over a 2D-plate domain, i.e.,

$$T(x, y, z) = T_p(z)\ T_\Omega(x, y) \tag{11}$$

T_p is the temperature profile along with the plate thickness, T_Ω is the temperature distribution over the reference surface. T_p may assume various distributions, the simplest being constant over the thickness, $T_p = T_0$. In that case, considering an isotropic plate, the normal strains induced by the heat distribution are the following:

$$\epsilon_{xxT} = \epsilon_{yyT} = \epsilon_{zzT} = \alpha\ T_0 \tag{12}$$

α is the coefficient of thermal expansion, and the subscript T indicates the strains induced by the thermal load, i.e., for the sake of simplicity, the other sources of strains are not considered. According to Eq. 12, the three components of strains have the same order of magnitude. Via the linear geometrical relations,

$$\epsilon_{zzT} = u_{zT,z} \tag{13}$$

$u_{zT,z}$ is the partial derivative of the transverse displcement with respect to z. By combining Eqs. 12 and 13 and integrating,

$$u_{zT} = u_{z1} + \alpha\ T_0\ z \tag{14}$$

In other words, a plate theory with at least a linear transverse displacement field is necessary to capture the normal transverse strain caused by the constant distribution of temperature across the thickness. CLT and FSDT are inadequate for this purpose. With a similar procedure, the linear distribution of temperature leads to a quadratic distribution of transverse displacement. In the most general case, T_P stems from a heat-conduction problem and, in the case of thick multilayered structures, may assume a layer-wise nonlinear distribution and require higher-order layer-wise displacement fields [85, 86].
Similar conclusions hold for 1D cases in which the displacement field over x and z should be at least linear. In general, the components of displacements usually constant in classical theories must be refined for the consistent inclusion of improved descriptions of temperature profiles in the transverse direction. Furthermore, as shown in [84], thin or thick structures equally require such refinements.

3.2 Electro-mechanical and electro-thermo-mechanical problems

The electro-mechanical modeling of intelligent structures embedding piezo-layers requires appropriate mechanical and electrical fields in the layers to carry out reliable simulations of direct and inverse effect and prevent failure mechanisms. The mechanical and electrical properties of multilayered piezoelectric structures change in the thickness direction. Furthermore, anisotropic multilayered composites usually have higher transverse shear and normal flexibilities than traditional homogenous isotropic ones [87]. In this section, issues related to composites' modeling are not considered as the focus is on the electro-mechanical features. The proper modeling of the electrical stiffness requires, at least, a linear distribution of the electrical field along with the thickness, leading to a parabolic electric potential. Numerical assessments on typical plate benchmarks have shown that natural frequencies from a model with equivalent single layer (ESL) capabilities and moderate expansion orders are sufficiently accurate. On the other hand, the analysis of local responses with coupling between the electrical and mechanical systems requires a layer-wise (LW) description of the displacement or, at least, the separated modeling of the piezoelectric and the structural layers [87]. If the focus is on accurate transverse electrical displacements and electrical charges for the sensor case, the modeling approach's minimum requirements are more severe. The direct, a-priori modeling of the electrical displacement via approaches such as the Reissner Mixed Variational Theorem (RMVT) is necessary [88]. Similar remarks hold for electro-thermo-mechanical analyses [89].

3.3 Local effects

Local effects may stem from various sources, such as geometrical boundary conditions, loadings, contact, and impact problems. Local effects are usually of 3D nature, as all three displacement and six stress components may be equally significant. A class of problems in which local effects are particularly important involves contact, e.g., rectangular blocks subjected to indentations, three-point bending tests, and contact between beams [90]. Features leading to more difficult modelings are thin walls and the material nonlinearities, such as plasticity. The latter implies the use of nonlinear solvers with potentially high computational costs. Higher-order 2D or 1D models can attenuate the computational overhead but must provide the proper modeling of non-classical effects such as transverse deformability and warping.

Another example of local effects is elastoplastic analysis. For instance, in the case of beams, local plasticity over the cross-section requires the use of refined models with higher than linear distributions of axial stress [91].

3.4 Deterministic and random vibration problems

Structural dynamics problems include various applications in which the 3D deformability of the body is essential. Examples are vibrations of thin-walled structures, rotordynamics, viscoelasticity, and wave propagation problems.

The free vibrations characteristics of thin-walled structures and their dynamic response require the modeling of the cross-sectional deformability. The classical approach exploits the use of a 2D plate or shell model. However, 1D refined models can succeed as well and with lower computational costs [92].

Samuels and Eringen [93] were the first authors to study random vibrations via TEBT under "rain-under-roof" excitation and both time-wise and space-wise white noise. They concluded that the results produced by using either the TEBT or EBBT differed by less than 5 %. This finding perhaps discouraged further investigations in this field. It will be shown in the next sections that the effects of rotary inertia and shear deformation may have a profound impact on this class of problems. Crandall and Yildiz [94] carried out studies on random vibrations via TEBT with the focus on the effects of various damping mechanisms. They considered, as an excitation, the time-wise band-limited white noise and investigated the growth pattern of the response characteristics with the increase of the cutoff frequency. It should be stressed that the above studies concentrated on the excitation with a constant spectral density. As a result, only the low end of the frequency spectrum contributes significantly to the response within the modal analysis approach. However, the shear deformation and rotary inertia only considerably affect the higher frequencies. Because the contribution of jth mode and its oscillatory frequency (ω_j) is inversely proportional to ω_j^4 for the viscously damped beam, the significantly affected frequencies contributed very little to the response. This is the reason for a small numerical difference between EBBT and TEBT. Elishakoff and Lubliner [95] found different results in the general case of excitations. They considered the band-limited white noise excitation with two cutoff frequencies: the lower one, ω_{c1}, and the upper one, ω_{c2}. The spectral density was taken constant within the range. It was demonstrated that when ω_{c1} is zero or close to zero, both TEBT and EBBT produce similar results. However, when ω_{c1} increases, the higher modes become more important. Because the higher natural frequencies are reduced by the effects of shear deformation and rotary inertia, the response predicted by TEBT is in excess of that predicted by EBBT. In such circumstances, one must utilize TEBT. In several example cases, Elishakoff and Lubliner [95] demonstrated that the application of EBBT would yield an error on the order of 50% or more.

Rotors are the primary parts of many machines as shafts, disks, turbine blades, and propellers. Although in many cases the classical 1D models can provide reasonably reliable results, several features of rotors require the use of refined models or 3D approaches; e.g., short and stubby bodies in which rotary inertia and shear deformations are significant, the presence of complex shapes, unconventional boundary conditions, and centrifugal stiffening [96].

The viscoelastic components in structures are useful for damping purposes. The typical architecture consists of viscoelastic layers embedded on surfaces or constrained between stiffer components. The structural modeling of such structures has to consider their layer-wise properties while keeping the computational costs low as nonlinear solvers are necessary [97]. Given that damping treatments exploit shear deformations of the viscoelastic layer, an accurate prediction of the stress distribution is essential and gains importance as the layer's thickness increases.

The study of wave propagation has many applications, such as the localization of damages for structural health monitoring. A class of waves adopted for these purposes are referred to as Lamb waves and induce displacements both in the propagation direction and in the transverse one. The domain to be modeled is usually large, and fine discretizations are necessary to the short wavelengths, i.e., high computational costs may arise. Furthermore, thin-walled structures with complex geometries are usually considered [98].

The conclusions of this brief overview of structural dynamics problems can be extended to composite structures by mentioning other effects. Examples are the coupling introduced by lamination schemes of the transverse deformability of orthotropic materials. [99, 100].

3.5 Thickness, shear and membrane locking

Thickness, shear, and membrane lockings may arise while using structural models and can be due or worsened by lower-order models. Thickness locking is a consequence of plane-strain/plane-stress hypotheses in plate and beam theories, and its magnitude depends on the thickness of the structure. The locking arises for contradictory hypotheses concerning transverse axial stress and strain and prevents 3D solutions as the thickness decreases. Thickness locking is present whenever the normal transverse strain is constant. In other words, the direct approach to avoid it is the use of, at least, parabolic distribution of transverse displacements [101].

The use of the FE approach for the numerical solution can lead to other locking mechanisms, i.e., shear and membrane locking. Locking stems from the overestimation of stiffness in thin structures and leads to poor convergence rates and may require very refined meshes. Shear locking is typical in the bending of a thin structure due to the difficulties in modeling the bending deformation appropriately with the strain energy erroneously absorbed by the shear modes. In shell elements, membrane locking can arise as well as the bending deformation leads - incorrectly - to the stretching of the mid-surface [102]. Common strategies to alleviate locking are based on modified integration and interpolation schemes. The use of higher-order models reduces the magnitude of the locking independently of the use of other alleviating measures.

3.6 Discussion

This section aims to provide some final remarks regarding the role of HOST in various problems through the prism of numerical examples. Table 1 shows the minimum requirements for the order of various variables for some problems involving metallic structures. The thermo-mechanical analysis, as stated above, requires, at least, a linear distribution of the transverse displacement over the thickness in the case of constant temperature profiles. In electro-mechanical analyses, one of the most critical variables is the electric potential and good accuracy requires quadratic expansions. The proper detection of the onset of plasticity over the thickness or cross-section demands quadratic distributions of axial stresses; similarly,

Table 1: Minimum requirements for structural theories for various applications involving isotropic structures

Problem	Variable	Requirement
Thermo-mechanical	Transverse displacement	Linear
Electro-mechanical	Electric potential	Quadratic
Elastoplaticity	Axial stress	Quadratic
Viscoelasticity	In-plane discplacements	Cubic
Thickness locking	Transverse displacement	Parabolic

the adequate modeling of viscoelastic effects depending on shear stresses requires cubic expansions of the in-plane displacements. The last row of Table 1 indicates the minimum requirement to avoid thickness locking without modified material coefficients. Parabolic distributions of transverse displacements are necessary. Table 2 shows the error on variables of interest for specific problems. The aim is to provide quantitative examples of the role of HOST. Depending on the case, different pairs of models were compared. Lower-order ones are either classical ones or theories with insufficiently refined expansions. On the other hand, higher-order models are those with the minimum expansion order to obtain acceptable accuracy. The error refers to either exact solutions or very accurate numerical models, e.g., refined 3D FE. The first problem considered is the thermo-mechanical analysis of a metallic plate with a linear temperature profile, and the maximum transverse displacement is reported. FSDT leads to some 37 % error whereas the use of a second-order expansion for all three displacement components leads to minor errors. In the elastoplastic analyses of a beam, two models are compared, and the plastic strain is considered. The lower-order one is a second-order beam model. The higher-order one is based on a Lagrange expansion of the displacement field over the cross-section- The Lagrange expansion uses nine-node (LE9) polynomials. Two structural dynamics problems are then considered. The first one involves the time response of a thin-walled cylinder under a local traveling load leading to severe cross-sectional deformations. TEBT is compared with a seventh-order model. The second problem concerns wave propagation in a plate, and two beam models were compared regarding the transverse displacement caused by waves. In all cases, the proper selection of the theory's order can lead to very accurate results. For a more comprehensive analysis, including the effect of the order on other variables, the reader can refer to the cited papers.

Elishakoff et al. [103] investigated the response of the space shuttle's weather protection system to random excitation. They neglected the fourth-order derivative of the displacement to time, appearing in the original Timoshenko-Ehrenfest equation, as discussed in [104]. The typical weather protection systems for launch vehicles are made of thin corrugated metal sheets, with a much larger bending stiffness

Table 2: Error of lower- and higher-order models for various applications involving isotropic structures

Problem	Variable	Lower-order	Higher-order
Thermo-mechanical, linear temperature profile, plate with a/h = 10 [86]	Transverse displacement	FSDT, 37 %	$N = 2$, 0.3 %
Elastoplastic, bending of a bi-metallic cantilever beam [91]	Plastic strain	$N = 2$, 30 %	LE9, 6 %
Dynamic response of a thin-walled cylinder [92]	Transverse displacement	TEBT, 80 %	$N = 7$, 5 %
Wave propagation in a thin plate [98]	Transverse displacement	$N = 1$, 30 %	$N = 4$, 0.1 %

perpendicular to the corrugation's direction. Thin beams support these sheets. Such a weather protection system is susceptible to excessive shear deformation, and, therefore, it is reasonable to model it via TEBT. They considered the cross-spectral densities of response for displacements and moments as defined in [105, 106]. For the displacements' spectral density, the difference of values of the first peak for EBBT and TEBT is minimal. Hence the value of areas under the spectral density curves, or the mean square values of displacements, are very close, with a difference of about 1.2 %. However, for the spectral density of moments, the higher-order modes dominate in formulating the structure's response. The difference between the values of spectral densities calculated via TEBT and EBBT and their associated contributions to the moment's mean square value is around 50 %.

4 HOST for laminated structures

4.1 Primary requirements for structural theories

The development of structural theories for composites originated from the already available models for metallic structures and incorporated mechanical features typical of commonly used layered structures. Composites have in-plane anisotropy, i.e., the properties of each layer can change significantly depending on directions. The anisotropy usually coexists with high transverse shear and normal flexibility and couplings between axial and shear strains. There may be one order of magnitude of difference between Young's moduli for different directions and two orders between Young's and shear moduli. Further couplings can arise between in-plane and out-of-plane strains, e.g., in the case of nonsymmetric stacking sequences.

Composites are transversely anisotropic as they have layer-wise discontinuous physical properties

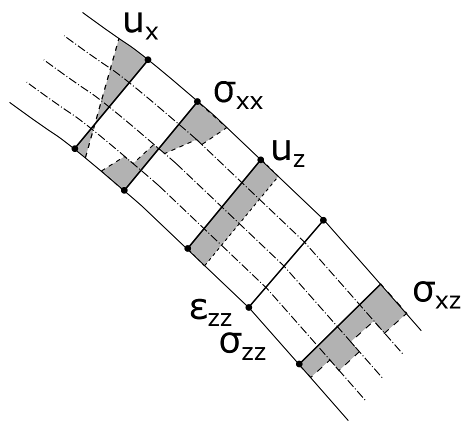


Figure 4: Through-the-thickness distributions via FSDT in composite structures; normal stretching and transverse stress are null

along with the thickness. Such a feature leads to compatibility and equilibrium conditions between layers challenging to model. The compatibility needs the discontinuity of the slopes of displacement components at the layer interfaces; such a requirement is referred to as the displacement field's zig-zag shape. Zig-zag must coexist with the equilibrium conditions leading to the continuity of the transverse stresses at the interfaces. The literature refers to the sum of the two conditions as to the C_0^z requirement.

The modeling challenges concerning composites can grow further in many scenarios, among others, the analysis of residual stress from the curing process, the analysis of damaged structures, the presence of singularities such as free-edges, the use of tens of layers, and the use of sandwich structures with soft cores. Reviews of laminated composite plate theories were furnished by Grigolyuk et al. [107, 108], Khandan et al. [109], and others.

4.2 Structural theories for laminates

The limitations of classical theories may grow further in the case of composite structures. Figure 4 shows some typical distributions via FSDT. As in the metallic case, in-plane displacement and stress are usually well-captured. On the other hand, the transverse shear stress fails to fulfill the top/bottom and interlaminar boundary conditions. The refinement techniques presented in previous sections remain valid for composites and may lead to improved results. Figure 5 shows examples of distributions via higher-order expansions and an equivalent single-layer approach (ESL). The inclusion of transverse deformability is decisive in many cases, but further refinements are necessary as, for instance, the interlaminar continuity of transverse stresses is not achieved. This section presents the most used techniques to improve structural

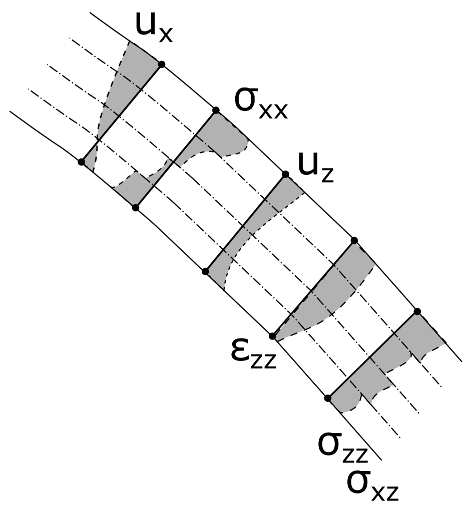


Figure 5: Through-the-thickness distributions via HOST in composite structures; the shape of the distributions depends on the expansion and its order

theories for composites: the zig-zag expansions, the layer-wise approach, and RMVT.

4.2.1 Zig-zag expansions

Zig-zag models are necessary to overcome the difficulties of higher-order expansions in fulfilling the C_z^0 requirements. The increase of the order is not enough to meet C_z^0, and transverse stresses distributions may be wrong if calculated via the constitutive law. Zig-zag theories can provide C_z^0. Zig-zag models were first proposed in the first half of the twentieth-century [110]. A complete review of zig-zag is available in [52] providing the main features of the three fundamental zig-zag approaches, namely, the Lekhnitskii [110], Ambartsumian [111], and Reissner-Murakami [112]. Other contributions are those by Rzhanitsyn [113–115], and Bolotin et al. [116, 117].

An example of a zig-zag model enhancing FSDT is the following:

$$
\begin{aligned}
u_\alpha &= u_{\alpha_1} + z\, u_{\alpha_2} + (-1)^k \tfrac{2z_k}{h_k}\, u_{\alpha Z} \\
u_\beta &= u_{\beta_1} + z\, u_{\beta_2} + (-1)^k \tfrac{2z_k}{h_k}\, u_{\beta Z} \\
u_z &= u_{z_1}
\end{aligned}
\tag{15}
$$

where $u_{\alpha Z}$ and $u_{\beta Z}$ are two additional variables, referred to as Murakami's zig-zag functions, and the amount of DOF is still independent of the number of layers. Similarly, the zig-zag terms can enhance higher-order theories, e.g., the $N = 3$ one,

$$
\begin{aligned}
u_\alpha &= u_{\alpha_1} + z\, u_{\alpha_2} + z^2\, u_{\alpha_3} + z^3\, u_{\alpha_4} + (-1)^k\, \tfrac{2z_k}{h_k}\, u_{\alpha Z} \\
u_\beta &= u_{\beta_1} + z\, u_{\beta_2} + z^2\, u_{\beta_3} + z^3\, u_{\beta_4} + (-1)^k\, \tfrac{2z_k}{h_k}\, u_{\beta Z} \\
u_z &= u_{z_1} + z\, u_{z_2} + z^2\, u_{z_3} + z^3\, u_{z_4} + (-1)^k\, \tfrac{2z_k}{h_k}\, u_{zZ}
\end{aligned}
\tag{16}
$$

4.2.2 Layer-wise approach

In an LW model, the unknown variables depend on the number of layers [118]. Such an approach leads to higher computational costs, but its adoption is necessary in many cases, e.g., the precise detection of stress in each layer, high-stress gradients along with the thickness, free-edge, and local effects. The use of a layer-wise approach leads to zig-zag distributions with the additional benefit of having independent layers. A layer-wise displacement field is the following:

$$
\begin{aligned}
u_\alpha^k &= L_1\, u_{\alpha_1}^k + L_2\, u_{\alpha_2}^k + \textstyle\sum_{i=3}^N\, L_i u_{\alpha_i}^k \\
u_\beta^k &= L_1\, u_{\beta_1}^k + L_2\, u_{\beta_2}^k + \textstyle\sum_{i=3}^N\, L_i u_{\beta_i}^k \\
u_z^k &= L_1\, u_{z_1}^k + L_2\, u_{z_2}^k + \textstyle\sum_{i=3}^N\, L_i u_{z_i}^k
\end{aligned}
\tag{17}
$$

Typical choices for the expansions are Lagrange or Legendre polynomials. Using Legendre polynomials as expansion functions along the thickness coordinate was suggested apparently by Vekua [119]. Interested readers can consult also works by Khoma [120], Ulukhanyan [121], and Zhavoronok [72, 122].

To facilitate the application of compatibility conditions at the interfaces, '1' and '2' variables should have the physical meaning of displacements at the top and bottom of the interface, respectively. The other additional terms serve as higher-order ones to enrich each layer's kinematics—usually, the use of third- or fourth-order terms guarantees quasi-3D accuracy. Figure 6 shows typical distributions via LW. Displacements are continuous, but their derivatives can change at the interface and leading to discontinuous strains. In turn, transverse stresses are very well detected with minor interlaminar discontinuities.

4.2.3 Reissner Mixed Variational Theorem

In mixed formulations, displacements and stresses are primary variables [77]. The RMVT has emerged as one of the most powerful tools for composites and has as primary variables the displacements and the transverse stresses [123, 124]. The use of RMVT is independent of the type of expansion adopted as

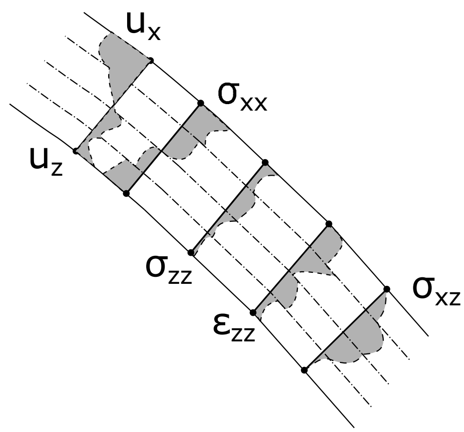


Figure 6: Through-the-thickness distributions via lw in composite structures; perfect continuity of transverse stress can be achieved via RMVT

Taylor expansions, zig-zag, layer-wise approaches, and combination thereof can be used. For instance, a layer-wise model can have the following distributions of displacements and transverse stresses:

$$\begin{aligned}
u_{\alpha}^{k} &= L_1\, u_{\alpha_1}^{k} + L_2\, u_{\alpha_2}^{k} + \textstyle\sum_{i=3}^{N} L_i u_{\alpha_i}^{k} \\
u_{\beta}^{k} &= L_1\, u_{\beta_1}^{k} + L_2\, u_{\beta_2}^{k} + \textstyle\sum_{i=3}^{N} L_i u_{\beta_i}^{k} \\
u_{z}^{k} &= L_1\, u_{z_1}^{k} + L_2\, u_{z_2}^{k} + \textstyle\sum_{i=3}^{N} L_i u_{z_i}^{k} \\
\sigma_{\alpha z}^{k} &= L_1\, \sigma_{\alpha z_1}^{k} + L_2\, \sigma_{\alpha z_2}^{k} + \textstyle\sum_{i=3}^{N} L_i \sigma_{\alpha z_i}^{k} \\
\sigma_{\beta z}^{k} &= L_1\, \sigma_{\beta z_1}^{k} + L_2\, \sigma_{\beta z_2}^{k} + \textstyle\sum_{i=3}^{N} L_i \sigma_{\beta z_i}^{k} \\
\sigma_{zz}^{k} &= L_1\, \sigma_{zz_1}^{k} + L_2\, \sigma_{zz_2}^{k} + \textstyle\sum_{i=3}^{N} L_i \sigma_{zz_i}^{k}
\end{aligned} \tag{18}$$

RMVT is convenient for multilayered structures as the interlaminar continuity of transverse stresses and zig-zag displacements are easily implementable. The post-processing for transverse stresses is not necessary. Numerical investigations proved that using an RMVT third-order zig-zag, equivalent single-layer model, can provide 3D-like accuracy in most cases except for severe local effects in which the layer-wise version is preferable [82].

5 Examples of problems on laminates in which the 3D stress field is necessary

5.1 Transverse normal and shear stress

As Koiter states, the accurate refinement of shell theories is meaningless unless the effects of shear and normal stresses are taken into account at the same time [125]. As indicated above, in composite structures, Koiter's recommendation is not enough as zigzag effects, and interlaminar continuity are additional requirements. In orthotropic materials, the intrinsic coupling between in-plane and out-of-plane normal stresses makes the a priori fulfillment of the σ_{zz} interlaminar continuity particularly difficult. A theory's accuracy depends on multiple factors, such as the thickness and orthotropic ratios, the number of layers, laminations – symmetric or not –, and the outputs, e.g., global ones such as natural frequencies, or local ones, such as through-the-thickness stress distributions.

Other complicating effects may lead to further requirements for the refined model. Accurate evaluations of the static and vibrational response of highly anisotropic, thick, and very thick shells require an LW description and mixed formulations [126, 127]. The use of mixed formulations, both ESL or LW, leads to the fulfillment of the interface continuity and, with at least second-order expansions for shear stress, homogenous conditions at the top and bottom surfaces. The use of unsymmetric laminates requires, at least, quadratic expansions [128]. The inclusion of σ_{zz} is essential to capture the unsymmetric distributions of stress along with the thickness in the case of thick structures.

A typical example of a 3D stress state in composites can be found in the free-edge regions. The Poisson effect and the various elastic moduli of the material over different plies let them behave differently in the in-plane direction. To satisfy the compatibility of the displacements at the interfaces, transverse stresses appear near the free edge, see Fig. 7. A correct evaluation of these stresses is necessary, for instance, to prevent damage mechanisms such as delamination [129]. Free-edge 3D stress fields have strong gradients - as the region is roughly equal to the total thickness of the laminate - and may lead to remarkably high peak values. The use of refined models can alleviate the computational overhead due to very refined 3D meshes over the thickness.

In another, earlier article, Ambartsumian [130] writes like the best version of the general theory of highly anisotropic shells is some symbiosis of some refined theory (and sometimes classical theory) together with

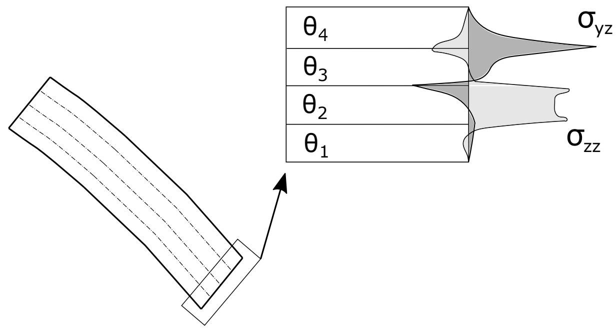


Figure 7: Free-edge effects in a composite structures

a theory for the edge on the level of the 3D theory of elasticity. In this case, the boundary conditions should be formulated from the perspective of 3D elasticity theory, which is not simple to do. There is no unified method of constructing three-dimensional mathematical models of realizable boundaries of shells in general. Comparisons with 3D theories were provided by Nigul [131] and numerous other authors.

Transverse stress can be computed via three approaches. First, stress components are evaluated using the constitutive laws. The second method exploits the indefinite equilibrium equations of 3D elasticity. In the 2D case, the in-plane stress components are integrated over the thickness to obtain the transverse ones. Mixed formulations lead to mixed cases as assumed transverse stress is directly available - a priori - as primary variables or can be evaluated - a posteriori, as in the previous cases - via constitutive laws or by integration [132].

The use of constitutive laws has the advantage of simplicity, but its accuracy is satisfactory in LW models with, at least, third-order expansions. The integration of the indefinite equilibrium equations provides good accuracy, but it may sometimes fail to fulfill boundary conditions. Furthermore, such a procedure may be inaccurate as the number of layers increases. LW mixed formulations provide the best results.

5.2 Local effects

As seen in previous sections, localized loadings require refined models. LW appears to be mandatory to model the localized stress field near the loading area. In the case of sandwich panels with soft cores, the need for refined models grows as the core deformability can be, at least, one order of magnitude higher than that of the faces, and the normal transverse deformation is necessary to model the sandwich core [133]. In the presence of point loads, fourth-order expansions and mixed formulations are necessary. 3D

FE may fail as massively refined meshes would be necessary to capture singularities due to point-loads.

Contact and impact problems are of high interest for composite structures as they can trigger various damage mechanisms [134]. Classical theories remain reliable to predict the global transverse displacement distribution but cannot model localized deformations that often occur in the contact region, causing 3D deformation and stress fields. LW is necessary for accurate through-the-thickness strain and stress distributions. Also, the use of 1D or 2D models can significantly reduce the computational costs as 3D FE may require multiple elements along the thickness to capture transverse distributions.

5.3 Damage and failure

Failure in composite structures is due to complex phenomena and depends on various features, such as material properties, geometries, stacking sequences, and fiber orientations. The onset of damage can be estimated via linear analyses and failure indexes. Among the others, a factor influencing the accuracy of failure indices is the quality of the structural model's strain or stress field. Also, the 3D stress field's availability, as compared to the 2D one from a classical plate or shell models, can be decisive for obtaining reliable results. In turn, the accuracy of 3D failure indices strongly depends on the accuracy of the transverse stresses [135], and such dependency grows in various cases, such as free-edges, matrix, and interface failures. The modeling of σ_{zz} peaks nearby free-edges in the presence of 90 layers is very demanding. Other mechanical behaviors requiring refined models are stress reversals observed along free surfaces, local effects on stress peaks due to stacking sequences, shear stress gradients in cores [136].

The propagation of damage after the onset of failure requires nonlinear analyses. To contrast the growth of computational costs without undermining accuracy, a common approach for the modeling of progressive damage in composites is a combination of continuum models for the intralaminar damage within the ply and discrete approaches - such as cohesive elements - for delamination [137]. As in the linear case, the use of higher-order terms and LW capabilities can be decisive in problems with highly localized stress peaks and gradients, e.g., edge effects, or with the presence of tens of layers. One of the principal advantages of refined models is the weaker size-dependency of the computational model than 3D FE or standard 2D models. In other words, the increase in computational costs due to the meshing of larger specimens is lower in higher-order models.

5.4 Micromechanics and multiscale

The microstructure of composite structures can be very complex, and its physical behaviors, e.g., failure and bridging the effect accurately to upper scales, can significantly enhance the accuracy of simulations. An efficient micromechanics model should provide accurate homogenized properties and detailed local fields with low computational costs. The computation of the local fields is particularly relevant for refined models as it requires the proper modeling of various constituents, e.g., fibers and matrix, with different material properties. The inclusion of shear deformability and transverse axial stress is essential to capture the behavior of representative volume elements [138]. The computational efficiency is decisive in the nonlinear regime [139], and multiscale frameworks [140, 141]. The use of refined models can lower the computational costs dramatically as current frameworks require 3D FE to model the complex behavior of the microstructure and limit the use of multiscale analyses to simple geometries.

5.5 Virtual manufacturing

The manufacturing of composite parts involves high pressure and temperature cycles inducing free-strains. The mismatch of free-strains at various scales and the evolution of mechanical properties may lead to residual stresses and dimensional changes in the manufactured part. Such a mismatch occurs at the microlevel between constituents, i.e., fiber and matrix, at the mesolevel between plies with different orientations, and the macrolevel between the part and the tool via friction and other geometrical constraints [142]. From the structural modeling standpoint, it is essential to consider the complex thermo-mechanical strain and stress state along with the thickness. Also, the changes in mechanical properties over the curing process require significantly different transverse deformability. The use of simplified models is acceptable to obtain global responses, such as the spring-in angle. Refined models are necessary for the proper computation of residual stresses or to tackle complex configurations with couplings.

5.6 Discussion

Some examples of minimum requirements for structural theories are in Table 3. The use of asymmetric lamination demands quadratic expansions for the transverse displacement. Many loading conditions, e.g., a distributed load over one face of a plate, may induce asymmetric stress distributions over the thickness. The proper detection of such behavior requires accurate transverse axial stress; usually, third-

Table 3: Minimum requirements for structural theories for various applications involving composite structures

Problem	Variable	Requirement
Asymmetric lamination	Transverse displacement	Quadratic
Asymmetric stress distributions	Transverse axial stress	Second-order
Local effects	Displacement field	Layer-wise fourth-order
Localization in micromechanics	Displacement field	Component-wise second-order

Table 4: Error of lower- and higher-order models for various applications involving composite structures

Problem	Variable	Lower-order	Higher-order
Thermo-mechanical, plate with a/h = 10 and 0/0/0 [85]	Shear stress	FSDT, 53 %	$N = 3$, 6 %
Electro-mechanical, plate with a/h = 50 and piezo/0/90/0/piezo [87]	First natural frequency	FSDT, 12 %	$N = 2$, 6 %
Localized load on a sandwich panel, a/h = 10[133]	Transverse displacement in the loaded area	$N = 1$, 12 %	$N = 4$, 5 %
Process-induced deformations [142]	Spring-in angle	$N = 2$, 5 %	LE9, 0.3 %

order displacement fields and LW are necessary. LW is mandatory to detect local effects, e.g., due to localized loads. In particular, fourth-order models are required. In the modeling of representative volume elements (RVE) for micromechanics applications, the accurate evaluation of the 3D stress field necessitates second-order Lagrange polynomials leading to component-wise models in which multiple second-order fields are assembled. Table 4 shows some examples concerning the accuracy of various models. As well-known, the computation of transverse stresses is among the most demanding requirements. The use of Lagrange-based models, e.g., LE9, leading to layer-wise descriptions of the displacement field remains a powerful method to incorporate multifield effects as in the analysis of process-induced deformations. The numerical values from this table offer a general overview of the capabilities of selected models. More comprehensive analyses of numerical results are available in the cited papers.

6 Problem requirements and capabilites of classical, HOST and 3D theories

This section aims to enlist structural models' primary requirements for a set of problems and considerations on accuracy and computational efficiency. Also, indications of the proper approaches to use are given. For the sake of brevity, FE solutions are considered; therefore, 1D, 2D, and 3D refer to beam, plate/shell, and solid elements, respectively. Furthermore, the opening subsection provides a brief historical overview on the debate concerning the opportunity of developing HOST.

6.1 Introduction to the debate between HOST and 3D

One of the authors of this paper (I. E.) recalls attending as an undergraduate student the All-Union Congress in Applied Mechanics held in Moscow, Russia in 1964. There were many illustrious Russian and Western scientists in the section discussing the current and future status of the beam, plate, and shell theories. Namely, Professors Aleksey Goldenveiser, Warner Koiter, Bernard Budiansky, Eric Reissner, and Ilya Vekua. During the panel discussion, one of the participants, Dr. Boris Kuranov, asked: "Why is there the need for theories of beams, plates, and shells, both classical and refined ones, when one can resort to 3D theories ab initio?" The answer was furnished by Koiter, in the spirit, that there was no need to use the tank to kill the mosquito, and we shall use the fact that usual shells are thin. When they are moderately thick, we can resort to simpler theories than 3D analysis. Vekua argued that even in an elementary problem of the Laplace equation, there are several solutions. Then, as the shell is a more complex construction, we ought to be tolerant of other approaches, of which there could be an assortment. As Ambartsumian wrote [48] "Apparently, the creation of a general theory of anisotropic laminar shells within the 3D problem of the theory of elasticity, which will be efficient for applications, is a rather difficult task, due to enormous mathematical complications. Thus, researchers are focusing on refined theories, correcting the classical ones. The main idea behind such an approach is reducing the 3D equations of an anisotropic body to 2D equations of the theory of shells." Other authors have recently developed some 3D techniques. For example, Fan et al. [143] stated: "In an entirely different way, Dong et al. [144, 145] directly developed quadrilateral 4-node, and hexahedral 8-node finite element models, for functionally graded and laminated structures based on the theory of 2D and 3D solid mechanics, respectively. Because traditional displacement-based lower-order elements suffer from shear locking, a technique of locking-alleviation was used by independently assuming locking-free element strains. Over-integration was also adapted in the thickness direction to evaluate the stiffness matrix. Similar work on smart composite structures was also presented in [146]. However, for very thick laminated structures with only a few layers, the computational accuracy is slightly compromised if only one linear finite element is used in the thickness direction. Fan et al. [143, 147] further developed very simple displacement-based 32-node hexahedral elements for static and dynamic analyses of composite laminates. Moreover, a stress-recovery approach was used to compute the distribution of transverse stresses by considering the equations of 3D elasticity." They conclude their work with the following statement: "Through a large number numerical results of static, free vibration, and transient analyses of various laminated plates and shells, it is demonstrated that the proposed CEH8 and DPH32 are capable of accurately and efficiently predicting the static and dynamical behaviors of composite laminates in a very simple and cost-effective manner. Because higher-order and layer-wise plate and shell theories involve (1) postulating very complex assumptions of plate/shell kinematics in the thickness direction, (2) defining generalized variables of displacements, strains, and stresses, and (3) developing very complex governing equilibrium, compatibility, and constitutive equations in terms of newly-defined generalized variables, while the currently proposed CEH8 and DPH32 merely involve displacement DOFs at each node, and rely only on the simple theory of solid mechanics, it is thus concluded by the authors that the development of higher-order or layer-wise theories are not entirely necessary for analyses of laminated plates and shells."

The present authors do not share the above strong claim about higher-order and zig-zag theories and that "Their industrial applications are thus hindered by their inherent complexity and the fact that it is difficult for end-users (front-line structural engineers) to completely understand all the newly-defined FEM DOF in higher-order and layer-wise theories." It appears that the two approaches, the higher-order, and zig-zag theories and their FEM implementation, on the one hand, and 3D finite approaches on the other, are complementary and can peacefully coexist. Indeed, as shown by Manea [148–153], Soler [154], and Zhavoronok [155] the 2D theories stem from 3D considerations. Likewise, the solution of 3D problems with expansions of Legendre polynomials leads to refined theories [155–158].

6.2 Multidimensional and multiscale problems

A typical example of a multidimensional problem is the analysis of reinforced shell structures for aerospace applications [159, 160]. In such structures, 1D, 2D, and 3D components coexist. Global responses are of interest for fast, preliminary design considerations or aeroelasticity and 1D and 2D classical models provide sufficient accuracy for such analyses. On the other hand, the accurate stress analysis with high-fidelity local fields requires the combined use of 1D, 2D, and 3D models. Particular attention appears to be necessary for modeling the interfaces between components of different dimensionalities, such as stringers and skins or spar caps and webs. To this purpose, structural models having pure displacements as primary unknowns should be preferred to ensure physically meaningful connections.

One of the most common examples of a multiscale problem in structural analysis is a composite component or bio-structures. At least three scales are of interest: the macro or structural scale, the mesoscale, and the microscale. For the macroscale, the guidelines for global responses are still valid, although shear deformation and couplings from stacking sequences should be considered carefully. The mesoscale considers the individual plies, and its modeling is necessary to compute the stress fields through the thickness, for instance, to detect the onset of damage. 3D provides good accuracy, but the number of degrees of freedom can be larger than 10^6 as at least three linear elements per layer are needed. The use of LW 2D or 1D models can provide the same accuracy with considerably lower costs. The microscale considers the individual constituents, e.g., fibers and matrix. 3D is the natural choice for this scale but limits the analysis to RVE with reduced fibers and makes nonlinear multiscale analyses extraordinarily cumbersome and limited to coupons. 2D classical models are usually used by neglecting the depth of the RVE and provide good accuracy, for instance, in the progressive failure within the matrix. The use of 1D HOST is promising as it allows the modeling of complete RVE with some 10^4 degrees of freedom [141].

6.3 Composite structures

The assessment of structural models for the analysis of composite structures follows the approach proposed in [128] and is based on five key capabilities.

Zig-zag displacements along the thickness

The computationally cheapest option is the ESL model with zig-zag functions. Mixed models, both ESL and LW, provide such capabilities as well as 3D.

Interlaminar continuity of transverse shear and normal stress

Mixed ESL and LW models are the only models providing this capability. Otherwise, increasing the expansion order can improve the interface stresses, although the continuity is not guaranteed. Similarly, the use of refined 3D meshes can reach similar results.

Accurate description of transverse normal stress

LW models or 3D are essential to obtain accurate distributions of σ_{zz}. As stated above, only mixed models can guarantee the interface continuity of this stress component.

Accurate transverse stress without integration of equilibrium equations

The use of LW is fundamental to obtain this capability. Mixed models provide the best accuracy but increasing the order of LW models without mixed features can improve the results significantly.

Minimization of the number of degrees of freedom

3D is the most expensive option due to the aspect ratio constraints in which the thickness of each ply plays a decisive role. 2D or 1D LW models are less costly but are still dependent on the number of layers. ESL is the cheapest solution as the number of variables is independent of the number of layers. As a general guideline based on a trade-off between accuracy and efficiency, a third-order ESL mixed model with zig-zag functions can be a robust option unless severe local effects are present as they require LW capabilities.

The order of the expansion is significant in any case. A third-order in-plane expansion and second-order transverse one may be considered the minimum requirement for a robust solution. The increase in the number of layers makes ESL less effective and, consequently, LW more necessary.

6.4 Thermo-electro-mechanical problems

In thermo-mechanical problems, the distribution of temperature inside the structure is decisive in the structural model's choice. The simplest case, in which a constant temperature profile is assigned, requires a linear distribution of the three displacement components and makes the classical models ineffective. In general, the distribution of temperature stems from the solution of a heat conduction problem, can have higher-order distributions, and can be layer-wise. The most demanding case is when thick structures are given as LW models, or 3D is necessary.

When electro- or thermo-electro-mechanical problems are considered, the considerations made above are still valid, but others are necessary as well. The use of piezo-patches or analysis of local responses with coupling between the electrical and mechanical systems requires a layer-wise description of the displacement or, at least, the separated modeling of the piezoelectric and the structural layers. If the focus is on accurate transverse electrical displacements and electrical charges for the sensor case, the modeling approach's minimum requirements are more severe. The direct, a-priori modeling of electrical displacement via mixed models is necessary.

In virtual manufacturing, in addition to the requirements stemming from thermal properties, LW capabilities are essential to capture residual transverse stress and estimating their effect on the geometry of the manufactured parts and their effective mechanical properties.

6.5 Local effects

The evaluation of local effects can be necessary because of the design of a given component within a large structure or for the presence of particular boundary conditions such as in contact and impact problems. Regardless of the problem, the proper evaluation of local effects demands accurate 3D stress fields. 3D models are the prime choice in this case, although there may be problems in which their use may be questionable. For instance, the presence of concentrated loads, leading to severe gradients of stress or singularities, requires extremely refined meshes on outstanding computational costs. HOST with LW capabilities can attenuate such drawbacks as the accuracy is increased via refined kinematics description without increasing mesh density. Moreover, the aspect ratio constraints in 2D and 1D models are less restrictive.

6.6 Dynamics

Among the others, the following cases were considered in this paper: thin-walled structures, rotordynamics, viscoelasticity, random vibrations, and wave propagation problems. FE for such problems may not be the best choice in some cases, but the considerations on the structural theories are still valid.

The global analysis of thin-walled structures can be carried you with classical 2D models. Complicating effects may be the presence of longitudinal and transverse stiffeners, the use of composite materials, complex geometries, and nonlinear effects such as the centrifugal stiffening in rotordynamics. All these effects may require the use of HOST, LW capabilities, or 3D. In most cases, the best practice may be a combination of multidimensional models with attention paid to the interface regions.

Viscoelasticity demands explicit modeling of layers with viscoelastic problems and the proper modeling of their shear deformability. Therefore, HOST with LW capabilities or 3D is necessary. In wave-propagation problems, accurate 3D deformations are essential to computing the 3D propagations of waves. HOST may be interfaced with 3D in the case of complex geometries or the presence of defects.

EBBT may not be safe for the design of structures considering random vibrations. At least TEBT should be chosen under acoustic loading. For the other spectral densities extending at higher frequencies, the higher-order theories ought to be resorted to. The study of random vibrations within higher-order theories is still awaiting its resolution.

6.7 Damage, failure and multiscale analysis

The prediction of the onset of damage via linear analyses and failure indexes is as accurate as the stress fields used for the index calculation. The availability of 3D stress fields allows the use of more advanced indexes and the analysis of regions with very complex local phenomena. A typical example is a free-edge analysis in which transverse stresses have severe gradients and peaks in areas as wide as the structure's thickness. 3D models or LW HOST are mandatory for meaningful results.

Similarly, the nonlinear analysis to estimate the propagation of damage exploits stress fields to update damage variables and the damaged region. Such analyses may be particularly demanding in composites as multiple damage mechanisms can coexist, and the component-wise modeling of meso- or micro-constituents is necessary. Although 3D can provide good accuracy, its computational costs are still too high to model complex structures. 1D and 2D HOST can alleviate such overheads without significant effects on the accuracy.

In the case of full multiscale nonlinear analyses, 1D HOST is very promising. They allow the 3D modeling of macro-, meso-, and microscale components with two orders of magnitude fewer degrees of freedom than 3D.

6.8 Discussion on computational costs and accuracy

The computational cost of a structural model can be evaluated via multiple parameters - e.g., runtime of the analysis and memory requirements - and stems from various features of the model - e.g., the number of DOF, parallelization, and integration procedures. Furthermore, the impact of computational costs can grow dramatically when moving from linear to nonlinear analyses. In this work, the primary parameter considered to draw guidelines is the number of DOF.

The use of DOF allows one to measure the computational cost independently of the machine used to run the analysis and the algorithms implemented. The latter can affect to a great extent the runtime in the nonlinear regime or when iterative procedures are needed as in contact analyses. Runtime is proportional to the size of the problem, although other aspects may affect it. For instance, the use of HOST may require more Gauss points for the integration procedures or may lead to ill-conditioning issues due to higher-order polynomials. For a given problem, the number of DOF tends to increase by one order of magnitude as the dimensionality increases. In other words, moving from 1D to 3D, there is an increase in DOF of two orders of magnitude [101]. Such behavior is valid when considering HOST for 1D and 2D models.

Table 5 presents numerical examples to compare HOST and 3D concerning the size of the FE models and their accuracy. The first column reports the problem considered and the type of HOST adopted. The second and third columns show the DOF of models used for comparisons, whereas, in the fourth one, the ratio between the two values is given. The fifth and sixth columns consider accuracy. The error is computed considering the more accurate model between HOST and 3D. When the convergence pattern is HOST $\rightarrow$ 3D, 3D is the reference and vice versa. The numerical values indicate that

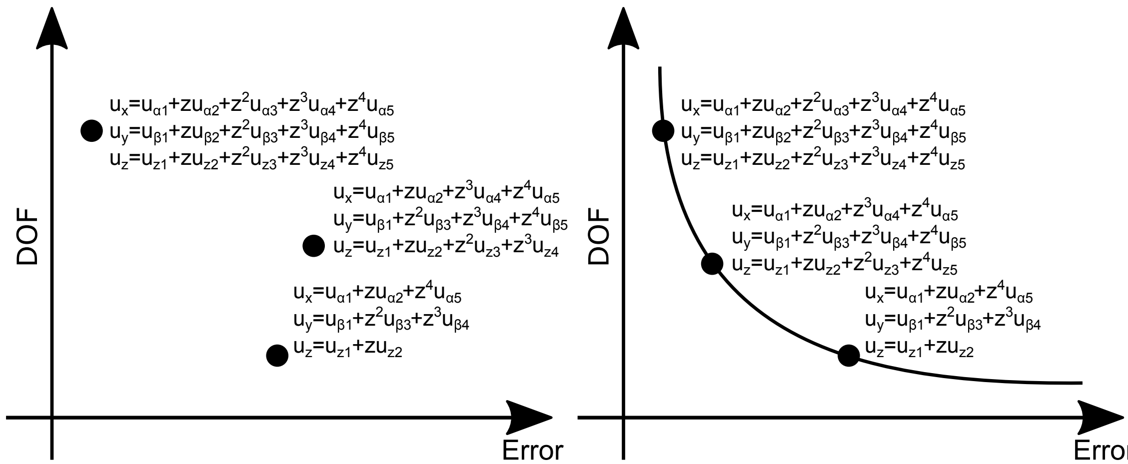
- In most cases, the HOST size is less than 10% the 3D one, and the differences regarding the accuracy minor than 5 %.
- In four cases, the 3D model tends to HOST. In other words, in those cases, the 3D model refinement is not enough to provide results as accurate as HOST. All cases considered multilayered structures.
- The analysis of free-edge effects seems particularly demanding for 3D with significant errors concerning stress and failure indexes.

7 Towards best theory diagrams and use of neural networks to assess structural theories

Over the last decade, a new method, referred to as the axiomatic/asymptotic method (AAM), has emerged as a tool to evaluate the accuracy and efficiency of any refined structural theory [62, 63, 166]. AAM obtains asymptotic-like results evaluating the unknown variables' relevance as problem parameters vary, e.g., thickness, orthotropic ratio, stacking sequence, and boundary conditions. A set of unknowns is set and, then, AAM evaluates the accuracy of all theories generated by their combinations. CUF generates the governing equations for the theories considered. Two parameters can identify a theory: the number of active terms and the error or accuracy provided. The use of two parameters allows the insertion of each expansion in a Cartesian reference frame, as in Fig. 8a. The Best Theory Diagram (BTD) is the curve composed of all models providing the minimum error with the least number of variables, see Fig. 8b.

Table 5: Comparison of computational costs and accuracy of HOST and 3D

Problem	HOST DOF	3D DOF	HOST/3D	Error	Variable and Convergence Pattern
Non-homogeneous atherosclerotic plaque [161] - Beam	23529	761244	3 %	3 %	Tranverse axial strain, HOST → 3D
Composite C-section spar [129] - Beam	89175	2276739	4 %	25 %	Free-edge peeling stress, 3D → HOST
Composite Omega stringer [135] - Beam	142476	4560150	3 %	30 %	Free-edge failure index, 3D → HOST
Composite notched specimen [137] - Plate	10000	10000000	0.1 %	3 %	Tensile peak stress, HOST → 3D
Multilayered beam [91] - Beam	23595	63210	37 %	0.4 %	Plastic strain, HOST → 3D
Double-swept blade [160] - Beam	13200	203808	6 %	1 %	Natural frequencies, HOST → 3D
Viscoelastic beam [97] - Beam	5475	56400	10 %	5 %	Modal loss factor, HOST → 3D
Randomly distributed RVE [140] - Beam	13642	31524	43 %	2 %	Local shear strain, HOST → 3D
Lattice structure [162] - Beam	13584	617580	2 %	1 %	Displacement, HOST → 3D
Three-point bending of a sandwich beam [134] - Beam	14229	201504	1 %	0 %	Transverse stress, HOST → 3D
Low-velocity impact on a bimetallic plate [163] - Plate	10659	856251	1 %	16 %	Plastic strain, 3D → HOST
Large deflections in asymmetric cross-ply beams [164] - Beam	5124	573675	1 %	7 %	Shear stress, HOST → 3D
Disbonding in sandwich beams [165] - Beam	41160	171888	24 %	1 %	Peak load, HOST → 3D
Curing of a composite part [142] - Beam	16569	599571	3 %	0 %	Spring-in angle, 3D → HOST



(a) Three shell theories (b) BTD

Figure 8: Representation of structural theories in a 2D Cartesian reference frame and an example of BTD

Given the accuracy, models with fewer variables than those on the BTD do not exist. Given the number of variables, models with better accuracy than those on the BTD do not exist.

The synergy between CUF and AAM is convenient as a tool for machine learning training [167, 168]. Neural networks are surrogate models for the fast mapping between given inputs and outputs. Structural theories and problem features are inputs, and accuracies serve as outputs. For instance, a fourth-order model with inactive terms and h/a = 0.1 has the following coding:

$$
\begin{aligned}
&u_\alpha = u_{\alpha_1} + z\, u_{\alpha_2} + z^4\, u_{\alpha_5} \\
&u_\beta = u_{\beta_1} + z\, u_{\beta_2} + z^3\, u_{\beta_4} \quad => \quad [1\,1\,1\,1\,1\,1\,0\,0\,1\,0\,1\,0\,1\,0\,0\,0.1] \\
&u_z = u_{z_1} + z\, u_{z_2} + z^2\, u_{z_3}
\end{aligned}
\tag{19}
$$

The first fifteen terms of the array refer to the generalized displacement variables, and the last one to the thickness ratio. Figure 9 shows the classical approach based on CUF and FE to assess the accuracy and efficiency of a structural theory and the one in which NN substitutes FE, namely:

- CUF generates the governing FE equations for all the shell theories stemming from subsets of the fourth-order expansions. Given that the expansion has fifteen terms, overall, 2^{15} FE shell models are available. For instance, FSDT is one of these models in which five terms are active - u_{α_1}, u_{β_1}, u_{z1}, u_{α_2}, and u_{β_2} - and ten inactive.

- The FE approach runs 2^{15} FE analyses and reports the error and number of active terms of each case in a 2D plot.

- The NN approach runs one-tenth of the FE analyses and uses them for training. Then, the 2D plot stems from querying the trained NN with all 2^{15} shell models.

- If a/h is a training variable, and, e.g., three a/h values are available, the overall number of analyses is 3×2^{15}. The query of the NN includes the shell model and the thickness ratio.

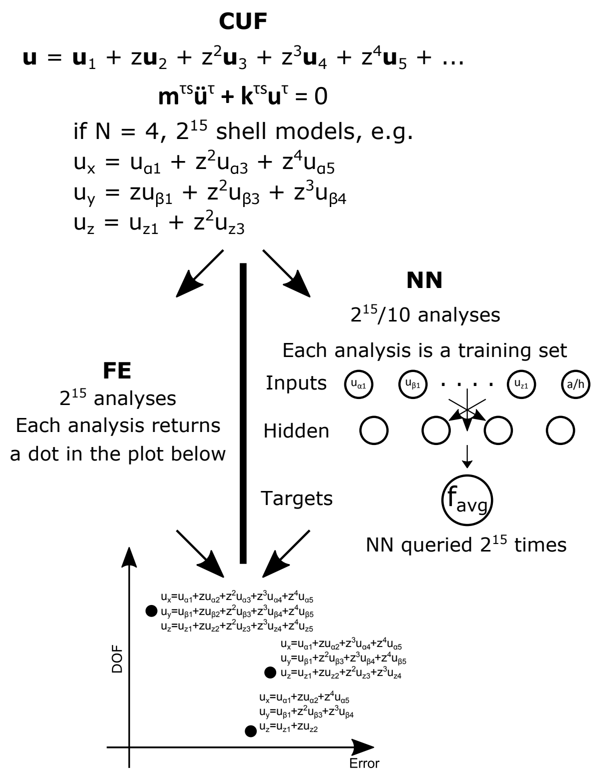


Figure 9: CUF and NN.

NN's use can facilitate the evaluation and development of refined structural theories as it can provide estimations on the influence of unknown variables and design parameters with reduced computational efforts.

8 Conclusion

This paper provides insights on the role of higher-order structural theories (HOST) for accurate predictions and efficient implementations and considering multiple structural problems. A brief historical overview of the primary contributions to structural theories' development is given together with indications on the major approaches that emerged over the last decades. Some of the most used methods to build HOST are presented both for metallic and composite structures. The requirements of structural theories to ensure reliable results for various applications are enlisted with estimations on accuracy. The convenience of 1D and 2D HOST as compared to 3D models is discussed. The main findings can be summarized as follows:

- The use of classical 1D and 2D theories remains valid in many cases, but there are numerous cases of engineering interest in which their use may lead to significant errors.
- The convenience of 1D or 2D HOST over 3D models is evident in most cases and due to superior efficiency regarding the computational model's size.
- There are not cases in which 3D cases are unavoidable, although efforts are required to make highly refined 2D or 1D models appealing for industrial applications.
- In a few cases, e.g., multiscale analyses, the use of 1D HOST seems the only option to tackle complex structures with bearable computational costs.

The last section of this paper presented a new approach to estimate the accuracy and efficiency of any structural theory via the use of machine learning algorithms.

References

[1] J.A.C. Bresse. *Cours de mécanique appliquée – Résistance des matériaux et stabilité des constructions.* Paris, Gauthier-Villars, 1859.

[2] J.W. Strutt (Lord Rayleigh). *The Theory of Sound.* Dover, New York, 1945.

[3] J.V. Poncelet. *Mécanique appliquée aux machines.* Paris, 1874.

[4] A.N. Bogoliubov. *Jean-Victor Poncelet.* Moscow: "Nauka" Publishing House, 1988.

[5] A.R.E. Oliveira. Poncelet and the development of applied mechanics. In Teun Koetsier and Marco Ceccarelli, editors, *Explorations in the History of Machines and Mechanisms*, pages 475–490, Dordrecht, 2012. Springer Netherlands.

[6] W.Y.M. Rankine. *A Manual of Applied Mechanics.* London: R. Griffin and Company Limited, 1858.

[7] W.Y.M. Rankine. *Miscellaneous Scientific Papers.* London: R. Griffin and Company Limited, 1881.

[8] R.V. Southwell. W.J. Rankine, a commemorative lecture, delivered on 12 december, 1955, in Glasgow. In *Proceedings of the Institution of Civil Engineers, Part I*, volume 5, pages 177–193, 1956.

[9] H.B. Sutherland. Prof. William John Macquorn Rankine. In *Proceedings of the Institution of Civil Engineers*, volume 132, pages 181–187, 1999.

[10] F. Grashof. *Die Festigkeitslehre mit besonderer Rüchsicht auf die Bedürfnisse des Maschinenbaues.* Berlin: Verlag von R. Gärtner, 1866.

[11] F. Grashof. *Theorie der Elasticität und Festigkeit mit Bezug auf ihre Anwendngen in der Technik.* Berlin, 1878.

[12] F. Grashof. *Elasticität und Festigkeit.* Berlin, 1878.

[13] F. Engesser. Über die knickfestigkeit gerader stäbe. *Zeitschrift für Architekten und Ingenieur Vereins zu Hannover*, 35:455–462, 1889.

[14] F. Engesser. Über knickfestigkeit gerader stäbe. *Zeitschrift des Architekten – und Ingenieur Vereins zu Hannover*, 35:733–744, 1889.

[15] F. Engesser. Die knickfestigkeit gerader stäbe. *Zentralbatt des Bauverwaltung*, 11:483–486, 1889.

[16] F. Engesser. Über die berechnung auf knickfestigkeit beanspruchten stäbe aus scheuss-und gusseisen. *Zeitschrift für Österreichische Ingenieur Archiv Vereins*, 45:506–508, 1893.

[17] F. Engesser. Über knickfragen. *Schweizarische Bauzeitung*, 26:24–33, 1895.

[18] F. Engesser. Über die berechnung gerader stäbe. *Zentralblatt Bauverwaltung*, 27:606–607, 1907.

[19] F. Engesser. Knicksicherheit von gitterstäben. *VDI-Zeitschrift*, 52:359–360, 1908.

[20] E. Gaber. Friedrich Engesser. *Die Bautechnik*, 45:664–664, 1931.

[21] O. Steinhardt. Errinerungen an Friedrich Engesser. *Die Bautechnik*, 75(2):109 – 111, 1998.

[22] J. Wauer. *Die Mechanik und ihre Fachvertreter an der Universität Karlsruhe*. KIT Science Publishing, 2017.

[23] A. Föppl. *Vorlesungen über Technische Mechanik – Dritter Band: Festigkeitslehre*. Leipzig: B.G. Teubner, 1897.

[24] A. Föppl. *Theory of Strength of Materials and Theory of Elasticity*. St. Petersburg, 1901.

[25] S.P. Timoshenko. *A Course of Elasticity Theory. Part 2: Rods and Plates*. St. Petersburg: A.E. Collins Publishers, 1916.

[26] S.P. Timoshenko. On the additional deflection due to shearing. *Herald of the Croatian Nature Association*, 33(1):50 – 52, 1921.

[27] S.P. Timoshenko. On the transverse vibrations of bars of uniform cross section. *Philosophical Magazine*, 43:125–131, 1922.

[28] E.I. Grigolyuk. *S.P. Timoshenko: Life and Destiny*. Moscow: Aviation Institute Press, 2002.

[29] I. Elishakoff. *Handbook on Timoshenko-Ehrenfest Beam and Uflyand-Mindlin Plate Theories*. WORLD SCIENTIFIC, 2019.

[30] I. Elishakoff. Who developed the so-called Timoshenko beam theory? *Mathematics and Mechanics of Solids*, 25(1):97–116, 2020.

[31] J.N. Reddy and C.F. Liu. A higher-order shear deformation theory of laminated elastic shells. *International Journal of Engineering Science*, 23(3):319 – 330, 1985.

[32] J.N. Reddy and A.R. Srinivasa. Misattributions and misnomers in mechanics: Why they matter in the search for insight and precision of thought. *Vietnam Journal of Mechanics*, 42(3):283–291, 2020.

[33] A.B. Basset. Vi. on the extension and flexure of cylindrical and spherical thin elastic shells. *Philosophical Transactions of the Royal Society of London. (A.)*, 181:433–480, 1890.

[34] E. Reissner. On the theory of bending of elastic plates. *Journal of Mathematics and Physics*, 23(1-4):184–191, 1944.

[35] E. Reissner. The effect of transverse shear deformation on the bending of elastic plates. *Journal of Applied Mechanics*, 12:69–76, 1945.

[36] E. Reissner. On bending of elastic plates. *Quarterly of Applied Mathematics*, 5(1):55–68, 1947.

[37] H. Hencky. Über die berücksichtigung der schubverzerrung in ebenen platten. *Ingenieur-Archiv*, 16:72–76, 1947.

[38] D. Bolle. Contribution au problème linéaire de flexion d'une plaque èlastique. *Bulletin. Technique de la Suisse Romande*, 21-22, 1947.

[39] F.B. Hildebrand, E. Reissner, and G.B. Thomas. *Notes on the Foundations of the Theory of Small Displacements of Orthotropic Shells, Technical Note 1833*, 1949.

[40] Y.S. Uflyand. Wave propagation by transverse vibrations of beams and plates. *PMM: Journal of Applied Mathematics and Mechanics*, 12:287–300, 1948.

[41] R.D. Mindlin. Influence of rotatory inertia and shear in flexural motions of isotropic elastic plates. *Journal of Applied Mechanics*, 18:1031–1036, 1951.

[42] A.K. Galinsh. Computation of plates and shells using refined theories. *Issledovaniia po Teorii Plastin i Obolochek*, 6-7:23–64, 1970.

[43] E.I. Grigolyuk and I.T. Selezov. Nonclassical theoreis of vibration for beams, plates and shells. *Advances in Science and Technology, Series. Mechanics of Deformable Solids*, 5:69–69, 1973.

[44] J.N. Reddy. A review of refined theories of laminated composite plates. *The Shock and Vibration Digest*, 22(7):3–17, 1990.

[45] K. Kapania and S. Raciti. Recent advances in analysis of laminated beams and plates, part I: Shear effects and buckling. *AIAA Journal*, 27(7):923–935, 1989.

[46] Mallikarjuna and T. Kant. A critical review and some results of recently developed refined theories of fiber-reinforced laminated composites and sandwiches. *Composite Structures*, 23(4):293 – 312, 1993.

[47] Y.M. Ghugal and R.P. Shimpi. A review of refined shear deformation theories for isotropic and anisotropic laminated beams. *Journal of Reinforced Plastics and Composites*, 20(3):255–272, 2001.

[48] S.A. Ambartsumian. Nontraditional theories of shells and plates. *Applied Mechanics Reviews*, 55(5):R35–R44, 2002.

[49] E. Carrera. Theories and finite elements for multilayered, anisotropic, composite plates and shells. *Archives of Computational Methods in Engineering*, 9(2):87–140, 2002.

[50] J.N. Reddy and R.A. Arciniega. Shear deformation plate and shell theories: From stavsky to present. *Mechanics of Advanced Materials and Structures*, 11(6):535–582, 2004.

[51] B.D. Annin and Y.M. Volchkov. Nonclassical models of the theory of plates and shells. *Journal of Applied Mechanics and Technical Physics*, 57:769–776, 2016.

[52] E. Carrera. Historical review of zig-zag theories for multilayered plates and shells. *Applied Mechanics Review*, 56:287–308, 2003.

[53] G.J. Simitses. Buckling of moderately thick laminated cylindrical shells: a review. *Composites Part B: Engineering*, 27(6):581 – 587, 1996.

[54] P.R. Heyliger. When beam theories fail. *Journal of Mechanics of Materials and Structures*, 8(1):15–35, 2013.

[55] A.L. Cauchy. Sur l'équilibre et le mouvement d'une plaque solide. *Exercises de Matematique*, 3:328–355, 1828.

[56] S.D. Poisson. Memoire sur l'équilibre et le mouvement des corps elastique. *Mem. Acad. Sci.*, 1829.

[57] A.E.H. Love. *The Mathematical Theory of Elasticity.* Cambridge Univ Press, fourth edition, 1927.

[58] G. Kirchhoff. Uber das gleichgewicht und die bewegung einer elastischen scheibe. *Journal fur reins und angewandte Mathematik*, 40:51–88, 1850.

[59] I. Elishakoff, J. Kaplunov, and E. Nolde. Celebrating the Centenary of Timoshenko's Study of Effects of Shear Deformation and Rotary Inertia. *Applied Mechanics Reviews*, 67(6), 12 2015.

[60] N. Challamel and I. Elishakoff. A brief history of first-order shear-deformable beam and plate models. *Mechanics Research Communications*, 102:103389, 2019.

[61] K. Washizu. *Variational Methods in Elasticity and Plasticity.* Pergamon, Oxford, 1968.

[62] E. Carrera and M. Petrolo. Guidelines and recommendation to construct theories for metallic and composite plates. *AIAA Journal*, 48(12):2852–2866, 2010.

[63] E. Carrera and M. Petrolo. On the effectiveness of higher-order terms in refined beam theories. *Journal of Applied Mechanics*, 78, 2011.

[64] A.L. Gol'denweizer. *Theory of thin elastic shells.* International Series of Monograph in Aeronautics and Astronautics. Pergamon Press, New York, 1961.

[65] P. Cicala. *Systematic approximation approach to linear shell theory.* Levrotto e Bella, Torino, 1965.

[66] A.L. Goldenveiser. Construction of the approximate theory of plate bending via the method of asymptotic integration of theory of elasticity equations. *PMM: Journal of Applied Mathematics and Mechanics*, 26:662–686, 1962.

[67] A.L. Goldenveiser, J.D. Kaplunov, and E.V. Nolde. On Timoshenko-Reissner type theories of plates and shells. *International Journal of Solids and Structures*, 30:675–694, 1993.

[68] A.L. Aghalovyan. Application of asymptotic integration method of the construction of approximate theory of anisotropic shells. *PMM: Journal of Applied Mathematics and Mechanics*, 30(3), 1966.

[69] A.L. Aghalovyan. *Asymptotic Theory of Anisotropic Plates and Shells.* Moscow: "Nauka" Publishing House, 1997.

[70] V.L. Berdichevsky. *Variational Principles in Continuum Mechanics.* Berlin: Springer, 2009.

[71] M.F. Mekhtiev. *Vibrations of Hollow Elastic Bodies.* Springer Verlag, Berlin, 2018.

[72] S.I. Zhavoronok. A Vekua-type linear theory of thick elastic shells. *ZAMM: Zeitschrift für angewandte Mathematik und Mechanik*, 94(1–2):164–184, 2014.

[73] H. Kraus. *Thin elastic shells.* John Wiley, 1967.

[74] S.P. Timoshenko. On the corrections for shear of the differential equation for transverse vibrations of prismatic bars. *Philosophical Magazine*, 41:744–746, 1922.

[75] A.K. Noor and J.M. Peters. A posteriori estimates for the shear correction factors in multilayered composite cylinders. *Journal of Engineering Mechanics*, 115(6):1225–1244, 1989.

[76] L. Euler. *De curvis elasticis*, chapter Methodus Inveniendi Lineas Curvas Maximi Minimive Proprietate Gaudentes, Sive Solutio Problematis Isoperimetrici Lattissimo Sensu Accept. Bousquet, 1744.

[77] K. Washizu. *Variational Methods in Elasticity and Plasticity.* Pergamon, Oxford, 1968.

[78] C. Ossadzow and M. Touratier. An improved shear-membrane theory for multilayered shells. *Composite Structures*, 52(1):85 – 95, 2001.

[79] S.I. Zhavoronok. Kinematics of normal modes of elastic layer for some wavenumbers investigation based on N-th order three-dimensional shells' theory. *Composite Mechanics and Design*, 181:45–56, 2012.

[80] O.B. Egorova, S.I. Zhavoronok, and A.S. Kurbatov. About variational equations of the expanded theory of N-th order of elastic shells and their application to some dynamics problems. *PNRPU Mechanics Bulletin*, 2:36–59, 2015.

[81] L. Hadji, T.H. Daouadji, A. Tounsi, and E.A. Bedia. A N-order refined theory for bending and free vibration of functionally graded beams. *Structural Engineering and Mechanics*, 54(5):923–936, 2015.

[82] E. Carrera. Theories and finite elements for multilayered plates and shells: a unified compact formulation with numerical assessment and benchmarking. *Archives of Computational Methods in Engineering*, 10(3):216–296, 2003.

[83] V.L. Berdichevsky. Variational-asymptotic method of shell theory construction. *PMM*, 43:664–667, 1979.

[84] E. Carrera. Transverse normal strain effect on thermal stress analysis of homogeneous and layered plates. *AIAA Journal*, 43(10):2232–2242, 2005.

[85] E. Carrera. An assessment of mixed and classical theories for the thermal stress analysis of orthotropic multilayered plates. *Journal of Thermal Stresses*, 23(9):797–831, 2000.

[86] E. Carrera. Temperature profile influence on layered plates response considering classical and advanced theories. *AIAA Journal*, 40(9):1885–1896, 2002.

[87] D. Ballhause, M. D'Ottavio, B. Kröplin, and E. Carrera. A unified formulation to assess multilayered theories for piezoelectric plates. *Computers and Structures*, 83(15):1217 – 1235, 2005.

[88] E. Carrera and P. Nali. Mixed piezoelectric plate elements with direct evaluation of transverse electric displacement. *International Journal for Numerical Methods in Engineering*, 80(4):403–424, 2009.

[89] E. Carrera, M. Boscolo, and A. Robaldo. Hierarchic multilayered plate elements for coupled multifield problems of piezoelectric adaptive structures: Formulation and numerical assessment. *Archives of Computational Methods in Engineering*, 14:383–430, 2007.

[90] M.H. Nagaraj, I. Kaleel, E. Carrera, and M. Petrolo. Nonlinear analysis of compact and thin-walled metallic structures including localized plasticity under contact conditions. *Engineering Structures*, 203:109819, 2020.

[91] I. Kaleel, M. Petrolo, E. Carrera, and A.M. Waas. On the effectiveness of higher-order one-dimensional models for physically nonlinear problems. *In: Advances in Predictive Models and Methodologies for Numerically Efficient Linear and Nonlinear Analysis of Composites*, 2019.

[92] E. Carrera and A. Varello. Dynamic response of thin-walled structures by variable kinematic one-dimensional models. *Journal of Sound and Vibration*, 331(24):5268 – 5282, 2012.

[93] J.C. Samuels and A.C. Eringen. Response of a simply supported Timoshenko beam to a purely random Gaussian process. *Journal of Applied Mechanics*, 25:496–500, 1958.

[94] S.H. Crandall and A. Yildiz. Random Vibration of Beams. *Journal of Applied Mechanics*, 29(2):267–275, 06 1962.

[95] I. Elishakoff and E. Lubliner. Random vibration of a structure via classical and nonclassical theories. In Sigge Eggwertz and Niels C. Lind, editors, *Probabilistic Methods in the Mechanics of Solids and Structures*, pages 455–467, Berlin, Heidelberg, 1985. Springer Berlin Heidelberg.

[96] E. Carrera, E. M. Filippi, and E. Zappino. Analysis of Rotor Dynamic by One-Dimensional Variable Kinematic Theories. *Journal of Engineering for Gas Turbines and Power*, 135(9), 07 2013. 092501.

[97] M. Filippi, E. Carrera, and A.M. Regalli. Layerwise Analyses of Compact and Thin-Walled Beams Made of Viscoelastic Materials. *Journal of Vibration and Acoustics*, 138(6), 07 2016. 064501.

[98] A.G. de Miguel, A. Pagani, and E. Carrera. Higher-order structural theories for transient analysis of multi-mode lamb waves with applications to damage detection. *Journal of Sound and Vibration*, 457:139 – 155, 2019.

[99] E. Carrera and M. Filippi. Variable Kinematic One-Dimensional Finite Elements for the Analysis of Rotors Made of Composite Materials. *Journal of Engineering for Gas Turbines and Power*, 136(9), 04 2014. 092501.

[100] A. Varello and E. Carrera. Free Vibration Response of Thin and Thick Nonhomogeneous Shells by Refined One-Dimensional Analysis. *Journal of Vibration and Acoustics*, 136(6), 08 2014. 061001.

[101] E. Carrera, M. Cinefra, M. Petrolo, and E. Zappino. *Finite Element Analysis of Structures through Unified Formulation.* John Wiley & Sons, Chichester, 2014.

[102] G. Li, E. Carrera, M. Cinefra, A.G. de Miguel, G.M. Kulikov, A. Pagani, and E. Zappino. Evaluation of shear and membrane locking in refined hierarchical shell finite elements for laminated structures. *Advanced Modeling and Simulation in Engineering Sciences*, 6(8), 2019.

[103] I. Elishakoff, M. Baruch, L. Zhu, and R. Caimi. Random vibration of space shuttle weather protection systems. *Shock and Vibration*, 2, 1995.

[104] I. Elishakoff. *An Equation Both More Consistent and Simpler Than the Bresse-Timoshenko Equation*, pages 249–254. Springer Netherlands, Dordrecht, 2009.

[105] Y.K. Lin. *Probabilistic Theory of Structural Dynamics.* New York: McGraw-Hill, 1967.

[106] I. Elishakoff. *Probabilistic Methods in the Theory of Structures.* Singapore: World Scientific, 2017.

[107] E.I. Grigoliuk and E.A. Kogan. State of the art of the theory of multilayered shells. *International Applied Mechanics*, 8(6):583–595, 1972.

[108] E.I. Grigoliuk and G.M. Kulikov. General direction of development of the theory of multilayered shells. *Mechanics of Composite Materials*, 24(2):231–241, 1988.

[109] R. Khandan, S. Noroozi, P. Sewell, and J. Vinney. The development of laminated composite plate theories: A review. *Journal of Material Sciences*, 47(16):5901–5910, 2012.

[110] S.G. Lekhnitskii. Strength calculation of composite beams. *Vestnik inzhenerov i tekhnikov (Herald of Engineers and Technologists)*, 9, 1935.

[111] S.A. Ambartsumian. On a general theory of anisotropic shells. *Journal of Applied Mathematics and Mechanics*, 22(2):305 – 319, 1958.

[112] H. Murakami. Laminated composite plate theory with improved in-plane response. *Journal of Applied Mechanics*, 53:661–666, 1986.

[113] A.R. Rzhanitsyn. *Theory of Compound Bars.* Moscow: Stroiizdat Publishing, 1948.

[114] A.R. Rzhanitsyn. *Compound Bars and Plates.* Moscow: Stroiizdat Publishing, 1986.

[115] V.V. Filatov. Design of beams via the theory of compound bars by A.R. Rzhanitsyn. *Vestnik MGSU (Herald of Moscow State University of Civil Engineering)*, 9:23–29, 2013.

[116] V.V. Bolotin. Vibration of layered elastic plates (bending, vibration of alternate layers of soft and hard elastic plates). *Proceedings of Vibration Problems*, 4(4):331–346, 1963.

[117] V.V. Bolotin and Y.N. Novichkov. *Mechanics of Multilayered Structures.* Moscow: Mashinostoenie Publishing, 1980.

[118] J.N. Reddy. *Mechanics of laminated composite plates and shells. Theory and Analysis.* CRC Press, 2^{nd} edition, 2004.

[119] I.N. Vekua. *Shell Theory, General Methods of Construction.* London: Pitman Advanced Publishing Program, 1985.

[120] I.Y. Khoma. *Generalized Theory of Anisotropic Shells.* Kiev: "Naukova Dumka" Publishers, 1987.

[121] A.R. Ulukhanyan. Dynamic equations of the theory of thin prismatic bodies with expansion in the system of Legendre polynomials. *Mechanics of Solids*, 46(3):467–479, 2011.

[122] S.I. Zhavoronok. On the variational formulation of the extended thick anisotropic shells theory of I.N. Vekua type. *Procedia Engineering*, 111:888–895, 2015.

[123] E. Reissner. On a mixed variational theorem and on shear deformable plate theory. *International Journal for Numerical Methods in Engineering*, 23(2):193–198, 1986.

[124] E. Reissner. On a certain mixed variational theorem and on laminated elastic shell theory. In I. Elishakoff and H. Irretier, editors, *Refined Dynamical Theories of Beams, Plates and Shells and Their Applications*, pages 17–27, Berlin, Heidelberg, 1987. Springer Berlin Heidelberg.

[125] W.T. Koiter. A consistent first approximation in the general theory of thin elastic shells. *Proceedings of Symposium on the Theory of Thin Elastic Shells, August 1959, North-Holland, Amsterdam*, pages 12–23, 1959.

[126] E. Carrera. Transverse Normal Stress Effects in Multilayered Plates. *Journal of Applied Mechanics*, 66(4):1004–1012, 12 1999.

[127] E. Carrera. A study of transverse normal stress effect on vibration of multilayered plates and shells. *Journal of Sound and Vibration*, 225(5):803 – 829, 1999.

[128] E. Carrera. An assessment of mixed and classical theories on global and local response of multilayered orthotropic plates. *Composite Structures*, 50(2):183 – 198, 2000.

[129] A.G. de Miguel, A. Pagani, and E. Carrera. Free-edge stress fields in generic laminated composites via higher-order kinematics. *Composites Part B: Engineering*, 168:375 – 386, 2019.

[130] S.A. Ambartsumian. Specific of theory of shells of modern materials. *Proceedings of the Armeinian Academy of Sciences*, 21(4):1–19, 1968.

[131] U. Nigul. Contrasting of analysis results of transient processes in shells and plates via theory of elasticity and approximate theories. *PMM: Journal of Applied Mathematics and Mechanics*, 33(2):308–322, 1969.

[132] E. Carrera. A priori vs. a posteriori evaluation of transverse stresses in multilayered orthotropic plates. *Composite Structures*, 48(4):245 – 260, 2000.

[133] E. Carrera and A. Ciuffreda. Bending of composites and sandwich plates subjected to localized lateral loadings: a comparison of various theories. *Composite Structures*, 68(2):185 – 202, 2005.

[134] M.H. Nagaraj, I. Kaleel, E. Carrera, and M. Petrolo. Contact analysis of laminated structures including transverse shear and stretching. *European Journal of Mechanics - A/Solids*, 80:103899, 2020.

[135] A.G. de Miguel, I. Kaleel, M.H. Nagaraj, A. Pagani, M. Petrolo, and E. Carrera. Accurate evaluation of failure indices of composite layered structures via various fe models. *Composites Science and Technology*, 167:174 – 189, 2018.

[136] S.E. Stapleton, B. Stier, S. Jones, A. Bergan, I. Kaleel, M. Petrolo, E. Carrera, and B.A. Bednarcyk. A critical assessment of design tools for stress analysis of adhesively bonded double lap joints. *Mechanics of Advanced Materials and Structures*, 0(0):1–21, 2019.

[137] M.H. Nagaraj, J. Reiner, R. Vaziri, E. Carrera, and M. Petrolo. Progressive damage analysis of composite structures using higher-order layer-wise elements. *Composites Part B: Engineering*, 190:107921, 2020.

[138] I. Kaleel, M. Petrolo, A.M. Waas, and E. Carrera. Computationally efficient, high-fidelity micromechanics framework using refined 1d models. *Composite Structures*, 181:358 – 367, 2017.

[139] I. Kaleel, M. Petrolo, A. M. Waas, and E. Carrera. Micromechanical Progressive Failure Analysis of Fiber-Reinforced Composite Using Refined Beam Models. *Journal of Applied Mechanics*, 85(2), 12 2017. 021004.

[140] I. Kaleel, M. Petrolo, E. Carrera, and A. M. Waas. Computationally efficient concurrent multiscale framework for the linear analysis of composite structures. *AIAA Journal*, 57(9):4019–4028, 2019.

[141] I. Kaleel, M. Petrolo, E. Carrera, and A. M. Waas. Computationally efficient concurrent multiscale framework for the nonlinear analysis of composite structures. *AIAA Journal*, 57(9):4029–4041, 2019.

[142] E. Zappino, N. Zobeiry, M. Petrolo, R. Vaziri, E. Carrera, and A. Poursartip. Analysis of process-induced deformations and residual stresses in curved composite parts considering transverse shear stress and thickness stretching. *Composite Structures*, 241:112057, 2020.

[143] Q. Fan, Y. Zhang, L. Dong, S. Li, and S.N. Atluri. Are higher-order theories and layer-wise zig-zag theories necessary for N-layer composite laminates? *Computer Modeling in Engineering & Sciences*, 107(2):155–186, 2015.

[144] L. Dong, A.S. El-Gizawy, K.A. Juhany, and S.N. Atluri. A simple locking-alleviated 4-node mixed-collocation finite element with over-integration, for homogeneous or functionally-graded or thick-section laminated composite beams. *Computers, Materials & Continua*, 40(1):49–78, 2014.

[145] L. Dong, A.S. El-Gizawy, K.A. Juhany, and S.N. Atluri. A simple locking-alleviated 3D 8-node mixed-collocation C0 finite element with over-integration, for functionally-graded and laminated thick-section plates and shells, with & without Z-pins. *Computers, Materials & Continua*, 41(3):163–192, 2014.

[146] S.N. Atluri M.C. Ray, L. Dong. Simple efficient smart finite elements for the analysis of smart composite beams. *Computers, Materials & Continua*, 47(3):143–177, 2015.

[147] Y. Zhang, Q. Fan, L. Dong, and S.N. Atluri. Are Higher-Order and Layer-wise Zig-Zag Plate and Shell Theories Necessary for Functionally Graded Materials and Structures? *CMES: Computer Modeling in Engineering and Sciences*, 111(6):567–585, 2016.

[148] V. Manea. Calcul placilor circulare elastice subtiri in teoria fara ipoteza Love — Kirchhoff. *Studii si cercetari mec. apl. Acad. RPR*, 14(6), 1963.

[149] V. Manea. Consideratii a supra teoriei placilor plane elastice subtiri. *Studii si cercetari mec. apl. Acad. RPR*, 14(4), 1963.

[150] V. Manea. Construirea solutiei ecuatiilor placilor plane elastice subtiri, din teoria fara ipoteza Love — Kirchhoff, prin metoda functiilor analitice. *Studii si cercetari mec. apl. Acad. RPR*, 14(5), 1963.

[151] V. Manea. On the theory of elastic plane plates of mean thickness. *Mecanique Appliquee, Rev. Roumaine des Sci. techn.*, 9(6), 1964.

[152] V. Manea. Vibratiile placilor, plane subtiri elastice in teoria fara ipoteza Love — Kirchhoff. *Studii si cercetari mec. apl. Acad. RPR*, 15(2), 1964.

[153] V. Manea. Cateva probleme ale teoriei placilor plane elastice. *Bucuresti, Acad. RSR*, 1966.

[154] A.I. Soler. Higher-order theories for structural analysis using Legendre polynomial expansions. *Journal of Applied Mechanics*, 36(4):757 – 762, 1969.

[155] S.I. Zhavoronok. Variational formulations of Vekua-type shell theories and some their applications. In *Shell Structures: Theory and Applications (W. Pietraszkiewicz and J. Gorski, eds.)*, volume 3, pages 341–344, 2013.

[156] V.V. Poniatovsky. Theory of plates and shells of moderate thickness. *Journal of Applied Mathematics and Mechanics*, 26(2), 1962.

[157] V.V. Poniatovsky. Towards the theory of bending of anisotropic plates. *Journal of Applied Mathematics and Mechanics*, 28(6), 1964.

[158] E. Carrera and V.V. Zozulya. Carrera unified formulation (cuf) for the micropolar plates and shells. i. higher order theory. *Mechanics of Advanced Materials and Structures*, 2020.

[159] E. Zappino and E. Carrera. Multidimensional model for the stress analysis of reinforced shell structures. *AIAA Journal*, 56(4):1647–1661, 2018.

[160] M. Filippi, E. Zappino, and E. Carrera. Multidimensional models for double-swept helicopter blades. *AIAA Journal*, 57(6):2609–2616, 2019.

[161] A. Varello and E. Carrera. Nonhomogeneous atherosclerotic plaque analysis via enhanced 1d structural models. *Smart Structures and Systems*, 13(4):659–683, 2014.

[162] E. Carrera, A. Pagani, D. Giusa, and R. Augello. Nonlinear analysis of thin-walled beams with highly deformable sections. *International Journal of Non-Linear Mechanics*, 128:103613, 2021.

[163] M.H. Nagaraj, E. Carrera, and M. Petrolo. Progressive damage analysis of composite laminates subjected to low-velocity impact using 2D layer-wise structural models. *International Journal of Non-Linear Mechanics*, 127:103591, 2020.

[164] A. Pagani and E. Carrera. Large-deflection and post-buckling analyses of laminated composite beams by Carrera unified formulation. *Composite Structures*, 170:40 – 52, 2017.

[165] L. Qi, M.H. Nagaraj, E. Carrera, C.F. Gao, and M. Petrolo. Numerical analysis of disbonding in sandwich structures using 1d finite elements. *Composite Structures*, 252:112717, 2020.

[166] M. Petrolo and E. Carrera. Methods and guidelines for the choice of shell theories. *Acta Mechanica*, 231:395–434, 2020.

[167] M. Petrolo and E. Carrera. On the use of neural networks to evaluate performances of shell models for composites. *Adv. Model. and Simul. in Eng. Sci.*, 7(31), 2020.

[168] M. Petrolo and E. Carrera. Selection of element-wise shell kinematics using neural networks. *Computers and Structures*, 244:106425, 2021.